Dynamics of physical systems , normal forms and Markov chains

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Mis en page avec la classe thesul.

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Albert Einstein.

À mes parents

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Introduction

Merci d'avoir entamé la lecture de cette thèse qui contient cinq chapitres indépendants, néanmoins tous appartenant au domaine des systèmes dynamiques.

Nous allons découvrir le comportement asymptotique de certains systèmes dynamiques de provenance physique (chapitres 1 et 2), étudier la géométrie du billard elliptique (chapitre 3) ainsi qu'étudier des questions de la théorie ergodique des actions du groupe libre (chapitre 4) et de la théorie des formes normales (chapitre 5).

Malgré la diversité des sujets il y a un esprit commun à tous ces chapitres, notre but étant de comprendre le comportement asymptotique des systèmes dynamiques. Y a-t-il un moyen d'exprimer l'évolution d'un système dynamique sur un temps long ? Est-il possible de quantifier ce comportement asymptotique ?

Dans la première moitié de cette thèse (chapitres 1, 2 et 3) nous nous concentrons sur des systèmes dynamiques individuels. Nous étudions trois systèmes dynamiques particuliers et nous considérons les problèmes géométriques liés à leur comportement limite.

$\boxed{1.}$ Dans le premier chapitre nous étudions une famille spéciale de champs de vecteurs sur le tore bi-dimensionnel que nous appelons l'***équation de Josephson***. Nous étudions la structure géométrique d'un ensemble associé à sa dynamique : les langues d'Arnold de l'application de premier retour sur un méridien fixé.

$\boxed{2.}$ Dans le deuxième chapitre nous nous intéressons à un système de ***bras articulés*** introduit par Joseph-Louis Lagrange. C'est un mécanisme qui bouge sur une surface en faisant des tours autour de son point de départ. Nous nous intéressons au nombre de tours qu'il réalise en moyenne pendant une longue durée, quand il est sur le plan euclidien, hyperbolique, une sphère ou n'importe quelle autre surface.

$\boxed{3.}$ Dans le troisième chapitre, nous revenons au ***billard elliptique.*** Nous étudions un problème de géométrie plane : prenons les orbites 3-périodiques de ce billard. Elles sont représentées par des triangles. Prenons les cercles inscrits dans ces triangles. Ils s'avère que les centres de ces cerles décrivent une courbe qui est de nouveau une ellipse ! Notre but était de trouver une preuve plus simple de ce fait, quitte à passer par le monde complexe.

Dans la deuxième moitié de cette thèse (chapitres 4 et 5) nous élaborons des méthodes générales pour l'étude asymptotique de différentes classes de systèmes dynamiques : des actions du groupe libre et des produits croisés.

$\boxed{4.}$ Dans le quatrième chapitre, nos objets d'étude sont des ***actions du groupe libre*** sur des espaces probabilisés. Nous prouvons un théorème de convergence des moyennes sphériques données par une chaîne de Markov sur un graphe dont les sommets sont les générateurs du groupe. Cela permet d'avoir de nouveaux outils de travail avec des actions de groupes de type fini comme les groupes de surfaces et, plus généralement, les groupes hyperboliques.

$\boxed{5.}$ Dans le cinquième et dernier chapitre, nous trouvons une ***forme normale*** d'un produit croisé au-dessus d'un difféomorphisme linéaire d'Anosov du tore. Ce travail s'inscrit dans une longue tradition de compréhension des feuilletages normalement hyperboliques transversalement hölderiens qui forment un ensemble ouvert de l'espace des systèmes dynamiques.

La liste des travaux de l'auteur

Les résultats de cette thèse font l'objet de cinq publications.

1. Yu. Ilyashenko, O. Romaskevich *Sternberg linearization theorem for skew products*, Journal of Dynamical and Control systems, article, DOI 10.1007/s10883-016-9319-6 (2016)
2. L. Bowen, A. Bufetov, O. Romaskevich *On convergence of spherical averages for Markov operators*, Geometriae Dedicata, **181** :1, pp. 293-306 (2016),article, DOI : 10.1007/s10711-015-0124-2
3. O. Romaskevich, *On the incenters of triangular orbits on elliptic billiards*, L'Enseignement Mathématique **60** :2, pp. 247–255 (2014), article, DOI : 10.4171/LEM/60-3/4-2
4. A. Klimenko, O. Romaskevich *Asymptotic properties of Arnold tongues and Josephson effect*, Moscow Mathematical Journal, **14** :2, pp. 367–384 (2014), article
5. V. Kleptsyn, O. Romaskevich, I. Schurov *L'effet Josephson et les systèmes lents-rapides*, Nanostructures. Physique mathématique et modèles, **8** :1, pp. 31-46, en russe (2013)

Comment lire cette thèse.

Grâce au fait que les chapitres sont indépendants, la lecture des chapitres peut être effectuée dans n'importe quel ordre. Les références bibliographiques sont regroupées à la fin de chaque chapitre (et il n'y a donc pas de bibliographie générale en fin de thèse).

La thèse est écrite en anglais. Cette thèse a une sœur russe qui peut être consultée sur ma page personelle : https://sites.google.com/site/olgaromaskevich/research

Les chapitres 1, 3, 4 et 5 de cette thèse contiennent les versions préliminaires des articles publiés (tous en anglais) liés au sujet. Pour les versions publiées, consulter les sites des éditions.

Dans l'introduction qui va suivre, nous allons présenter chacun de nos chapitres, l'un après l'autre, avec ses motivations et ses résultats.

Bonne lecture !

Chapitre 1. Une équation qui modèlise le contact de Josephson et ses langues d'Arnold.

Ce chapitre s'appuie principalement sur deux résultats. Le premier a été obtenu en collaboration avec Alexey Klimenko [1], et le deuxième en collaboration avec Ilya Schurov et Victor Kleptsyn [2] . Nous étudions une famille de champs de vecteurs sur le tore $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, muni des coordonnées $(x,t) \in \mathbb{T}^2$, donnée par un système d'équations différentielles de la forme

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = \cos x + a + b\cos t, \\ \dfrac{\partial t}{\partial \tau} = \mu, \end{cases} \tag{1}$$

avec trois paramètres réels $(a,b,\mu) \in \mathbb{R}^2 \times \mathbb{R}_+$, $\mu > 0$. Nous appelons cette famille **l'équation de Josephson.** Nous considérons l'application $P_{a,b,\mu} : \mathbb{S}^1 \to \mathbb{S}^1$ de premier retour sur la transversale $\{t = 0\}$ et son nombre de rotation [3] $\rho \in \mathbb{R}/\mathbb{Z}$, qui est une fonction des paramètres : $\rho = \rho(a,b,\mu)$. Nous nous intéressons au changement du nombre de rotation en fonction des paramètres.

Plus particulièrement, nous nous intéressons au **nombre de rotation du flot** $\bar{\rho} = \bar{\rho}_{a,b,\mu}$ de l'équation (1), qui est une valeur réelle telle que $\bar{\rho} = \rho \mod 1$. Ce nombre peut être défini comme la limite $\bar{\rho} := \lim_{n\to\infty} \frac{x(2\pi n)}{2\pi n}$ où $x(t)$ est une solution de l'équation (1) [4]. Cette valeur donne plus d'informations sur le flot que le nombre de rotation ρ de l'application $P_{a,b,\mu} : \mathbb{S}^1 \to \mathbb{S}^1$ parce qu'elle caractérise le nombre de fois que les orbites du champ de vecteurs (1) s'enroulent autour du tore et pas seulement l'endroit où elles atterrissent sur la transversale $\{t = 0\}$.

Définition 1. *Nous appelons **langues d'Arnold** de l'équation de Josephson, notées $\mathcal{A}_\alpha$, les ensembles de niveau de $\bar{\rho}$ d'intérieur non-vide :*

$$\mathcal{A}_\alpha := \left\{(a,b,\mu) \in \mathbb{R}^2 \times \mathbb{R}_+ : \bar{\rho}(a,b,\mu) = \alpha\right\}, \mathring{\mathcal{A}}_\alpha \neq \emptyset \tag{2}$$

Le but de ce chapitre est de comprendre la structure des langues d'Arnold $\mathcal{A}_\alpha$ définies par (2) : pour quelles valeurs de $\alpha \in \mathbb{R}$ existent-elles et quelle forme ont-elles ?

L'idée d'étudier le nombre de rotation comme une fonction des paramètres a été formulée

1. Klimenko A., Romaskevich O. *Asymptotic properties of Arnold tongues and Josephson effect*, Moscow Mathematical Journal, **14** :2, pp. 367–384 (2014)

2. Kleptsyn V. , Romaskevich O., Schurov I. *L'effet Josephson et les systèmes lents–rapides*, Nanostructures. Physique mathématique et modèles, **8** :1, pp. 31-46, en russe (2013)

3. Chaque homéomorphisme $P : \mathbb{S}^1 \to \mathbb{S}^1$ du cercle $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ respectant l'orientation peut être relevé de la façon unique modulo un entier à une application de la droite $\bar{P} : \mathbb{R} \to \mathbb{R}$. Alors **le nombre de rotation** $\rho(P) \in \mathbb{R}/\mathbb{Z}$ de P est défini comme une limite

$$\rho(P) := \lim_{n\to\infty} \frac{\bar{P}^{\circ n}(x)}{2\pi n}$$

pour un point $x \in \mathbb{R}$. Henri Poincaré a demontré que cette limite existe et est indépendante de x. Cette limite $\rho(P)$ caractérise la rotation moyenne effectuée par le homéomorphisme P.

4. Cette définition ne dépend pas de la condition initiale $x(0)$

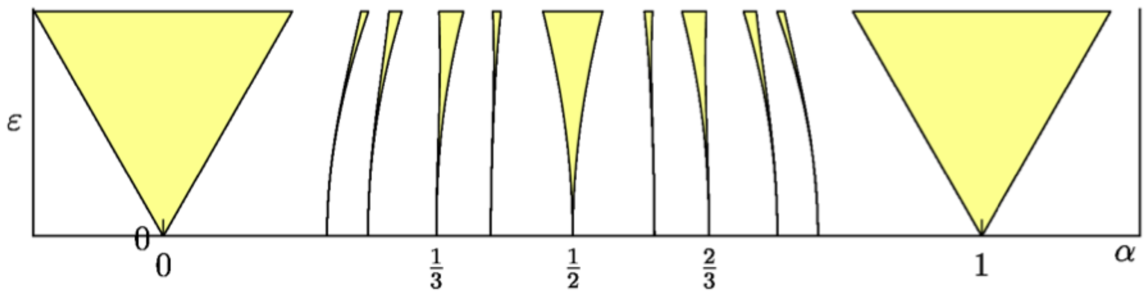

FIGURE 1 – L'ensemble des langues d'Arnold de la famille (3) pour toutes les valeurs du nombre de rotation $\rho = \frac{p}{q}$ avec dénominateur $q \leq 5$. Chaque langue a une « racine » au point $(\frac{p}{q}, 0)$ correspondant à la rotation pure avant de grossir petit à petit avec ε qui grandit. Pour cette famille, les langues d'Arnold existent pour toutes les valeurs rationnelles de $\rho = \frac{p}{q}$ et leur largeur diminue quand q augmente.

pour la première fois par A. Mayer[5]. Plus tard, en 1959, cette idée a été reprise par V. Arnold[6], qui a considéré une famille $f_{a,\varepsilon} : \mathbb{S}^1 \to \mathbb{S}^1$ de difféomorphismes du cercle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, plus précisément, une famille de perturbations des rotations :

$$x \mapsto x + 2\pi a + \varepsilon \sin x, \quad a \in \mathbb{R}, \varepsilon \in (0,1). \tag{3}$$

Arnold a considéré la fonction $\rho = \rho(a, \varepsilon)$ – le nombre de rotation du difféomorphisme $f_{a,\varepsilon}$ et il a démontré que les ensembles $\mathcal{A}_\alpha = \{(a, \varepsilon) : \rho(a, \varepsilon) = \alpha\}$ de niveau de la fonction ρ sont d'intérieur non-vide si et seulement si $\alpha \in \mathbb{Q}$. Il a appelé ces ensembles $\mathcal{A}_\alpha$, $\alpha \in \mathbb{Q}$ du plan des paramètres (a, ε) des **zones de captation de phase**, plus tard baptisés **langues d'Arnold**, voir la Fig. 1.

L'existence de langues d'Arnold pour toutes les valeurs rationnelles de ρ est générique pour les familles paramétrées de difféomorphismes : en effet, la rationnalité du nombre de rotation correspond à l'existence d'une orbite périodique. Si cette orbite est hyperbolique alors elle persiste sous perturbations et le nombre de rotation ne change pas. Alors, pour une famille au hasard des langues d'Arnold vont apparaître pour toutes les valeurs rationnelles du nombre de rotation.

La famille des applications de Poincaré $P_{a,b,\mu}$ correspondant à l'équation de Josephson (1) est exceptionnelle : les langues d'Arnold pour le nombre de rotation $\bar{\rho}(a, b, \mu)$ n'apparaissent que pour les valeurs *entières* de $\bar{\rho}$. Ce phénomène s'explique par le fait que l'application $P_{a,b,\mu}$ est conjuguée à l'application de Mœbius. Récemment A. Glutsuyk et L. Rybnikov[7] ont prouvé que cette famille (et ses analogues) est la seule qui a cette propriété d'absence de langues. Plus précisément, pour un champ de vecteurs sur le tore $\mathbb{T}^2$ de la forme $\frac{dx}{dt} = v(x) + a + bf(t)$ et pour une fonction $v(x)$ analytique fixée qui n'est pas de la forme $v(x) = \alpha \sin(mx) + \beta \cos(mx) + \gamma$, avec $\alpha, \beta, \gamma \in \mathbb{R}, m \in \mathbb{Z}$, il existe une fonction f analytique

5. A.G. Mayer *Rigid transformation of a circle into a circle*, Science Notes of GGU **12**, pp. 215–229 (1939)
6. V. I. Arnold *Geometrical Methods in the Theory of Ordinary Differential Equations*, Grundlehren der mathematischen Wissenschaften, **250**, p.110, Fig. 80 (1988)
7. A. Glutsyuk, L. Rybnikov *On families of differential equations on two-torus with all Arnold tongues* Cornell University, **15** (2015)

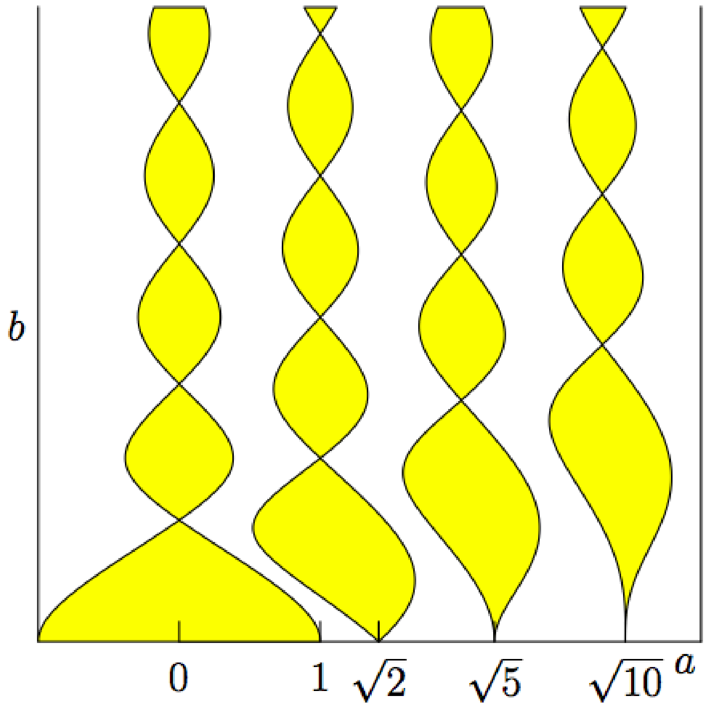

FIGURE 2 – La structure géométrique des langues d'Arnold pour l'équation de Josephson (1) est assez originale par rapport au dessin correspondant à la famille standard d'Arnold, voir Figure 1 : les frontières des langues s'auto-intersectent en formant des « tresses » . Ici sont dessinées les sections des langues avec un plan $\{\mu = 1\}$. L'image représente les sections des langues (sur le plan $(a, b) \in \mathbb{R}^2$) correspondant aux valeurs entières du nombre de rotation : $\bar{\rho} = 0, 1, 2, 3$. Notons que la langue d'Arnold correspondant à $\rho = 0$ contient tout le segment $[-1, 1] \times 0$ (l'équation correspondante a une solution constante). Les sections des langues correspondant à $\bar{\rho} = k \in \mathbb{Z}$ « commencent » aux points $\sqrt{1 + k^2}$.

telle que les langues d'Arnold pour l'application de Poincaré correspondante existent pour *toutes* les valeurs rationnelles du nombre de rotation.

Chacune des langues d'Arnold pour l'équation de Josephson a deux frontières (deux courbes analytiques) qui s'intersectent mutuellement en formant des « tresses », voir Figure 2. Le but de ce chapitre est d'expliquer et de quantifier cette structure géométrique.

Notre motivation principale était la beauté de cette image mathématique. Il faudrait néanmoins mentionner les motivations physiques qui lient l'équation (1) avec la modélisation de la dynamique des contacts de Josephson, qui sont des constructions passionnantes provenant de la théorie de la supraconductivité. Ces contacts sont utiles dans la construction de voltmètres extrêmement précis, dans la recherche en géologie, dans la détection de sous-marins et dans l'étude de l'activité du cerveau humain. Le comportement des contacts de Josephson est de nature quantique mais peut être mesuré par des fonctions macroscopiques. Les graphes de ces fonctions sont des escaliers de Cantor. Les marches de ces escaliers sont appelées par les physiciens **marches de Shapiro**. Dans les termes du modèle (1) elles correspondent aux sections des langues d'Arnold par les droites $\{\mu = \text{const}, b = \text{const}\}$. L'intérêt principal des physiciens est de pouvoir localiser le plus précisément possible les trous entre ces marches. Notons que le cas le plus intéressant pour les physiciens est celui où $\mu << 1$.

Dans la première partie de ce chapitre nous expliquons le comportement ondulatoire des frontières des langues. Plus précisément, nous prouvons que quand $b \to \infty$ les frontières des sections des langues d'Arnold (par des plans $\{\mu = \text{const}\}$) peuvent être bien approchées par les fonctions entières de Bessel. Nous prouvons que les frontières de chaque langue (correspondant à un nombre de rotation entier : $\bar{\rho} = k \in \mathbb{Z}$) sont données par les graphes des fonctions que nous appelons $a_{0,k}(b)$ et $a_{\pi,k}(b)$ (pour les raisons de ce choix de noms, nous vous invitons à lire la partie du chapitre 1 sur les symétries de l'équation) et que ces fonctions ont le comportement asymptotique suivant : quand $b \to \infty$,

$$a_{0,k}(b) \sim k\mu + J_k\left(-\frac{b}{\mu}\right), \tag{4}$$

$$a_{\pi,k}(b) \sim k\mu - J_k\left(-\frac{b}{\mu}\right), \tag{5}$$

où $J_k(z)$ est la k-ième fonction entière de Bessel, $J_k(z) = \frac{1}{2\pi}\int_0^{2\pi}\cos(kt - z\sin t)dt$. Dans la première partie de ce chapitre nous donnons des estimations plus précises sur la vitesse de convergence dans les approximations (4) et nous déterminons la zone dans laquelle ces estimations sont valables, voir la Figure 3. Cette convergence des frontières des langues d'Arnold vers les fonctions de Bessel a été remarquée empiriquement dans le contexte des contacts de Josephson par les physiciens S. Holly, A. Janus et S. Shapiro [8]. Nous l'avons rigoureusement prouvé ce fait avec A. Klimenko.

Notre résultat sur l'approximation des langues d'Arnold par les fonctions de Bessel s'avère utile parce qu'il donne un corollaire important : les langues ont des points d'intersection que nous appelons les **points d'adjacence**. A. Glutsyuk, V. Kleptsyn, D.Filimonov, I Schurov prouvent [9] grâce à la théorie classique des équations linéaires complexes que ces points d'adjacence se situent sur une même droite verticale : $a = \bar{\rho}\mu$, où $\bar{\rho}$ est la valeur du nombre de rotation pour la langue en question [10].

Dans la deuxième partie de ce chapitre nous regardons la même équation dans un autre régime limite, où $\mu \to 0$. Dans ce cas, l'étude de cette équation s'inscrit dans la théorie des systèmes lents-rapides : la variable t est une variable lente, elle change considérablement pendant les intervalles de temps très longs (proportionnels à $\frac{1}{\mu}$) tandis que x est une variable rapide. Nous appliquons des résultats classiques de la théorie des systèmes lents-rapides pour prouver la proximité des langues dans une zone compacte fixée (quitte à diminuer μ) audessus de la droite l_2, voir Figure 3. Nous prouvons que la distance entre deux langues voisines décroît exponentiellement quand $\mu \to 0$. Aussi, dans cette deuxième partie, nous présentons un algorithme numérique pour la construction des langues d'Arnold de cette équation, pour les valeurs de μ assez petites (jusqu'à $\mu = 0.01$).

Les résultats de cette deuxième partie font partie de notre travail avec V.Kleptsyn et I. Schurov. Dans la présente thèse nous avons retravaillé nos preuves et amélioré la présentation.

8. S. Holly, A. Janus, S. Shapiro *Effect of Microwaves on Josephson Currents in Superconducting Tunneling*, Rev. Mod. Phys. **36** , pp. 223–225 (1964)

9. A. Glutsyuk, V. Kleptsyn, D. Filimonov, I. Schurov *On the adjacency quantization in the equation modelling the Josephson effect*, Functional Analysis and Its Applications **48** :4, pp. 272-285 (2014)

10. Ce résultat est prouvé juste pour les valeurs de μ assez grands, $\mu > 1$ mais il reste une conjecture vraisemblable pour les $\mu \in (0,1)$

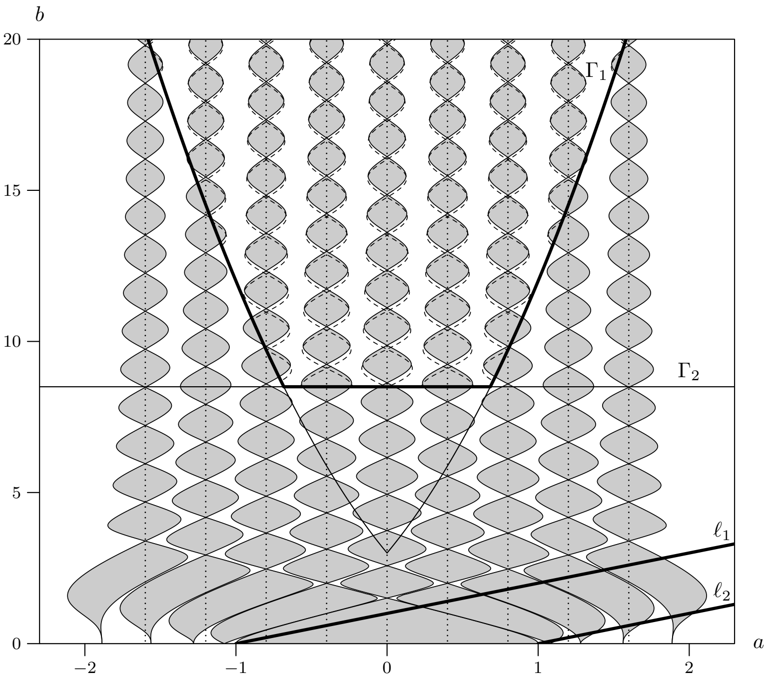

FIGURE 3 – L'image représentant les sections des langues d'Arnold de l'équation de Josephson (1) à μ fixé, $\mu = 0.4$. Les langues representées correspondent aux valeurs entières du nombre de rotation $\bar{\rho} = -4, \ldots, 4$. Dans la première partie du chapitre 1 nous démontrons la proximité des frontières des langues et des fonctions entières de Bessel quand $b \to \infty$: notre preuve fonctionne dans une zone au-dessus des courbes Γ_1 et Γ_2, définies plus précisement au cours du chapitre 1. Les frontières de cette zone sont marquées en gras, et les graphes des fonctions de Bessel en pointillés. Dans la deuxième partie du chapitre 1 nous étudions un autre régime limite : $\mu \to 0$, qui est plus pertinent du point de vue physique. La zone étudiée dans ce régime est la zone entre deux droites l_1 et l_2 ainsi que la zone bornée au-dessus de la droite l_1. Nous prouvons la proximité des langues au sein de ces deux zones quand $\mu \to 0$; déjà sur ce dessin nous pouvons voir que les langues sont très proches autour de la droite l_1.

La famille (1) a déjà été dans d'autres contextes avant le début des études concernants les langues d'Arnold, en liaison avec la dynamique des contacts de Josephson, initiée par V. Buchstaber, O. Karpov et S. Tertychnyi en 2006 [11]. Cette équation apparaît pour la première fois dans la littérature dans l'article de R. Foote, dans le contexte du planimètre de Prytz [12]. Elle apparaît aussi dans les études des traces de vélo [13]. En 2001 Yu. Ilyashenko et J. Guckenheimer ont commencé l'étude de cette équation dans le contexte des systèmes lents-rapides (ce qui a inspiré la partie 2 de ce chapitre) sans connaître les connexions avec les contacts de Josephson. Ils étaient intéressés par la recherche de solutions spéciales de cette équation (solutions du type « canard » qui restent pendant les intervales de temps assez longs près de la courbe lente, même dans la zone d'instabilité de cette courbe).

Cette thèse s'inscrit dans un cycle des travaux sur la famille (1) qui a été commencé par V. Buchstaber, O. Karpov et S. Tertycniy. Parallèlement à Yu. Ilyashenko [14], ils ont redécouvert les propriétés de base de l'application de premier retour $P_{a,b,\mu}$ (comme la propriété de Mœbius, découverte initialement par R. Foote) et ont donné les premières descriptions empiriques des langues d'Arnold pour l'équation (1). L'étude de l'équation de Josephson continue jusqu'à ce jour avec des travaux de V. Buchstaber, A.Glutsuyk, Yu. Ilyashenko, V. Kleptsyn, A. Klimenko, I. Schurov, D. Filimonov. D.Ryzhov et d'autres. La plupart de ces travaux ont pour but de comprendre la Figure 3.

Chapitre 2. Le problème de Lagrange : le calcul de la vitesse angulaire asymptotique du point extremal d'un bras articulé.

Dans le deuxième chapitre de cette thèse, nous nous inspirons du problème classique qui a été formulé par J.-L. Lagrange [15] dans le cadre d'une étude du mouvement des planètes. Lagrange a étudié le mouvement du système formé de N segments attachés en une chaîne que J.-C. Hausmann appelle dans ses articles [16] un **bras articulé** (voir la Figure 4).

En supposant que chacun des segments tourne autour de l'extremité du segment précédent avec une vitesse angulaire constante ω_j, Lagrange s'intéressait à la vitesse angulaire asymptotique de l'extrémité du système. Autrement dit, Lagrange étudiait le comportement de la fonction complexe $z(t) : \mathbb{R} \to \mathbb{C}$ suivante :

$$z(t) = \sum_{j=1}^{N} l_j \exp(i\omega_j t + i\beta_j^0), \tag{6}$$

où $l_1, \ldots, l_N \in \mathbb{R}_+$ correspondent aux longueurs des intervalles, $\omega_j \in \mathbb{R}$ aux vitesses asymptotiques locales de chacun des intervalles, et $\beta_j^0 \in [0, 2\pi)$ aux conditions initiales de la position

11. V. M. Buchstaber, O. V. Karpov, S. I. Tertychnyi *Features of the dynamics of a Josephson junction biased by a sinusoidal microwave current*, Journal of Communications Technology and Electronics, **51** :6, pp. 713–718 (2006)

12. R.L. Foote *Geometry of the Prytz planimeter*, Reports on mathematical physics,**42**, pp. 249–271 (1998)

13. M.Levi, S.Tabachnikov *On bicycle tire tracks geometry, hatchet planimeter, Menzin's conjecture and oscillation of unicycle tracks*, Experimental Mathematics **18** :2 ,pp. 173–186 (2009) ; D.Finn *Can a bicycle create a unicycle track ?*, The Mathematical Association of America, pp. 283–292 (2002)

14. Yu. Ilyashenko *Lectures on dynamical systems*, Summer School. manuscript (2009)

15. J. L. Lagrange *Théorie des variations séqulaires des éléments des planètes, I, II*, Nouveaux Mémoires de l'Académie de Berlin (1781, 1782), Oeuvres de Lagrange, **5**, Gauthier-Villars, Paris, pp. 123-344 (1870)

16. J.-C. Hausmann *Sur la topologie des bras articulés*, Algebraic Topology Poznań, Lecture Notes in Mathematics, pp. 146–159 ; J.-C. Hausmann *Contrôle des bras articulés et transformations de Möbius*, L'Enseignement Mathématique **51**, pp.87–115 (2005)

du bras articulé. Il s'intéressait à la limite

$$\lim_{t\to\infty} \frac{\arg z(t)}{t}, \tag{7}$$

que nous allons appeler la **vitesse angulaire asymptotique** du bras articulé et noter ω .

Lagrange a résolu ce problème dans le cas le plus simple où la longueur l_j de l'un des segments est plus grande que la somme des longueurs des autres : dans ce cas la limite (7) existe et est égale à la vitesse angulaire de ce segment, $\omega = \omega_j$. Dans le cas général, ce problème a été résolu beaucoup plus tard, dans les travaux de P. Hartman, E. van Kampen, A. Wintner, H. Weyl, B. Jessen et H. Tornehave[17].

Pour $N = 3$ le cas le plus intéressant est celui où les vitesses $\omega_1, \omega_2, \omega_3$ sont irrationellement indépendantes : le mouvement du bras articulé n'est pas périodique. En dehors du cas de Lagrange (quand une des longueurs est plus longue que la somme des deux autres), ce problème a été résolu par P. Hartman, E. van Kampen et A. Wintner qui ont demontré, en utilisant le théorème ergodique, que dans ce cas la vitesse asymptotique existe et s'écrit comme une somme convexe

$$\omega = \frac{\alpha_1}{\pi}\omega_1 + \frac{\alpha_2}{\pi}\omega_2 + \frac{\alpha_3}{\pi}\omega_3 \tag{8}$$

où α_j sont les *angles positifs* du triangle de côtés l_1, l_2 et l_3. La preuve de ce résultat se présente sous forme d'un calcul direct (qui est applicable pour tout N).

Dans ce chapitre nous proposons un nouveau regard géométrique sur la formule (8) qui permet d'obtenir le résultat analogue dans le cas d'une surface générale et, entre autres, de donner la réponse au problème de Lagrange sur la sphère et sur le plan hyperbolique.

Il est important de noter que le problème de Lagrange peut être étudié sous beaucoup d'angles différents : premièrement, il s'inscrit dans l'étude beaucoup plus générale des fonctions presque périodiques (voir le survol de B. Jessen sur la question[18]). Deuxièmement, il est intéressant de regarder les ensembles de niveau de $z^{-1}(w)$ pour chaque valeur complexe $w \in \mathbb{C}$ de l'extrémité d'un bras articulé qui est vue comme une fonction sur le tore $\mathbb{T}^n$ muni des coordonées $(\theta_1, \ldots, \theta_N)$:

$$z(\theta_1, \ldots, \theta_N) = \sum_{j=1}^{N} l_j \exp(i\theta_j).$$

Cette question a été étudiée, entre autres, par J.-C. Hausmann.

17. Pour le cas de trois intervalles, $N = 3$: P. Bohl *Über ein in der Theorie der säkularen Störungen vorkommendes Problem*, J. reine angew. Math. **135**, pp. 189–283 (1909) ; pour le cas général avec la supposition que les vitesses angulaires ω_j sont rationellement indépendantes voir P. Hartman, E. R. Van Kampen, A. Wintner *Mean Motions and Distribution Functions*, Amer. J. Math. **59** :2, pp.261–269 (1937) ; pour les remarques sur l'application du théorème ergodique H. Weyl *Mean Motion*, Amer. J. Math. **60**, pp. 889-896 (1938) ; pour le cas général de N arbitraire voir le survol B. Jessen and H. Tornehave, Mean motions and zeros of almost periodic functions, Acta Math. **77**, pp. 137–279 (1945)

18. B.Jessen *Some Aspects of the theory of almost periodic functions*, ICM (1954)

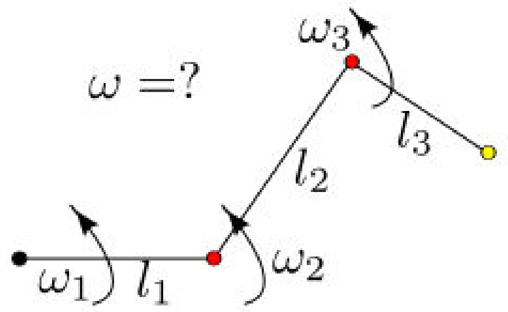

FIGURE 4 – Le problème de Lagrange consiste à étudier le mouvement du bras articulé. Ce bras consiste en N segments de longueurs $l_1, \ldots, l_N \in \mathbb{R}_+$, attachés en une chaîne. Chacun des segments tourne avec une vitesse angulaire constante $\omega_j, j = 1, \ldots, N$ autour de l'extrémité du segment précédent. L'intérêt de Lagrange se portait sur la vitesse angulaire asymptotique ω de l'extrémité de ce système, définie comme la limite $\frac{N(T)}{T}$ où $N(T)$ est le nombre de tours effectués en temps T. Lagrange voulait prouver l'existence de cette limite et trouver son expression en terme des longueurs l_j et des vitesses angulaires ω_j.

Chapitre 3. Le problème des centres des orbites triangulaires du billard elliptique.

Ce chapitre est inspiré par une remarque de D. Reznik, ingénieur à UC-Berkeley, qui s'intéresse à la géométrie plane et a remarqué empiriquement le fait suivant :

Dans le billard elliptique les centres des cercles inscrits dans les triangles correspondant aux orbites 3-périodiques décrivent une courbe qui est une ellipse.

Notre défi était initialement de trouver une preuve simple de ce résultat en utilisant des méthodes de géométrie plane. N'ayant pas réussi, nous avons elaboré des méthodes de géométrie algébrique complexe qui nous ont permis de prouver ce résultat. Après coup, la preuve géométrique « réelle » a, elle aussi, été trouvée (toujours assez élaborée, elle utilise des notions de conjugaison isogonale et de point de Gergonne – nous incitons le lecteur à trouver une preuve plus simple). Pendant l'été 2016 une autre preuve « réelle » est apparue dans un article de R. Garcia [19], dans lequel l'auteur calcule explicitement l'équation de l'ellipse en question.

Le cœur de ce chapitre est le résultat de notre article paru dans *l'Enseignement Mathématique* [20] qui se concentre sur la preuve complexe du résultat empirique de D. Reznik.

L'idée de notre preuve complexe est classique [21] : complexifier et projectiviser le problème, c'est-à-dire passer de la géométrie de $\mathbb{R}^2$ à la géométrie de $\mathbb{CP}^2$, regarder les complexifications de l'ellipse initiale, et définir la loi de réflexion complexe. L'argument géométrique est très simple une fois le bon cadre établi.

La complexification de la loi de réflexion dans le billard est assez subtile en raison de l'apparition de directions isotropes, c'est-à-dire de vecteurs de longueur nulle dans la nouvelle métrique $ds^2 = dz^2 + dw^2$ sur $\mathbb{CP}^2$. La réflexion par rapport aux tangentes isotropes à l'ellipse peut quand même être définie pour la trajectoire de billard. Pour ce faire nous avons suivi les

19. R.A. Garcia, *Centers of inscribed circles in triangular orbits of an elliptic billiard* (2016)

20. Romaskevich O., *On the incenters of triangular orbits on elliptic billiards*, L'Enseignement Mathématique **60** :2, pp. 247–255 (2014)

21. voir par exemple, Ph. Griffiths and J. Harris, Cayley's explicit solution to Poncelet's porism. L'Enseign. Math., pp. 31–40. **24** (1978)

idées des travaux d'A. Glutsyuk et Yu. Kudryashov [22] dans le cadre de l'étude de la conjecture d'Ivrii sur la mesure de l'ensemble des orbites périodiques dans un billard.

Chapitre 4. Les moyennes sphériques markoviennes pour les actions du groupe libre.

Ce chapitre reproduit notre travail avec Alexander Bufetov et Lewis Bowen [23] et s'inscrit dans le domaine de la théorie ergodique des actions du groupe libre par les applications qui préservent la mesure d'un espace mesuré (X, μ). Le mot ergodique se traduit littéralement du grec comme *le travail sur un chemin* : en effet, la théorie ergodique étudie les actions des groupes sur l'espace en mesurant les valeurs moyennes des fonctions sur les chemins que parcourent les points de l'espace sous ces actions.

Nous continuons dans la voie commencée par J. von Neumann et G. Birkhoff [24], qui ont étudié les moyennes temporaires des fonctions sur X le long des orbites des transformations de X. Le théorème ergodique classique affirme que pour l'application inversible $T : X \to X$ préservant la mesure μ sur X, $\mu(X) < \infty$ et pour la fonction integrable $\varphi \in L^1(X, \mu)$ et pour μ- presque tout $x \in X$ la limite des moyennes temporaires

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \varphi(T^k x)$$

existe. Dans le cadre de l'étude des actions de groupes, ce théorème peut être vu comme un théorème de convergence des moyennes sur les boules pour l'action du groupe abélien $\mathbb{Z}$ (dont le générateur est la transformation T).

Dans ce chapitre nous prouvons le théorème analogue pour les actions du *groupe libre* $\mathbb{F}_r$ à r générateurs dans l'espace (X, μ) muni de la mesure finie $\mu(X) < \infty$.

Définition 2. *Nous définissons la* **norme** $||\cdot||$ *sur le groupe* $\mathbb{F}_r =< a_1, \ldots, a_r >$ *de façon standard : la norme* $||g||, g \in \mathbb{F}_r$ *est égale à la longueur minimale du mot répresentant l'élément g dans l'alphabet* $\left\{a_1, \ldots, a_r, a_1^{-1}, \ldots a_r^{-1}\right\}$.

Alors, la **sphère** $S(n)$ dans le groupe libre se définit comme l'ensemble des éléments de même norme : $S(n) := \{g \in \mathbb{F}_r : ||g|| = n\}$. Notons que le nombre d'éléments de la sphère dans le groupe libre $\mathbb{F}_r$ grandit exponentiellement avec son rayon : $|S(n)| = (2r)(2r-1)^{n-1}$.

Une action du groupe $\mathbb{F}_r$ sur l'espace X est donnée par un homomorphisme

$$T : \mathbb{F}_r \to \mathrm{Aut}(X, \mu). \tag{9}$$

Nous définissons l'opérateur S_n des **moyennes sphériques** :

22. A. Glutsyuk *On quadrilateral orbits in complex algebraic planar billiards*, Mosc. Math. J. **14** , pp.239–289 (2014) ; A. Glutsyuk, Yu. Kudryashov *No planar billiard possesses an open set of quadrilateral trajectories*, J. Mod. Dyn. **6**, pp. 287–326 (2012)

23. Bowen L., Bufetov A., Romaskevich O. *On convergence of spherical averages for Markov operators*, Geometriae Dedicata, **181** :1, pp. 293-306 (2016)

24. J. von Neumann*Physical Applications of the Ergodic Hypothesis*, Proc. Natl. Acad. Sci. USA, **18** :3, pp. 263–266 (1932) ; J. von Neumann *Proof of the Quasi-ergodic Hypothesis*, Proc. Natl. Acad. Sci. USA, **18** :1), pp. 70–82 (1932) ; G. D. Birkhoff *Proof of the ergodic theorem*, Proc. Natl. Acad. Sci. USA, **17** :12, pp. 656–660 (1931) ; G.D. Birkhoff *What is the ergodic theorem ?*, Amer. Math. Monthly, **49** :4, pp. 222–226 (1942) ; voir aussiI.P. Kornfeld, G. Sinai Ya, S.V. Fomin *Ergodic Theory*, Springer (1982)

Définition 3. *Soit $\varphi \in L^1(X,\mu)$ une fonction, ses* ***moyennes sphériques*** *par rapport à l'action T du groupe $\mathbb{F}_r$ sur X se définissent comme*

$$S_n\varphi := \frac{1}{|S(n)|} \sum_{g\in\mathbb{F}_r:||g||=n} \varphi\circ T(g), \quad n = 1,2,3,\dots \tag{10}$$

Dans la théorie ergodique des actions du groupe libre, la convergence des moyennes sphérique se présente comme une question naturelle : la frontière $S(n)$ de la boule $B(n)$ dans un groupe $\mathbb{F}_r$ présente le poids principal des éléments dans ce groupe non moyennable.

L'étude des moyennes sphériques pour le groupe libre a été débutée par V. Arnold et A. Krylov [25] : ils ont prouvé l'analogue du théorème de H. Weyl [26] au sujet de l'équidistribution des orbites d'une rotation irrationelle sur le cercle. Plus exactement, ils ont demontré, que pour une action du groupe libre $\mathbb{F}_2 =< a,b >$ par les rotations $T(a), T(b) \in SO(3)$ de la sphère $\mathbb{S}^2$, si l'orbite $\mathbb{F}_2 x$ d'un point $x \in \mathbb{S}^2$ sur la sphère est dense, alors elle est équidistribuée. Cela signifie que pour chaque sous-ensemble $P \subset \mathbb{S}^2$ borné par une courbe lisse par parties, la partie des éléments de la sphère $S(n) \subset \mathbb{F}_r$ dans le groupe qui envoient x dans l'ensemble P est asymptotiquement proportionnelle à l'aire de P lorsque $n \to \infty$:

$$\lim_{n\to\infty} \frac{|S(n)x \cap P|}{|S(n)|} = \frac{\text{mes } P}{\text{mes } \mathbb{S}^2}.$$

En généralisant le résultat de V. Arnold et A. Krylov sur l'action générale du groupe $\mathbb{F}_2$, Y. Guivarc'h à demontré [27] un théorème de convergence des moyennes sphériques (10) en norme L^2 : pour chaque fonction $\varphi \in L^2(X,\mu)$, les moyennes sphériques $S_{2n}\varphi$ convergent dans L^2. Notons qu'il n'y a pas de raison de s'attendre à la convergence des moyennes sphériques $S_n\varphi$, parce que φ peut être une fonction propre des opérateurs $T(a), T(b)$ de valeur propre -1. A. Nevo et E. Stein ont par la suite généralisé [28] le résultat de Guivarc'h sur la convergence dans L^p pour les fonctions $\varphi \in L^p, p \in (1,\infty)$ en utilisant des méthodes spectrales. Récemment, T. Tao a demontré [29] l'impossibilité de prouver l'analogue du théorème de Guivarc'h pour la convergence presque partout : il a donné l'exemple d'une action de $\mathbb{F}_2 \curvearrowright (X,\mu)$ et d'une fonction $\varphi \in L^1(X,\mu)$ telle que $\sup_{x\in X} |S_n\varphi(x)| = \infty, n \to \infty$.

Dans ce chapitre, nous considérons la généralisation de la définition (10) : les **moyennes sphériques markoviennes** pour lesquelles les différents éléments de la sphère $S(n)$ sont pris avec des poids différents.

Ces poids vont être donnés par une chaîne de Markov avec un nombre d'états fini qui correspondra à l'ensemble des éléments du groupe $\mathbb{F}_r$. En se promenant sur cette chaîne, le processus probabiliste va créer des mots de longueur n, et la moyenne de la fonction φ sur

25. V.I. Arnold, A. L. Krylov *Equidistribution of points on a sphère and ergodic properties of solutions of ordinary differential equations in a complex domain*, Dokl. Akad. Nauk SSSR **148**, pp. 9–12 (1963)

26. H. Weyl *Über die Gibbs'sche Erscheinung und verwandte Konvergenzphänomene*, Rendiconti del Circolo Matematico di Palermo, **330**, pp. 377–407 (1910) ; H. Weyl *Ueber die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** :3, pp. 313–352 (1916)

27. Y. Guivarc'h *Généralisation d'un théorème de von Neumann*, C. R. Acad. Sci. Paris Sér. A-B **268**,pp.1020–1023 (1969)

28. A. Nevo, E. Stein *A generalization of Birkhoff's pointwise ergodic theorem*, ActaMath. **173**, pp.135–154 (1994)

29. T. Tao *Failure of the L1 pointwise and maximal ergodic theorems for the free group*, Forum of Mathematics, Sigma, **3** (2015)

l'orbite du groupe calculée avec les poids correspondants à ce processus va donner la nouvelle définition de la moyenne sphérique.

Cette chaîne est donnée par un graphe fini orienté $\Gamma = (V, E)$ où V est un ensemble des sommets, et L - l'ensemble des arêtes. Les sommets sont representés par les éléments du groupe grâce à l'application injective $\mathcal{L} : V \to \mathbb{F}_r$.

L'espace des états de la chaîne de Markov coïncide avec les sommets V de Γ. Les probabilités des sauts sont définies par une matrice stochastique $(\Pi_{v,w})$ dont les lignes et les colonnes sont numérotées par les éléments de V. Nous supposons aussi que cette matrice a une distribution stationnaire $(\nu_v)_{v\in V} : \Pi^T\nu = \nu$. L'ensemble E des arêtes du graphe Γ est défini comme

$$E := \{(w, v)|\Pi_{v,w} > 0\}.$$

À chaque chemin orienté $\mathbf{s} = (s_1, \dots, s_n) \in V^n, (s_j, s_{j+1}) \in E$ dans le graphe Γ correspond un automorphisme de X :

$$T_\mathbf{s} := T(\mathcal{L}(s_1)) \circ \dots \circ T(\mathcal{L}(s_n)),$$

et la probabilité de ce chemin dans le graphe : $\Pi_\mathbf{s} := \Pi_{s_n s_{n-1}} \dots \Pi_{s_2 s_1}$.

Définition 4. *Les* ***moyennes sphériques markoviennes*** *pour l'action T donnée par* (9) *du groupe libre $\mathbb{F}_r$ correspondant à la matrice stochastique Π sont les opérateurs $S_n : L^1(X, \mu) \to L^1(X, \mu)$, qui sont définis comme les moyennes de la fonction $\varphi \circ T$ sur tous les chemins $\mathbf{s} = (s_1, \dots, s_n)$ dans le graphe $\Gamma = (V, E)$ (correspondant à la matrice Π comme décrit dessus) de longueur n :*

$$S_n\varphi(x) := \sum_{\mathbf{s}=(s_1,\dots,s_n)} \nu_{s_n}\Pi_\mathbf{s}\varphi\left(T_\mathbf{s}x\right). \tag{11}$$

Remarquons tout d'abord qu'en choisissant comme sommets du graphe les générateurs et leurs inverses $V = \{a_1, \dots, a_r, a_1^{-1}, \dots, a_r^{-1}\}$ ainsi que les coefficients de la matrice Π comme $\Pi_{v,w} = \frac{1}{2r-1}$ si $v \neq w^{-1}$ et $\Pi_{v,w} = 0$ sinon, nous retrouvons la définition classique des moyennes sphériques. Si nous changeons la matrice Π pour qu'elle ait des coefficients différents du cas classique, nous obtenons de nouvelles moyennes sphériques. L'idée générale de cette approche est que les moyennes sphériques markoviennes pour une matrice Π bien choisie puissent correspondre aux moyennes sphériques uniformes d'autres groupes de type fini différents du groupe libre. Bien sûr, cette approche se limite au cas des groupes qui possédent un codage markovien. Cependant, cette classe est très large et contient les groupes des surfaces et même, plus généralement, tous les groupes hyperboliques de M. Gromov.

Une question très intéressante (et toujours ouverte) est celle de la convergence des moyennes sphériques pour les groupes hyperboliques de Gromov. Cette convergence a été prouvée par K. Fujivara et A. Nevo [30] sous l'hypothèse que l'action soit fortement mélangeante.

Même si la convergence des moyennes sphériques pour les groupes hyperboliques (sans aucune hypothèse sur l'action) n'est pas encore établie, la convergence de Cesàro des moyennes sphériques a été prouvée par A. Bufetov, A. Klimenko et M. Khristoforov [31] pour toutes les actions des semi-groupes markoviens. Dans le cas particulier des groupes hyperboliques, il

30. K. Fujiwara, A. Nevo *Maximal and pointwise ergodic theorems for word-hyperbolic groups*,Ergod. Theory Dyn. Syst. **18**, pp. 843–858 (1998)

31. A. Bufetov, M. Khristoforov, A. Klimenko *Cesàro convergence of spherical averages for measure- preserving actions of Markov semigroups and groups*, Int. Math. Res. Not. IMRN **21** , pp. 4797–4829 (2012)

existe une preuve courte et élégante [32] par M. Pollicott et R. Sharp qui utilise des méthodes de D. Calegari et K. Fujiwara [33].

La théorie ergodique des actions de groupes non-commutatifs étant un sujet très vaste, nous renverrons le lecteur (et la lectrice !) motivé(e !) à un survol de A. Nevo [34] ainsi qu'aux références dans le corps du chapitre 5.

Dans ce chapitre nous nous sommes intéressés à la preuve de la convergence dans L^1 des moyennes sphériques markoviennes (11) sans aucune condition sur l'action du groupe, quitte à restreindre la classe des matrices Π étudiées. Notre résultat généralise celui de A. Bufetov [35], qui a prouvé la convergence des moyennes sphériques pour les chaînes de Markov symétriques, c'est-à-dire en imposant des conditions de type « égalité » sur les coefficients de la matrice Π. Nous allons présenter un théorème qui permet de prouver l'analogue du théorème de Bufetov pour l'ensemble ouvert des matrices stochastiques Π.

Notre approche est nouvelle mais elle se base sur des méthodes d'opérateurs de Markov élaborées par R. Grigorchuk [36] et J.-P. Thouvenot et utilisées par A. Bufetov dans la démonstration de son théorème de convergence. Le point clé dans la preuve de Bufetov est la trivialité de la tribu correspondant au comportement limite de l'opérateur de Markov lié au processus probabiliste défini plus haut. Cette trivialité est prouvée grâce au théorème de G.-C. Rota [37] concernant les opérateurs de Markov et se fonde sur la réversibilité de la chaîne de Markov. Nous trouvons un moyen de contourner la condition de réversibilité pour prouver un théorème plus général.

Chapitre 5. Le théorème de linéarisation des produits croisés à la Sternberg.

Ce dernier chapitre reproduit notre article publié avec Yulij Ilyashenko dans *Journal of Dynamical and Control Systems* [38]. Ce travail s'inscrit dans le domaine de la dynamique partiellement hyperbolique et dans l'étude des phénomènes génériques, c'est-à-dire les phénomènes qui ont lieu pour des sous-ensembles ouverts dans l'espace des systèmes dynamiques.

Ce chapitre est consacré plus particulièrement à un théorème de normalisation pour les **produits croisés Hölder**. Nous allons d'abord expliciter la motivation de l'étude de cette classe de systèmes dynamiques.

Les produits croisés sont des applications de l'espace-produit $M = B \times I$ de la base B et de la fibre I de la forme suivante :

$$F : M \to M, F : (b, x) \mapsto (\alpha(b), f_b(x)), \;\; b \in B, x \in I.$$

32. M. Pollicott, R. Sharp *Ergodic theorems for actions of hyperbolic groups*, Proc.Am.Math.Soc.**141**, pp. 1749–1757 (2013)
33. D. Calegari, K. Fujiwara *Combable functions, quasimorphisms, and the central limit theorem*, Ergod. Theory Dyn. Syst. **30** :5, pp. 1343–1369 (2010)
34. A. Nevo *Harmonic analysis and pointwise ergodic theorems for noncommuting transformations*, J.Am. Math. Soc. **7** :4, pp. 875–902 (1994)
35. A. Bufetov *Convergence of spherical averages for actions of free groups*, Ann. Math. **155**, pp. 929–944 (2002)
36. R. I. Grigorchuk *Ergodic theorems for actions of free semigroups and groups*, Math.Notes **65**, pp. 654–657 (1999)
37. G.-C. Rota *An "Alternierende Verfahren" for general positive operators*, Bull.Am.Math.Soc. **68**, pp. 95–102 (1962)
38. Ilyashenko Yu., Romaskevich O. *Sternberg linearization theorem for skew products*, Journal of Dynamical and Control systems, **22** :3, pp. 595–614 (2016)

Ici $\alpha(b)$ est une transformation de la base B [39] et $f_b(x)$ une application sur les fibres qui est *a priori* différente sur chaque fibre. Donc les produits croisés donnent une famille d'applications possédant un feuilletage invariant (le feuilletage vertical $\{b = \text{const}\}$). La compréhension de la dynamique des produits croisés est le premier pas vers la compréhension de la dynamique des applications possédant un feuilletage invariant, et la littérature sur ce sujet est vaste.

Les produits croisés ayant la propriété de **dominated splitting condition** (DSC) sont les systèmes partiellement hyperboliques pour lesquels la dynamique dans la fibre est moins forte que la dynamique dans la base [40]. Un premier exemple d'un tel système est le produit d'un difféomorphisme linéaire d'Anosov dans la base et de l'identité dans la fibre. Un tel produit croisé a une propriété remarquable de stabilité : leurs perturbations de classe C^1 vont toujours avoir un feuilletage invariant lisse le long des fibres.

Les produits croisés DSC sont un cas spécial de systèmes *normalement hyperboliques*, c'est-à-dire de systèmes avec un feuilletage invariant le long duquel la dynamique est moins forte que la dynamique dans la direction transversale.

Définition 5. [41] *Le feuilletage $\mathcal{F}$ de la variété compacte M, invariant par le difféomorphisme f, est appelé **normalement hyperbolique** si le fibré tangent de M peut être representé comme une somme directe de sous-fibrés invariants par df :*

$$TM = E^u \oplus E^c \oplus E^s$$

tels que pour une certaine metrique sur M : $df|_{E^s} < 1 < df|_{E^u}$ et $df|_{E^s} < df|_{E^c} < df|_{E^u}$ en chaque point de la variété M.

D'après les résultats classiques de M. Hirsch, C. Pugh et M. Shub [42], les feuilletages normalement hyperboliques persistent sous des perturbations C^1. Une application $g : M \to M$ qui est C^1-proche d'une application f normalement hyperbolique (ayant un feuilletage invariant $\mathcal{F}$), va elle aussi avoir un feuilletage invariant $\mathcal{G}$, proche de $\mathcal{F}$, et va être normalement hyperbolique par rapport à ce feuilletage. De plus, f et g vont être conjuguées le long des feuilles de $\mathcal{F}$ et $\mathcal{G}$. Par contre, même si le feuilletage $\mathcal{F}$ est lisse, le feuilletage $\mathcal{G}$ préserve la régularité dans la direction des feuilles mais pourra ne pas être différentiable dans la direction transversale. En 2012, C. Pugh, M. Shub et A. Wilkinson ont prouvé que ce feuilletage est hölderien dans la direction transversale [43]. Leur théorème s'accorde bien avec le principe heuristique énoncé par J. Moser :

La régularité des objets qui apparaissent naturellement dans la dynamique lisse est la régularité Hölder.

Un autre exemple de cette dépendance hölderienne est la dépendance des variétés (lisses en direction des feuilles) stable et instable $\mathcal{W}^s(m), \mathcal{W}^u(m)$ du difféomorphisme hyperbolique $F : M \to M$ par rapport au point de base $m \in M$.

39. Dans toutes nos considerations $\alpha(b)$ sera un difféomorphisme d'Anosov.

40. Pour une définition précise voir, par exemple, Yu. Ilyashenko, A. Negut *Hölder properties of perturbed skew products and Fubini regained*, Nonlinearity, **25**, pp. 2377–2399

41. Pour la définition plus précise (comparaisons de deux operateurs en termes de leurs normes) et le survol, voir par exemple, C. Pugh, M. Shub, A. Wilkinson *Hölder foliations, revisited*, J. Modern Dyn. **6**, pp. 835–908 (2012)

42. M. Hirsch, C. Pugh, M. Shub *Invariant manifolds*, Lecture Notes in Mathematics, **583** (1977)

43. Voir A. Gorodetskii *Regularity of central leaves of partially hyperbolic sets and applications* (en russe), Izv. Ross. Akad. Nauk Ser. Mat. **70** :6, pp. 19–44 (2006) ; traduction Izv. Math. **70** :6 (2006), pp. 1093–1116 ; et Yu. Ilyashenko, A. Negut *Hölder properties of perturbed skew products and Fubini regained*, Nonlinearity, **25**, pp. 2377–2399 pour le cas des produits croisés avec la condition de DSC et C. Pugh, M. Shub, A. Wilkinson *Hölder foliations, revisited*, J. Modern Dyn. **6**, pp. 835–908 (2012) pour le cas général des systèmes dynamiques normalement hyperboliques

La compréhension de la régularité des perturbations des produits croisés permet de construire des exemples de sous-ensembles ouverts dans l'espace des difféomorphismes qui possèdent des attracteurs étranges [44]. Le processus de construction suit souvent le même schéma : l'attracteur étrange se construit d'abord dans l'espace des produits croisés [45] et elle est répandue (par perturbation) sur les systèmes dynamiques normalement hyperboliques, hölderiens en direction transversale. Ceci prouve la généricité de l'effet trouvé.

Ce programme se retrouve suivi dans bon nombre de travaux qui ont pour but de construire des applications avec des exposants de Lyapounov non-nuls [46], des feuilletages pathologiques [47], des applications des variétés à bord avec un attracteur de Milnor de mesure positive [48] ou avec des bassins qui s'intersectent [49], et pour la construction des attracteurs osseux [50]. Dans cette thèse, dans le cadre de ce programme, nous établissons une forme normale locale pour les produits croisés hölderiens afin de pouvoir travailler avec ces produits dans les voisinages de leurs points fixes. La motivation initiale de cette recherche était le souhait de donner des outils pour simplifier le travail de 2011 de Yu. Ilyashenko lié à la construction des gros attracteurs sur des variétés à bord [51].

Nous considérons ici un produit croisé hyperbolique $F : M \to M$ au-dessus d'un difféomorphisme linéaire d'Anosov $A : B \to B$ du tore $B = \mathbb{T}^d$ dans la base et avec une feuille unidimensionelle. Notre forme normale est linéaire $F_0 : (b, x) \mapsto (Ab, \lambda(b)x)$ et la conjugaison entre le difféomorphisme initial et cette forme normale [52] $H : M \to M$ ne transforme pas la base : $H(b, x) = (b, h_b(x))$. Cette conjugaison sur les feuilles permet de garder la structure de produit croisé lors des transformations. Nous prouvons que cette conjugaison est hölderienne par rapport au point de base et nous calculons explicitement son exposant de Hölder, qui s'avère être lié au rapport de la dynamique dans la base et dans la fibre de l'application initiale F.

44. Le mot *étrange* n'est pas strictement défini et change d'un article à l'autre. L'idée est de construire un ensemble ouvert dans l'espace des difféomorphismes qui possède une propriété inattendue dont on penserait qu'elle va disparaîre sous perturbation.

45. Il y a souvent un pas qui précède la construction dans l'espace des produits croisés avec la base - variété, c'est le pas des produits croisés avec la base qui est l'espace des suites. Ces produits croisés sont souvent appelés dans la littérature *les systèmes des fonctions itérées (iterrated functions systems)*.

46. A. Gorodetski, Yu. Ilyashenko, V. Kleptsyn, M. Nalski *Non-removable zero Lyapunov exponents*, Funct. Anal. Appl. **39** :1, pp. 27–38 (2005)

47. Feuilletages dont l'holonomie n'est pas absolument continue : M. Shub, A. Wilkinson *Pathological foliations and removable zero exponents* (1999)

48. Yu. Ilyashenko *Thick attractors of step skew products*, Regular Chaotic Dyn., **15**, pp. 328– 334 (2015) pour le cas de la base $\Sigma^2 = \{0,1\}^{\mathbb{N}}$ et Yu. Ilyashenko *Thick attractors of boundary preserving diffeomorphisms*, Indagationes Mathematicae, **22** :(3-4), pp. 257–314 (2011) pour le cas de la base $\mathbb{T}^2$

49. Yu. Ilyashenko *Diffeomorphisms with intermingled attracting basins*, Funkts. Anal. Prilozh, **42** :4, pp. 60–71 (2008)

50. Yu. Kudryashov *Bony attractors*, Funkts. Anal. Prilozh, **44** :3, pp. 73–76 (2010)

51. Yu. Ilyashenko *Thick attractors of boundary preserving diffeomorphisms*, Indagationes Mathematicae, **22** :(3-4), pp. 257–314 (2011)

52. La conjugaison $H : M \to M$ est un homéomorphisme tel que $F \circ H = H \circ F_0$

Première partie

Dynamics and ergodic averages of physical systems

1

Dynamics of the equation modeling a Josephson junction

This chapter is devoted to the study of a so-called *Josephson equation*, the three-parametric family of ODEs on a torus. This family arises from a special model of Josephson effect in the physics of superconductivity. We are interested in the structure of *Arnold tongues* for this equation since first, their sections correspond to the domains of the *phase-lock*, relevant for physical experiments and second, because they have a beautiful mathematical structure.
We study these Arnold tongues in two critical regimes : in the first of them, we prove that on the parametric plane (one of the three parameters is fixed) the boundaries of the tongues are asymptotically close to Bessel functions. This is a joint work wih Alexey Klimenko. In the second regime, the system has one slow and one fast variable, and the techniques elaborated in the slow-fast systems theory help us to give a qualitative description of the tongues. This is a joint work with Ilya Schurov and Victor Kleptsyn.

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This Chapter is organized as follows : first, we give an introduction to the theory of the three-parametric vector field modeling the Josephson contact (that we will call from now on **the Josephson equation**) and we give all the basic definitions. Then, we describe two different limit regimes that will be of interest to us in the following and treat them one by one. In both of these regimes we will be studying Arnold tongues corresponding to the rotation number of the flow defined by the considered vector field and some fixed transverse section.

The first regime is studied in our article with Alexey Klimenko published in *Moscow Mathematical Journal* (that we join untouched). For this regime we have established a theorem about the limiting behavior of the tongues in terms of integer Bessel functions. The second regime is much more difficult to understand and there are still lots of open questions : we describe a qualitative behavior of the tongues in this regime in our article (in Russian) with Ilya Schurov and Victor Kleptsyn [21] and we introduce some ideas on how computer simulations of the Arnold tongues could be done. Here we will present the ideas and theorems from our joint article, with some improved presentation.

1.1 Introduction

1.1.1 Set-up and definitions

Let us first explain what kind of dynamics we will be studying and formulate precisely the questions that will be of interest. We will give some physical as well as mathematical motivations in the following Subsection 1.1.2. We will be studying the following differential equation :

$$\frac{dx}{dt} = \frac{\cos x + a + b\cos t}{\mu}, \tag{1.1}$$

where $a, b, \mu \in \mathbb{R}$ are real parameters, $\mu > 0$. Since the right-hand side is a 2π-periodic function in t and x, this equation has a quotient which is a system of equations on the two-dimensional torus $R^2/2\pi\mathbb{Z}^2$ with coordinates x and t :

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = \cos x + a + b\cos t, \\ \dfrac{\partial t}{\partial \tau} = \mu. \end{cases} \tag{1.2}$$

Note that the vector field corresponding to the equation (1.2) can also be considered as a vector field on a cylinder $\mathbb{R}^2/((x,t) \sim (x, t+2\pi))$. In both cases the Poincaré map from the transversal line $\{t = 0 \bmod 2\pi\}$ to itself can be defined, we denote it as $P_{a,b,\mu} : \mathbb{S}^1 \to \mathbb{S}^1$ for the torus, and $\widetilde{P}_{a,b,\mu} : \mathbb{R} \to \mathbb{R}$ for the cylinder. So $P_{a,b,\mu}$ is a homeomorphism of the circle while $\widetilde{P}_{a,b,\mu}$ is its lift on the line $\mathbb{R}$.

Let us remind the reader that for any circle homeomorphism P its rotation number can be defined as following :

Definition 1.1. *Let the* rotation number ρ *of the map* $P : \mathbb{S}^1 \to \mathbb{S}^1$ *be the limit*

$$\rho := \lim_{n\to\infty} \frac{\widetilde{P}^{\circ n}(x) - x}{2\pi n}.$$

where $\widetilde{P} : \mathbb{R} \to \mathbb{R}$ *is a lift of* P *on the universal cover.*

The rotation number $\rho(P)$ shows how the points on the circle rotate on average under the homeomorphism P. It is well known that for any homeomorphism of a circle such a limit exists and does not depend on the point $x \in \mathbb{R}$ (see, for example, [19]). The value of the rotation number is an important characteristic of a map : for instance, it is invariant under conjugation by homeomorphisms.

In our particular case, the rotation number ρ is a function of three real parameters a, b, μ : we will denote it $\rho_{a,b,\mu}$ correspondingly.

We will be interested in the question how the rotation number of the homeomorphism changes while the parameters change in the three-dimensional parameter space.

Definition 1.2. *We say that the* phase lock *occurs for the value* $k \in \mathbb{R}$ *of rotation number if the level set*

$$E_k := \{(a, b, \mu) | \rho_{a,b,\mu} = k\} \tag{1.3}$$

in the space of parameters $\mathbb{R}^2 \times \mathbb{R}_+$ *has nonempty interior. In this case the level set* E_k *is called an* Arnold tongue.

From now on, all along this Chapter, the main question for us is for what values of ρ do these Arnold tongues occur, and, more specifically, how do they look like ?

1.1.2 Physical motivations

The equation (1.1) comes from the physics of superconductivity and can be found in physical works [24, 27, 18] as a model for the dynamics of Josephson junction in its sine-form which is

$$\frac{dx}{dt} = \frac{\sin x + a + b\sin t}{\mu}. \tag{1.4}$$

Substitutions $x \to x \pm \pi/2$, $t \to t \pm \pi/2$ transform one variant (1.1) to another (1.4) and none of our results is affected by such a change. We prefer to work with the form (1.1).

The properties of this equation (1.4) were studied in the context of Josephson junction modelling [3, 4, 5, 6, 7, 12, 17, 20, 30] as well as in some different contexts as the geometry of Prytz planimeter [11] and the study of bicycle tracks [10, 22].

Let us give a physical interpretation [24, 25, 27] of the equation (1.4). This equation gives a dynamical model of a so-called Josephson junction, a small device named after Brian Josephson who theoretically predicted the existence of super-current component in a current going through a weak electrical contact between two superconducting electrodes in 1962. Josephson received a Nobel Prize in Physics ten years after because of the experimental approval of his conjecture.

Now Josephson junctions are widely used in technology to built electronic circuits. There is a significant research on ultrafast computers using Josephson logic. Maybe even more importantly, Josephson junctions can be used to build circuits called SQUIDs (superconducting quantum interference devices). These devices serve to construct extremely sensitive magnetometers and voltmeters (one thousand times better than any other available voltmeters). Since SQUID's feel even small changes in a magnetic field, they are even used in the sensing of these fields created by neurological currents and serve to the monitoring the activity of the brain (or the heart). Other applications of SQUIDs are geological research and submarine detection .

The Josephson junction is made up by putting a very thin barrier of a non-superconducting material (**a weak link**) between two layers of superconducting material. This barrier can be made of different materials, for example it can be an insulator or another non-superconducting metal. The size of the barrier is several microns. Josephson predicted that in this tiny system the *tunneling effect* is possible : the pairs of superconducting electrons could go right through the barrier from one superconductor to another.

We know that for many metals the extreme cooling draws them to a completely different state : there exists a critical temperature (which depends on a metal but is in general very low, around minus 250 Celsius) at which metal goes from the electrical resistance state to the superconducting state, in which it gives essentially no resistance to the flow of the electrical current [53].

The explanation for this sort of behavior is that at some point because of the interaction of the electrons with the ionic lattice of the metal, two electrons start to slightly attract, although above the critical temperature the interaction between those electrons was repulsive. This attraction allows the electrons to drop into a lower energy state and though there is a possibility for the electrons to move through the ionic lattice, and hence, the current can flow. Hence in this state the material gives no electrical resistance and, at the same time, there is a super-current that can flow and which is called the *critical current*.

In the junction, until a critical current is reached, electron pairs can tunnel across a non-superconducting barrier without any resistance but once this critical current is exceeded, the voltage develops across the junction. This voltage depends on time and on the current, and as long as the current through the junction is less than the critical current, the voltage is zero.

53. Let us remark, that recently the existence of high-temparature supeconductivity was established, for example for some ceramic materials that exhibit the same behavior at warmer temperatures. The record of the highest temperature at which superconductive state can occur is now at -70 Celsius [8].

As soon as the current exceeds the critical current, the voltage will oscillate in time. This is exactly the changes in voltage that are interesting for physicists.

Let us suppose that the current that goes through the junction has a form $I(t) = \bar{I}(t) + \tilde{I}(t)$, i.e. it is a sum of a constant term $\bar{I}$ and a periodic term $\tilde{I}$ with a zero mean (we can suppose that it is generated by an external electromagnetic signal). The voltage on the electrodes of Josephson junction is given as a derivative with respect to time of the function x which has a quantum nature. The function x is a difference of phases of wave functions that describe the properties of the "liquid mass" of couples of electrons in superconductive materials. So, although the function is quantum, it's derivative is a macro-physical characteristic – the voltage between the superconductive plates.

For Josephson junction description the so-called resistive and capacitively shunted model is used and it is given by the equation [24, 27]

$$\dot{x} + F(x) = I(t) \tag{1.5}$$

where F is an odd 2π-periodic function which represents the relation between the current and the phase. For the most of mathematical models it can be represented as a sum $F(x) = \sin x + H(x)$ where H is either zero or small. What is important is that such a model coincides well with the results of the experiments [18].

Let us remark that strictly speaking, all the fuctions and variables here are dimensionless quantities corresponding to their physical analogues. For more precise explanation of the equation (1.5), see the books of K. Likharev, B. Ulrich and M. Tinkham [23, 24, 31].

As said before, the important function for us is a volt-ampere (V-I) characteristics of a junction showing the dependance of the voltage across the junction on the external current. In terms of the equation (1.5) this is a function that shows the relation between the average value of $\dot{x}$ (wih respect to time) and the average value $\int I(t)dt$ of the current. For the junction described by the equation (1.4) the current is sinusoidal and equal to $I(t) = a + b\sin t$. Hence a volt-ampere characteristic corresponds to the graph of a rotation number $\rho_{a,b,\mu}$ which is studied as a function of parameter a with fixed b and μ. And the study of volt-ampere characteristics is translated in our model to the study of the rotation number $\rho_{a,b,\mu}$ of a circle diffeomorphism $P_{a,b,\mu}$ as a function of parameters. Here the parameter μ plays a role of the ratio between the frequence of exterior signal to the internal frequency of the junction, see [17] for details.

In generic families of circle diffeomorphisms rational rotation numbers exist for intervals in the space of parameters (since small perturbations do not destroy a quality of having a hyperbolic periodic point). For Josephson equation, the corresponding sections of Arnold tongues (by the lines $\{b = \text{const}, \mu = \text{const}\}$) are called *Shapiro steps* by physicists. On Figure 1.1 one can see a picture of the steps from the original article of Sidney Shapiro in 1963 who first noticed the existence of the steps experimentally [29].

We will although see that Josephson equation doesn't satisfy the general paradigm of the existence of steps for each rational value of rotation number ρ : the tongues for the equation (1.2) exist only for *integer* values of ρ. This has a nice mathematical explanation, to which we pass immediately.

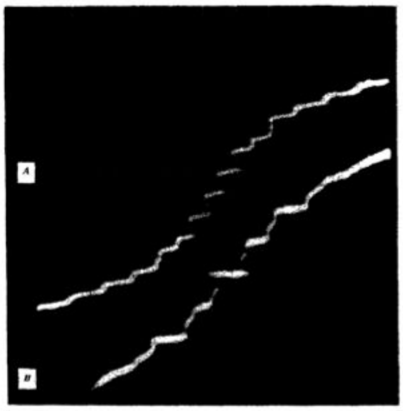

FIG. 3. Microwave power at 9300 Mc/sec (A) and 24850 Mc/sec (B) produces many zero-slope regions spaced at $h\nu/2e$ or $h\nu/e$. For A, $h\nu/e = 38.5\ \mu$V, and for B, 103 μV. For A, vertical scale is 58.8 μV/cm, horizontal scale is 67 nA/cm; for B, vertical scale is 50 μV/cm, horizontal scale is 50 μA/cm.

FIGURE 1.1 – A Cantor staircase with Shapiro steps from the original article [29]

1.1.3 Mathematical motivations

The first time the rotation numbers of families of circle diffeomorphisms were considered by V. I. Arnold for the following two-parametric family of diffeomorphisms of the circle (for ε small enough)

$$x \mapsto x + 2\pi a + \varepsilon \sin x, \tag{1.6}$$

where $x \in \mathbb{S}^1$ is a point on the circle, $a \in \mathbb{R}$ and $\varepsilon \in [0, 1)$ are parameters. Note first that for $\varepsilon = 0$ the family is just a family of rotations and hence $\rho_{a,0} = a$.

Arnold considered the level sets of the rotation number in the plane of parameters (a, ε) and he obtained the picture that can be seen on Figure 1.2. He was interested in the **phase-lock areas** (later baptized Arnold tongues) : level sets of the rotation number of *non-empty interior.* Arnold noticed that for the family (1.6) the tongues do not exist for irrational values of rotation number because of the Denjoy theorem and monotonicity arguments : it is easy to see that the set $\{\rho(a, \varepsilon) = \alpha \neq \mathbb{Q}\}$ is a continuous curve starting from the point $(\alpha, 0)$, see [1]. Then, for each $\alpha \in \mathbb{Q}$ the corresponding tongue $\{\rho(a, \varepsilon) = \alpha\}$ exists and "grows" from the point $(\alpha, 0)$. One can note that for a fixed $\varepsilon > 0$ the rotation number as a function of parameter a is a Cantor staircase, see Figure 1.3. But, contrary to the classical Cantor staircase, in this case the Cantor set of the points of growth (the closure of the set of parameters a corresponding to the irrational rotation numbers) has a positive Lebesgue measure.

The study of Arnold tongues of Josephson equation continues the process initiated by Arnold. But in the case of the equation (1.2) and the corresponding circle map $P_{a,b,\mu}$ the situation is drastically different. We will describe this situation in the next Section.

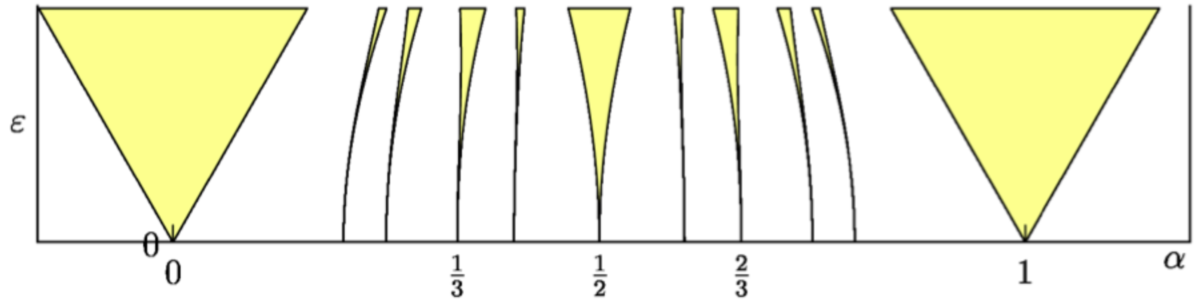

FIGURE 1.2 – Arnold tongues for a classical family (1.6) on the plane of parameters (a, ε) : the tongues that are drawn on the picture are those corresponding to rational values with denominator no bigger than 5.

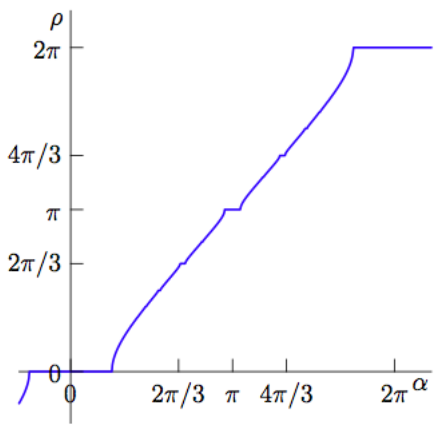

FIGURE 1.3 – A Cantor staircase for a standard Arnold family of circle diffeomorphisms (1.6) : the graph is a section of the picture on Figure 1.2 by a line $\{\varepsilon = \text{const}\}$

1.2 Why this equation is special ?

1.2.1 Only integer tongues exist : Riccati equation

First note that since the right-hand side of the equation (1.2) (and thus the map $\widetilde{P}_{a,b,\mu}$) grows monotonically with a, there is no phase lock for $k \notin \mathbb{Q}$. As said before, this happens generically to Arnold tongues : they are absent for irrational values of rotation number. But the specificity of the equation (1.1) gives that for $k \in \mathbb{Q} \setminus \mathbb{Z}$ there is no phase lock as well. This follows from the following lemma that was proven by R. Foote [11] and then rediscovered independently by Buchstaber-Karpov-Tertychnyi and Ilyashenko in [5] and in [16, 17] in the context of Josephson equation.

Lemma 1.3. *The Josephson equation* (1.2) *is conjugated to the Riccati equation with* 2π*- periodic coefficients and its Poincaré map* $P_{a,b,\mu}$ *from the transversal* $\{t = 0\}$ *to itself is conjugated to a Mœbius map.*

Proof. After a change of variables

$$u = \tan\frac{x}{2}, \frac{2\dot{u}}{u^2+1} = \dot{x}, \cos x = \frac{1-u^2}{1+u^2} \tag{1.7}$$

the system (1.2) becomes :

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \alpha(t)u^2 + \beta(t)u + \gamma(t), \\ \dfrac{\partial t}{\partial \tau} = \mu. \end{cases} \tag{1.8}$$

where $\alpha(t) = a + b\cos t - \frac{1}{2}, \beta(t) = 0, \gamma(t) = a + b\cos t + \frac{1}{2}$. The equation (1.8) is a Riccati equation with periodic coefficients – for this equation its Poincaré map from the transversal $\{t = 0\}$ to itself is Mœbius, see for example [9] for the proof.□

This Lemma gives immediately the result of the absence of the non-integer Arnold tongues.

Proposition 1.4. *The Arnold tongues for the rotation number* $\rho_{a,b,\mu}$ *corresponding to the Poincaré map* $P_{a,b,\mu}$ *of Josephson equation* (1.2) *do not exist for non-integer values of rotation number* $\rho \notin \mathbb{Z}$.

Proof. In the coordinates of Lemma 1.3 Josephson family corresponds to the family of Riccati equations with periodic coefficients. Hence the Poincaré map $P_{a,b,\mu}$is a Mœbius map in coordinates defined in (1.7).

But any Mœbius map has zero, one or two fixed points and is called in these cases respectively elliptic, parabolic or hyperbolic. Suppose $\rho_{a,b,\mu} = \frac{p}{q} \notin \mathbb{Z}$, then the Poincaré map $P_{a,b,\mu}$ is elliptic in some coordinates. Then, considering this map P_ρ as a map of a circle, after a change of coordinates it is just a circle rotation. And in this case, rotation number is destroyed by a small perturbation. □

So, Arnold tongues exist only for integer rotation numbers, and moreover, for a fixed μ a point (a, b) lies in the interior of the tongue if and only if the corresponding map $P_{a,b,\mu}$ is hyperbolic, and on the boundary of the tongue if and only if $P_{a,b,\mu}$ is parabolic (or identical). Indeed, the boundary corresponds to the case of fixed points of the map that disappear by small perturbation : those points are called parabolic. When one moves on the curve in the space of parameters starting inside the Arnold tongue in the direction of the boundary, the real solutions of the equation $P_{a,b,\mu}(z) = z$ collapse in one solution and on the exit the map to complex conjugated points.

This property of Josephson family (1.1) of not having the Arnold tongues for all rational values of the rotation number is unique : as has been proven recently by A. Glutsyuk and L. Rybnikov in [13] the family of equations on the torus of the form $\dot{x} = v(x) + A + Bf(t)$ won't have all of the rational Arnold tongues for all analytic functions f if and only if the function $v(x)$ is a sum of harmonics with the same frequencies $v(x) = a\sin(mx) + b\cos(mx) + c$. In this case the Arnold tongues exist for the values of rotation number lying in the set $\frac{1}{m}\mathbb{Z}$.

1.2.2 Dynamical description of the boundaries

In addition to the Mœbius property the additional important property of the Josephson equation (1.2) which is the symmetry of phase curves under the map

$$\tau : (t, x) \mapsto (-t, -x). \tag{1.9}$$

This symmetry property is easier to work with in the cosine form (1.1) than in sine form (1.4) – this is the main practical reason we passed from physical convention to this form.

This central symmetry property together with the Mœbius property of Poincaré map give an analytic description of the boundaries of Arnold tongues in terms of Poincaré map. This is a very useful fact : for the theoretical study of the geometry of Arnold tongues as well as for computer simulations of their boundaries.

Indeed, suppose the Poincaré map is not identical $P_{a,b,\mu} \neq \mathrm{id}$ and a point (a, b) is lying on a boundary of some Arnold tongue E_k defined by (1.3) corresponding to some integer value of rotation number $\rho_{a,b,\mu} = k, k \in \mathbb{Z}$ for a fixed μ. Then as we know from Lemma 1.3, $k \in \mathbb{Z}$. In this case the map $P_{a,b,\mu}$ has a fixed point : but since phase curves are preserved by the central symmetry this fixed point has to be mapped to another fixed point of the map $P_{a,b,\mu}$ by the mapping $\tau : x \mapsto -x$ on the circle $\mathbb{S}^1$. But a parabolic map has a unique fixed point hence this fixed point has to map to itself. There are two solutions of the equation $x = -x \mod 2\pi$ and hence two fixed points of the mapping τ : 0 and π. So the boundaries of an Arnold tongue E_k with the rotation number equal to $k \in \mathbb{Z}$ are two analytic curves $a_{0,k}(b)$ and $a_{\pi,k}(b)$ that are given correspondingly by the conditions

$$a = a_{0,k}(b) \Leftrightarrow P_{a,b,\mu}(0) = 0$$
$$a = a_{\pi,k}(b) \Leftrightarrow P_{a,b,\mu}(\pi) = \pi$$

Numerical experiments show that these curves $\{(b, a_{0,k}(b)) \,|b \in \mathbb{R}\}$ and $\{(b, a_{\pi,k}(b)) \,|b \in \mathbb{R}\}$ (for a fixed μ) intersect in a countable number of points that we will call **adjacency points**, see Figure 1.4. The mathematical proof of this fact is an easy consequence of our Theorem with Alexey Klimenko that we will present later. On the Figure 1.4 one can see that the boundaries "oscilate" : we prove that this oscilation in vertical direction (when $b \to \infty$) is close to the oscilation of integer Bessel functions, see our article in Subsection 1.3 for the precise statement. This effect of approximation of the boundaries by Bessel functions was first empirically observed in [15] by physicists Shapiro, Janus and Holly and we prove this result rigorously for the model (1.2).

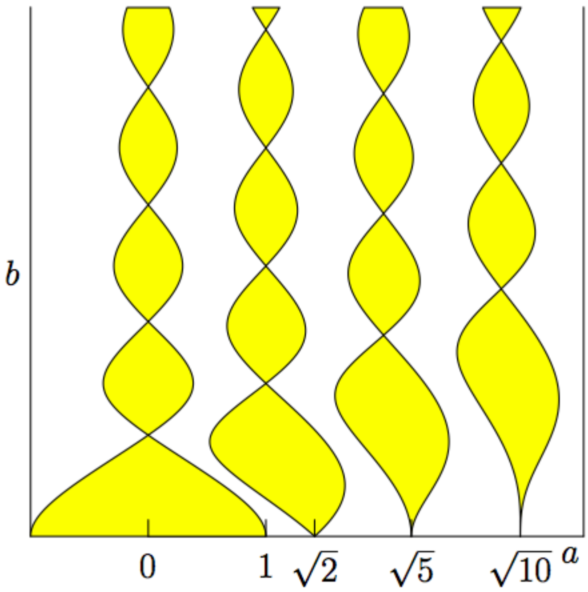

FIGURE 1.4 – Arnold tongues for Josephson equation on the plane of parameters (a, b) for a fixed μ, here $\mu = 1$. One can see that each Arnold tongue looks like a braid and has adjacency points - points of the intersection of its two analytic boundaries.

One can see from the Figure 1.4 that the adjacency points seem to occur on the same vertical line $\{a = k\mu\}$ where k is a number of the tongue (corresponding rotation number). IThis nice fact was proven (for $\mu \geq 1$) recently in [12] using the Stokes phenomena. For $\mu < 1$ this is not yet proven but stays a reasonable conjecture. The difficulty resides in a study of adjacency points near the line $\{b = 0\}$.

1.2.3 The roots of the tongues

We want to give two small remarks about the geometrical structure of the tongues that will be useful for the following : first, for the reasons of symmetry discussed in Subsection 1.2.2 we will be interested in the structure of the tongues in the first quadrant $\{a, b > 0\} \in \mathbb{R}^2$ of the parameter plane. The second remark is that we actually know the *roots* of the tongues (the points on the $\{b = 0\}$ axis where the tongues "start"). In other words, the exact value of rotation number $\rho_{a,0,\mu}$ can be calculated since for $b = 0$ the equation (1.1) can be explicitly integrated. After some calculations [17], one can find that the tongue number k intersects the line $\{b = 0\}$ in the point $(\text{sgn}\, k \cdot \sqrt{k^2\mu^2 + 1}, 0)$ if $k \neq 0$ and on the whole interval $[-1, 1]$ for $k = 0$. So the empirical picture is the following : except for the tongue number 0, the tongues for $k > 0$ root slightly on the right and then incline to the left and oscillate around the line $\{a = k\mu\}$ with Bessel asymptotic.

In this chapter, we present two main results. We study the sections of Arnold tongues by the planes with fixed μ in two different regimes :

- *First regime : big amplitude.* We suppose that the amplitude of the current is big enough. We call this case $b \to \infty$. It is considered in our article with Alexey Klimenko, see

Subsection 1.3, and we describe an interesting asymptotic form of the tongues in this regime. Although in this case μ will be fixed, we still take it into account. Particularly, the dependence of estimating functions on μ will be explicit.

Second regime : small external frequency. This case corresponds to the limit $\mu \to 0$ and is more relevant for applications for Josephson junction dynamics. We explain the qualitative behavior of the system in this case and show the connection between the geometrical structure of the tongues with slow-fast properties of the equation (1.2). We show the zones in the space of parameters which are almost covered by the carpet of Arnold tongues (except for the gaps, exponentially small in μ). We also exhibit an effective algorithm of construction of the boundaries of those tongues for small values of $\mu, \mu \sim 0.01$).

1.3 First regime : big amplitude

This case is completely described in our article that is joined untouched (references in the article correspond to the bibliography of the article and not the bibliography of the Chapter).

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ASYMPTOTIC PROPERTIES OF ARNOLD TONGUES AND JOSEPHSON EFFECT

A. KLIMENKO AND O. ROMASKEVICH

To our dear teacher Yu. S. Ilyashenko on his 70th birthday

Abstract. A three-parametrical family of ODEs on a torus arises from a model of Josephson effect in a resistive case when a Josephson junction is biased by a sinusoidal microwave current. We study asymptotics of Arnold tongues of this family on the parametric plane (the third parameter is fixed) and prove that the boundaries of the tongues are asymptotically close to Bessel functions.

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Key words and phrases. Arnold tongues, Josephson effect, dynamical system on the torus, rotation number.

1. Introduction

We will deal with a family of differential equations on a circle $\mathbb{R}/2\pi\mathbb{Z}$

$$\frac{dx}{dt} = \frac{\cos x + a + b\cos t}{\mu}, \tag{1}$$

which arises in physics of the Josephson effect[1]. In the paper we refer to (1) as to *the Josephson equation*.

Here a, $b \in \mathbb{R}$, and $\mu > 0$ are parameters. Such a family was studied in the context of Prytz planimeter [8] as well as in the context of bicycle track trajectories [7], [15]. For the first time the techniques of slow-fast systems (for $\mu \ll 1$) were applied to this equation by J. Guckenheimer and Yu. Ilyashenko in [10] but in the context of Josephson effect the family (1) has not been studied from a mathematical point of view till the series of works [2], [3] by V. M. Buchstaber, O. V. Karpov and S. I. Tertychnyi. Now this subject has become quite popular, see for instance [4], [5], [12], [14], [9], [6].

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[1]In physical literature this equation is often written with sines instead of cosines. Substitutions $x \to x \pm \pi/2$, $t \to t \pm \pi/2$ transform one variant to another; one can see that these substitutions do not affect any results of this paper.

The family (1) can be generalized to the following form:

$$\frac{dx}{dt} = \frac{f(x) + a + bg(t)}{\mu}, \tag{2}$$

where f and g are 2π-periodic functions with zero averages:

$$\int_0^{2\pi} f(x)\, dx = 0, \qquad \int_0^{2\pi} g(t)\, dt = 0. \tag{3}$$

Any equation of the form (2) defines a vector field on a two-dimensional torus $\mathbb{R}^2/2\pi\mathbb{Z}^2$ with coordinates x and t. Namely, introducing a new time variable τ, we can express this vector field as

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = f(x) + a + bg(t), \\ \dfrac{\partial t}{\partial \tau} = \mu. \end{cases} \tag{4}$$

The same vector field can also be considered as a vector field on the cylinder $\mathbb{R}^2/((x,\, t) \sim (x,\, t + 2\pi))$. In both cases the Poincaré map from the transversal line $\{t = 0 \bmod 2\pi\}$ to itself can be defined, we denote it as $P_{a,b,\mu}$ for the torus, and $\widetilde{P}_{a,b,\mu}$ for the cylinder. Clearly, $\widetilde{P}_{a,b,\mu}$ is a lift of $P_{a,b,\mu}$.

Consider the *rotation number* $\rho_{a,b,\mu}$ of the map $P_{a,b,\mu}$, which is, by definition, the limit

$$\rho_{a,b,\mu} := \lim_{n\to\infty} \frac{\widetilde{P}^{\circ n}_{a,b,\mu}(x) - x}{2\pi n}.$$

It is well known that this limit exists and does not depend on the point $x \in \mathbb{R}$ (see, for example, [13]). The value of the rotation number is an important characteristic of a map $P_{a,b,\mu}$: for instance, it is invariant under conjugation by homeomorphisms.

Definition 1. We say that the *phase lock* occurs for the value $k \in \mathbb{R}$ of rotation number if the level set

$$E_k := \{(a,\, b,\, \mu)\colon \rho_{a,b,\mu} = k\}$$

in the space of parameters $\mathbb{R}^2 \times \mathbb{R}_+$ has nonempty interior. In this case the level set E_k is called an *Arnold tongue*.

The structure of Arnold tongues for the equation (1) and its generalizations is of a great interest for physical applications as well as from a purely mathematical point of view. We study sections of Arnold tongues by the planes with fixed μ. Nevertheless, we still take μ into account, particularly, constants in the $O(\,\cdot\,)$'s do not depend on μ.

Since the right-hand side of the equation (2) (and thus the map $\widetilde{P}_{a,b,\mu}$) grows monotonically with a there is no phase lock for $k \notin \mathbb{Q}$. This happens generically to Arnold tongues: they are absent for irrational values of rotation number. Moreover, the specificity of the equation (1) gives that for $k \in \mathbb{Q} \setminus \mathbb{Z}$ there is no phase lock as well.

It's easy to see that the substitution $u = \tan \frac{x}{2}$ conjugates the equation (1) to a Riccati equation. This fact was noticed by R. Foote in [8] in the context of Prytz planimeter then rediscovered independently by Yu. Ilyashenko [11], [12] and

V. Buchstaber, O. Karpov, S. Tertychnyj [4] in the context of Josephson effect. This simple but important remark gives that the Poincaré map $P_{a,b,\mu}$ is conjugated to a Möbius transformation. Lots of uncommon properties of Josephson equation follow from this fact, the absence of phase lock for non-integer rotation numbers as one of the examples.

Indeed, if $\rho_{a,b,\mu} = p/q$, $q > 1$, then the Poincaré map has a periodic point of period q. But a Möbius transformation with periodic non-fixed points should be periodic itself. Therefore, $(\widetilde{P}_{a,b,\mu})^q(x) = x + p$. Monotonicity in a yields that this identity can appear for only one value of a provided b and μ are fixed, hence the level set has empty interior.

So for a fixed μ there is a countable number of tongues on the plane of parameters (a, b), corresponding to integer rotation numbers. From now on we will consider the half-plane $b > 0$; another half-plane could be studied using symmetries of the equation.

The previous argument uses only the fact that $f(x) = \cos x$, imposing no conditions on $g(t)$. But when g is even (in particular, when $g(t) = \cos t$) the equation (4) possesses an additional symmetry: the map $(x, t) \mapsto (-x, -t)$ brings phase curves to themselves with orientation reversed. This means that $-P_{a,b,\mu}(-x) = P^{-1}_{a,b,\mu}(x)$. Hence, if x_0 is fixed point of $P_{a,b,\mu}$, then $-x_0$ is also a fixed point. If the point (a, b, μ) lies on the boundary of an Arnold tongue then the Möbius map $\widetilde{P}_{a,b,\mu}$ is either parabolic or identity. In the parabolic case its only fixed point $\hat{x}$ should satisfy $\hat{x} \equiv -\hat{x} \pmod{2\pi}$, hence $\hat{x}$ is either 0 or π.

For any fixed b and μ the set $E_k^{b,\mu} = \{a \in \mathbb{R}\colon (a, b, \mu) \in E_k\}$ is a closed interval $E_k^{b,\mu} = [a^-_{b,\mu}, a^+_{b,\mu}]$. When a varies from left end $a^-_{b,\mu}$ of the interval $E_k^{b,\mu}$ to its right end $a^+_{b,\mu}$, the set $\{x\colon \widetilde{P}_{a,b,\mu}(x) > x + k\}$ grows monotonically since the right-hand side of the equation (2) is monotonic in a. Thus, if $\hat{x}$ is a fixed point of $P_{a,b,\mu}$ for $a = a^-_{b,\mu}$, then $\widetilde{P}_{a,b,\mu}(\hat{x}) > \hat{x} + k$ for all $a \in E_k^{b,\mu}$, except $a = a^-_{b,\mu}$, and $\hat{x}$ can not be a fixed point of $P_{a,b,\mu}$ for $a = a^+_{b,\mu}$. Hence $P_{a,b,\mu}$ has a fixed point 0 at one end of the segment $E_k^{b,\mu}$ and π on its other end.

Therefore, for a fixed μ, boundary of the Arnold tongue with rotation number equal to $k \in \mathbb{Z}$ can be presented as a union of two graphs of analytic functions denoted by $a_{0,k}(b)$ and $a_{\pi,k}(b)$, where 0 (respectively, π) is fixed by Poincaré map when $a = a_{0,k}(b)$ (respectively, $a = a_{\pi,k}(b)$). These graphs can intersect, and the Poincaré map $\widetilde{P}_{a,b,\mu}$ is identical at the intersection points.

2. Main Results

We are interested in the asymptotics of the boundaries $a_{0,k}(b)$ and $a_{\pi,k}(b)$ of Arnold tongues for (1) as $b \to \infty$. These estimates will be established in two steps. First, in Theorem 1 we show that the boundaries $a_{0,k}(b)$ and $a_{\pi,k}(b)$ are close to the line $a = k\mu$. Thereupon we show in Theorem 2 that the functions $a_{0,k}(b) - k\mu$ and $a_{\pi,k}(b) - k\mu$ are asymptotically close to normalized integer Bessel functions. This fact was noticed for the first time in [17], right after the discovery of the Josephson effect in 1962 with the first explanation on a physical level of rigor; see also Chapter 5 in [16], Section 11.1 in [1], and [4]. In this paper we give a complete proof of this statement, as well as the estimates on the difference.

Theorem 1. *There exist positive constants C_1, C_2, K_1, K_2 such that the following holds.*

If the parameters a, b, μ are such that

$$|a| + 1 \leqslant C_1\sqrt{b\mu}, \qquad b \geqslant C_2\mu \tag{5}$$

then

$$\left|\frac{a}{\mu} - \rho_{a,b,\mu}\right| \leqslant \frac{K_1}{\sqrt{b\mu}} + \frac{K_2}{b\mu}\ln\left(\frac{b}{\mu}\right) \leqslant \frac{K_1}{\sqrt{b\mu}} + \frac{2K_2}{\sqrt{b\mu^3}}. \tag{6}$$

Theorem 2. *There exist positive constants C_1', C_2', K_1', K_2', K_3' such that the following holds.*

For the parameters b, μ and a number $k \in \mathbb{Z}$ satisfying inequalities

$$|k\mu| + 1 \leqslant C_1'\sqrt{b\mu}, \qquad b \geqslant C_2'\mu \tag{7}$$

the following estimates hold

$$\begin{aligned} \left|\frac{a_{0,k}(b)}{\mu} - k + \frac{1}{\mu}J_k\left(-\frac{b}{\mu}\right)\right| &\leqslant \frac{1}{b}\left(K_1' + \frac{K_2'}{\mu^3} + K_3'\ln\left(\frac{b}{\mu}\right)\right), \\ \left|\frac{a_{\pi,k}(b)}{\mu} - k - \frac{1}{\mu}J_k\left(-\frac{b}{\mu}\right)\right| &\leqslant \frac{1}{b}\left(K_1' + \frac{K_2'}{\mu^3} + K_3'\ln\left(\frac{b}{\mu}\right)\right). \end{aligned} \tag{8}$$

Theorem 2 is our main result — it shows how the boundaries of Arnold tongues could be approximated by Bessel functions if b is sufficiently large; this is illustrated by Figure 1.

Recall that the Bessel function of the first kind can be defined as

$$J_k(-z) = \frac{1}{2\pi}\int_0^{2\pi}\cos(kt + z\sin t)dt. \tag{9}$$

It has the following asymptotics for large z (see [18]):

$$J_k(-z) = \sqrt{\frac{2}{\pi z}}\cos\left(-z - \frac{k\pi}{2} + \frac{\pi}{4}\right) + O\left(\frac{1}{z^{3/2}}\right) \quad \text{as } z \to +\infty.$$

Applying this to (8) we obtain

$$a_{\dots,k}(b) = k \pm \sqrt{\frac{2}{\pi b\mu}}\cos\left(\frac{b}{\mu} - \frac{k\pi}{2} + \frac{\pi}{4}\right) + O_\mu(b^{-1}\ln b). \tag{10}$$

(Here $O_\mu(\,\cdot\,)$ is $O(\,\cdot\,)$ with the constant depending on μ.) Therefore, the Bessel asymptotics is indeed the main term for $a_{\dots,k}(b)$. In particular, (10) means that the graphs of $a_{0,k}(b)$ and $a_{\pi,k}(b)$ do have infinitely many intersections, that is, each Arnold tongue has infinitely many horizontal sections of zero width. The points (a, b) on the plane of parameters corresponding to the intersections of the boundaries of some Arnold tongue are clearly very special. Poincaré map $P_{a,b,\mu}$ corresponding to such points is an identity map.

Definition 2. Point $(a, b) \in \mathbb{R}^2$, $b \neq 0$, on the boundary of the Arnold tongue with $\rho_{a,b,\mu} = k \in \mathbb{Z}$ is called an *adjacency point* if it lies on the intersection of the boundaries, i.e., $a = a_{0,k}(b) = a_{\pi,k}(b)$.

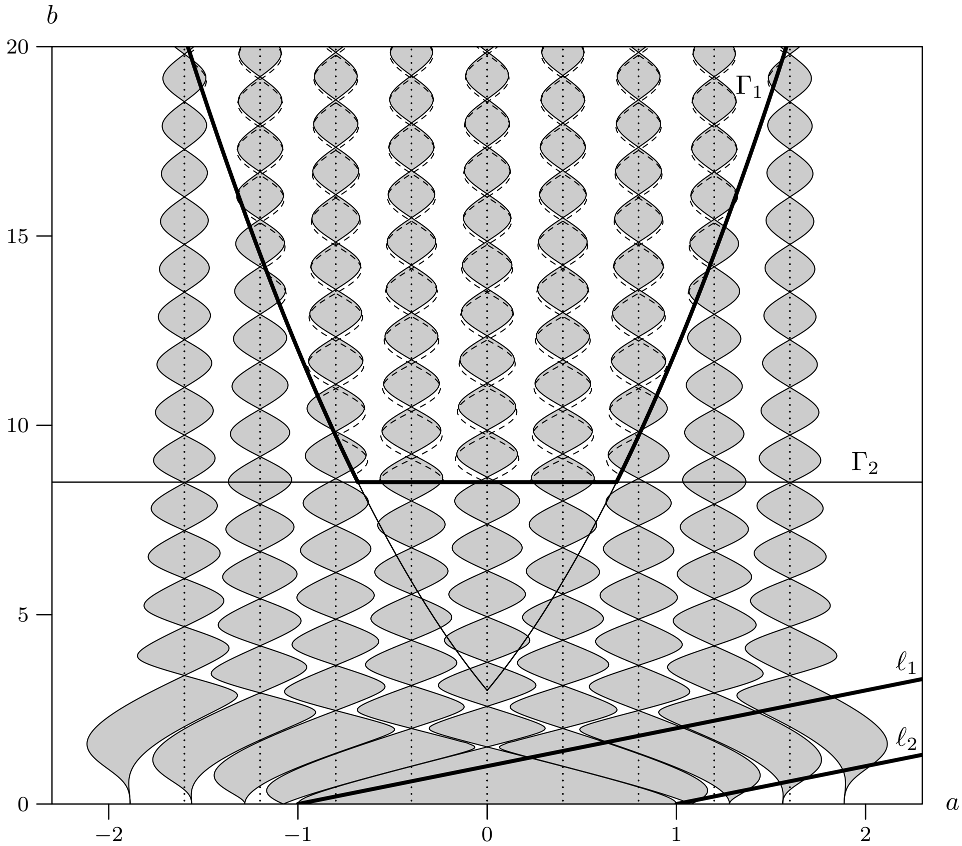

FIGURE 1. Arnold tongues of Josephson equation in the domain on the plane of parameters $(a,\, b)$ for a fixed $\mu = 0.4$

Grey domains are Arnold tongues E_k for $k = -4, \ldots, 4$, their boundaries (solid lines) are curves $a = a_{0,k}(b)$ and $a = a_{\pi,k}(b)$. Curves Γ_1 and Γ_2 are defined by conditions of the form (5). The estimates of Theorem 2 are applicable in the domain above both Γ_1 and Γ_2 (contoured with bold line). Relations between conditions of the form (5) and the conditions (7) of Theorem 2 are discussed in the first part of the proof of this theorem. Dashed lines in this domain represent Bessel approximations given by Theorem 2.
Dotted lines are the lines $a = k\mu$, which contain all adjacency points of Arnold tongues [9].
Domain between the lines ℓ_1 and ℓ_2 is where the slow-fast techniques work [14].
The computer simulations for this picture were done by I. Schurov.

Recently many interesting results on the structure of Arnold tongues for the Josephson equation were discovered. Here we present a brief summary.

First of all, the k-th Arnold tongue E_k intersect the line $b = 0$ at one point $(\operatorname{sgn} k \cdot \sqrt{k^2\mu^2 + 1}, 0)$ if $k \neq 0$, and E_0 intersects this line by the segment $[-1, 1]$ (for $b = 0$ the equation (1) does not depend on time and can be easily integrated). As we have noted above, Theorem 2 implies that each tongue has infinitely many adjacency points.

What is more surprising, Figure 1 suggests the following conjecture: the adjacency points of a k-th Arnold tongue lie on the same line $a(b) \equiv k\mu$ (dotted lines on Fig. 1). This is proven in [9] for $\mu \geqslant 1$ and the proof uses the classical theory of non-autonomous linear equations of complex variable. For $\mu < 1$ this fact is not yet proven and rests a reasonable conjecture. The difficulty resides in a study of adjacency points near the line $b = 0$.

The result in [9] is the only global non-trivial result on the structure of Arnold tongues of (1), other results concentrate attention on the behaviour in some domains of the parameter plane.

For instance, for μ small enough the techniques of slow-fast systems can be used to show that the domain between the lines $\ell_1 = \{b = a + 1\}$ and $\ell_2 = \{b = a - 1\}$ is filled up tightly with Arnold tongues and the distances between the tongues diminish exponentially in μ. For the review of slow-fast techniques for (1) see [14].

The overall picture of the behaviour of Arnold tongues is the following: in any finite domain around the line $b = 0$ the tongues fill up tightly the space [14] and for b tending to infinity (when "b is bigger than μ is smaller") Bessel behaviour prevails over the slow-fast one. This is, however, very sketchy, and many questions still can be asked on a local behaviour of Arnold tongues. As an example, it seems from the picture that the right boundaries of Arnold tongues E_k, $k > 0$, have inflection points on the line ℓ_2. We have no idea if it is true or not and how to prove it.

Now let us sketch proofs of Theorem 1 and Theorem 2. First of all, we rewrite (1) as an integral equation

$$x(t) - x(0) = \frac{at + b\sin t + \int_0^t \cos x(\tau)\, d\tau}{\mu} \tag{11}$$

and use the fact that for the most part of the segment $[0, 2\pi]$ the function $\cos x(t)$ oscillates very fast, since dx/dt is large if only $|\cos t|$ is not very small. It will be shown that this implies that the integral in (11) is quite small, hence for all solutions of (1) the difference $x(2\pi) - x(0) = \widetilde{P}_{a,b,\mu}(x(0)) - x(0)$ is close to $2\pi a/\mu$. But if the circle map is uniformly $2\pi\varepsilon$-close to the rigid rotation by the angle $2\pi\alpha$, then its rotation number is ε-close to α. Therefore, inside the k-th Arnold tongue a/μ should be close to k, hence a is close to $k\mu$.

For the second theorem, we expand the integral in (11) using the formula (11) itself:

$$x(2\pi) - x(0) = \frac{2\pi a}{\mu} + \frac{1}{\mu}\int_0^{2\pi} \cos\left(\frac{a\tau + b\sin\tau + \int_0^\tau \cos x(s)\, ds}{\mu} + x(0)\right) d\tau. \tag{12}$$

On the boundary of the Arnold tongue, where either $a = a_{0,k}(b)$ or $a = a_{\pi,k}(b)$, the left-hand side equals $2\pi k$ if either $x(0) = 0$ or $x(0) = \pi$. We will show that the inner integral is small and its influence on the value of the outer integral is also small so it can be dropped. Then, we replace $a\tau$ with $k\mu\tau$ inside the outer integral (since $a - k\mu$ is small due to Theorem 1). This yields a change of the outer integral in (12) by the amount of the next order of magnitude. Therefore

$$2\pi k \approx \frac{2\pi a}{\mu} + \frac{1}{\mu}\int_0^{2\pi} \cos\left(k\tau + \frac{b}{\mu}\sin\tau + x(0)\right) d\tau.$$

The integral on the right-hand side can be expressed in terms of $J_k(z)$ by (9) and we thus obtain

$$k \approx \frac{a}{\mu} \pm \frac{1}{\mu} J_k(-b/\mu),$$

where the sign is "+" if $x(0) = 0$ and "−" if $x(0) = \pi$.

The remaining part of this paper is organized as follows. In the next section we obtain several estimates for the integral $\int_0^\tau \cos x(s)\, ds$ and related values. In Section 4 we deduce Theorems 1 and 2 from these estimates. Finally, in Section 5 we discuss partial generalizations of these results to the equations of type (2).

3. Estimations of the Integrals

In what follows in the next section we will need estimates for the integral expressions contained both in (11) and (9). Fortunately, these estimates can be obtained simultaneously. Indeed, consider an equation

$$\frac{dx}{dt} = \frac{\gamma\cos x + a + b\cos t}{\mu}. \tag{13}$$

If $\gamma = 1$, we obtain the standard Josephson equation (1), while if $\gamma = 0$ we obtain integrable differential equation with solutions

$$x(t) = x(0) + \frac{at + b\sin t}{\mu}.$$

Therefore, if $\hat{x}(t)$ is the solution of an equation corresponding to $\gamma = 0$ with an initial condition $\hat{x}(0) = 0$, then $\int_0^{2\pi} \cos\hat{x}(\tau)\, d\tau$ coincides with the integral in (9) for $k = \frac{a}{\mu}$ and $z = -\frac{b}{\mu}$.

Below we always assume that

$$|\gamma| \leqslant 1.$$

The main instrument in our proof is the following Lemma 1. Informally speaking, it states that if $x(t)$ is moving with almost constant speed, then the time average of a bounded function ψ and its space average along the same arc of a trajectory are close to each other.

Lemma 1. *Suppose that $\dot{x}(t)$ is of the constant sign for $t \in [t_0, t_1]$. Denote*

$$|\dot{x}|_{\min} = \min_{t\in[t_0,t_1]} |\dot{x}(t)|\,, \qquad |\dot{x}|_{\max} = \max_{t\in[t_0,t_1]} |\dot{x}(t)|\,, \qquad \operatorname*{osc}_{[t_0,t_1]}(\dot{x}) = |\dot{x}|_{\max} - |\dot{x}|_{\min}\,.$$

Then for any bounded integrable function ψ on a circle we have

$$\left|\frac{1}{t_1-t_0}\int_{t_0}^{t_1}\psi(x(t))\,dt-\frac{1}{x_1-x_0}\int_{x_0}^{x_1}\psi(x)\,dx\right|\leqslant\frac{\operatorname{osc}_{[t_0,t_1]}(\dot{x})}{|\dot{x}|_{\min}}\cdot\|\psi\|_{C^0}\,,\tag{14}$$

where $x_0=x(t_0)$, $x_1=x(t_1)$, $\|\psi\|_{C^0}=\sup_{x\in\mathbb{R}/2\pi\mathbb{Z}}|\psi(x)|$.

Proof. Indeed,

$$\left|\frac{1}{t_1-t_0}\int_{t_0}^{t_1}\psi(x(t))\,dt-\frac{1}{x_1-x_0}\int_{x_0}^{x_1}\psi(x)\,dx\right|=$$
$$=\left|\frac{1}{x_1-x_0}\int_{x_0}^{x_1}\left[\frac{x_1-x_0}{t_1-t_0}\cdot\frac{dt}{dx}-1\right]\psi(x)\,dx\right|.$$

It remains to show that the absolute value of the expression in square brackets is not more than $\operatorname{osc}_{[t_0,t_1]}(\dot{x})/|\dot{x}|_{\min}$. Suppose that $\dot{x}(t)$ is positive on $[t_0,\,t_1]$. Then $(x_1-x_0)/(t_1-t_0)$ and dx/dt belong to $[|\dot{x}|_{\min}\,,\,|\dot{x}|_{\max}]$, hence

$$\frac{|\dot{x}|_{\min}}{|\dot{x}|_{\max}}-1\leqslant\frac{x_1-x_0}{t_1-t_0}\cdot\frac{dt}{dx}-1\leqslant\frac{|\dot{x}|_{\max}}{|\dot{x}|_{\min}}-1,$$

and finally, we obtain that

$$\left|\frac{x_1-x_0}{t_1-t_0}\cdot\frac{dt}{dx}-1\right|\leqslant\frac{\operatorname{osc}_{[t_0,t_1]}(\dot{x})}{|\dot{x}|_{\min}},$$

so inequality (14) is proven. The case of negative $\dot{x}(t)$ is treated similarly. □

Consider a solution $x(t)$ of equation (13) on some interval $[0,\,t^*]$. Take all points $0=t_0<t_1<\cdots<t_k\leqslant t^*$ such that $x(t_k)\equiv x(0)\pmod{2\pi}$ and split the interval $[0,\,t^*]$ by these points into subsegments $I_i=[t_{i-1},\,t_i]$, $i=1,\,\ldots,\,k$, and $I^*=[t_k,\,t^*]$.

As it was said before, the subintervals with "small" and "not so small" values of $|\dot{x}|$ are treated differently. Consider a set

$$M_\delta=\{\tau\in[0,\,t]\colon\ |\cos\tau|<\delta\},$$

where δ will be chosen later.

However, from now on we assume that

$$A:=|a|+1\leqslant\frac{b\delta}{C_a},\tag{15a}$$

$$\delta\geqslant C_b\sqrt{\frac{\mu}{b}},\tag{15b}$$

$$\delta\leqslant 1,\tag{15c}$$

where positive constants C_a and C_b are sufficiently large.

The subsegments I_i and I^* thus fall into the following categories:

type 1 *segments*: the subsegments that are fully covered by M_δ;

type 2 *segments*: the subsegments I_i that are partially covered by M_δ and I^* in the case if it is not fully covered by M_δ;

type 3 *segments*: the subsegments I_i that are not intersecting with M_δ.

Note that there is no more than five segments of type 2 since any such segment is either I^*, or contains one of the four points τ with $|\cos\tau| = \delta$ in its interior. Let us also denote by $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ the union of all the segments of corresponding type.

We start with an estimate for the length of the segment of type 2 or 3.

Remark. In the exposition below we use the notation $u(s) = O(v(s))$ in the following precise sense: there exist a constant C such that $|u(s)| \leqslant Cv(s)$ (here $v(s)$ is always positive), and this constant is assumed to be independent from parameters a, b, μ, from the values of δ, C_a, C_b (but we still suppose that (15) holds), and from any other variables. Informally speaking, one can fix some large explicit values for C_a and C_b (say, one million) and then replace all $O(\,\cdot\,)$'s in the text below with some explicit estimates. We prefer not to make these hindsight substitutions in order not to hide dependencies between constants in different estimates.

Proposition 1. *If C_a and C_b in (15) are sufficiently large, then the following holds.*

Let I be any segment of type 2 or 3. Let $\hat t$ be any point in $I \setminus M_\delta$. Then the length $|I|$ of this segment satisfies the following estimate:

$$|I| = O\biggl(\frac{\mu}{b\cos\hat t}\biggr).$$

Proof. The proof for type 3 segments is trivial: $x(t)$ travels distance not more than 2π with its speed bounded from below, thus the time of the travel is bounded from above. However, for the type 2 segments we need a sort of bootstrapping argument: the lower bound for the speed holds only for initial moment $\hat t$ and it worsens as time goes; nevertheless, it worsens so slowly that we cannot travel distance of 2π for such a long time that the speed estimate is totally ruined.

Let us pass to the formal proof. Denote $I = [t_-,\, t_+]$, $L = |I| = t_+ - t_-$. Inequality $|x(t_+) - x(t_-)| \leqslant 2\pi$ and the mean value theorem yields that

$$|I| \cdot \min_I |\dot x| \leqslant |x(t_i) - x(t_{i-1})| \leqslant 2\pi. \tag{16}$$

For any $\tau \in I$ we have $\tau = \hat t + s$ for some s with $|s| \leqslant L$. Therefore,

$$\begin{aligned}|\dot x(\hat t + s)| &\geqslant \left|\frac{b\cos(\hat t + s)}{\mu}\right| - \left|\frac{\gamma\cos x + a}{\mu}\right| \\ &\qquad\geqslant \frac{b\bigl(|\cos\hat t| - |\cos(\hat t + s) - \cos\hat t|\bigr)}{\mu} - \frac{A}{\mu} \geqslant \frac{b\,|\cos\hat t| - A - b\cdot L}{\mu}\end{aligned}$$

since the cosine is a Lipschitz function with constant equal to one. Now (16) yields

$$L \cdot \frac{b\,|\cos\hat t| - A - b\cdot L}{\mu} \leqslant 2\pi.$$

The same argument works for any subsegment $\tilde I \subset I$ such that $\hat t \in \tilde I$. We can choose such $\tilde I$ to be of any length between zero and L, so

$$by^2 - (b\,|\cos\hat t| - A)y + 2\pi\mu \geqslant 0 \quad \text{for any } y \in [0,\, L].$$

Since $|\cos \hat{t}| > \delta$, one can see that if $C_a \geqslant 2$ and $C_b \geqslant 32\pi$ then it follows from (15a) and (15b) that this quadratic polynomial has two positive real roots. Therefore, L does not exceed its smaller root:

$$L \leqslant \frac{b\,|\cos \hat{t}| - A - \sqrt{(b\,|\cos \hat{t}| - A)^2 - 8\pi b\mu}}{2b}$$
$$= \frac{4\pi\mu}{b\,|\cos \hat{t}| - A + \sqrt{(b\,|\cos \hat{t}| - A)^2 - 8\pi b\mu}} \leqslant \frac{4\pi\mu}{b\,|\cos \hat{t}| - A} \leqslant \frac{8\pi\mu}{b\,|\cos \hat{t}|}.$$

The last inequality here uses (15a) with $C_a \geqslant 2$. The proposition is proven. □

This yields the estimate of a Lebesgue measure of $\mathcal{I}_1 \cup \mathcal{I}_2$, which we denote by the symbol $\operatorname{mes}(\,\cdot\,)$.

Proposition 2. *If C_a and C_b in (15) are sufficiently large, then*

$$\operatorname{mes}(\mathcal{I}_1 \cup \mathcal{I}_2) = O\left(\frac{\mu}{b\delta} + \delta\right).$$

Proof. The set $\mathcal{I}_2$ consists of not more than five segments, and the length of each of them is bounded by Proposition 1 (we choose $\hat{t}$ with $|\cos \hat{t}| = \delta$):

$$\operatorname{mes} \mathcal{I}_2 = 5 \cdot O(\mu/b\delta).$$

The set $\mathcal{I}_1$ is a subset of M_δ, hence

$$\operatorname{mes} \mathcal{I}_1 \leqslant \operatorname{mes} M_\delta \leqslant 4 \arcsin \delta \leqslant 4 \cdot \frac{\pi}{2}\delta = O(\delta).$$ □

Now, let us estimate the integral over any subsegment I_k of type 3.

Proposition 3. *If C_a and C_b in (15) are sufficiently large then for any bounded function $h\colon \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ with zero average:*

$$\int_0^{2\pi} h(\xi)\, d\xi = 0,$$

and for any segment I_j of the type 3 we have

$$\left| \int_{I_j} h(x(\tau))\, d\tau \right| \leqslant \|h\|_{C^0} \int_{I_j} \left[O\left(\frac{1}{b\,|\cos \hat{t}|}\right) + O\left(\frac{\mu}{b \cos^2 \hat{t}}\right) \right] d\hat{t}.$$

Proof. It follows from Lemma 1 that

$$\frac{1}{|I_j|} \left| \int_{I_j} h(x(\tau))\, d\tau \right| \leqslant \|h\|_{C^0} \cdot \frac{\operatorname{osc}_{I_j}(\dot{x})}{\min_{I_j} |\dot{x}|}. \tag{17}$$

Here we use that $x(t_j) - x(t_{j-1}) = \pm 2\pi$, hence $\int_{x(t_{j-1})}^{x(t_j)} h(x)\, dx = 0$.

In order to estimate expressions in the right-hand side of (17), we take any $\hat{t} \in I_j$; Proposition 1 and Lipschitz property of cosine then give us that

$$\operatorname{osc}_{I_j}(\cos t) \leqslant |I_j| \leqslant O\left(\frac{\mu}{b\,|\cos \hat{t}|}\right).$$

Further,

$$\operatorname{osc}_{I_j}(\dot{x}(t)) \leqslant \frac{1}{\mu}\big(\operatorname{osc}_{I_j}\cos x(t) + b\operatorname{osc}_{I_j}(\cos t)\big) \leqslant \frac{2}{\mu} + O\biggl(\frac{1}{|\cos\hat{t}|}\biggr),$$

$$\begin{aligned}\min\nolimits_{I_j}|\dot{x}(t)| &\geqslant \frac{1}{\mu}\big(\min\nolimits_{I_j}|b\cos t| - A\big) \geqslant \frac{1}{\mu}\big(b\,|\cos\hat{t}| - \operatorname{osc}_{I_j}(b\cos t) - A\big)\\ &= \frac{1}{\mu}\biggl(b\,|\cos\hat{t}| - O\biggl(\frac{\mu}{|\cos\hat{t}|}\biggr) - A\biggr).\end{aligned} \tag{18}$$

For sufficiently large C_a and C_b, (15a) and (15b) make the second and the third terms in the right-hand side of (18) to be smaller than $b\,|\cos\hat{t}|\,/3$, hence

$$\min_{I_j}|\dot{x}(t)| \geqslant \frac{b\,|\cos\hat{t}|}{3\mu}.$$

Therefore,

$$\frac{1}{|I_j|}\left|\int_{I_j} h(x(\tau))\,d\tau\right| \leqslant \|h\|_{C^0}\cdot O\biggl(\frac{1}{b\,|\cos\hat{t}|} + \frac{\mu}{b\cos^2\hat{t}}\biggr),$$

and it remains to integrate the last inequality over $\hat{t}\in I_j$. □

4. Proofs of Theorems

Proof of Theorem 1. Note that if a circle map is uniformly $2\pi\varepsilon$-close to the rigid rotation by the angle $2\pi a/\mu$, then its rotation number is ε-close to a/μ. For any solution $x(t)$ of (1) we have

$$\left|\frac{x(2\pi)-x(0)}{2\pi} - \frac{a}{\mu}\right| = \left|\frac{1}{2\pi\mu}\int_0^{2\pi}\cos x(t)\,dt\right|,$$

hence the first inequality in (6) follows from the next proposition. The second inequality in (6) uses simple estimate $\ln z < 2\sqrt{z}$. □

Proposition 4. *There exist positive constants C_1, C_2, K_1, K_2 such that the following holds.*

If parameters a, b, μ satisfy (5) then for any $t^*\in[0,\,2\pi]$ and any solution $x(t)$ of (1) we have

$$\left|\int_0^{t^*}\cos x(t)\,dt\right| \leqslant K_1\sqrt{\frac{\mu}{b}} + \frac{K_2}{b}\ln\biggl(\frac{b}{\mu}\biggr). \tag{19}$$

Proof. 1. Fix values of C_a and C_b such that Propositions 1, 2, and 3 hold for them. Let us also assume that $C_b \geqslant 2$. Set

$$\delta = C_b\sqrt{\frac{\mu}{b}},\qquad C_1 = \frac{C_b}{C_a},\qquad C_2 = C_b^2. \tag{20}$$

One can see that if a, b, and μ satisfy (5) with these values of C_1 and C_2 then all inequalities in (15) hold.

2. Split the integral in (19) into the integrals over subintervals I_i and I^*. For the subintervals of types 1 or 2 we use Proposition 2 and bound the integrand by 1. For the subintervals of type 3 we use Proposition 3. Hence

$$\left|\int_0^{2\pi} \cos x(t)\,dt\right| \leqslant O\left(\frac{\mu}{b\delta}+\delta\right) + \int_{\mathcal{I}_3} O\left(\frac{1}{b\,|\cos\hat{t}|}+\frac{\mu}{b\cos^2\hat{t}}\right)dt.$$

Since $\mathcal{I}_3 \subset [0,\,2\pi]\setminus M_\delta$, the last integral is not more than the corresponding integral over $[0,\,2\pi]\setminus M_\delta$, which equals

$$\begin{aligned} 4\int_0^{\arccos\delta} & O\left(\frac{1}{b\,|\cos\hat{t}|}+\frac{\mu}{b\cos^2\hat{t}}\right)d\hat{t} \\ &= O\left(\frac{1}{b}\right)\cdot\ln\frac{1+\sqrt{1-\delta^2}}{\delta} + O\left(\frac{\mu}{b}\right)\cdot\frac{\sqrt{1-\delta^2}}{\delta} \\ &\leqslant O\left(\frac{1}{b}\ln\frac{2}{\delta}\right)+O\left(\frac{\mu}{b\delta}\right) = O\left(\frac{1}{b}\left[\ln\sqrt{\frac{b}{\mu}}+\ln\frac{2}{C_b}\right]\right)+O\left(\sqrt{\frac{\mu}{b}}\right). \end{aligned} \tag{21}$$

As $C_b \geqslant 2$, the second term in square brackets is negative and can be discarded. This yields (19). □

Proof of Theorem 2. The proof contains two parts. The core part (see items 2–6 below) shows that if some conditions similar to those of Theorem 1 hold for $a = a_{0,k}(b,\,\mu)$ (or $a = a_{\pi,k}(b,\,\mu)$), b, and μ, then a is close to the Bessel function as stated in (8). However, *a priori* we do not know that for a given b and μ the boundaries $a_{\dots,k}(b,\,\mu)$ of k-th Arnold tongue satisfy these estimates. Thus we start with preliminary part (item 1 below) showing that under some conditions on k, b, and μ the triples $(a_{0,k}(b,\,\mu),\,b,\,\mu)$ and $(a_{\pi,k}(b,\,\mu),\,b,\,\mu)$ satisfy conditions needed for the core part of the proof.

1. First of all, fix C_a and C_b such that Propositions 1, 2, and 3 hold for them. Now fix values of C_1, C_2 and $\delta = C_b\sqrt{b/\mu}$ defined by (20).

Let us show that if C_1' and C_2' are appropriately chosen, then for any b, μ, and k that satisfy (7), each one of the triples

$$(k\mu,\,b,\,\mu),\qquad (a_{0,k}(b,\,\mu),\,b,\,\mu),\qquad (a_{\pi,k}(b,\,\mu),\,b,\,\mu) \tag{22}$$

satisfies (5). For the first triple this obviously holds for any $C_1' \leqslant C_1$, $C_2' \geqslant C_2$. Consider the second triple (the argument for the third one is exactly the same). If C_1' is sufficiently small and C_2' is sufficiently large, then the following inequalities hold:

$$\frac{K_1}{\sqrt{C_2'}}+K_2C_1' < 1,\qquad C_1' \leqslant C_1/2,\qquad C_2' \geqslant C_2, \tag{23}$$

Take any constants C_1' and C_2' that satisfy (23). We now show that for any b, μ, k satisfying (7) we have

$$|a_{0,k}(b,\,\mu)-k\mu| < 1. \tag{24}$$

Indeed, the inequality (24) holds for all sufficiently large b due to Theorem 1. Therefore, if it fails for some b', μ', and k' satisfying (7) then by continuity there exist $b'' \geqslant b'$ such that (24) "almost holds": $|a_{0,k'}(b'',\,\mu')-k'\mu'| = 1$. Clearly, the triple (b'', μ', k') also satisfies (7), and the triple $(a_{0,k'}(b'', \mu'), b'', \mu')$ satisfies conditions (5) of Theorem 1 because

$$|a_{0,k'}(b'', \mu')| + 1 \leqslant |k'\mu'| + 2 \leqslant 2C_1'\sqrt{b''\mu'} \leqslant C_1\sqrt{b''\mu'}.$$

Therefore, Theorem 1 yields

$$|a_{0,k'}(b'', \mu') - k'\mu'| \leqslant K_1\sqrt{\frac{\mu'}{b'}} + \frac{K_2}{\sqrt{b'\mu'}} \leqslant \frac{K_1}{\sqrt{C_2'}} + K_2 C_1' < 1,$$

this contradicts our assumption $|a_{0,k'}(b'', \mu') - k'\mu'| = 1$.

2. From now on we fix $C'_{1,2}$ that satisfy (23). In particular this means that Propositions 1, 2, and 3 hold for all triples in (22), where b, μ, and k satisfy (7).

Consider a point $a_{0,k}(b, \mu)$ with such values of b, μ, and k. Let $x_0(t)$ be the solution of (1) with $a = a_{0,k}(b, \mu)$ such that $x_0(0) = 0$. As it was said before, then $x_0(2\pi) - x_0(0) = 2\pi k$, and (11) yields

$$k = \frac{x_0(2\pi) - x_0(0)}{2\pi} = \frac{a_{0,k}(b, \mu)}{\mu} + \frac{1}{\mu}\int_0^{2\pi} \cos x_0(\tau)\, d\tau.$$

Therefore,

$$a_{0,k}(b, \mu) - k\mu + J_k\left(-\frac{b}{\mu}\right) = -\frac{1}{2\pi}\int_0^{2\pi} \cos\left(kt + \frac{b}{\mu}\sin t + \psi(t)\right) - \cos\left(kt + \frac{b}{\mu}\sin t\right) dt, \tag{25}$$

where

$$\psi(t) = \left(\frac{a_{0,k}(b, \mu)}{\mu} - k\right)t + \frac{1}{\mu}\int_0^t \cos x_0(\tau)\, d\tau. \tag{26}$$

Denote also $\hat{x}(t) = kt + (b/\mu)\sin t$. Then the right-hand side in (25) equals

$$-\frac{1}{2\pi}\int_0^{2\pi} \cos(\hat{x}(t)) \cdot (\cos\psi(t) - 1)\, dt + \frac{1}{2\pi}\int_0^{2\pi} \sin(\hat{x}(t)) \cdot \sin\psi(t)\, dt.$$

Denote the summands here as S_1 and S_2 respectively.

3. Let us start by estimating the norm of ψ. The triple $(a_{0,k}(b, \mu), b, \mu)$ satisfies conditions (5), hence we may apply Theorem 1 for the first summand in (26) and Proposition 4 for the second one. Then we obtain

$$\|\psi\|_{C^0} = O\left(\frac{1}{\sqrt{b}}\left(\frac{1}{\mu^{1/2}} + \frac{1}{\mu^{3/2}}\right)\right), \tag{27}$$

In order to estimate S_1, we bound the first cosine by 1 and the second multiplier by $\|\psi\|_{C^0}^2/2$. This yields

$$|S_1| = O\left(\frac{1}{b}\left(\frac{1}{\mu} + \frac{1}{\mu^3}\right)\right).$$

4. The estimation of S_2 goes along the lines of proof of Proposition 4. We split $[0, 2\pi]$ into subsegments J_j and J^* by the points where $\hat{x}(t) \equiv 0 \pmod{2\pi}$, consider the set M_δ, and classify these subsegments into types 1, 2, or 3 as above.

Recall that $\hat{x}(t)$ is a solution of the equation (13) with $\gamma = 0$, and the parameters equal to $\hat{a} = k\mu$, $\hat{b} = b$, $\hat{\mu} = \mu$. As it was said before, we can apply Propositions 1, 2, and 3 to it.

The integral in S_2 splits into the sum of integrals over subintervals J_j and J^*. We denote the part of this sum corresponding to the segments of types 1 and 2 by $S_2^{(1,2)}$ and the part corresponding to the segments of type 3 by $S_2^{(3)}$. Proposition 2 applies to $S_2^{(1,2)}$:

$$|S_2^{(1,2)}| \leqslant \sum_{\substack{J=J_j,J^* \\ \text{of types 1 or 2}}} \left|\frac{1}{2\pi}\int_J \sin\hat{x}(t)\cdot\sin\psi(t)\,dt\right| \leqslant \frac{\|\sin\psi\|_{C^0}}{2\pi} \sum_{\substack{J=J_j,J^* \\ \text{of types 1 or 2}}} |J|$$
$$\leqslant \left(\frac{1}{\sqrt{b}}\left(\frac{1}{\mu^{1/2}}+\frac{1}{\mu^{3/2}}\right)\right)\cdot O\left(\frac{\mu}{b\delta}+\delta\right) = O\left(\frac{1}{b}\left(1+\frac{1}{\mu}\right)\right).$$

5. The part $S_2^{(3)}$ is estimated as follows. Fix any point t_j in each I_j. Then

$$|S_2^{(3)}| \leqslant \sum_{J_j \text{ of type 3}} \left|\frac{1}{2\pi}\int_{J_j} \sin\hat{x}(t)\cdot\sin\psi(t_j)\,dt\right|$$
$$+ \sum_{J_j \text{ of type 3}} \left|\frac{1}{2\pi}\int_{J_j} \sin\hat{x}(t)\cdot\left[\sin\psi(t)-\sin\psi(t_j)\right]dt\right|.$$

Denote the two sums on the right-hand side by $S_2^{(3)\star}$ and $S_2^{(3)\star\star}$, respectively. The first sum, $S_2^{(3)\star}$ is estimated by Proposition 3:

$$S_2^{(3)\star} \leqslant \|\psi\|_{C^0}\int_{\mathcal{J}_3}\left[O\left(\frac{1}{b\,|\cos\hat{t}|}\right)+O\left(\frac{\mu}{b\cos^2\hat{t}}\right)\right]d\hat{t}.$$

The integral is managed exactly in the same way as the integral over $\mathcal{I}_3$ in the proof of Proposition 4; together with inequality $\ln z \leqslant 2\sqrt{z}$ and (27) this yields

$$S_2^{(3)\star} = O\left(\frac{1}{b}\left(1+\frac{1}{\mu^2}\right)\right).$$

6. In the sum $S_2^{(3)\star\star}$ we bound $\sin\hat{x}(t)$ by 1 and the difference in square brackets by $\operatorname{osc}_{J_j}\psi \leqslant |J_j|\cdot\max_{J_j}|\psi'| \leqslant |J_j|\cdot(|a-k\mu|+1)/\mu$:

$$S_2^{(3)\star\star} \leqslant \sum_{J_j \text{ of type 3}} |J_j|\operatorname{osc}_{J_j}\psi \leqslant \sum_{J_j \text{ of type 3}} |J_j|^2\cdot\left(\left|\frac{a}{\mu}-k\right|+\frac{1}{\mu}\right).$$

We have already seen in (24) that $|a-k\mu| = O(1)$ hence the last bracket is $O(1/\mu)$. Proposition 1 yields

$$|J_j|^2 \leqslant \int_{J_j} O\left(\frac{\mu}{b\,|\cos\hat{t}|}\right)d\hat{t},$$

therefore by (21) we obtain

$$S_2^{(3)\star\star} \leqslant \int_{[0,2\pi]\setminus M_\delta} O\left(\frac{d\hat{t}}{b\,|\cos\hat{t}|}\right) = O\left(\frac{\ln(b/\mu)}{b}\right).$$

Joining together the estimates for S_1, $S_2^{(1,2)}$, $S_2^{(3)\star}$, and $S_2^{(3)\star\star}$, we complete the proof. $\square$

5. Generalizations

Let us now discuss some possible generalizations of Theorems 1 and 2. Theorem 1 can be straightforwardly generalized to any equation of the form (2) such that the graph of the function g transversely crosses the line $\{t = 0\}$. More precisely, the proof given above uses only the following properties of the functions f and g:

(1) functions f and g are bounded by 1;
(2) g is Lipschitz with constant 1;
(3) the graph $y = g(t)$ transversely intersects the line $y = 0$.

(Recall that also $\int_0^{2\pi} f(x)\,dx = 0$, $\int_0^{2\pi} g(t)\,dt = 0$.)

Constants equal to one in these properties can be easily replaced by any other constants by the means of the substitutions

$$(f,\, g,\, a,\, b,\, \mu) \to (f/D,\, g/D,\, a/D,\, b,\, \mu/D),$$
$$(f,\, g,\, a,\, b,\, \mu) \to (f,\, g/D,\, a,\, bD,\, \mu)$$

with some $D > 0$. As for the last condition, it is used in two parts of the proof: (1) estimates of $\operatorname{mes} M_\delta$ and (2) estimates of the integrals $\int_{[0,2\pi]\setminus M_\delta} d\hat{t}/\,|g(\hat{t})|$ and $\int_{[0,2\pi]\setminus M_\delta} d\hat{t}/g^2(\hat{t})$ in (21). Let us express transversality condition in the following quantitative way: there exists $\varepsilon_0 > 0$ and $L > 0$ such that for any $\varepsilon \leqslant \varepsilon_0$ we have

$$\operatorname{mes} M_\varepsilon := \operatorname{mes}\{t\colon\ |g(t)| \leqslant \varepsilon\} \leqslant L\varepsilon.$$

Suppose that $\delta \leqslant \varepsilon_0$ (this is a required modification of condition (15c)), then $\operatorname{mes} M_\delta$ is estimated exactly in the same way as in the proof, and for integrals we use the following estimate:

$$\int_{[0,2\pi]\setminus M_\delta} \frac{d\hat{t}}{g^2(\hat{t})} = \int_0^\infty \operatorname{mes}\left\{\hat{t} \in [0,\, 2\pi] \setminus M_\delta\colon \frac{1}{g^2(\hat{t})} \geqslant y\right\} dy$$
$$= \int_0^\infty \operatorname{mes}\left\{\hat{t} \in [0,\, 2\pi]\colon \delta \leqslant g(\hat{t}) \leqslant \frac{1}{\sqrt{y}}\right\} dy.$$

The set is empty if $y > 1/\delta^2$, otherwise we bound its measure by $\operatorname{mes} M_{1/\sqrt{y}}$, which is estimated via transverality condition:

$$\int_0^{1/\delta^2} \operatorname{mes} M_{1/\sqrt{y}}\, dy \leqslant \int_0^{1/\varepsilon_0^2} 2\pi\, dy + \int_{1/\varepsilon_0^2}^{1/\delta^2} \frac{L}{\sqrt{y}} dy \leqslant O(1) + O\left(\frac{1}{\delta}\right).$$

Another integral is bounded similarly, and (21) preserves its form. Therefore, we obtain the following generalization of Theorem 1.

Theorem 3. *Fix any positive constants L_0, L_1, L_2, L_3. Then there exist positive constants C_1, C_2, K_1, K_2 depending on $L_{0,1,2,3}$ such that the following holds. Consider any functions f and g with zero averages such that*

(1) *their continuous norms are bounded:* $\|f\|_{C_0} \leqslant L_1$, $\|g\|_{C_0} \leqslant L_1$,
(2) *g is Lipschitz with constant L_2:* $|g(t_1) - g(t_2)| \leqslant L_2\,|t_1 - t_2|$,
(3) *for any $\delta < 1/L_0$ there is a bound* $\operatorname{mes}\{|g(t)| < \delta\} \leqslant L_3\delta$.

Then if the parameters a, b, μ of the equation (2) *are such that*

$$|a| + 1 \leqslant C_1\sqrt{b\mu}, \qquad b \geqslant C_2\mu$$

we have

$$\left|\frac{a}{\mu} - \rho_{a,b,\mu}\right| \leqslant \frac{K_1}{\sqrt{b\mu}} + \frac{K_2}{b\mu}\ln\left(\frac{b}{\mu}\right) \leqslant \frac{K_1}{\sqrt{b\mu}} + \frac{2K_2}{\sqrt{b\mu^3}}.$$

As for Theorem 2, we have seen in Section 1 that the reduction to a Riccati equation and identification of fixed point of $\widetilde{P}_{a,b,\mu}$ for the Arnold tongue boundaries with 0 and π works only if $f(x) = \cos x$ and $g(t)$ is even. These conditions cannot be significantly extended (trivial extension is obtained by coordinate change $x' = x+x_0$, $t' = t+t_0$; the conditions take form $f(x') = \cos(x'-x_0)$, $g(t') = g(2t_0-t')$). Under these assumptions and transversality condition discussed above the following analogue of Theorem 2 holds. Modifications in its proof are exactly the same as above.

Theorem 4. *Fix any positive constants L_0, L_1, L_2, L_3. Then there exist positive constants C_1', C_2', K_1', K_2', K_3' depending on $L_{0,1,2,3}$ such that the following holds.*

Consider any function g with zero average that satisfies conditions 1–3 *of Theorem* 3 *and the condition $g(t) = g(-t)$. Let $a_{0,k}(b, \mu)$ and $a_{\pi,k}(b, \mu)$ be the boundaries of k-th Arnold tongue of the equation* (2) *with this g and $f(x) = \cos x$. Then if the parameters b, μ and a number $k \in \mathbb{Z}$ satisfy inequalities*

$$|k\mu| + 1 \leqslant C_1'\sqrt{b\mu}, \quad b \geqslant C_2'\mu$$

the following estimates hold

$$\left|\frac{a_{0,k}(b)}{\mu} - k + \frac{1}{\mu}\tilde{J}_k\left(-\frac{b}{\mu}\right)\right| \leqslant \frac{1}{b}\left(K_1' + \frac{K_2'}{\mu^3} + K_3'\ln\left(\frac{b}{\mu}\right)\right),$$
$$\left|\frac{a_{\pi,k}(b)}{\mu} - k - \frac{1}{\mu}\tilde{J}_k\left(-\frac{b}{\mu}\right)\right| \leqslant \frac{1}{b}\left(K_1' + \frac{K_2'}{\mu^3} + K_3'\ln\left(\frac{b}{\mu}\right)\right),$$

where

$$\tilde{J}_k(-z) = \frac{1}{2\pi}\int_0^{2\pi}\cos(kt + zG(t))\,dt, \qquad G(t) = \int_0^t g(\tau)\,d\tau.$$

The function $\tilde{J}_k$ stems from integral representations (11) (12), which now have the form

$$x(t) - x(0) = \frac{at + bG(t) + \int_0^t \cos x(\tau)\,d\tau}{\mu},$$
$$x(2\pi) - x(0) = \frac{2\pi a}{\mu} + \frac{1}{\mu}\int_0^{2\pi}\cos\left(\frac{a\tau + bG(\tau) + \int_0^\tau \cos x(s)\,ds}{\mu} + x(0)\right)d\tau$$

(note that $G(2\pi) = 0$ due to (3)). The function $\tilde{J}_k$ also has asymptotic representation similar to the one for J_k:

$$\tilde{J}_k(-z) \sim \sum_j \frac{1}{\sqrt{2\pi z\,|g'(t_j)|}}\cos\left(zG(t_j) + kt_j + \frac{\pi}{4}\operatorname{sgn}(g'(t_j))\right) \quad \text{as } z \to +\infty, \quad (28)$$

where the sum is taken over all the zeros t_j of the function g on a circle.

Recall that these zeroes are simple (and hence the denominators in (28) are nonzero) due to transversality condition 3 of Theorems 3 and 4.

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1.4 Second regime : small external frequency

Let us consider the equation (1.2) in a different from the Subsection 1.3 mode, i.e. when μ is small. The totally different world opens to us : the world of *slow-fast systems*, and the techniques of the study in this Subsection differ from the techniques of Subsection 1.3. This Chapter is based on our paper [21] with Victor Kleptsyn and Ilya Schurov written in Russian. The presentation was improved and some pictures and proofs were added.

1.4.1 Zonal behavior of the tongues

Thanks to the use of the algorithm that is described later in Subsection 1.4.5 we obtained the pictures of boundaries of Arnold tongues for different values of μ, see Figures 1.5 and 1.6. On these pictures one can see that when μ diminishes, the tongues are approaching each other and three domains (zones) with qualitatively different behavior of the tongues emerge :

- Zone A : $A = \{(a,b)|b < a-1\}$.The tongues are thin.
- Zone B : $B = \{(a,b)|a-1 < b < a+1\}$. The tongues fill up almost all the space of parameters, and there are no adjacency points for each fixed tongue in this zone.
- Zone C : $C = \{(a,b)|b > a+1\}$. The tongues are organized in a lattice-form structure, and they pave almost all of the space of parameters, adjacency points are present.

We should note that zones B and C exist only in the bounded neighborhood of zero (not depending on μ). For example, as we see from theorems in Subsection 1.3 for b big enough the boundaries stop approaching each other and do not give a lattice-form structure : the better approximation for the boundaries in this case is Bessel approximation. The zone C is dispersing in this case and the theorems of Subsection 1.3 give some estimates on when the lattice structure starts to disappear and Bessel behavior wins over it.

The goal of this Subsection is two formulate mathematically some of the descriptions of zonal behavior stated above and to prove them as well as to establish an effective algorithm that permits to calculate the boundaries of Arnold tongues and draw pictures as those from Figues 1.5 and 1.6.

1.4.2 Slow-fast systems reminder

The structure of zones B and C described in previous Subsection can be understood with the help of the theory of slow-fast systems. Let us remind its basic notions.

Definition 1.5. *Consider a family of differential equations*

$$\begin{cases} \dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{y}, \mu) \\ \dot{\boldsymbol{y}} = \mu g(\boldsymbol{x}, \boldsymbol{y}, \mu) \end{cases} \tag{1.10}$$

where $\mu \in \mathbb{R}_{>0}, \mu << 1$ and $\boldsymbol{x}, \boldsymbol{y}$ are multidimensional variables, $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{y} \in \mathbb{R}^m$, and f, g : $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are some functions.

Such a family of differential equations is called a slow-fast system. *The variables $\boldsymbol{x}$ are called* fast variables, *and the variables $\boldsymbol{y}$ – a* slow variables.

FIGURE 1.5 – The black lines correspond to the boundaries of Arnold tongues for Josephson equation on the parametric plane (a, b) corresponding to rotation numbers $\rho = 0, 1, 2, \ldots, 10$ for $\mu = 0.2$

FIGURE 1.6 – By diminishing μ and taking $\mu = 0.1$ one can see that the tongues become closer to each other, and the gaps between the tongues are impossible to detect with an eye

The slow-fast system terminology comes from the idea that the rates of change of the variables **y** and **x** differ drastically. The variables y change approximately μ times slower than the variables **x** (at least, in a generic point of a phase space where $f \neq 0$).

For $\mu = 0$ the system (1.10) becomes a system of equations on the variable x since y in this case can be considered as a parameter. This system is called a *fast system*.

Definition 1.6. *A set of fixed points*

$$M = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m \mid f(\boldsymbol{x}, \boldsymbol{y}, 0) = 0\}$$

of a fast system is called a slow hypersurface*, and in two-dimensional case, a* slow curve*.*

A generic trajectory of a generic slow-fast system with one fast and one slow variable admits the following description [26] : it has alternating phases of slow (with a velocity of order $O(\mu)$) motion near the slow curve and fast (with a velocity of order $O(1)$) jumps along the trajectories $\{y = \text{const}\}$ of a fast system. Those jumps occur near the points where the tangent line to a slow curve is parallel to the axis of slow motion, the so-called **fold points**.

In the case of our study we see that when $\mu << 1$ the system of equations (1.2) can be considered as a slow-fast system with one fast variable x and one slow variable t. The study of Josephson equation as a slow-fast system has been started by J. Guckenheimer and Yu. Ilyashenko [14] even if the authors didn't use this terminology. The Josephson equation was chosen by them in order to produce some duck solutions for slow systems.

1.4.3 Slow curve for Josephson equation

The slow curve for (1.2) is the following subset M of the torus

$$M = \{(x, t) \mid \cos x + a + b \cos t = 0\} \tag{1.11}$$

By simple calculation one can prove a following

Proposition 1.7. *In the zone A the slow curve is an empty set, in the zone B the slow curve is a contractible convex curve with two fold points and in the zone C the curve M is split into two non-contractible curves with homotopy type $(1, 0)$each of them having two fold points.*

The little stop-motion film of how the slow curve changes is depicted in Figure 1.7.

1.4.4 Description of the behavior

Theorem 1.8. *Let B' be some open bounded set in the space of parameters (a, b) for which $\bar{B}' \Subset B = \{(a, b) | a - 1 < b < a + 1\}$. Then for μ small enough there exist constants $C_1, C_2 > 0$ such that the distance between two neighboring Arnold tongues in B' is bounded by $C_1 \exp\left(-\frac{C_2}{\mu}\right)$.*

Proof. From the considerations of Subsection 1.2.2 we know that the boundaries of Arnold tongues are given by a following condition : a trajectory with an initial condition $(x_0, 0) \in \mathbb{T}^2$ should be passing through a point $(x_0, 2\pi) \in \mathbb{T}^2$ on the torus with coordinates (x, t), where

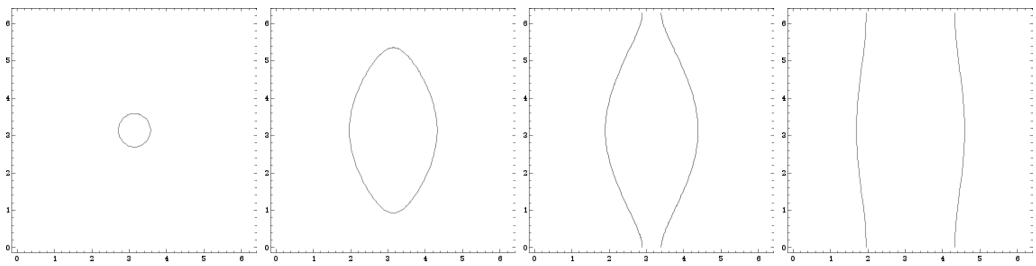

FIGURE 1.7 – A stop–motion movie showing the change of a slow curve with a fixed a and growing b. There is no slow curve for small b then it appears when $b = a - 1$ (it is a point in this case). The the slow curve continues growing and at $b = a + 1$ there is a new bifurcation : the curve starts intersecting itself. Then it falls into two non-intersecting homotopic circles that tend to torus meridians while $b \to \infty$

$x_0 = 0$ or $x_0 = \pi$. With the use of the solutions $x(t)$ of the equation (1.2), the points x_0 can be lifted to the universal covering of the circle $\mathbb{S}^1$ and the conditions above can be rewritten as

$$\tilde{x}_0(2\pi) = 2\pi k, \tag{1.12}$$
$$\tilde{x}_\pi(2\pi) = \pi + 2\pi k, \tag{1.13}$$

where $k \in \mathbb{Z}$ is a number of Arnold's tongue whose boundaries we are describing. And $\tilde{x}_0(t)$ ($\tilde{x}_\pi(t)$ is a phase curve of the equation (1.2) with an initial condition $\tilde{x}_0(0) = 0$ (correspondingly, $\tilde{x}_\pi(0) = \pi$ correspondingly), lifted to the covering (a cylinder in terms of phase space, a line in terms of Poincaré map).

The property of symmetry of phase curves under the central symemetry τ (see formula (1.9) in Subsection 1.2.2) gives us the following. If the Poincaré map for a full period of time ($T = 2\pi$) displaces some point by a distance of $D = 2\pi k, k \in \mathbb{Z}$ then this same point will be displaced by a Poincaré map corresponding to the half of the period $\frac{T}{2} = \pi$ by twice a smaller distance, $\frac{D}{2} = \pi k$. Hence the conditions described above in (1.12) can be rewritten as

$$\tilde{x}_0(\pi) = \pi k, \tag{1.14}$$
$$\tilde{x}_\pi(\pi) = \pi + \pi k. \tag{1.15}$$

In other words, the boundaries of the tongues are described by one of these two conditions :

Zero boundary. A point 0 maps to 0 or $\pi \mod 2\pi$) after a half-period

Pi boundary. A point π maps to 0 or $\pi \mod 2\pi$) after a half-period

The 0 or π response depends on the oddity of $k \in \mathbb{Z}$.

Note that when the parameters of the equation (1.2) a and b change continuously, the value $x_0(\pi)$ ($x_\pi(\pi)$ correspondingly) changes continuously from 0 to π (correspondingly, from π to 2π), so the shift for the half of the period is one half of the turn on the circle. Consequently, for the full period the shift will give the full circle and the rotation number will grow by 1. This corresponds to the passage to the neighboring tongue.

Now suppose that the parameters $(a_0, b_0) \in B'$ belong to the boundary of some tongue. Without loss of generality, we will suppose that this is the *zero boundary* and that the condition 0 maps to π holds. Other cases are treated by analogy.

Let us consider an arc $J^u = [(\pi, \pi), (2\pi, \pi)] \subset \{t = \pi\}$ containing a point $x = \frac{3\pi}{2}$, see Figure 1.8. This arc intersects the repelling part of the slow curve. Let us reverse time for a

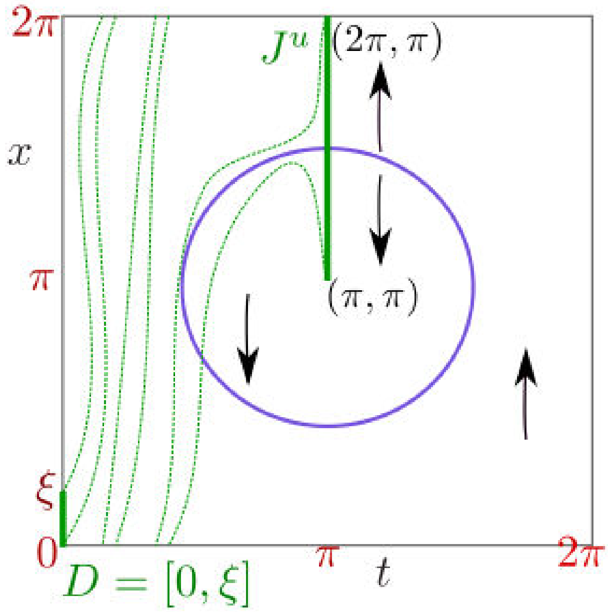

FIGURE 1.8 – The flow of the vector field (1.2) corresponding to the parameters (a, b, μ) belonging to the zero boundary of the tongue, $a = a_{0,k}(b)$ for a fixed μ. The interval J^u corresponds to the half of the circle $\{t = \pi\}$ and covers the repelling part of the slow curve. Its preimage D under the flow of Josephson equation is exponentially small in μ. The slow curve is the violet curve.

moment : in this case, the repelling part becomes an attractive one. The image D of the arc J^u under the Poincaré map from the transversal $\{t = \pi\}$ to the transversal $\{t = 0\}$ (in the inversed time) has a length $O\left(\exp\left(-\frac{C}{\mu}\right)\right)$. This follows from the fact that while moving near the stable part of a slow curve the trajectories of a slow-fast system exponentially attract to each other, see a detailed proof by Ilyashenko-Guckenheimer [14], Proposition 4. The precise bounds for C can be found following the works of I.Schurov on canard cycles in generic slow-fast systems [28], Lemma 5.4.

Note that since the condition "0 maps to π after the half of the period" holds then the lower extremity of the interval J^u will map to 0. Hence, its preimage D can be written as $D = [0, \xi]$, where $\xi = O\left(\exp -\frac{C}{\mu}\right)$.

This is not hard to show that the derivative of the solution with respect to parameters a and b in the domain $t \in [0, \pi]$ is bounded away from zero (and in fact has order of $O\left(\frac{1}{\mu}\right)$). Hence, by changing the parameter a or b up to the size of the order of $O\left(\frac{1}{\mu}\right)$ one can map the higher extremity of D to zero. In the course of this continuous change the value $x_0(\pi)$ will be continuously changing from π to 2π, what corresponds to the changing in rotation number by 1, i.e. to the jump to the next Arnold tongue. $\square$

The analogous statement can be proven also for the domain C and the lattice-like behavior of Arnold tongues :

Theorem 1.9. *Let C' be some open bounded set in the space of parameters (a, b) for which $\bar{C}' \Subset C = \{(a, b)|b > a + 1\}$. Then for μ small enough there exist constants $C_1, C_2 > 0$ such that the distance*

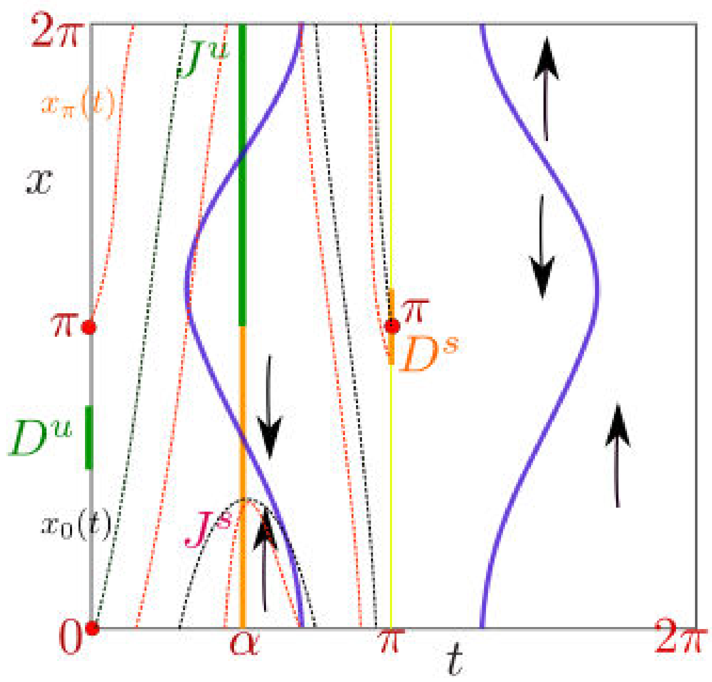

FIGURE 1.9 – The idea of the proof of Theorem 1.9 is shown on this Figure. The slow curve in the domain C (in violet) has two components – two circles. Under the supposition we make in the proof of Theorem 1.9 the picture is the following.The circle $\{t = \alpha\}$ is divided into two parts : an attractive part J^s and a repelling part J^u. The trajectories of all the points of the transversal $\{t = 0\}$ except for an exponentially small interval D^u will cross the section J^s : so the trajectories $x_0(t)$ and $x_\pi(t)$ will be very close once they have stayed some time near the attracting part of the slow curve. And since $x_0(\pi) = \pi$ then, by a small change of parameters one can obtain $x_\pi(\pi) = \pi$ and pass to the neighboring tongue.

between two neighboring Arnold tongues in C' is bounded by $C_1 \exp\left(-\frac{C_2}{\mu}\right)$.

Proof. Basic ideas of the proof are analogous to those from the proof of Theorem 1.8. Suppose that a point (a_0, b_0) lies on a boundary of some Arnold tongue. As in the proof of Theorem 1.8 we can suppose that the condition "0 maps to π after a half of the period" holds.

Let us consider a transversal $\Gamma = \{t = \alpha\}$, where $\alpha \in [0, \pi]$ is chosen in such a way that Γ intersects a slow curve in two points (in other words, Γ is far away from fold points). Let us break Γ into two intervals. One of them J^s intersects a stable part of a slow curve, and another one J^u – an unstable part :

$$J^s = \{(x,t) \mid t = \alpha,\ x \in [0, \pi)\} \subset \Gamma,$$
$$J^u = \{(x,t) \mid t = \alpha,\ x \in [\pi, 2\pi)\} \subset \Gamma.$$

See Figure 1.9 for the illustration.

Let us also denote as D^u the image of the interval J^u under the Poincaré map from the transversal Γ to the transversal $\{t = 0\}$ in the negative time, and as D^s — the image of the interval J^s under the Poincaré map from the transversal Γ to the transveral $\{t = \pi\}$ in the positive time. Following the same arguments as in Theorem 1.8, one can prove that the intervals D^u and D^s are exponentially thin.

Let us now consider a trajectory $x_0(t)$ (correspondingly, $x_\pi(t)$), that passes through a point $(0,0)$ (correspondingly, $(\pi, 0)$). There are two possible cases :

1. The point 0 lies in the interval D^u and hence, $x_0(\alpha) \in J^u$.

2. The point 0 doesn't belong to the interval D^u and in this case $x_0(\alpha) \in J^s$ and $x_0(\pi) = \pi \in D^s$ (remember : we supposed that the condition 0 maps to π» holds).

Let us suppose that the case 2 holds. Suppose $\pi \notin D^u$ and then $x_\pi(\alpha) \in J^s$ and $x_\pi(\pi) \in D^s$. In this case the distance between $\pi = x_0(\pi)$ and $x_\pi(\pi)$ is exponentially small, and after an exponentially small change of parameters one can obtain the condition «π maps to π». In this case rotation number either grows by 1 either diminishes by 1. And if $\pi \in D^u$, then one can exponentially change one of the parameters a or b so that this condition breaks.

The case 1 can be considered in an analogous way (we just have to exchange indices u and s). $\square$

Remark 1.10. From the proof of Theorem 1.9 follows that any point on the boundary of the tongue verifies one of these two conditions : $\{0, \pi\} \cap D^u \neq \varnothing$ or $\{0, \pi\} \cap D^s \neq \varnothing$. These conditions give a family of the exponentially thin tubes in the space of parameters to which the boundaries of tongues belong. These are those tubes that one can see on the images of numerical experiments, as on Figures 1.6 and 1.5 in the zone C.

When one moves along the boundary of a tongue, the corresponding characteristic trajectory passes near stable or unstable part of the slow curve : it depends on which one of the cases 1 or 2 holds. This corresponds to the "left" or "right" movement of the boundary of the tongue on the plane of parameters. In the moment, when the boundary makes a turn, the trajectory passes the comparable times near stable and unstable parts of slow curve. Those solutions are colled *duck* solutions, see [2]. One says that the name *duck solutions* is related to the form of solutions as well as to the fact that their discovery was unexpected.

1.4.5 Boundary programming and Newton's method

This Subsection is dedicated to the problem of constructing Arnold tongues for the equation (1.2) with numerical methods. Such a problem is highly difficult if one approaches it directly. Indeed, in order to calculate a rotation number with the formula (1.1) one needs to integrate the equation (1.2) for some long periods of time. To find the rotation number with a mistake not more than ε one needs to integrate the equation on the period of time of the length of the order $(\varepsilon\mu)^{-1}$. And for the construction of Arnold tongues, one needs to know the rotation number on a considerably dense net in the space of parameters.

Although, if one uses the special properties of Josephson equation described in Section 1.2, one can propose a much more effective algorithm of construction of the Arnold tongues. We will describe this algorithm in the rest of this Subsection.

As already explained in a previous Subsection, the boundaries of Arnold tongues can be defined by fixing the images of points 0 and π under the Poincaré map on the half of the period, see (1.14).

FIGURE 1.10 – The picture of domains A, B and C described euristically in Subsection 1.4.1. The domains B and C correspond to non-empty slow curves : the techniques of slow-fast systems are applied to study Arnold tongues, the domain A is much harder to study and almost nothing is known about the behaviour of the tongues in this domain.

Let us suppose that condition "0 *maps to* 0" holds (in this case the number of the tongue k is even, $k = 2l$). Other conditions can be considered in the same manner. Suppose $x = x_0(t; a, b, \mu)$ gives a phase trajectory passing by the point $(0, 0)$.

Let us fix some parameter μ and put

$$Q(a, b) = x_0(\pi; a, b, \mu) \tag{1.16}$$

We are interested in the level line of the function $Q(a, b)$, corresponding to the value πk :

$$L_k := \{(a, b) \mid Q(a, b) = \pi k = 2\pi l\}.$$

Suppose the boundary of the tongue L_k is given by the graph of the function $a = a(b)$. When $b = 0$, the equation (1.2) can be integrated explicitly (as mentionned before in Subsection 1.2.3) and the value $a(0)$ is known :

$$a(0) = \sqrt{1 + l^2\mu^2}$$

Now let us suppose that the value $a_0 = a(b_0)$ for some b_0 is known. Let us take some small step of size h and find an approximate value of $a(b_0 + h)$. This is equivalent to solving the equation

$$Q(a, b_0 + h) - \pi k = 0 \tag{1.17}$$

with respect to a. As a zero-approximation solution let us put $a = a_0$. Now consider the system

$$\begin{cases} x' = \cos x + a + b\cos t, \\ t' = \mu, \\ u' = -u\sin x + 1. \end{cases} \tag{1.18}$$

The third equation in the system corresponds to the equation in variations with respect to a parameter, $u = \frac{\partial x}{\partial a}$.

After the numerical calculation of the system of differential equations (1.18), we can find $Q_0 = Q(a_0, b_0 + h)$ as well as $Q' = \frac{\partial Q}{\partial a}|_{a_0, b_0+h}$. Now we replace Q as a function of a by a tangent in the point a_0 (in other words, we apply Newton's method to find out the solution of the equation (1.17)), and as a first approximation to a we find :

$$a_1 = a_0 - (Q_0 - \pi k)/Q'.$$

After finding out a_1, we make a substitution $b_0 + h \mapsto b_0$, $a_1 \mapsto a_0$ and we repeat the procedure. In such a way, one can find the value $a(b)$ for any value b on the net with a step h.

On each step with respect to b we do only one step of Newton's method. But the mistake won't grow : one can show by induction that on each step the mistake of the zero-order approximation is $O(h)$ and of the first-order approximation is $O(h^2)$. This assures that on the next step the mistake in the 0-order approximation will be equal to $O(h^2 + h) = O(h)$ etc.

The described algorithm works effectively for μ of order 1 but for smaller μ (around 0.1) the problems of convergence of Newton's method reappear. The secret is in the fact that the trajectory comes closer near the repelling part of the slow curve M. In this case a big derivative with respect to the initial condition is stocking up, and it gives the computational instability of our method.

When one computes the tongues in the domain B this problem can be solved by considering an inverse map : Q^{-1} instead of Q. In other words, one integrates the system (1.18) in the reversed time. In this case the considered trajectory goes near the attracting part of the slow curve and the problem that was described doesn't occur anymore.

Unfortunately, in the domain C this simple trick doesn't work : when one passes near the adjacency point of the neighboring tongue the trajectory we consider is a duck trajectory, in other words it passes near the repelling part of the slow curve in positive and in negative time as well. So, in this case we adapt our Newton's algorithm and we make several steps of Newton's method which uses a method of bisection of intervals (stable but less effective) in the case when there is no convergence of initial method.

In this way, we manage to construct the tongues for $\mu = 0.01$ with this method. As far as we know, before the algorithms for the construction of Arnold tongues for Josephson equation were known only for μ of order 0.1 (see [23, 3]).

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2

Lagrange problem and the asymptotic angular velocity of a swiveling arm

In this Chapter we study a classical problem of finding an asymptotic angular velocity of the motion represented as a sum of circular motions. This problem was first considered by Lagrange who was studying the movement of celestial bodies. We give a solution of this problem for a complete oriented riemannian surface of a curvature that is close to constant. As a particular case, we obtain an explicit answer to the initial Anatoly Stepin's question for the value of the asymptotic angular velocity of a swiveling arm on the hyperbolic plane.

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The problem to which this Chapter is devoted was first formulated by Joseph-Louis Lagrange in the XVIIIth century. He has been interested in the understanding of the asymptotic behavior of the end of the chain of intervals, each one of which is turning around its first vertex with a constant angular velocity.

The organization of the Chapter is the following : in Section 2.1 we formulate the classical case of this problem – the movement on the euclidian plane. This case was studied in a series of works by P. Bohl, P. Hartman, E. R. Van Kampen, A. Wintner and H. Weyl. After giving some bibliographical information, we give a new proof of the Lagrange problem for this classical case. In Section 2.2 we formulate the problem for the case of an arbitrary riemannian surface, and we present a solution for this case based on a new proof from Section 2.1.

2.1 Classical Lagrange problem : motion of the swiveling arm on the euclidian plane

2.1.1 Setting : definitions and history of the problem

For the fixed numbers $l_1, l_2, \dots, l_N \in \mathbb{C}$ consider the map Ψ from the N-torus to the complex plane, $\Psi : \mathbb{T}^N \to \mathbb{C}$ that sends a point $(\theta_1, \dots, \theta_N) \in \mathbb{T}^N = \mathbb{R}^N / (2\pi\mathbb{Z})^N$ to the point

$$\sum_{j=1}^{N} l_j e^{2\pi i \theta_j}. \tag{2.1}$$

We will call Ψ **a swiveling arm** of type $l = (l_1, \dots, l_N)$ on the complex plane. Note that one can think of such a map as of a set of intervals on the complex plane, attached one to another in a chain. Note that in the given definition the complex numbers $l_j \in \mathbb{C}$ give the following information about this chain : their modules $|l_j|$ correspond to the lengths of the intervals in a chain and their arguments $\arg l_j$ correspond to the angles that the intervals have with the horizontal direction in the position $\Psi(0,\dots,0)$.

The topology of $\Psi^{-1}(z)$ for some fixed z is an interesting question, considered, among others, by Jean-Claude Haussmann in [4, 5]. We will spice up this geometrical construction with some dynamics.

Let us consider the linear flow $\mathbb{T}^t$ on $\mathbb{T}^N$ given by a vector field

$$X = \sum_{j=1}^{N} \omega_j \frac{\partial}{\partial \theta_j}, \omega_j \in \mathbb{R}. \tag{2.2}$$

This linear flow $\mathbb{T}^t$ gives the dynamics of a swiveling arm : each joint of an arm is turning with a constant angular speed ω_j around the end of a previous joint, see Figure 2.1.

The question that interests us is the following :

Let us fix an initial position of the arm. Does the end of it (the end of the Nth joint) have an asymptotic angular velocity in such a movement and if yes, is it possible to calculate its value ω as a function of the lengths $|l_j|$ and angular speeds ω_j ? And does the answer depend on the initial position ?

Let us consider an initial condition for the vector field be $(\theta_1^0,\dots,\theta_N^0)$. Then the extremity of a swiveling arm of type $(l_1,\dots,l_n)$ moving in a vector field (2.2) at the time t will be given by a point

$$z(t) = l_1 e^{i\theta_1^0} e^{i\omega_1 t} + \dots + l_N e^{i\theta_N^0} e^{i\omega_N t} \in \mathbb{C}. \tag{2.3}$$

We will suppose from now on that $l_j = |l_j| e^{i\theta_j^0}$, in other words that $\arg l_j = \theta_j^0$. So l_j encodes the initial position of the swiveling arm and the equation (2.3) can be rewritten in a simpler form, encoding the information about initial conditions :

$$z(t) = l_1 e^{i\omega_1 t} + \dots + l_N e^{i\omega_N t} \tag{2.4}$$

Definition 2.1. *For a swiveling arm of type $l = (l_1,\dots,l_N)$ in dynamics defined by the flow of the vector field* (2.2)*, the* asymptotic velocity *of the endpoint of the system ω is defined as a limit*

$$\omega = \lim_{T\to+\infty} \frac{\arg z(T)}{T}, \tag{2.5}$$

where arg *is a continuous determination of the argument $z(t)$ and $z(t) \in \mathbb{C}$ is the end of the system of swiveling arms, see* (2.3).

Remark 2.2. For the time being we assume that $z(t) \neq 0$. In this case one can choose a continuous determination of the argument function along the path $z(t), t \in \mathbb{R}_+$.

The question of finding the angular asymptotic velocity (2.5) of a swiveling arm in the vector field (2.2) was first stated by Lagrange in his two volume work [7] on celestial mechanics. The formulation of the problem makes one think about the theory of epicycles developed by Hipparchus of Rhodes in the 2nd century BC who was decomposing the motion of planets in a superposition of periodic motions. But naturally, Lagrange was wide aware (after the works of Kepler, Newton and others) that this theory was not appropriate to describe the solar system.

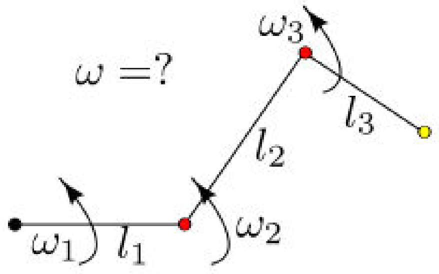

FIGURE 2.1 – A swiveling arm of type (l_1, l_2, l_3) rotating in a vector field (2.2) with $N = 3$.

He knew that the movement of the planets under the gravitational forces was complicated although close to an elliptical one. The swiveling arm motion appears in Lagrange's calculations as an approximation of the variation of the longitude of the perihelion for the orbit of a planet in the N-body problem.

From now on, we will call the question of finding the angular asymptotic velocity of a swiveling arm in the vector field (2.2) **a Lagrange problem.**

Lagrange solved this problem in a simple case when one of the intervals is longer than the sum of the lengths of all others. In this case, $\omega = \omega_j$ where j is the number of the longest joint. In this case we will say that a swiveling arm is **Lagrangian** : its end point never passes by zero and the asymptotic velocity ω is equal to the angular velocity corresponding to the longest of the joints. Moreover, the angle $\varphi(t)$ chosen as a continuous branch of the argument $\varphi(t) = \arg z(t)$ has the linear asymptotics of the form

$$\varphi(t) = \omega_j t + O(1).$$

For the general case, Lagrange didn't consider it and he even added : "*Il est fort difficile et peut-être même impossible de se prononcer, en général, sur la nature de l'angle φ.*" [54]. The existence of the asymptotic angular velocity was obtained for a general case in a sequence of works by P. Bohl, P. Hartman, E. R. Van Kampen, A. Wintner and H. Weyl in [1, 3, 10] by some straightforward calculation coupled with the use of ergodic theorem but the explicit formula for this velocity ω is still not established.

One can note some subtleties : the question of Lagrange is asked for all **all** values of asymptotic velocitites ω_j. As we will see later, the proof simplifies in the case when ω_j are rationally independent because of the possibility of use of ergodic theorem. This proof was obtained by Hartman-Van Kampen-Wintner and Weyl. The general case was studied by Jessen and Tornehave, see [9] for a very nice survey of the question.

Let us also note that Lagrange problem was studied in a much more wider context of almost periodic functions, see [8].

54. It is hard and maybe even impossible to understand, in general, the nature of the angle φ

2.1.2 What happens if the swiveling arm that passes by 0

If there is no preponderant term in the exponential polynomial (2.3) (**non-Lagrangian case**) then a swiveling arm considered by Lagrange may pass by zero. Although in this case, one still can find a way to define the limit (2.5). First of all, if $z(t)$ passes by 0 only a finite number of times, one gives the sense to the limit (2.5) when $T \to \infty$ in an obvious way.

Although, it can happen that the set $\{t : z(t) = 0\}$ is not finite (note for example that even in the simplest case $N = 2$ it always occurs when $l_1 = l_2$). But even in this case, one can define the limit (2.5) in the following way.

Since $z(t)$ is an analytic function then if $z(t) = 0$, the tangent line to $z(t)$ is still well defined and the argument $\arg z(t)$ is then well defined $\mod \pi$. So the argument of the function $z(t)$ at 0 can be defined as an angle corresponding to the inclination of this tangent line.

By considering the argument as a function $\mod \pi$ and not 2π and if one accepts that $r(t)$ can be negative (it changes sign if t passes a zero of $z(t)$ of the odd order), one can give sense to the formula (2.5). In what follows we place ourselves in this setting.

So for example, in the case $N = 2$ and $l_1 = l_2$ this improved definition will give the asymptotics for $\varphi(t) = \frac{1}{2}(\omega_1 + \omega_2)t + o(t)$ that can be checked by direct calculation, see [9, 2].

2.1.3 The answer for Lagrange problem for $N = 3$ and rationally independent angular velocities

The case $N = 2$ being completely treated, the first non-trivial case of the Lagrangian problem is $N = 3$. This Chapter is dedicated to a new purely geometric proof of the classical result

Theorem 2.3. *[3, 6] For the dynamics of a swiveling arm of type $l = (l_1, l_2, l_3)$ in the plane $\mathbb{C} \cong \mathbb{R}^2$ governed by a vector field* (2.2) *such that l_j satisfy all of three triangle inequalities and for $\omega_1, \omega_2, \omega_3$ rationally independent, the asymptotic velocity for the movement exists and is equal to the convex sum*

$$\omega = \frac{\alpha_1}{\pi}\omega_1 + \frac{\alpha_2}{\pi}\omega_2 + \frac{\alpha_3}{\pi}\omega_3 \tag{2.6}$$

where α_j are the angles in the triangle formed by intervals with sides l_1, l_2, l_3. The angle $\alpha_j > 0$ is the angle corresponding to the side l_j, $j = 1, 2, 3$.

Note that the formulation of the Theorem contains that the resulting asymptotic angular velocity doesn't depend on the initial conditions $\arg l_j$. This is true in the case when ω_j are rationally independent but is not true in general case.

2.1.4 Formulation of the Hartman-van Kampen-Wintner theorem for general N

The result for $N = 3$ stated in previous Subsection can be deduced from a more general theorem for any N :

Theorem 2.4. *[3, 6] For the dynamics of a swiveling arm of type $l = (l_1, \ldots, l_N)$ in the plane $\mathbb{C} \cong \mathbb{R}^2$ governed by a vector field* (2.2) *and for $\omega_1, \omega_2, \ldots, \omega_N$ rationally independent, the asymptotic velocity*

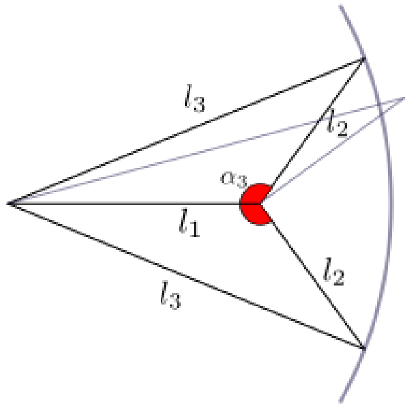

FIGURE 2.2 – For a fixed position of the first interval, the angles θ_2 corresponding to the position of the second interval that correspond to the inequality $|l_1 e^{i\theta_1} + l_2 e^{i\theta_2}| < |l_3|$, are contained in the range $(-\alpha_3, \alpha_3)$ where α_3 is the angle in the triangle with the sides $|l_j|$. On the picture one can see two positions of the swiveling arm in which $|l_1 e^{i\theta_1} + l_2 e^{i\theta_2}| = |l_3|$.

for the movement exists and is equal to the convex sum $\omega = \sum_{j=1}^{N} \omega_j q_j$ *where the coefficients* q_j *correspond to the following volumes of the subsets of the torus* $\mathbb{T}^N$ *with coordinates* $(\theta_1, \ldots, \theta_N)$ *:*

$$q_k = \text{mes}_N \left\{ (\theta_1, \ldots, \theta_N) \in \mathbb{T}^N \mid |l_1 e^{i\theta_1} + \ldots + l_{k-1} e^{i\theta_{k-1}} + l_{k+1} e^{i\theta_{k+1}} + \ldots + l_N e^{i\theta_N}| < |l_k| \right\}, \tag{2.7}$$

where mes_N *is a normalised Lebesgue measure of the torus* $\mathbb{T}^N$*.*

Let us first remark how Theorem 2.3 can be deduced from Theorem 2.4. First of all, one can think that $\omega_1 = 0$ (after passing to the rotating system of coordinates, see Proposition 2.5 for more precision). Then, from formula (2.7) for the coefficients one has

$$q_3 = \text{mes}_2 \left\{ (\theta_1, \theta_2) \in \mathbb{T}^N \mid |l_1 e^{i\theta_1} + l_2 e^{i\theta_2}| < |l_3| \right\}. \tag{2.8}$$

For a fixed θ_1 one can easily see that the measure in question is equal to $\frac{\alpha_3}{\pi}$ and doesn't depend on θ_1, see Picture 2.2. Then the integration with respect to θ_1 gives the same answer. Since all the coefficients are obviously symmetrical with respect to the change of the order of intervals then the final answer is given by formula (2.6).

2.1.5 Classical proof of Hartman-van Kampen-Wintner theorem for general N

Let us remind the reader of the classical proof of Hartman, van Kampen and Wintner of Theorem 2.4. The ideas of the proof are given in [6] but we find it useful to give a detailed argument here. We are interested in the study of the asymptotic behavior of the argument of the function $z(t) : \mathbb{R}_+ \to \mathbb{C}$ defined by (2.4). For now, let us suppose that $\arg z(t) \neq 0 \; \forall t \in \mathbb{R}_+$ and the swiveling arm doesn't pass through $0 \in \mathbb{C}$. Then we can write $z(t)$ in its polar form, $z(t) = r(t) \exp \varphi(t)$. Let us proceed with the following computation :

$$\ln z(t) = \ln r(t) + i\varphi(t) \Rightarrow \varphi(t) = \mathrm{Re}\left(\frac{\ln z(t)}{i}\right) \Rightarrow \frac{d\varphi}{dt} = \mathrm{Re}\left(\frac{1}{i}\frac{z'(t)}{z(t)}\right). \quad (2.9)$$

The last expression can be calculated explicitly using (2.4). We then obtain

$$\frac{d\varphi}{dt} = \mathrm{Re}\left(\frac{1}{i}\frac{\sum_{j=1}^{N} l_j i\omega_j e^{i\omega_j t}}{\sum_{j=1}^{N} l_j e^{i\omega_j t}}\right) = \mathrm{Re}\left(\frac{\sum_{j=1}^{N} |l_j|\omega_j e^{i(\omega_j t+\theta_j^{(0)})}}{\sum_{j=1}^{N} |l_j| e^{i(\omega_j t+\theta_j^{(0)})}}\right). \quad (2.10)$$

Here $l_j = |l_j| e^{i\theta_j^{(0)}}$ where $\left(\theta_1^{(0)}, \ldots, \theta_N^{(0)}\right) \in \mathbb{T}^N$ is a vector corresponding to the initial position of the swiveling arm.

The asymptotic angular velocity in which we are interested can be represented as

$$\omega = \lim_{T\to\infty} \frac{\varphi(T)}{T} = \lim_{T\to\infty} \int_0^T \frac{d\varphi(t)}{dt} dt. \quad (2.11)$$

The idea of the proof of Hartman, van Kampen and Wintner was that this expression is a time average of some function and that in this case the ergodic theorem can be applied to obtain ω as an explicit mean of this function. Indeed, let $f : \mathbb{T}^N \to \mathbb{R}$ be a following function on the torus :

$$f(\theta_1, \ldots, \theta_N) := \mathrm{Re}\left(\frac{\sum_{j=1}^{N} |l_j|\omega_j e^{i\theta_j}}{\sum_{j=1}^{N} |l_j| e^{i\theta_j}}\right). \quad (2.12)$$

Note that now we can express $\frac{d\varphi}{dt}$ as a value of this function at the point which is the image of the initial condition $\theta^0 = (\theta_1^0, \ldots, \theta_N^0) \in \mathbb{T}^N$ by the flow T^t of the linear vector field (2.2) on the torus $\frac{d\varphi}{dt} = f\left(T^t\theta^0\right)$.

Hence $\forall t_1, t_2 \in \mathbb{R}$ we have $\varphi(t_2) - \varphi(t_1) = \int_{t_1}^{t_2} f\left(T^\tau \theta^{(0)} d\tau\right)$ and the formula (2.11) gives

$$\omega = \lim_{T\to\infty} \frac{1}{T}\int_0^T f\left(T^\tau\theta^{(0)}\right) d\tau. \quad (2.13)$$

Now let us remind ourselves that the swiveling arm may pass by 0. In this case the function $f : \mathbb{T}^N \to \mathbb{R}$ defined by (2.12) is not continuous and takes infinite values when the end of the swiveling arm $z(t)$ passes by zero. But this function is integrable, $f \in L^1(\mathbb{T}^N, \mathrm{mes})$.

Take some $T_0 \in \mathbb{R}$ and denote $\tilde{f}(\theta) := \frac{1}{T_0}\int_0^{T_0} f \circ T^t(\theta) dt$. This new function being an average of f is a continuous function on the torus, $f \in C(\mathbb{T}^N)$.

Note also that the time averages as well as space averages of these two functions f and $\tilde{f}$ coincide.

Indeed, for the space averages since T^t is a measure-preserving flow,

$$\int_{\mathbb{T}^N} \tilde{f}(\theta) = \int_{\mathbb{T}^N} \frac{1}{T_0}\int_0^{T_0} f \circ T^t(\theta) dt d\theta = \int_0^{T_0} \frac{1}{T_0}\int_{\mathbb{T}^N} f \circ T^t(\theta) d\theta dt =$$
$$= \int_0^{T_0} \frac{1}{T_0}\int_{\mathbb{T}^N} f(\theta) d\theta dt = \int_{\mathbb{T}^N} f(\theta) d\theta. \quad (2.14)$$

And for the time averages $f_\infty(\theta)$ and $\tilde{f}_\infty(\theta)$, analogically, we get

$$\tilde{f}_\infty(\theta) := \lim_{T\to\infty}\frac{1}{T}\int_0^T \tilde{f}\circ T^t(\theta)dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{1}{T_0}\int_0^{T_0} f\circ T^{t+\tau}(\theta)d\tau dt =$$
$$= \frac{1}{T_0}\int_0^{T_0}\lim_{T\to\infty}\frac{1}{T}\int_0^T f\circ T^{t+\tau}(\theta)dtd\tau = f_\infty(\theta). \quad (2.15)$$

Note that the flow T^t is uniquely ergodic (since ω_j are rationally independent [55]) and $\tilde{f} \in C(\mathbb{T}^n)$ hence the space averages of $\tilde{f}$ coincide with time averages of $\tilde{f}$ for **all** values of $\theta \in \mathbb{T}^N$. Hence the same is true for f and the limit (2.13) can be written as a space average for all $\theta \in \mathbb{T}^N$. Hence we obtain that the limit (2.5) for any initial position of the swiveling arm $z(0) \in \mathbb{C}$ is just given by the space integral that can be explicitly calculated :

$$\int_{\mathbb{T}^N} f(\theta)d\theta = \mathrm{Re}\int_{\mathbb{T}^N}\frac{\sum_j \omega_j|l_j|e^{i\theta_j}}{\sum_j |l_j|e^{i\theta_j}}d\theta_1\dots d\theta_N = \sum_{j=1}^N \omega_j|l_j|\,\mathrm{Re}\int_{\mathbb{T}^N}\frac{e^{i\theta_j}d\theta_1\dots d\theta_N}{\sum_j |l_j|e^{i\theta_j}} =$$
$$= \sum_{j=1}^N \omega_j|l_j|\,\mathrm{Re}\int_{\mathbb{T}^{N-1}}\int_0^{2\pi}\frac{e^{i\theta_j}d\theta_j}{|l_j|e^{i\theta_j}+B(\theta_1,\dots,\theta_{j-1},\theta_{j+1},\dots,\theta_N)}d\theta_1\dots\theta_{j-1}\theta_{j+1}\dots\theta_N =$$
$$= \sum_{j=1}^N \omega_j|l_j|\,\mathrm{Re}\int_{\mathbb{T}^{N-1}}\int_0^{2\pi}\frac{1}{i|l_j|}\frac{\partial\ln(B_j+|l_j|e^{i\theta_j})}{\partial\theta_j}d\theta_1\dots\theta_{j-1}\theta_{j+1}\dots\theta_N =$$
$$\sum_{j=1}^N \omega_j\,\mathrm{Re}\int_{\mathbb{T}^{N-1}}\int_0^{2\pi}\frac{1}{i}\frac{\partial\ln(B_j+|l_j|e^{i\theta_j})}{\partial\theta_j}d\theta_1\dots\theta_{j-1}\theta_{j+1}\dots\theta_N, \quad (2.16)$$

where $B_j = B(\theta_1,\dots,\theta_{j-1},\theta_{j+1},\dots,\theta_N) := \sum_{k\in\{1,\dots,j-1,j+1,\dots,N\}}|l_k|e^{i\theta_k}$.

Now note that the internal integral over θ_j is equal to 1 if 0 is inside the circle of center B_j and radius $|l_j|$, in other words if $|l_j| > B_j$ and 0 otherwise. So from this we deduce that

$$\int_{\mathbb{T}^N} f(\theta)d\theta = \sum_{j=1}^N \omega_j \mathrm{mes}_{N-1}\left\{B(\theta_1,\dots,\theta_{j-1},\theta_{j+1},\dots,\theta_N) < |l_j|\right\} \quad (2.17)$$

so we obtain exactly the needed response. □

2.1.6 A new proof for $N = 3$: a dipolar form

In the previous Subsection we have shown the classical argument which is essentially a computation. In this Subsection we will give a new geometric proof of Theorem 2.3 that is based on the following idea : the argument of the swiveling arm changes significantly only when the arm passes by zero, and that happens (in case $N = 3$) when this arm closes up into a triangle.

First of all, let us note that we can suppose $\omega_1 = 0$ – it is sufficient to obtain the main result of Theorem 2.10 for arbitrary ω_1.

55. The same assumptions about ω_j hold for the Theorem 2.10 and the swiveling arm on the hyperbolic plane.

Proposition 2.5. *Suppose the limiting asymptotic velocity ω exists for a swiveling arm of type (l_1, l_2, l_3) with a corresponding vector field*

$$(\omega_2 - \omega_1)\frac{\partial}{\partial\theta_2} + (\omega_3 - \omega_1)\frac{\partial}{\partial\theta_3} \tag{2.18}$$

Then the limiting asymptotic velocity exists as well for a swiveling arm of the same type with a corresponding vector field

$$\omega_1\frac{\partial}{\partial\theta_1} + \omega_2\frac{\partial}{\partial\theta_2} + \omega_3\frac{\partial}{\partial\theta_3} \tag{2.19}$$

exists as well and is equal to $\omega_1 + \omega$.

Proof. This is almost obvious : the two systems described in the formulation, one with angular velocities $(\omega_1, \omega_2, \omega_3)$ and the other with angular velocities $(0, \omega_2 - \omega_1, \omega_3 - \omega_1)$ are related by the rotation. Indeed, the position of the first system at time t is just the image of the position of the second one at time t to which the rotation by the angle $\omega_1 t$ around the base point 0 is applied. $\square$

So now we suppose $\omega_1 = 0$ and we will think about Ψ as a map from $\mathbb{T}^2$ with coordinates (θ_2, θ_3) to $\mathbb{C}$. Also let us think about the flow T^t as a linear flow on the two-dimensional torus corresponding to the vector field

$$X = \omega_2\frac{\partial}{\partial\theta_2} + \omega_3\frac{\partial}{\partial\theta_3}.$$

Definition 2.6. *For any analytic curve γ let us define the* argument map $f_\gamma = \arg : \mathbb{C} \to \mathbb{R}$ *which is a multivalued map that defines an argument* $\arg\gamma(t)$ *of the point on this curve. Each time we use this notation we suppose that we take the continuous determination of the argument (the argument can be defined even when γ passes by* 0*, see Subsection 2.1.2).*

The limiting angular velocity (2.5) that interests us can be written as a time average

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T df_{\gamma(t)}$$

where $\gamma(t) = \Psi(T^t(\theta_2,^0 \theta_3^0))$ for some initial position (θ_2^0, θ_3^0) of the swiveling arm.

So for each trajectory $z(t)$ of the swiveling arm in the complex plane we have defined a 1-form $d\arg(z)$. Taking the limit of the time average of this form on the part of the path $z(t), t \in [0, T]$ is equivalent to the calculation of the change of the argument. The map $\Psi : \mathbb{T}^2 \to \mathbb{C}$ transports this form $d\arg(z)$ to the form on the torus. We will call this 1-form β **a Lagrange form** and we will study it carefully.

Let us note that the form β may be singular since Ψ sends some points on the torus $\mathbb{T}^2$ to $0 \in \mathbb{C}$. Indeed, there are two points $A, B \in \mathbb{T}^2$ that correspond to the values of θ_2 and θ_3 that make swiveling arms close up into triangles, see Figure 2.3.

Let us choose the coordinates on the torus $\mathbb{T}^2$ in a following way : the coordinates (θ_2, θ_3) define a swiveling arm such that the second (third) vector forms an angle θ_2 (θ_3) with the positive direction of horizontal axis (counted counterclockwise). From now on, we work in these coordinates.Then the points on the torus corresponding to the moments when the swiveling

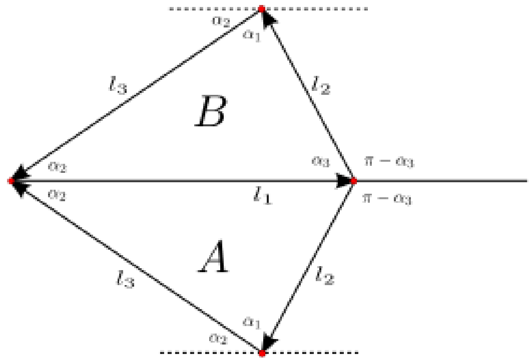

FIGURE 2.3 – Two positions of the swiveling arm that give singular points of the form β : the system forms one of two closed-up triangles A or B

arm closes up are $A = (-\pi + \alpha_3, \pi - \alpha_2)$ and $B = (\pi - \alpha_3, \pi + \alpha_2)$, see Figure 2.3. Now we will understand in more detail how the form β looks like and hence, how to take its time averages.

Let us first formulate a proposition that will be useful for us in the work with a non-singular part of the form β.

Proposition 2.7. *Consider a space M with a measure μ on it and a uniquely ergodic flow $T^t : M \to M$ of a vector field X on this space, the measure μ being the only invariant measure for the action.*

a. Then, for any point $x \in M$ and for any continuous function $f \in C(M, \mu)$ there exists a limit of time averages $\lim_{T\to\infty} \frac{1}{T} \int_0^T f \circ T^t(x)dt$ and this limit

- *doesn't depend on the point $x \in M$*
- *is equal to the space average $\int_M f d\mu$ of the function f.*

b. If one replaces f by $\bar{f} = f + X(g)$, where $g \in L^1(M, \mathbb{R})$, the time averages of f and $\bar{f}$ coincide.

c. For any closed 1-form β on M define the function $f := \beta(X)$. Then its space average $\int_M f d\mu$ is well defined on the cohomology class of β, in other words, it doesn't change if β is remplaced by $\bar{\beta} = \beta + dg$ where $g \in C^1(M, \mathbb{R})$.

d. For the case when $M = \mathbb{T}^N$ and X is a vector field given by (2.2), for any smooth 1-form β holds $\int_M \beta(X) = [\beta][\omega_1, \ldots, \omega_N]$, where $[\beta] \in H^1(\mathbb{T}^N, \mathbb{R})$ and $[\omega_1, \ldots, \omega_N] \in H_1(\mathbb{T}^N, \mathbb{R})$ corresponds to the sum of the coordinate circles with coefficients $\omega_j \in \mathbb{R}$. Note that $[\beta]$ has a representative with $\beta_{\text{reg}} \in [\beta]$ with constant coefficients $\beta_j \in \mathbb{R} : \beta_{\text{reg}} = \sum_{j=1}^N \beta_j d\theta_j$.

Proof. a. The existence of the limit and its independence from the initial point $x \in M$ follows from the ergodic theorem.

b. Now let us replace f by $\bar{f}$. In this case, the difference of limits for f and $\bar{f}$ is equal to

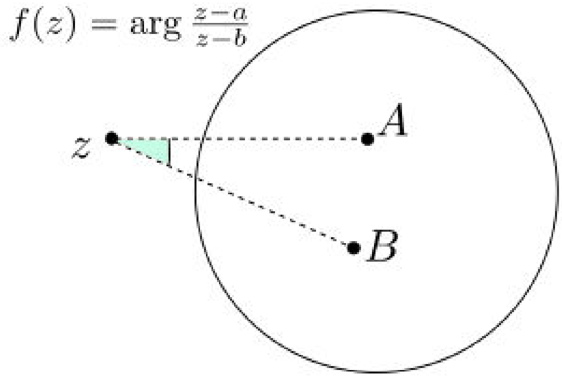

FIGURE 2.4 – Multivalued function $f(z) = \arg \frac{z-A}{z-B}$ is well-defined outside the big ball containing points A and B as an angle between two rays connecting z to A and to B correspondingly (there is a continuous determination for this function)

(we apply Newton-Leibnitz formula here)

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(g) \circ T^t(x)dt = \lim_{T\to\infty} \frac{g(T^T(x)) - g(x)}{T} = 0 \tag{2.20}$$

since g is bounded.

c. The statement of this point can be deduced to the statement of the point b. To prove that the addition of dg doesn't change the space integral we can replace it by the time average using ergodic theorem, and then apply the argument in equation (2.20).

d. The first statement of this point is just the application of point c. to this particular case, $M = \mathbb{T}^n$, $X = \sum_j \omega_j \frac{\partial}{\partial \theta_j}$. The fact that each form has its representative with constant coefficients follows from the fact that $H^1(\mathbb{T}^N, \mathbb{R}) \cong \mathbb{R}^N$. In conclusion, to find the integral $\int_{\mathbb{T}^N} \beta(X)$ for a smooth form β we just have to find its cohomology representative with constant coefficients. $\square$

The form β which is measuring the change of the argument in the system is not smooth, as stated above. Although we know how its singular part looks like.

Definition 2.8. *Fix two distinct points $A, B \in \mathbb{C}$. Let us consider a following multifunction f on the complex plane : $f(z) = \arg \frac{z-A}{z-B}$. This multifunction can not be defined on all of the plane in a continuous way although it is well defined outside a large enough ball $B(R)$ around A and B, see Figure 2.4. Then let us choose a function $\bar{f} : \mathbb{C} \to \mathbb{R}$ such that $\bar{f} = f$ in $\mathbb{C} \setminus B(R)$ and $\bar{f} \in C^\infty$. Then $g = f - \bar{f}$ is a multifunction that $g = 0$ in $\mathbb{C} \setminus B(R)$ and its differential dg defines a singular one-form that we will call a* dipolar form.

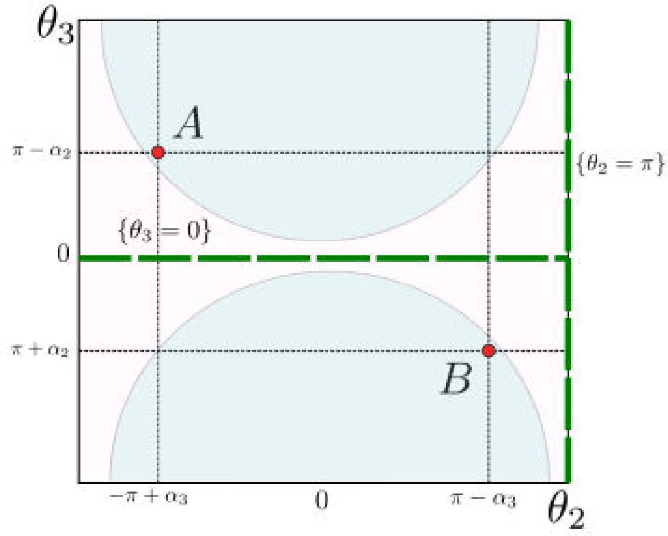

FIGURE 2.5 – The torus $\mathbb{T}^2$ corresponding to the position of the swiveling arm with three joints in the assumption $\omega_1 = 0$. Points $A, B \in \mathbb{T}^2$ corresponding to singularities of the form β are put inside the (blue) disk and hence the form β_{sing} is defined as a dipolar form on the torus.

The dipolar form defined on the complex plane can be transported to a form on the two-torus that has singularities in the points $A, B \in \mathbb{T}^2$ corresponding to the moments when the chain closes up into the triangles. For this, we will choose a disk on the torus containing the points A, B and transport the dipolar form on the plane to the form that we call β_{sing}, a dipolar form on the torus. Note that β_{sing} depends on the choice of this disk. We choose it as shown on Figure 2.5.

Then we have a following

Proposition 2.9. *For each triple of $|l_j|, j = 1, 2, 3$ satisfying all of the triangle inequalities, the Lagrange 1-form β associated to this triple can be represented as a sum of a regular part having constant coefficients, a singular part, corresponding to a dipolar form β_{sing} on the torus, defined before and a differential of a smooth function f. In other words, there exists a form $\beta_{\text{reg}} \in H^1(\mathbb{T}^2, \mathbb{R})$ with constant coefficients and a function $f \in C^1(\mathbb{T}^2)$ that $\beta = \beta_{\text{reg}} + \beta_{\text{sing}} + df$.*

Proof. First of all, let us note that $\beta - \beta_{\text{sing}}$ is a smooth 1-form on the torus. Indeed, when a point $\theta \in \mathbb{T}^2$ makes a loop around the point A (respectively, B) on the torus, the argument of the end of the swiveling arm grows (or, respectively, diminishes) by 2π exactly as a value of the dipolar form. This means hat A, B can't be the singularities of the difference. Neither any other point of course. The difference between the form and the corresponding dipolar form β_{sing} has its representative β_{reg} in a family of forms with constant coefficients since $H^1(\mathbb{T}^N, \mathbb{R}) \cong \mathbb{R}^N$ and the difference $\beta - \beta_{\text{sing}} - \beta_{\text{reg}}$ is a differential of a smooth function. □

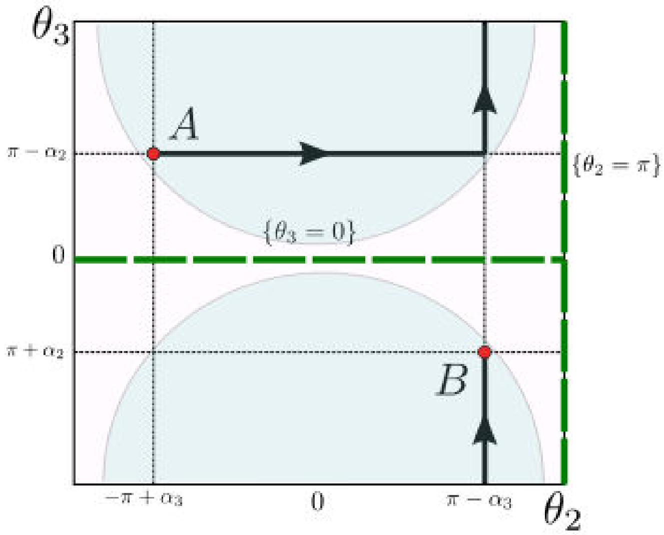

FIGURE 2.6 – A special path connecting the points A and B on the two-torus (chosen inside a blue disc used for a definition of the form β_{sing}. The flux of the vector field X through this path is equal to the evaluation of the dipolar part of the form β on X. The paths corresponding to the generators in cohomology of the torus correspond to the periods of the regular part of the form β.

2.1.7 A new proof for $N = 3$: evaluation of Lagrange form

As noticed before, to calculate the change of the argument, we have to calculate the time average of the form β. This average is the sum of three terms : the averages for $\beta_{\text{sing}}, \beta_{\text{reg}}$ and df. Since f is smooth, its time average is equal to the space integral and hence 0, see Proposition 2.7. Following the same Proposition, the time average of β_{reg} is its evaluation on the vector-field $X = \omega_2 \frac{\partial}{\partial \theta_2} + \omega_3 \frac{\partial}{\partial \theta_3}$, in other words, a space integral.

But let us first understand the time average of the dipolar singular part.

Let us choose a path γ connecting A and B on the torus that is contained in the disk used for the definition of β_{sing}. Note that all of such paths are homotopic. We will choose a special path connecting A and B as drawn on the Figure 2.6. This path corresponds to the passage from the first singular point A to the second point B by first rotating the second joint and then finishing the movement by rotating the third joint. Note that the time average of the dipolar form is equal to the flux of the vector field X through this path. The intuition under this statement is that the argument of $\arg z(t)$ changes by 2π (grows or diminishes in dependence of the direction) only if the trajectory $z(t)$ crosses the path between A and B.

Let us count the flux of the vector field $X = [\omega_2, \omega_3]$ through this path. On the first way, when θ_2 is changing and $\theta_3 = \pi - \alpha_2$ remains constant, the flux depends only on the vertical component of the field, ω_3. Also, the trajectories are transverse to the path and intersect it from the left to the right, it means we need to calculate the flux with the sign and obtain the response on this interval of the path equal to the $-\frac{2\pi - 2\alpha_3}{2\pi}\omega_3$. Analogously, the flux through the vertical component of the path is equal to $\frac{2\alpha_2}{2\pi}\omega_2$.

Now let us calculate the regular part β_{reg} of the Lagrange 1-form β : $\beta_{\text{reg}} = \beta_2 d\theta_2 + \beta_3 d\theta_3$ where $\beta_2, \beta_3 \in \mathbb{Z}$. Note that the regular part depends on the choice of the homotopy path of the path γ or, equivalently, of the disk containing singular points A and B.

The numbers β_2, β_3 are the periods of the form. Two calculate them, we can integrate this form on the paths in $\mathbb{T}^2$ which correspond to the first and second generator of cohomology $H^1(\mathbb{T}^2, \mathbb{R})$. What is important here is that those paths can not intersect the path γ that was connecting the singularities. Because only in this case the evaluation of a regular part will give us the correct quantity corresponding to the time average of the form $\beta - \beta_{\text{sing}}$. We choose these paths as shown on Figure 2.6 : geometrically, β_2 corresponds to the change of the argument of the end of the system when $\theta_3 = 0$. In this case, the argument of the end of the system of swiveling arms doesn't change when θ_2 makes one turn of the circle. Indeed, since triangle inequality holds true, $l_2 < l_1 + l_3$ and the turning second vector will never get around 0 if the first and the third one are pointing in one direction. Analogously, $\beta_3 = 1$ because the argument changes by 2π when the third interval is making one turn and the second is fixed, pointing in the direction $\theta_2 = \pi$. So we have that $\beta_{\text{reg}}[\omega_2, \omega_3] = \omega_3$.

Summing up the effect given by a dipolar part and an effect given by regular part (a differential gives no effect), we obtain the answer for the asymptotic speed : $\omega = \frac{\alpha_2}{\pi}\omega_2 + \frac{\alpha_3}{\pi}\omega_3$. Passing back to the system where the first interval is turning, we obtain the answer in a general case, $\omega = \omega_1 + \frac{\alpha_2}{\pi}(\omega_2 - \omega_1) + \frac{\alpha_3}{\pi}(\omega_3 - \omega_1) = \sum_j \frac{\alpha_j}{\pi}\omega_j$. The proof of Lagrange problem in the euclidian case is finished. $\square$

2.2 Lagrange problem on the riemannian surface with non-zero curvature

Let us now note that the similar problem can be formulated on any riemannian surface M which is *oriented* (in order to define the angular velocities and rotations) and *complete* (in order to be able to connect the points on this surface by geodesic paths).

Let us fix some point $0 \in M$ which will be the base for the swiveling arm. We will consider N geodesic intervals of lengths $|l_1|, |l_2|, \ldots, |l_N|$ on M that are forming a chain exactly in the same way as swiveling arms do on the euclidian plane. These intervals exist due to completeness. Let us also fix some initial positions for these intervals. Then, each of the intervals will turn with a constant angular speed around the end of the previous interval (or around $0 \in M$ for the first interval).

If the lengths $|l_j|, j = 1, \ldots, N$ are small enough then one may consider a local chart near $0 \in M$ corresponding to an open domain in the complex plane and to define the continuous determination of the argument. Then, analogically to the definition (2.5) one can define asymptotic angular velocity of the extremity of the system in a case of a swiveling arm on the complete riemannian surface and ask the same question : does a corresponding limit exist and what is it equal to ?

2.2.1 Redefining the angles

Note that there is one important complication in the case of a general riemannian surface. In the euclidian case the angles $\theta_1^0, \ldots, \theta_N^0$ were used to define the initial position of the swiveling arm on the euclidian plane (see Formula (2.3) as the angles of the respectful positions of the intervals, in comparison with the horizontal level. Note that this definition makes no

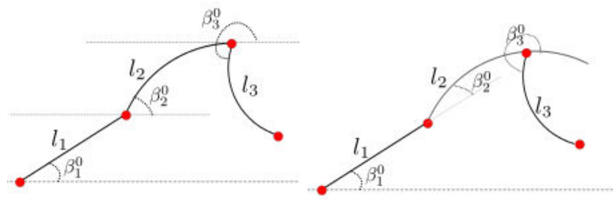

FIGURE 2.7 – Different ways to define the initial positions of the intervals : in the first case, with respect to the common horizontal direction (as an euclidian case where the translations exist) and with respect to the previous interval (in general case).

sense on the general riemmanian surface since there is no common horizontal level, in other words, global translations do not exist.

So we have to change the coordinates on the space of positions of a swiveling arm. We will count the angle now as with respect to the initial positions of the previous interval in the chain. So from now on θ_j^0 is an angle between the geodesic line continuing in the direction corresponding to l_{j-1} and the interval l_j, see Figure 2.7 for the comparison of two approaches.

Note that the map Ψ from the torus $\mathbb{T}^N$ to M that corresponds to the end of the system of swiveling arms of lengths $l_1, \ldots, l_N$ still can be defined with the use of these new coordinates. Indeed, by fixing the angular velocities ω_j, we can say that each interval is turning around the end of the previous one with the corresponding velocity since the notion of the *angle* is defined. Hence, with the use of the vector field (2.2) we define a movement of the sum of periodic motions on the oriented complete surface M.

To standardize our approach, let us rewrite the formulation of Theorem 2.3 in the case of these new coordinates first on the euclidian plane. Let us denote $(\theta_1, \theta_2, \theta_3) \in \mathbb{T}^3$ the coordinates that were used in Section 2.1 corresponding to the angles that the intervals make with a horizontal axis. And let us denote $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3) \in \mathbb{T}^3$. Then one can easily see that the connection between these two sets of coordinates is a linear transformation of a following form :

$$\begin{aligned}\tilde{\theta}_1 &= \theta_1 \\ \tilde{\theta}_2 &= \theta_2 - \theta_1 \\ \tilde{\theta}_3 &= \theta_3 - \theta_2.\end{aligned}$$

Its inverse is

$$\begin{aligned}\theta_1 &= \tilde{\theta}_1 \\ \theta_2 &= \tilde{\theta}_1 + \tilde{\theta}_2 \\ \theta_3 &= \tilde{\theta}_1 + \tilde{\theta}_2 + \tilde{\theta}_3.\end{aligned}$$

One can see that in the euclidian case the consideration of the second set of coordinates corresponds to the change of the set of angular velocities ω_j to $\omega_1, \omega_1 + \omega_2, \omega_1 + \omega_2 + \omega_3$. Hence in these coordinates the asymptotic angular velocity from Theorem 2.3 will take the following form :

$$\omega = \frac{\alpha_1}{\pi}\omega_1 + \frac{\alpha_2}{\pi}(\omega_1+\omega_2) + \frac{\alpha_3}{\pi}(\omega_1+\omega_2+\omega_3) = \;= \omega_1 + \omega_2\frac{\alpha_2+\alpha_3}{\pi} + \omega_3\frac{\alpha_3}{\pi}. \tag{2.21}$$

In the following we suppose that the coordinates on $\mathbb{T}^3$ that we are working with are $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$ and they correspond to the angles of the intervals made with a previous interval in the chain.

2.2.2 Constant curvature case

The question of finding the asymptotic velocity of such a system for $N = 3$ was communicated to us by Anatoly Stepin for the case of the hyperbolic plane $M = \mathbb{H}^2$. If we place ourselves in the Poincaré disc model then the end of the swiveling arm for $N = 3$ can be written out explicitely and actually the same classical argument (explained wonderfully in [6]) can be applied to this case to obtain the final answer but the formulas will take much more place. We actually have done the calculation, and it takes eight pages of double integrals. This motivated us to search for a simpler argument. The argument we presented in Subsections 2.1.6 and 2.1.7 is purely geometrical and once it is elaborated in the euclidian case, it can be expanded to the hyperbolic case to obtain the following

Theorem 2.10. *Consider a torus $\mathbb{T}^3$ with coordinates $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$. For a swiveling arm of type $l = (l_1, l_2, l_3)$ in the hyperbolic plane $\mathbb{H}^2$ such that l_j satisfy all of three triangle inequalities, and for $\omega_1, \omega_2, \omega_3$ rationally independent, the asymptotic velocity $\omega$for the movement governed by the vector field*

$$X = \sum_j \omega_j \frac{\partial}{\partial \tilde{\theta}_j} \tag{2.22}$$

exists and is equal to the sum

$$\omega = \omega_1 + \omega_2\frac{\alpha_2+\alpha_3+A}{\pi} + \omega_3\frac{\alpha_3}{\pi}, \tag{2.23}$$

where α_j are the angles in the triangle formed by intervals with sides $|l_1|, |l_2|, |l_3|$ and A is the area of this triangle. The angle α_j is the angle corresponding to the side l_j.

Here the angles $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$ correspond to the angles that the interval in a swiveling arm makes with the previous interval as explained in Subsection 2.2.1.

Proof. Note that the answer for the euclidian case was obtained in 2.1.7 without almost any calculation and is due only to the geometry of the movement.

First of all, by repeating the argument of Proposition 2.5 word by word for the hyperbolic geometry, we can suppose that $\omega_1 = 0$. Now, since we are studying a vector field (2.22) then the passage back to ω_1 will be just adding ω_1 to the final answer (for the limit angular velocity ω) for $\omega_1 = 0$.

We will apply the argument of Subsections 2.1.6 and 2.1.7 to the hyperbolic case. All the ideas and notations are preserved, the only change to make is a change of coordinates : instead of coordinates (θ_2, θ_3) the coordinates $(\tilde{\theta}_2, \tilde{\theta}_3)$ are considered. This changes the coordinates of the singular points for the form β_{sing}. But note that the principal properties still do hold :

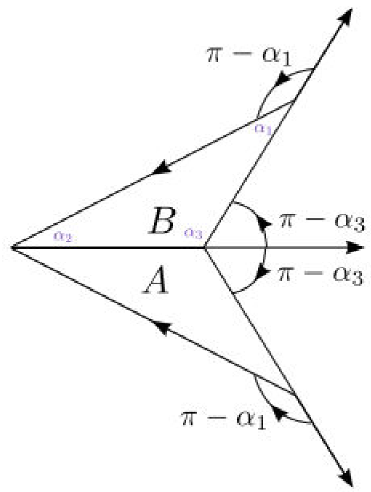

FIGURE 2.8 – Two positions corresponding to the singular points of the form β_{sing} on the hyperbolic plane : the positions in which the swiveling arm closes up into a triangle. This triangle has the sides of lengths $|l_j|, j = 1,2,3$ and the angles of values $\alpha_j, j = 1,2,3$. The corresponding to singularities points on the torus $\mathbb{T}^2$ with coordinates $(\tilde{\theta}_2, \tilde{\theta}_3)$ are the points $A(-\pi + \alpha_3, -\pi + \alpha_1)$ and $B(\pi - \alpha_3, \pi - \alpha_1)$. Note that now the coordinates are defined as angles between the present direction of the interval and the positive direction of the previous interval in the chain. We suppose that the coordinate is growing when the angle changes counterclockwise.

there are exactly two positions for a couple second and third joint for which the swiveling arm closes up. This is related to the fact that the angles of the hyperbolic triangle are defined by its sides in a unique way, and the triangles with the lengths verifying triangle inequalities do exist.

Let us reconsider the analogues of Figures 2.3, 2.5 and 2.6 in the new set of coordinates.

The singular points of the form β_{sing} that we will still call A and B now will have the coordinates $A(-\pi + \alpha_3, -\pi + \alpha_1)$ and $B(\pi - \alpha_3, \pi - \alpha_1)$, see Figure 2.8. Then, we will choose a path from A to B as on Figure 2.9 consisting of two straight intervals : one corresponding to the growth of $\tilde{\theta}_2$ for a constant $\tilde{\theta}_3 = -\pi + \alpha_1$ and one corresponding to the growth of $\tilde{\theta}_3$ for a constant $\tilde{\theta}_2 = \pi - \alpha_3$. Note that when $\tilde{\theta}_3$ grows (counterclockwise), the vertical part of this path from $-\pi + \alpha_1$ to $\pi - \alpha_1$ passes through π.

Analogically the proof in Subsection 2.1.7 we count the evaluation of Lagrange 1-form on the vector field (2.22). The singular part β_{sing} gives

$$-\frac{2\pi - 2\alpha_3}{2\pi}\omega_3 - \frac{2\alpha_1}{2\pi}\omega_2. \tag{2.24}$$

The regular part β_{reg} (with constant coefficients) is equal to $\beta_{\text{reg}} = \beta_2 d\tilde{\theta}_2 + \beta_3 d\tilde{\theta}_3$. Then β_2 is a period corresponding to the integration on the circle $\{\tilde{\theta}_3 = 0\}$: this integration will give $\beta_2 = 1$. Analogically, β_3 corresponds to the integration on the circle $\{\tilde{\theta}_2 = \pi\}$ that will give $\beta_3 = 1$ as well. Hence the evaluation of the regular part on the vector field X given by (2.22) is equal to $\omega_2 + \omega_3$ and by summing the two effects we have : $\omega = \frac{\pi - \alpha_1}{\pi}\omega_2 + \frac{\alpha_3}{\pi}\omega_3$. By adding ω_1 (passing to the rotating system of coordinates), we have the answer.$\square$

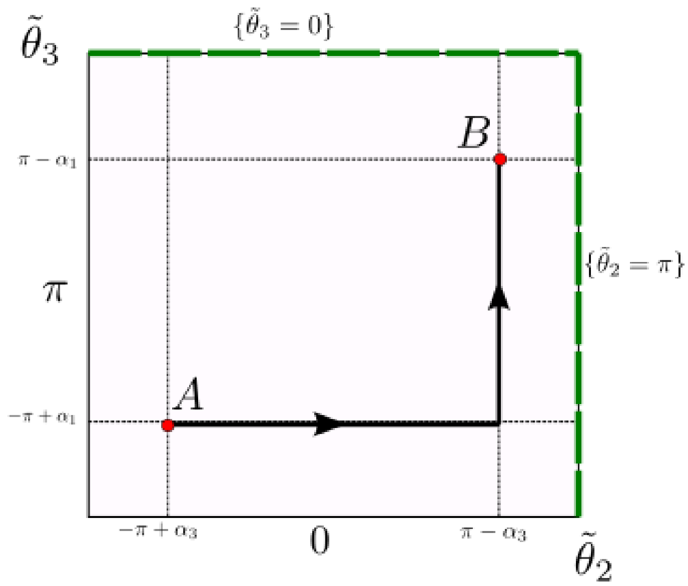

FIGURE 2.9 – A path of integration for a singular part of Lagrange form in the hyperbolic case on the two-torus with coordinates $(\tilde{\theta}_2, \tilde{\theta}_3)$. The path corresponds to turning first the second interval and then the third interval to come from one singular point to the other. Here the coordinates $\tilde{\theta}_j, j = 2, 3$ correspond to the relative positions of the intervals (with respect to a previous interval). The green paths $\{\tilde{\theta}_3 = 0\}$ and $\tilde{\theta}_2 = \pi$ are used after for the calculation of the periods of the regular part of Lagrangian 1-form.

So we see that the proof in the euclidian case extends to the hyperbolic case because hyperbolic geometry is a. isotropic (the geometry is the same in all directions, existence of local rotations) b. if one fixes three lengths of the sides of a triangle, such a triangle is uniquely defined (up to isometry) and its angles are the functions of lengths and nothing else.

Let us give an important

Remark 2.11. The possibility to pass to the case $\omega_1 = 0$ as explained in Proposition 2.5 reduces the configuration space of our system and we deal with a two-dimensional dynamics rather than three-dimensional. Note that after this remark and after reducing the system from the system with velocities $(\omega_1, \omega_2, \omega_3)$ to the system with velocities $(0, \tilde{\omega}_2, \tilde{\omega}_3)$ with $\tilde{\omega}_j = \omega_j - \omega_1$ we see that the only important condition for us is the rational independence of $\tilde{\omega}_2$ and $\tilde{\omega}_3$. For rationally dependent $\tilde{\omega}_2$ and $\tilde{\omega}_3$ the proof is also obvious since the system will be completely periodic. So, in general, for any N by this simple rotation one can diminish the dimension by 1 in case of euclidian as well as hyperbolic geometry. So one can say that the asymptotic velocity ω exists in any case (for N=3), not only if ω_j are rationally independent. This is true for euclidian geometry (as proven for any N by Jessen and Tornehave, and in case $N = 3$ by Bohl already) but also (and what is new) for hyperbolic geometry. In this case, the limiting velocity is obviously a ratio between a variation of the argument in a period to the length of the period.

Note that for the movement on the sphere the arguments of Theorems 2.3 and 2.10 will follow and the same theorem will be true modulo the remark that if the intervals in the swiveling arm are too big, the argument won't be defined on the sphere. So our theorem will hold only for small values of l_j. The formulation is the following (note a change in the sign) :

Theorem 2.12. *Consider a torus $\mathbb{T}^3$ with coordinates $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$. Consider a swiveling arm of type $l = (l_1, l_2, l_3)$ on the sphere $\mathbb{S}^2$ such that $|l_j|$ satisfy all of three triangle inequalities and are small enough : $|l_1| + |l_2| + |l_3|$ should be less than the distance between the north and south poles of the sphere. Then for $\omega_1, \omega_2, \omega_3$ rationally independent, the asymptotic velocity for the movement governed by the vector field* (2.22) *exists and is equal to the sum*

$$\omega = \omega_1 + \omega_2 \frac{\alpha_2 + \alpha_3 - A}{\pi} + \omega_3 \frac{\alpha_3}{\pi}, \tag{2.25}$$

where α_j are the angles in the triangle formed by intervals with sides $|l_1|, |l_2|, |l_3|$ and A is the area of this triangle. The angle α_j is the angle corresponding to the side l_j.

Here the angles $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$ correspond to the angles that the interval in a swiveling arm makes with the previous interval as explained in Subsection 2.2.1.

Remark 2.13. Let us remark that the answer for the constant curvature geometries (Theorems 2.3, 2.10 and 2.12) can be rewritten in a unified form as

$$\boxed{\omega = \omega_1 + \omega_2 \frac{\pi - \alpha_1}{\pi} + \omega_3 \frac{\alpha_3}{\pi}}$$

2.2.3 An arbitrary riemmanian surface : kite property

Let us now prove the Lagrange problem on the arbitrary oriented complete riemannian surface. The difference with a constant curvature case is that the isotropic property doesn't hold anymore : the geometry is different in dependence of the direction. This problem can easily be dealt with by taking the mean values of the functions considered in previous Section.

More precisely, let us fix some point 0 on the surface and let us consider a swiveling arm of type (l_1, l_2, l_3) with a base in the point 0. As one can see from the main argument in Subsections 2.1.6 and 2.1.7, the most important positions (the only that actually do influence the change of the argument) of a swiveling arm are those which correspond to the passages by 0 of the extremity of the system. In other words, the interesting configurations of the swiveling arm are related to the triangles formed by the intervals of lenghts $|l_1|, |l_2|, |l_3|$ with one of the vertices in 0 and the edges of lengths $|l_1|$ and $|l_3|$ passing by this vertex.

Note than in the euclidian $M = \mathbb{R}^2$ (as well as in hyperbolic $M = \mathbb{H}^2$ and spherical $M = \mathbb{S}^2$ geometries) geometry the following property holds :

Definition 2.14 (Kite[56] property for the surface M.)**.** *We will say that an orientable complete riemannian surface M has a kite property in the point $0 \in M$ if for any $l_1, l_2, l_3 \in \mathbb{R}_+$ – three numbers verifying all triangle inequalities and for any direction $\varphi \in \mathbb{S}^1$ there exist two triangles on M with sides of lengths l_1, l_2, l_3 such that*

- *both of these triangles have a vertex in* 0
- *for both triangles, the sides of lengths l_1 and l_3 pass by* 0
- *the side of length l_1 is the same for both triangles, and goes in the direction φ (formally, the corresponding tangent vector on the geodesic is equal to the vector defined by φ)*

This precise property that we use in the proof of Theorems 2.3 and 2.10. Even more, we use that the angles α_1, α_2 and α_3 in the triangles forming a kite do not change with $\varphi \in \mathbb{S}^1$. For a general surface the kite property may not hold in all generality and of course, the corresponding angles (if they exist) will be now the functions $\alpha_1^{\pm}(\varphi), \alpha_2^{\pm}(\varphi), \alpha_3^{\pm}(\varphi)$ of direction φ in which the first interval l_1 goes, see Fig. 2.10. But note that the following Proposition holds

Proposition 2.15. *For any complete oriented riemannian surface M the kite property holds for l_1, l_2, l_3 small enough.*

Proof. This almost obvious and follows from the fact that the small discs are convex on any riemannian surface. Hence the disc with radius l_2 and center at the end of the side of length l_1 will intersect the disc with radius l_3 and center 0 in exactly two points that correspond to the missing vertices of two triangles that we wish to find, see Figure 2.11. Of course, the angles in those triangles will depend on the direction φ. □

After making this remark, we can formulate the answer to the Lagrange problem in the general case which comes from the averaging of the answer in a constant curvature case.

2.2.4 Formulation and proof

Theorem 2.16. *Consider a torus $\mathbb{T}^3$ with coordinates $(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$ defined in Subsection 2.2.1. Consider an arbitrary oriented and complete riemmanian surface M and a swiveling arm of type (l_1, l_2, l_3) on it corresponding to the vector field*

$$X = \omega_1 \frac{\partial}{\partial \tilde{\theta}_1} + \omega_2 \frac{\partial}{\partial \tilde{\theta}_2} + \omega_3 \frac{\partial}{\partial \tilde{\theta}_3} \tag{2.26}$$

56. Kite is *cerf-volant* in French i.e. a flying deer. In Russian it is a flying snake though...

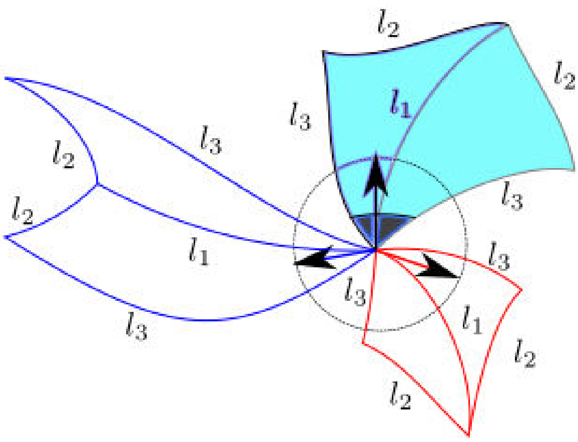

FIGURE 2.10 – In each direction $\varphi \in T_0^1 M$ of the unitary tangent bundle to the surface in the base point $0 \in M$ there is a corresponding geodesic going out of the point 0. To each of these geodesics corresponds a "kite" of two triangles $\Delta^+(\varphi)$ and $\Delta^-(\varphi)$ with the sides of the lengths l_1, l_2, l_3 as shown on the picture. The triangles $\Delta^+(\varphi)$ and $\Delta^-(\varphi)$ have a common side of length l_1. Those kites change their form hence the angles of these triangles are the functions of $\varphi : \alpha_1^-(\varphi), \alpha_2^-(\varphi), \alpha_3^-(\varphi)$, for Δ^- and $\alpha_1^+(\varphi), \alpha_2^+(\varphi), \alpha_3^+(\varphi)$, for Δ^-. The difference between the triangles Δ^- and Δ^+ is done with the help of orientation on the surface.

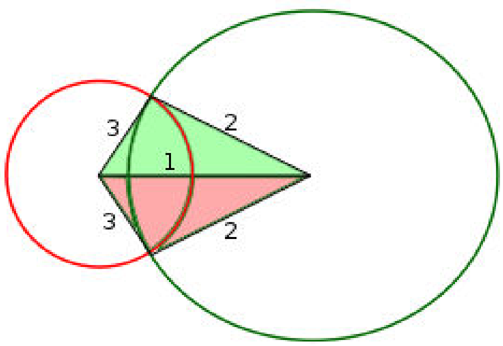

FIGURE 2.11 – The intersection of two convex discs gives two points that correspond to the vertices of two triangles with the sides of lengths l_1, l_2, l_3.

on the three torus $\mathbb{T}^3$. Suppose a base point of the arm is 0. Suppose also that angular velocities $\omega_j \in \mathbb{R}$ are rationally independent. Then for any swiveling arm with the lengths of the intervals l_j small enough such that the kite property holds, the asymptotic angular velocity exists and is equal to the linear combination of angular velocities ω_j

$$\omega = \omega_1 + \omega_2 \frac{\pi - \bar{\alpha}_1}{\pi} + \omega_3 \frac{\bar{\alpha}_3}{\pi}$$

where

$$\bar{\alpha}_j = \frac{\bar{\alpha}_j^+ + \bar{\alpha}_j^-}{2}, j = 1,2,3,$$
$$\bar{\alpha}_j^\pm = \frac{1}{2\pi} \int_{\mathbb{S}^1} \alpha_j^\pm(\varphi) d\varphi$$

are the means of the angles in triangles with the sides $|l_j|, j = 1,2,3$ in the kite property, see Pic. 2.10 with respect to the direction of the first interval. Here the parameter φ comes from the definition of a kite property.

Proof. Since the problem a priori doesn't have any rotational symmetry anymore we can't pass to the case $\omega_1 = 0$ as we did in Proposition 2.5. So we will consider the three-dimensional torus $\mathbb{T}^3$ with coordinates $(\theta_1, \theta_2, \theta_3)$. But still, the proof will repeat step by step the proof of Subsections 2.1.6, 2.1.7. There will be once more a Lagrange 1- form (now on a three-torus) encoding (with the vector field (2.26)) the dynamics of a swiveling arm. This form has singular and regular parts, $\beta = \beta_{\text{reg}} + \beta_{\text{sing}} + df$, f being a smooth function on $\mathbb{T}^3$.

The asymptotic velocity we are interested in is, as before, given by the evaluation of this form on the vector field X. The evaluation of a regular part is a space integral and for the calculation of the evaluation of its singular part we have to consider a flux through a surface with a boundary (in the proof of Subsection 2.1.7 we were dealing with paths because the dimension was 2 but here we will deal with surfaces).

Note that the points A and B corresponding to the positions of a swiveling arm in which it closes up into a triangle, still exist but they do now depend on the parameter $\varphi \in \mathbb{S}^1$ from the kite property, now we will have

$$A(\varphi) = \left(-\pi + \alpha_3^-(\varphi), -\pi + \alpha_1^-(\varphi)\right),$$
$$B(\varphi) = \left(\pi - \alpha_3^+(\varphi), \pi - \alpha_1^+(\varphi)\right).$$

So for each φ the plane $\tilde{\theta}_1 = \varphi$ intersects a singular set of a Lagrangian form in two points. So the singular set of the form β is represented (for small l_j) by two circles.

Hence for each value of θ_1 (or one can say φ, see the definition of a kite) one has two corresponding triangles with the sides of lengths $|l_j|, j = 1,2,3$ and the angles $\alpha_j^\pm(\varphi), j = 1,2,3$ that correspond to two points on $\mathbb{T}^3$ in which a form β has a singularity.

So to calculate the evaluation of β_{sing} on X, we need to calculate the flux of X through a cylinder whose boundary is the union of before-mentioned circles, see Picture 2.12. We will represent this cylinder as the union of the paths with fixed θ_1. Those are the paths analogical to the paths in the proof for constant curvature case.

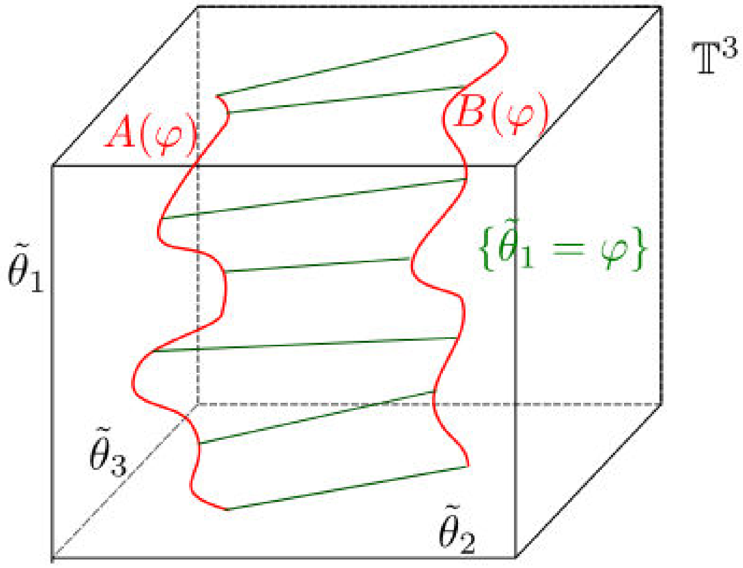

FIGURE 2.12 – The calculation of asymptotic velocity is related to the calculation of the flux of the vector field X through the cylinder foliated by intervals $\theta_1 = \text{const}$ depicted on the picture. The cube on the picture is a fondamental domain of the torus $\mathbb{T}^3$ so its opposite sides are identified.

Then the flux of the vector field in the components number 2 and 3 along the path with a fixed $\tilde{\theta}_1 = \varphi$ will be equal to the values analogous to the values before (modulo the fact that the angles now are not the same for $\Delta^-(\varphi)$ and $\Delta^+(\varphi)$, so the evaluation of a singular part on the vector field X will give, analogously to the formula (2.24) :

$$-\frac{2\pi - \alpha_3^+(\varphi) - \alpha_3^-(\varphi)}{2\pi}\omega_3 - \frac{\alpha_1^+(\varphi) + \alpha_1^-(\varphi)}{2\pi}\omega_2.$$

Note that since the $A(\varphi)$ and $B(\varphi), \varphi \in \mathbb{S}^1$ are two closed circles hence the $\tilde{\theta}_1$ component of the vector field X won't give any contribution to the evaluation of a singular part.

Now, to sum up the contributions for each φ, we will take a mean over φ when θ_1 is changing and so we will obtain that the evaluation of the singular part of β on X gives :

$$\beta_{\text{sing}}[X] = -\frac{\pi - \bar{\alpha}_3}{\pi}\omega_3 - \frac{\bar{\alpha}_1}{\pi}\omega_2.$$

To calculate the evaluation of the regular part of the form $\beta_{\text{reg}} = \beta_1 d\tilde{\theta}_1 + \beta_2 d\tilde{\theta}_2 + \beta_3 d\tilde{\theta}_3$ where $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ we need to find the periods of this form.

To do this we will integrate the form on three circles :

- $\{\tilde{\theta}_2 = \pi, \tilde{\theta}_3 = 0\}$ for the calculation of β_1
- $\{\tilde{\theta}_1 = 0, \tilde{\theta}_3 = 0\}$ for the calculation of β_2
- $\{\tilde{\theta}_1 = 0, \tilde{\theta}_2 = \pi\}$ for the calculation of β_3

Let us note that all of these three circles can be chosen in a way that they are disjoint from the cylinder of singularities (and even more, there is a torus containing this cylinder disjoint from these three circles). This is clear for the two last paths since $\theta_1 = \text{const}$ and this follows from the 2-dimensional pictures drawn before, see for example Figure 2.9. The first circle neither doesn't intersect the cylinder since this corresponds to a degenerate position that is never approached by continuous curves $A(\varphi)$ and $B(\varphi)$, $\varphi \in \mathbb{S}^1$.

One can easily see that in all of these cases, $\beta_j = 1, j = 1, 2, 3$ and hence the evaluation of the regular part will give $\beta_{\text{reg}}[X] = \omega_1 + \omega_2 + \omega_3$. By summing up two contributions we get the final answer. $\square$

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3

A problem of the incenters of triangular orbits in an elliptic billiard and complex reflection

This chapter reproduces our article in the Enseignement Mathématique devoted to the study of one planimetrical problem related to the elliptic billiard. We consider 3-periodic orbits in an elliptic billiard. Numerical experiments conducted by Dan Reznik have suggested that the locus of the centers of inscribed circles of the corresponding triangles is an ellipse. It turns out that methods of complex algebraic geometry are more effective in the study of this problem than real planimetry ones. We prove the fact observed by Reznik by the complexification of the problem coupled with the complex law of reflection.

In the end of the Chapter we have inserted a small bonus to the initial article – the simplest proof of the main theorem by the methods of plane geometry that are within a reach of a school program although not trivial.

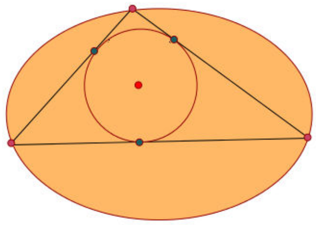

FIGURE 3.1 – The illustration for the main theorem : an ellipse, one of the periodic orbits, its incircle and the center of this circle. The locus of incenters is forming an ellipse.

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3.1 Introduction

Let us state a nice geometric fact that will be of interest for us in this Chapter.

Theorem 3.1. *Consider a set of three-periodic trajectories of an elliptic billiard. Then the set of the incenters (the centers of the inscribed circles) corresponding to the triangles formed by three points of the trajectory lying on the ellipse is an ellipse itself, see Figure 3.1 for illustration.*

In Section 3.2 of this chapter we will prove this theorem by somewhat unexpected methods in this context – methods of complex algebraic geometry. In Section 3.3 of this chapter we will give a plane geometry proof of this Theorem. The reader can compare those two proofs and choose which one she (or he) prefers. In our opinion although a plane geometry proof from Section 3.3 juggles with lots of nice plane geometry facts, the complex one is actually less complex and more conceptual.

Let us note that both of the proofs use in a crucial way a following corollary of Poncelet theorem and integrability of the elliptical billiard :

Proposition 3.2. *All 3-periodic orbits in an elliptical billiard are tangent to some ellipse confocal to the initial ellipse.*

3.2 A proof of the theorem with complex methods

ON THE INCENTERS OF TRIANGULAR ORBITS IN ELLIPTIC BILLIARD

by Olga ROMASKEVICH *)

ABSTRACT. We consider 3-periodic orbits in an elliptic billiard. Numerical experiments conducted by Dan Reznik have shown that the locus of the centers of inscribed circles of the corresponding triangles is an ellipse. We prove this fact by the complexification of the problem coupled with the complex law of reflection.

1. THE STATEMENT OF THE THEOREM AND THE IDEA OF THE PROOF

Elliptic billiards are at the same time classical and popular subject (see, for example [1], [2], [3] and [4]) since they continue to deliver interesting problems. We will consider an ellipse and a billiard in it with the standard reflection law: the angle of incidence equals the angle of reflection. Let the trajectory from a point on the boundary repeat itself after two reflections: this means that we obtained a triangle which presents a 3-periodic trajectory of the ball in the elliptic billiard. Poncelet's famous theorem [5] states that the sides of these triangles are tangent to some smaller ellipse confocal to the initial one.

We prove the following fact which was observed experimentally by Dan Reznik [10]:

THEOREM 1.1. *For every elliptic billiard the set of incenters (the centers of the inscribed circles) of its triangular orbits is an ellipse.*

*) Supported in part by RFBR grants $12-01-31241$ mol-a and $12-01-33020$ mol-a-ved.

The proof uses very classical ideas: complexify and projectivize, that is, replace the Euclidean plane by the complex projective plane. This approach was used by Ph. Griffiths and J. Harris in [6] and, more recently, by R. Schwartz in [9]. The main tool in the proof is that of complex reflection: we consider an ellipse as a complex curve and define a complex law of reflection off a complex curve. The locus of the incenters will be also a complex algebraic (even rational) curve. We will prove that the latter curve is a conic in $\mathbf{CP}^2$. Its real part will be a bounded conic – an ellipse.

The reasons for developping complex methods for the solution of a problem in planimetry are twofold: first of all, such an approach shows that sometimes complexification paradoxically simplifies things. We think that complex methods could be a useful tool in obtaining many results of this kind. Ideologically, this work is related with the recent work by A. Glutsyuk where he studies complex reflection, see for example [13] an the joint work with Yu. Kudryashov [14]. The second reason to develop the complex approach for this particular problem was the incompetence of the author to prove this fact with real tools besides computation. The reader is encouraged to find an alternative proof of the Theorem 1.1.

Complex reflection law and its basic properties needed here are reviewed in Section 2. Section 3 contains the proof of Theorem 1.1. In Section 4 we discuss the position of the foci of an obtained ellipse.

2. Complex reflection law

For our purposes it will be useful to pass from the Euclidean plane $\mathbf{R}^2$ to the complex projective plane $\mathbf{CP}^2$: the metric now is replaced, in local complex coordinates (z, w), by a quadratic form $ds^2 = dz^2 + dw^2$. In the following we will be interested in the geometry of this new space $\mathbf{CP}^2$ with quadratic form ds^2. One could have replaced the initial Euclidean metric by a pseudo-euclidean one: the geometry of such a space is also interesting and somewhat similar to our case. The best references here will be [7] and [8].

Definition 2.1. The lines with directing vectors that have zero length are called *isotropic*. All other lines are called *non-isotropic*.

Let us fix a point $x \in \mathbf{CP}^2$ and define complex symmetry with respect to a line passing through x as a map acting on the space $\mathcal{L}_x$ of all lines passing through x. There are two isotropic lines $L_x^{v_1}$ and $L_x^{v_2}$ in $\mathcal{L}_x$ with directing vectors $v_1 = (1, i)$ and $v_2 = (1, -i)$.

DEFINITION 2.2 (Complex reflection law). For a point $x \in \mathbf{CP}^2$, the *complex reflection (symmetry)* in a non-isotropic line $L_x \in \mathcal{L}_x$ is the mapping given by the same formula as in the case of standard real symmetry: it's a linear map that in the coordinates defined by the line L_x and its orthogonal line $L_x^{\perp}$ has a diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The image of any line L under reflection in an isotropic line $L_x^{v_1}$ (or $L_x^{v_2}$) is defined as a limit of its images under reflections with respect to a sequence of non-isotropic lines converging to $L_x^{v_1}$ (or $L_x^{v_2}$).

Moreover, the complex reflection in a curve is the reflection in its tangent line.

THEOREM 2.3 ([13]).

a. *The complex symmetry with respect to any isotropic line L at some point $x \in L$ is well defined for all non-isotropic lines (i.e. the latter limit of the images of a sequence of non-isotropic lines exists) and maps every non-isotropic line $X \ni x$ to L.*
b. *Under the reflection at the point x with respect to some isotropic line $L \in \mathcal{L}_x$, the line L itself may be mapped to any line passing through x (i.e. the mapping in this case is multivalued). In particular, it can stay fixed.*

The isotropic directions generated by the vectors v_1 and v_2 can be represented by the points $I_1 = (1 : i : 0) \in \mathbf{CP}^2$ and $I_2 = (1 : -i : 0) \in \mathbf{CP}^2$, respectively. We choose an affine coordinate z on the projective line $\mathbf{CP}^1 = \mathbf{C} \cup \infty$ at infinity, that is, the line through points I_1 and I_2 in such a way that $I_1 = 0$ and $I_2 = \infty$. The lemma below implies Theorem 2.3 and follows easily from the definition. It describes the reflection in a line close to isotropic.

LEMMA 2.4 ([13]). *For any $\varepsilon \in \bar{\mathbf{C}} \setminus \{0, \infty\}$, let L_ε be the line through the origin $(0,0) \in \mathbf{C}^2$ and having direction ε. Let $\tau_\varepsilon : \mathbf{CP}^1 \to \mathbf{CP}^1$ be the reflection in L_ε acting on the space $\mathbf{CP}^1$ of the lines through the origin. Then $\tau_\varepsilon(z) = \frac{\varepsilon^2}{z}$ in the above introduced coordinate z.*

Proof. The map τ_ε is a projective transformation that preserves L_ε as well as the set of isotropic lines. So $\tau_\varepsilon(\varepsilon) = \varepsilon$ and $\tau_\varepsilon\{0, \infty\} = \{0, \infty\}$. Let us show that τ_ε permutes 0 and ∞. Otherwise, it would have three fixed points on the infinity line $\mathbf{CP}^2 \setminus \mathbf{C}^2$ and therefore be identity map of the infinity line. Moreover, the points lying on L_ε are fixed for τ_ε. In this case τ_ε should be identity but it's a nontrivial involution, contradiction.

We see that the restriction of τ_ε is a nontrivial conformal involution of $\mathbf{CP^2} \setminus \mathbf{C}$ fixing ε and permuting 0 and ∞. So it should map z to $\frac{\varepsilon}{z^2}$.

3. The proof

Let us consider triangular orbits of the complexified elliptic billiard: the triangles inscribed into a complexified ellipse and satisfying the complex reflection law. Denote the initial ellipse from Theorem 1.1 by Γ, and the Poncelet ellipse tangent to all triangular orbits by γ. We use the same symbols for complexifications of these conics. The following classical fact will be used for Γ and γ, and for the inscribed circles.

LEMMA 3.1 ([11], p. 179, [12], p.334).

a. *Ellipses Γ and γ in the real plane are confocal if and only if their complexifications have 4 common isotropic tangent lines and their common foci lie on the intersections of these lines.*

b. *The two tangent lines to the complexified circle passing through its center are isotropic.*

DEFINITION 3.2 (Sides and degenerate sides of a triangle). A *side* of a triangle in $\mathbf{CP}^2$ with disctinct vertices is a complex line through a pair of its vertices. A triangle is called *degenerate* if all its vertices lie on the same line. A priori, a triangular orbit may have coinciding vertices. We will call *A the degenerate side* through two coinciding vertices if A is obtained as a limit of sides $A_\varepsilon, \varepsilon \to 0$ of non-degenerate triangular orbits. For such a side *A its image* under reflection is defined as a limit (which exists as the limit in Definition 2.2) of images of A_ε.

By taking a family A_ε of lines tangent to γ and converging to A, and computing their images (in fact, applying Lemma 3.3 below), one could deduce the structure of degenerate triangular orbits formulated in Lemma 3.4.

LEMMA 3.3. *Let A be a common isotropic tangent line to two analytic (algebraic) curves γ and Γ and let the tangency points be quadratic and distinct. If A is deformed in a family A_ε ($A = A_0$) of lines tangent to γ then the image of A_ε under the reflection in Γ tends to some non-isotropic line as $\varepsilon \to 0$.*

Proof. The more general case of this lemma is contained in [13], see Proposition 2.8 and Addendum 2 there.

The isotropic line A is deformed in a family A_ε : let us suppose that the family is chosen in such a way that the angle between A and A_ε is precisely ε. Suppose that A_ε intersects Γ in some point a_ε tending to the point a_0 of isotropic tangency. The simple computation shows that since the tangency points are quadratic, the tangent line T_ε to Γ at the point a_ε has the angle of the order $\sqrt{\varepsilon}$ with A. This with lemma 2.4 gives that the limit of the reflected lines is a non-isotropic one.

Now we can describe the degenerate triangles occuring in our problem.

LEMMA 3.4. *If a triangular orbit in the complexified ellipse Γ is degenerate then it has two coinciding non-isotropic non-degenerate sides B and one degenerate isotropic side A.*

Proof. Since $\deg \Gamma = 2$, two vertices should merge, so the degenerate side A through them is tangent to Γ and to γ, and hence is isotropic by Lemma 3.1. The other two sides are non-isotropic by Lemma 3.3 and they coincide.

LEMMA 3.5 (Main lemma). *The complex curve of incenters $\mathcal{C}$ intersects the complex line F through the foci of Γ at exactly two points with index 1.*

Proof. Let $c \in \mathcal{C} \cap F$ and suppose that the corresponding triangle is degenerate, see Figure 1. By Lemma 3.4 one of its sides is isotropic, and two other sides coincide and are non-isotropic. We will denote the isotropic line as A and non–isotropic line as B. Line A is tangent to the inscribed circle, so by Lemma 3.1, $c \in F \cap A$. Also c is a point of intersection of bisectors, so either $c \in B$ or $c \in B^{\perp}$. Note that B is tangent to the inscribed circle, hence if $c \in B$, then B should be isotropic, which is a contradiction. So $c \in B^{\perp}$, but by Lemma 3.1 c is a focus. $B^{\perp}$ is tangent to Γ and passes through the focus, so it should be isotropic which is impossible since B is not isotropic.

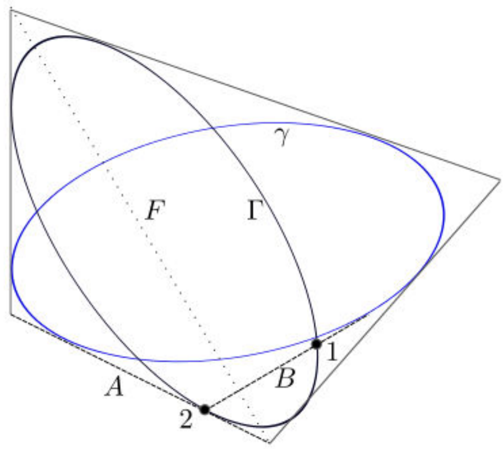

FIGURE 1

Two complex confocal ellipses Γ and γ having four common isotropic tangent lines. The line F of real foci passes through the intersections of isotropic lines. A degenerate trajectory for an elliptic billiard in Γ with caustic γ: the degenerate triangle is an interval between points 1 and 2 and its sides are lines A and B. Line A is isotropic while B is not.

Now consider the case of a not degenerate triangle corresponding to $c \in \mathcal{C} \cap F$. Consider the reflection in F: the inscribed circle, as well as its center c, are mapped to themselves. If the set of the sides of a triangle and their images under the reflection in F consists of *six* lines, then the inscribed circle and the ellipse γ should be tangent to all of them. But five tangent lines already define a conic, so γ must be a circle. But in this case, Theorem 1.1 is trivial and the locus under consideration is a point.

Therefore some sides of the triangle should map to some other sides. One needs to consider two cases: either there is a side which maps to itself, or there are two sides which map to each other. But the latter case reduces to the former: indeed, the points of intersection of the two exchanging lines with Γ (not lying on F) are mapped to each other, so the line connecting them is mapped to itself. Therefore, in the non-degenerate case, the corresponding triangle has a side which is symmetric with respect to F and tangent to γ. There are only two such lines, and hence only two intersections c_1 and c_2, both real (see Figure 2), and only two triangles corresponding to them, for each $c_i, i = 1, 2$.

Let us now prove that the intersections $\mathcal{C} \cap F$ have index 1. Let us

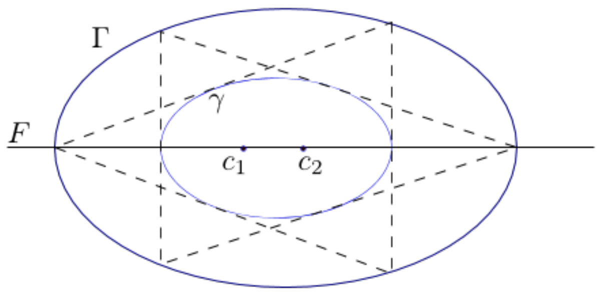

FIGURE 2
Two triangular orbits in Γ corresponding to the centers c_1, c_2 of inscribed circles lying on the foci line F

parametrize an ellipse γ by a parameter ε, and consider the corresponding center $c(\varepsilon) \in \mathcal{C}$, assuming that $c(0) \in F$. It suffices to prove that $\frac{\partial c}{\partial \varepsilon}(0) \neq 0$. Suppose the contrary: the centers of the circles do not change in the linear approximation: $c(\varepsilon) = c(0) + O(\varepsilon^2)$. Then the radius of the incircle $r(\varepsilon)$ has nonzero derivative at $\varepsilon = 0$, unless for $\varepsilon = 0$ both the incircle and the ellipse γ are tangent to the sides of the triangle at the same points. This is impossible if γ is not a circle, since two distinct conics can not be tangent at more than two distinct points. So we have that the radii of the incircles change linearly: $r(\varepsilon) = r(0) + \alpha\varepsilon(1 + o(1))$ for $\alpha \neq 0$. But this is not possible due to symmetry: indeed, the radius has to be an even function of ε.

Theorem 1.1 follows directly from the Lemma 3.5 since an algebraic curve intersecting some line in exactly two points (with multiplicities) is a conic.

4. FOCI STUDY

One could surmise that the ellipse $\mathcal{C}$ that is obtained in Theorem 1.1 is confocal to the initial one. It appears that it is not so. Picture 3 shows how the foci of the ellipse $\mathcal{C}$ move regarding the foci of the ellipse Γ.

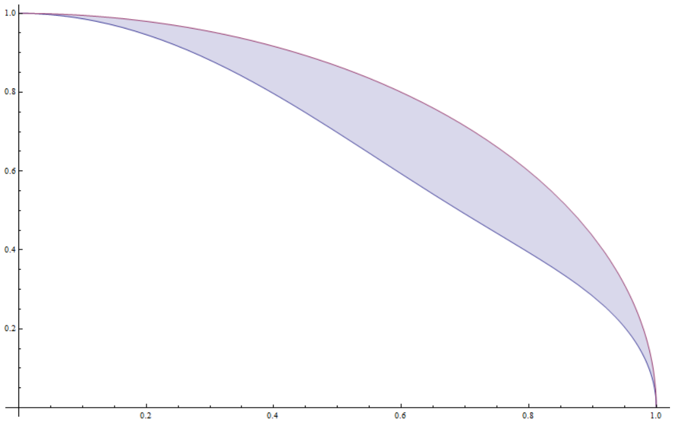

FIGURE 3

Distances between the common center of ellipses Γ and C and their foci as functions of the ratio of semi-axes of an initial ellipse

We suppose that the ratio between the semi-axis of the initial ellipse Γ is $t \in (0,1)$. The upper graph on Figure 3 is a graph of the distance from the center of Γ to its foci: just the arc of the circle $\left\{\left(t, \sqrt{1-t^2}\right), t \in (0,1)\right\}$. The lower graph is a graph of analogous (quite complicated) function for the ellipse $\mathcal{C}$. This graph was obtained by pure computation. The reader is encouraged to find the geometrical meaning for the position of the foci of $\mathcal{C}$.

5. ACKNOWLEDGMENTS

The author is grateful to Alexey Glutsyuk for stating the problem as well as for inspiring her with complex reflection ideas and for permanent encouragement. The author would also like to thank the referee for pointing out a considerable amount of references as well as making very helpful suggestions. The author is also grateful to A. Gorodentsev for providing the reference [11], to I. Schurov for his succor in making pictures and to É. Ghys and S. Tabachnikov for support. This work was done in the pleasant atmosphere of École Normale Supérieure de Lyon.

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(Reçu le 24 april 2013)

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3.3 A proof of the theorem with plane geometry methods

We set for ourselves a motivating goal to prove Theorem 3.1 by the use of simple plane geometry methods without any calculations. The proof stated here is not at all trivial but it is the simplest one we have found and we hope that maybe a reader can suggest an easier way. This proof will use some basic ideas of projective geometry as well as some theorems about Gergonne point and isogonal conjugacy.

Let us first remind the reader about some basic facts.

3.3.1 A polar and a pole

Definition 3.3. A polar *of a point P with respect to a non–degenarate curve γ of second order is the set of points N which are harmonically conjugated with the point P with respect to the points M_1 and M_2 of intersection of the curve γ by lines passing through P. Harmonic conjugation means that a cross-ratio of those* 4 *points is fixed and equal to* -1 *:*

$$(M_2, M_1, P, N) = \frac{M_2P}{M_2N} \cdot \frac{M_1N}{M_1P} = -1 \tag{3.1}$$

if we consider the segments with respectful orientations. See Figure 3.2.

One can prove that a polar is a line and for this line a point P is called a pole. Note that if one can draw two real tangent lines to the quadric from the point P then a polar line is passing through the points of tangency, see Figure 3.2.

One can also define a *polar transformation* with respect to a non–degenerate quadric of the projective space to its set of lines (which is in the bijection with the space itself) : each point is mapped to the corresponding polar line. So, for example, one can define an image under the polar transformation (with respect to some quadric) of a quadric by looking at the images of the tangent lines. It is a simple fact (that can be proven algebraically or geometrically) that the image of a quadric under a polar transformation is, once more, a quadric, see for example [1]. We will use these facts in the proof, for more information on polar transform see for example [1, 3, 5, 6].

3.3.2 Gergonne point and isogonal conjugacy

Our proof will be also using a notion of Gergonne point. It was discovered by Joseph Diaz Gergonne in the beginning of XIX century. The definition of this point comes with a theorem :

Theorem 3.4. *For any triangle ABC and for its inscribed circle denote by A_1, B_1, C_1 correspondingly the points of tangency of the inscribed circle with the sides of the triangle. Then the lines AA_1, BB_1 and CC_1 intersect in one point G that is called* the Gergonne point*, see Figure 3.3.*

Gergonne point has lots of nice properties : for around twenty of them discovered by a computer, see the article by Deko Dekov, [2]. The other interesting definition that will be important for us is *isogonal conjugacy*. It also comes along with a theorem :

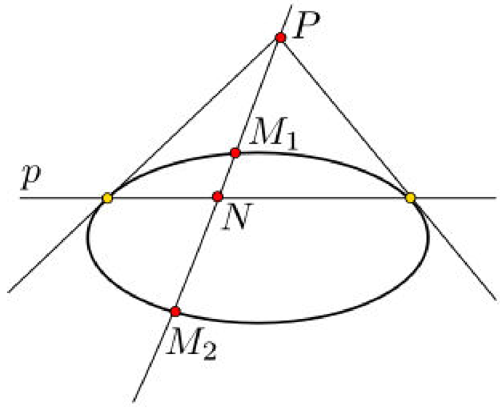

FIGURE 3.2 – For each point P one can define a corresponding line p : for any line intersecting the conic in two points M_1, M_2 the point N of the intersection of this line with a conic satisfies the cross-ratio equation (3.1). In the case of a point P outside an ellipse, its polar line is a line connecting two tangency points

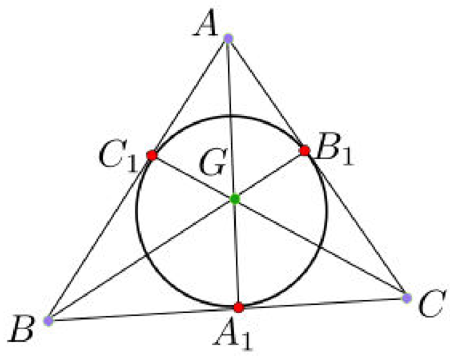

FIGURE 3.3 – Gergonne point in the triangle ABC : a point of the intersection of lines joining the vertices with the points of tangency of the inscribed circle.

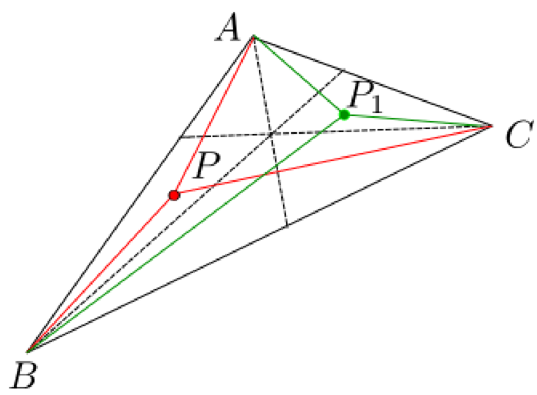

FIGURE 3.4 – Construction of an isogonally conjugated point P_1 to a point P in the triangle ABC : dotted lines are bisectors, and red lines are reflected to green ones under the reflections with respect to those bisectors.

Theorem 3.5. *Fix a triangle ABC. For any point P of the plane consider the lines AP, BP, CP. Then if one takes their reflections with respect to corresponding bisectors of the triangle ABC, the new reflected lines will intersect in one point. The point P_1 obtained in such a manner is called an isogonally conjugated point to P with respect to the triangle ABC, see Figure 3.4.*

Isogonal conjugacy has lots of beautiful properties but we will restrict ourselves to the property we will need. It is contained in a following :

Proposition 3.6. *Gergonne point is isogonally conjugated to the center of negative homothety of in-circle and circumcircle.*

For more general information on Gergonne point and isogonal conjugacy, see [1]. Now, after these reminders, we are ready to start the proof.

3.3.3 Proof of the main result with plane geometry methods

Let us call our initial ellipse Γ and a corresponding Poncelet ellipse γ (see Proposition 3.2), as we were already doing in Section 3.2. Consider some 3-periodic orbit and a corresponding triangle ABC. Let us follow the proof step by step :

1. Let us draw three tangent lines to the ellipse Γ in the points A, B, C. Those tangent lines will intersect in the points that we will call correspondingly A^*, B^*, C^*. Note that those points are actually the centers of excircles. These are the circles inscribed in the infinite domains, bounded by one of the sides and the continuations of two other sides. This is almost obvious since one can see that the line BA^* bisects the corresponding angle (that follows from the fact that the reflection in the point B preserves the angles. This explanation is easier to draw than to write down, so see Figure 3.5.
2. Then note that the line AB is a polar of the point C^* as well as BC is a polar of A^* and AC a polar of B^* (with respect to Γ). This follows from our remarks in Subsection 3.3.1.

3. Note that the lines AB, BC and AC are tangent to the so-called Poncelet ellipse γ. Consider now a polar transform of this ellipse γ with respect to the bigger ellipse $Gamma$: it will map to some conic that is passing by the points A^*, B^* and C^*. Let us denote this conic $\bar{\Gamma}$. So, once again, this conic $\bar{\Gamma}$ is conjugated to the conic γ with respect to the initial ellipse Γ.
4. So now we actually we can look at our problem in a new way. Before we were thinking that the triangle ABC was moving inside the ellipse Γ and outside the ellipse γ (when we say outside we mean that its sides were tangent to γ). Now we will be thinking of a triangle $A^*B^*C^*$ moving inside the conic $\bar{\Gamma}$ and, at the same moment, outside of the ellipse Γ. See Figure 3.6.
5. How to define a center of the inscribed circle of ABC in terms of $A^*B^*C^*$? It is actually the intersection point of AA^*, BB^* and CC^*. In other words, the intersection point of the lines connecting the vertices of $A^*B^*C^*$ with its tangency points with an ellipse Γ.
6. The problem is then reformulated as following : for two conics and a triangle moving "between" them in such a way that its vertices stay on one conic and that its sides are tangent to the other conic, prove that the intersection point of the lines connecting the vertices and tangent points a. exists and b. moves on an ellipse.
7. Note that the existence of such a point is the same statement as Theorem 3.4. Indeed, by a projective transformation, we can map an inside ellipse Γ to a circle. Actually, we can prove this theorem in the case when Γ and $\bar{\Gamma}$ are circles : indeed, by a projective transformation we can map two conics to the circles, and all the tangency and intersection properties as well as the class of conics are preserved. From now on, we think of Γ and $\bar{\Gamma}$ as of circles.
8. The fact that this point moves on an ellipse is much less trivial but it follows from the following beautiful Theorem by Alexander Skutin, [4] :
   Theorem 3.7. *Consider a family of triangles with a fixed inscribed and circumscribed circles. There are isogonal transformations related to each one of these triangles. Then, for any fixed point on the plane, the images of its isogonal transformations related to the family of these triangles, form a conic.*
   Note that this theorem is very powerful hence it has a free parameter – the position of a point on a plane. By applying this theorem in a particular case when this point is an inverse homothety center of Γ and $\bar{\Gamma}$, we prove the theorem.

□

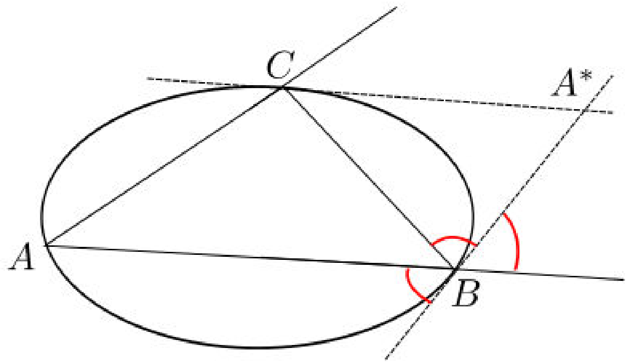

FIGURE 3.5 – The intersections of tangent lines to the ellipse in the points corresponding to the trajectories of a billiard define the centers of excircles : indeed, they are intersections of two outer bisectors.

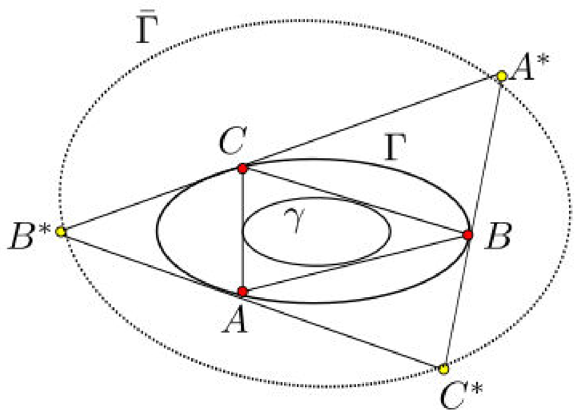

FIGURE 3.6 – The initial 3-periodic trajectory $A - B - C$ and a corresponding triangle $A^*B^*C^*$ with vertices – centers of excircles. The study of the triangle ABC between γ and Γ is replaced by the study of triangle $A^*B^*C^*$ between Γ and $\bar{\Gamma}$.

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Deuxième partie

General methods of asymptotic study of dynamical systems : normal forms and ergodic theory

4

Markovian spherical averages for measure-preserving actions of the free group

This chapter is devoted to the ergodic theory of measure-preserving actions of a finitely-generated free group. We establish the mean convergence of the spherical averages in the very general case of Markov chains under some mild nondegeneracy assumptions on the stochastic matrix defining our Markov chain. This convergence was previously known only for symmetric Markov chains, while the conditions ensuring convergence in our paper are inequalities rather than equalities, so mean convergence of spherical averages is established for a much larger class of Markov chains.
This is a joint work with Alexander Bufetov and Lewis Bowen, and the chapter here reproduces our article published in Geometriae Dedicata.

MEAN CONVERGENCE OF MARKOVIAN SPHERICAL AVERAGES FOR MEASURE-PRESERVING ACTIONS OF THE FREE GROUP

LEWIS BOWEN, ALEXANDER BUFETOV, AND OLGA ROMASKEVICH

Abstract. Mean convergence of Markovian spherical averages is established for a measure-preserving action of a finitely-generated free group on a probability space. We endow the set of generators with a generalized Markov chain and establish the mean convergence of resulting spherical averages in this case under mild nondegeneracy assumptions on the stochastic matrix Π defining our Markov chain. Equivalently, we establish the triviality of the tail sigma-algebra of the corresponding Markov operator. This convergence was previously known only for symmetric Markov chains, while the conditions ensuring convergence in our paper are inequalities rather than equalities, so mean convergence of spherical averages is established for a much larger class of Markov chains.

1. Introduction

Consider a finitely generated free group $\mathbb{F}$ and a probability space (X, μ).

Let $T : \mathbb{F} \to \mathrm{Aut}(X, \mu)$ denote a homomorphism of $\mathbb{F}$ into the group of measure-preserving transformations of (X, μ). We consider a finite alphabet V with a labeling map $\mathcal{L} : V \to \mathbb{F}$.

We will study an arbitrary Markov chain with V being its set of states. That is, take a stochastic matrix $\Pi = (\Pi_{v,w})_{v,w\in V}$ with rows and columns indexed by the elements of V (so $\sum_w \Pi_{v,w} = 1$ for every v). We assume that Π has a stationary distribution $\nu : V \to [0, 1]$ with $\nu(v) > 0$ for all $v \in V$. Stationarity means that $\sum_{v\in V} \Pi_{w,v}\nu(v) = \nu(w)$ for any w.

Let $G = (V, E)$ denote the directed graph on V with edge set

$$E := \{(w, v) : \ \Pi_{vw} > 0\}.$$

Note (w, v) is the reverse of (v, w) above. This is intentional.

By a *directed path* in G we mean a sequence $s = (s_1, \dots, s_n) \in V^n$ of vertices such that $(s_i, s_{i+1}) \in E$ for all i. The length of such a path is $|s| := n$. For any such path we denote

$$\mathcal{L}(s) = \mathcal{L}(s_1)\cdots\mathcal{L}(s_n) \in \mathbb{F}, \quad T_s = T_{\mathcal{L}(s)} \in \mathrm{Aut}(X, \mu), \quad \Pi_s = \Pi_{s_n s_{n-1}} \cdots \Pi_{s_2 s_1}.$$

Define spherical averages $S_n : L^1(X, \mu) \to L^1(X, \mu)$ by the formula

$$S_n(\phi)(x) := \sum_{s=(s_1,\dots,s_n)} \nu(s_n)\Pi_s \phi(T_s x) \tag{1}$$

The goal of this paper is to prove that, under mild additional conditions on Π, there is a constant k such that the averages $\frac{1}{2k}\sum_{i=0}^{2k-1} S_{n+i}$ are mean ergodic in L^1. To state these conditions properly, we need more notation.

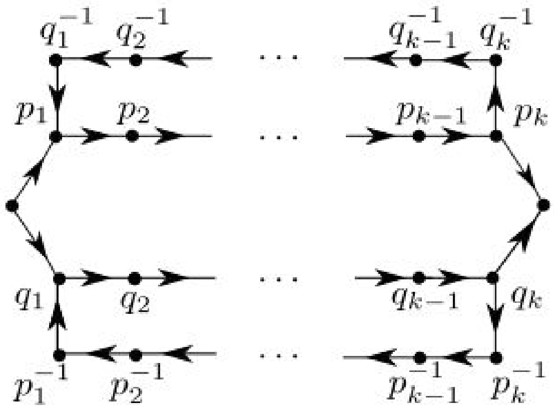

FIGURE 1. A good subgraph with paths $p = (p_1, \ldots, p_k)$ and $q = (q_1, \ldots, q_k)$ from Definition 1. We have used the notation $p^* = (p_k^{-1}, \ldots, p_1^{-1})$ and $q^* = (q_k^{-1}, \ldots, q_1^{-1})$.

Notation 1. If $p \in V^k$ and $q \in V^l$ then we let $pq \in V^{k+l}$ be their concatenation. So if $p = (p_1, \ldots, p_k)$ and $q = (q_1, \ldots, q_l)$ then $pq = (p_1, \ldots p_k, q_1, \ldots, q_l)$. We let $\mathcal{L}(p) = \mathcal{L}(p_1) \cdots \mathcal{L}(p_k) \in \mathbb{F}$ denote the product of the labels.

Definition 1. A subgraph $H \subset G$ is *good* of order k if it consists of vertices u, w and directed paths p, q, p^*, q^* of length k so that

- upw, uqw, pq^*p, qp^*q are directed paths in G
- $\mathcal{L}(p^*) = \mathcal{L}(p)^{-1}$, $\mathcal{L}(q^*) = \mathcal{L}(q)^{-1}$

Figure 1 illustrates the structure of a good subgraph. We do not require that a good subgraph be induced.

Definition 2. For each $v \in V$, let $\Gamma_v \leq \mathbb{F}$ be the subgroup generated by all elements of the form $\mathcal{L}(p)$ where pv is a directed path from v to itself in G. To be more precise, the condition on p is that it be a directed path of the form $p = (p_1, \ldots, p_n) \in V^n$ such that $p_1 = v$ and $(p_n, v) \in E$ is an edge of G.

Definition 3. We will say that Π is *admissible* of order k if

- its associated graph G contains a good subgraph of order k,
- G is strongly connected and
- there is some $v \in V$ such that $\Gamma_v = \mathbb{F}$.

Theorem 1.1. *Suppose* Π *is admissible of order* k. *Then for any probability-measure-preserving action* $\mathbb{F} \curvearrowright (X, \mu)$ *and any* $f \in L^1(X, \mu)$

$$\frac{1}{2k} \sum_{i=0}^{2k-1} S_{n+i} f$$

converges in L^1 *to* $\mathbb{E}[f|\mathbb{F}]$ *as* $n \to \infty$, *where* $\mathbb{E}[f|\mathbb{F}]$ *is the conditional expectation on the sigma algebra of* $\mathbb{F}$*-invariant measurable subsets.*

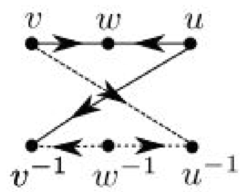

FIGURE 2. Sufficient condition for a graph G to contain a good subgraph, see Proposition 1.2

Remark 1. Note that the conditions on Π depend only on which entries are positive and which are zero. In particular, no relations are assumed between the entries of the Markov chain.

In practice, it is a straightforward task to check whether Π is admissible. We note for example, the following special case:

Proposition 1.2. *Suppose V is finite and $\mathcal{L} : V \to \mathbb{F}$ is injective, so that we may identify V as a subset of $\mathbb{F}$. Also suppose G is strongly connected and for every $(a, b) \in E$, $(b^{-1}, a^{-1}) \in E$ where the inverse is taken in the group $\mathbb{F}$. If there exist $v, w, u \in V$ such that $(v, w), (u, w), (u, v^{-1}) \in E$ (see Figure 2) then G contains a good subgraph. So if there is some $v \in V$ such that $\Gamma_v = \mathbb{F}$ then the conclusion to Theorem 1.1 holds.*

Proof. Note $(v, u^{-1}), (w^{-1}, v^{-1}) \in E$. Because G is strongly connected and finite, there exists a k so that for any ordered pair of vertices of G there exists a directed path between them of length k. In particular there exists a directed path $p := (p_1, \dots, p_k)$ from $p_1 := w$ to $p_k := v$ and a directed path $q := (q_1, \dots, q_k)$ from $q_1 := v^{-1}$ to $q_k := u$. It is now elementary to check that upw, uqw, pq^*p, qp^*q are directed paths in G where p^* is the unique directed path in G with $\mathcal{L}(p^*) = \mathcal{L}(p)^{-1}$. □

1.1. **Historical remarks.** For two rotations of a sphere, convergence of spherical averages was established by Arnold and Krylov [1], and a general mean ergodic theorem for actions of free groups was proved by Guivarc'h [23].

A first general pointwise ergodic theorem for convolution averages on a countable group is due to Oseledets [29] who relied on the martingale convergence theorem.

First general pointwise ergodic theorems for free semigroups and groups were given by R.I. Grigorchuk in 1986 [19], where the main result is Cesàro convergence of spherical averages for measure-preserving actions of a free semigroup and group. Convergence of the spherical averages themselves was established by Nevo [25] for functions in L_2 and Nevo and Stein [27] for functions in L_p, $p > 1$ using deep spectral theory methods. Whether uniform spherical averages of an integrable function under the action of a free group converge almost surely remains an open problem (it is tempting to speculate that a counterexample might be possible along the lines of Ornstein's example [28]). The method of Markov operators in the proof of ergodic theorems for actions of free semigroups and groups was suggested by R. I. Grigorchuk [20], J.-P. Thouvenot (oral communication), and in [10]. In [12] pointwise convergence is proved for Markovian spherical averages under the additional assumption that the Markov chain be reversible. The key step in [12] is the triviality of the tail sigma-algebra for the corresponding Markov operator; this is proved using Rota's "Alternierende Verfahren" [30], that is to say, martingale convergence. The reduction of powers of the Markov operator to Rota's "Alternierende Verfahren" in [12] essentially relies on the reversibility of the Markov chain. In this paper, we show that the triviality of the tail sigma-algebra still holds under much milder assumptions on the underlying chain.

The study of Markovian averages is motivated by the problem of ergodic theorems for general countable groups, specifically, for groups admitting a Markovian coding such as Gromov hyperbolic groups [22] (see e.g. Ghys-de la Harpe [18] for a detailed discussion of the Markovian coding for Gromov hyperbolic groups). First results on convergence of spherical averages for Gromov hyperbolic groups, obtained under strong exponential mixing assumptions on the action, are due to Fujiwara and Nevo [17]. For actions of hyperbolic groups on finite spaces, an ergodic theorem was obtained by Bowen in [3].

Cesàro convergence of spherical averages for all measure-preserving actions of Markov semigroups, and, in particular, Gromov hyperbolic groups, was established by Bufetov, Klimenko and Khristoforov in [13]. In the special case of hyperbolic groups, a short and very elegant proof of this theorem, using the method of Calegari and Fujiwara [15], was later given by Pollicott and Sharp [31]. Using the method of amenable equivalence relations, Bowen and Nevo [4], [5], [6], [7] established ergodic theorems for "spherical shells" in Gromov hyperbolic groups. The latter do not require any mixing assumptions.

1.2. **Examples.**

1.2.1. *Uniform spherical averages.* Consider the special case in which $\mathbb{F} = \langle a_1, \dots, a_r \rangle$ and $V = \{a_1, \dots, a_r\} \cup \{a_1^{-1}, \dots, a_r^{-1}\} \subset \mathbb{F}$. We let $\mathcal{L} : V \to \mathbb{F}$ be the inclusion map and $\Pi_{a,b} = \frac{1}{2r-1}$ if $a \neq b^{-1}$, $\Pi_{a,b} = 0$ otherwise. We let ν be the stationary distribution that is uniformly distributed on V. In this case, Π is admissible of order 1 and S_n is the uniform average on the sphere of radius n centered at the identity in $\mathbb{F}$. That is,

$$S_n(\phi)(x) = |\{g \in \mathbb{F} : \ |g| = n\}|^{-1} \sum_{|g|=n} \phi(T_g x)$$

for $\phi \in L^1(X, \mu)$ and $x \in X$. So Theorem 1.1 proves the mean ergodic theorem for the averages $\frac{S_n + S_{n+1}}{2}$. This result was first obtained by Guivarc'h [23].

1.2.2. *A surface group example.* Let $\Lambda = \langle a, b, c, d | [a, b][c, d] = 1 \rangle$ denote the fundamental group of the closed genus 2 surface. There is a natural Markov coding of this group, developed by Bowen-Series [8], that was used in [14] to prove a pointwise ergodic theorem for Cesàro averages of spherical averages (with respect to the word metric on this group). Using this coding and Theorem 1.1 we will show:

Corollary 1.3. *There exists a sequence π_n of probability measures on Λ such that*

- *π_n is supported on the union of the spheres of radius n and radius $n+1$ centered at the identity in Λ (with respect to the word metric);*

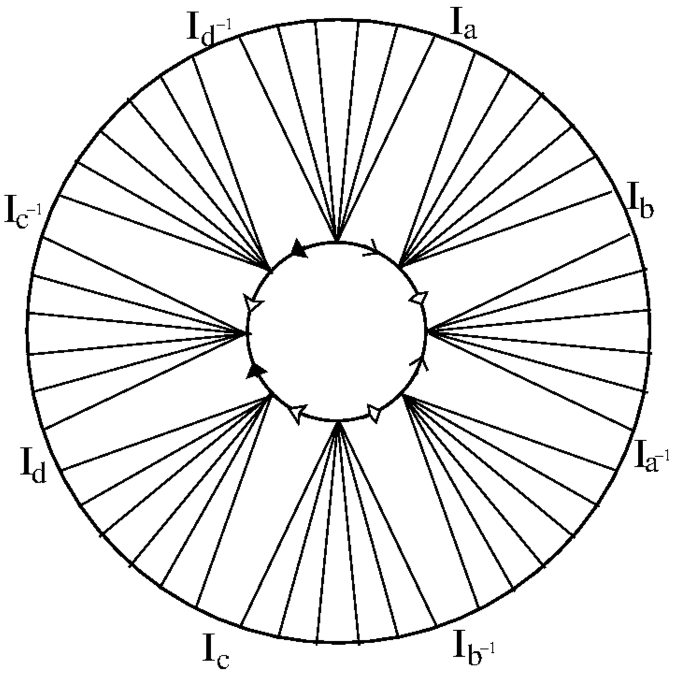

FIGURE 3. This is a distorted view of the region $\mathcal{R}$ in the hyperbolic plane together with all of the geodesics of the tesselation $\mathcal{T}$ incident to $\mathcal{R}$. Every interior angle incident to the inner circle in this diagram is $\pi/4$. There are 48 intervals in $\mathcal{I}$ total. Only 8 special intervals are labeled.

- π_n *is mean ergodic in* L^1 *in the sense that: if* $\Lambda \curvearrowright (X,\mu)$ *is any probability-measure-preserving action and* $f \in L^1(X,\mu)$ *then the averages* $\pi_n(f) \in L^1(X,\mu)$ *defined by*
$$\pi_n(f)(x) = \sum_{g\in\Lambda} \pi_n(g) f(g^{-1}x)$$
converge in $L^1(X,\mu)$ *to* $\mathbb{E}[f|\Lambda]$, *the conditional expectation of* f *on the sigma-algebra of* Λ*-invariant subsets.*

To explain the coding, let $\mathcal{R}$ denote a regular octogon in the hyperbolic plane (which we identify with $\mathbb{D}$ the unit disk in the complex plane) with all interior angles equal to $\pi/4$. This is a fundamental domain for an action of Λ on $\mathbb{D}$ by isometries. It can be arranged that if $S = \{a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1}\}$ then $\mathcal{R} \cap s\mathcal{R}$ is an edge of $\mathcal{R}$ for any $s \in S$.

Let $\mathcal{T} = \cup_{g\in\Lambda} g\partial\mathcal{R}$ be the union of the boundaries of Λ-translates of $\mathcal{R}$. We may think of $\mathcal{T}$ as a union of bi-infinite geodesics. Let $\mathcal{P} \subset \partial\mathbb{D}$ denote the collection of endpoints of those geodesics in $\mathcal{T}$ which meet $\mathcal{R}$ (crucially this includes lines which meet $\partial\mathcal{R}$ only in a vertex of $\mathcal{R}$). The points $\mathcal{P}$ partition $\partial\mathbb{D} - \mathcal{P}$ into connected open intervals; we denote the collection of all these intervals by $\mathcal{I}$. See figure 3.

For $s \in S$, consider the edge $\mathcal{R} \cap s\mathcal{R}$. This edge is contained in a bi-infinite geodesic that separates the hyperbolic plane into two half-spaces. Let $L(s)$ denote the open arc of $\partial\mathbb{D}$ bounding the half space that contains $s\mathcal{R}$. For each $I \in \mathcal{I}$ let $s_I \in S$ be an element such that $I \subset L(s_I)$. For each I there are either one or two choices for s_I. Define $f : \partial\mathbb{D} \to \partial\mathbb{D}$ by $f(x) = s_I^{-1}x$ for $x \in I$. As observed in [8, 32], the map f is Markov in the sense that for any $J \in \mathcal{I}$, $f(I) \cap J \neq \emptyset$ implies $f(I) \supset J$.

Let $V = \mathcal{I}$, $E = \{(I, J) \in V \times V : \ f(I) \supset J\}$, $G = (V, E)$ be the associated directed graph, $\mathbb{F} = \langle a, b, c, d\rangle$ be the rank 4 free group, and $\mathcal{L} : V \to \mathbb{F}$ be the map $\mathcal{L}(I) = s_I$. We extend $\mathcal{L}$ to the set of all finite directed paths in G as explained in the introduction. In [32, Theorem 5.10 and Corollary 5.11] (see also [2, Theorem 2.8]), the following is proven:

Lemma 1.4. *Let $\pi : \mathbb{F} \to \Lambda$ be the canonical surjection $\pi(s) = s$ for $s \in S$. Then for every $g \in \Lambda - \{e\}$ there is a unique element $w \in \mathbb{F}$ such that (a) $\pi(w) = g$ and (b) there exists some directed path p in G such that $\mathcal{L}(p) = w$. Moreover, the word length of w is the word length of g.*

Theorem 1.5. *If $\Pi = (\Pi_{v,w})_{v,w\in V}$ is any stochastic matrix with $\Pi_{v,w} > 0$ for all $(w, v) \in E$ then it is admissible of order 1.*

Proof. In [14], it is shown that the adjacency matrix of G is irreducible. Equivalently, G is strongly connected.

For $s \in S$, let $I_s \subset \mathcal{I} = V$ be the unique interval contained in $\mathcal{L}(s) \setminus \cup_{t\neq s}\mathcal{L}(t)$. By direct inspection we see that for any $s, t \in S$, $(I_s, I_t) \in E$ if and only if: $t \neq s^{-1}$ and I_t is not adjacent to $I_{s^{-1}}$. For example, there are directed edges from I_a to $I_c, I_{c^{-1}}, I_d$ and $I_{d^{-1}}$ but there are no directed edges from I_a to $I_{a^{-1}}, I_b$ or $I_{b^{-1}}$. There is also a loop from I_a to itself. So if $v = a$ then Γ_v contains $\mathcal{L}(I_a) = a$, $\mathcal{L}(I_a, I_c) = ac$, $\mathcal{L}(I_a, I_d, I_c) = adc$, $\mathcal{L}(I_a, I_d, I_b) = adb$. Since a, ac, adc, adb generate $\mathbb{F}_4$, we have $\Gamma_v = \mathbb{F}_4$.

Let $u = w = I_a$, $p = (I_a), q = (I_c), p^* = (I_{a^{-1}})$, $q^* = (I_{c^{-1}})$. Then

- upw, uqw, pq^*p, qp^*q are directed paths in G;
- $\mathcal{L}(p^*) = \mathcal{L}(p)^{-1}$, $\mathcal{L}(q^*) = \mathcal{L}(q)^{-1}$.

So G contains a good subgraph of order 1. □

Corollary 1.3 follows immediately from Lemma 1.4 and Theorems 1.5 and 1.1.

1.3. **Outline of the argument.** We consider the *synchronous tail equivalence relation* $\mathcal{R}_{sync}$ on $V^{\mathbb{N}}$ given by

$$\mathcal{R}_{sync} = \{(s, t) \in V^{\mathbb{N}} \times V^{\mathbb{N}} : \ \exists N \ (s_i = t_i \ \forall i \geq N)\}.$$

For a natural number $k > 0$ we also consider the *k-step asynchronous tail equivalence relation* on $V^{\mathbb{N}}$ given by

$$\mathcal{R}_k = \{(s, t) \in V^{\mathbb{N}} \times V^{\mathbb{N}} : \ \exists p \in \mathbb{Z}, N \in \mathbb{N} \ (s_{pk+i} = t_i \ \forall i \geq N)\}.$$

Let $\sigma : V^{\mathbb{N}} \to V^{\mathbb{N}}$ denote the shift map $\sigma(s)_i = s_{i+1}$. Observe that $\mathcal{R}_k$ is generated by $\mathcal{R}_{sync}$ and the orbit-equivalence relation of σ^k. So we have the following natural inclusions:

$$\mathcal{R}_{sync} \subset \mathcal{R}_k \subset \mathcal{R}_1.$$

More generally, $\mathcal{R}_k \subset \mathcal{R}_d$ if $d \mid k$. We also have a cocycle $\alpha : \mathcal{R}_1 \to \mathbb{F}$ defined by

$$\alpha(s, t) = \mathcal{L}(s_1) \cdots \mathcal{L}(s_{N+p}) \cdot (\mathcal{L}(t_1) \cdots \mathcal{L}(t_N))^{-1}$$

where N, p are such that $s_{p+i} = t_i \ \forall i \geq N$.

Given a measure-preserving action $\mathbb{F} \curvearrowright (X,\mu)$ on a probability space and a subequivalence relation $\mathcal{S}$ of $\mathcal{R}_1$, we let $\mathcal{S}^X$ denote the skew-product equivalence relation on $V^{\mathbb{N}} \times X$:

$$\mathcal{S}^X = \Big\{ \big((s,x),(t,y)\big) :\ s\mathcal{S}t, \alpha(t,s)x = y \Big\}.$$

Given a subequivalence relation $\mathcal{S} \subset \mathcal{R}_1$, let $\mathcal{F}^X_{\mathcal{S}}$ denote the sigma-algebra of measurable subsets of $V^{\mathbb{N}} \times X$ that are unions of $\mathcal{S}^X$-equivalence classes. In other words, $\mathcal{F}^X_{\mathcal{S}}$ is the $\mathcal{S}^X$-invariant sigma-algebra.

For convenience, we will let $\mathcal{F}^X_{sync}, \mathcal{F}^X_k$ denote the $\mathcal{R}^X_{sync}$ and $\mathcal{R}^X_k$-invariant sigma-algebras respectively. The main technical step in the proof of Theorem 1.1 is:

Theorem 1.6. *If the directed graph G contains a good subgraph (as in Definition 1) then $\mathcal{F}^X_{2k} = \mathcal{F}^X_{sync}$ (up to sets of measure zero).*

We prove this in the next section and in §3 use it to prove Theorem 1.1.

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2. Proof of Theorem 1.6

Let $u, w \in V$ and p, q, p^*, q^* be directed paths in G satisfying the requirements of Definition 1. We need more notation:

Notation 2. If $s \in V^{\mathbb{N}}$ and $n < m$ are natural numbers then we let $s_{[n,m]} = (s_n, s_{n+1}, \ldots, s_m) \in V^{m-n+1}$. We also write $s_{[n,\infty)} = (s_n, s_{n+1}, \ldots) \in V^{\mathbb{N}}$.

Let us define

- $\tau_n : V^{\mathbb{N}} \to \mathbb{N}$ so that $\tau_n(s)$ is the n-th time of occurrence of either upq or uqw. In other words, $\tau_n(s)$ is the smallest natural number so that there exist $i_1 < i_2 < \ldots < i_n$ with $i_n = \tau_n(s)$ so that for each j
$$s_{[i_j, i_j+k+1]} \in \{upw, uqw\}.$$
- $\omega_n : V^{\mathbb{N}} \to V^{\mathbb{N}}$ by
$$\omega_n(s) = \begin{cases} s_{[1,\tau_n(s)]} q s_{[\tau_n(s)+k+1,\infty)} & \text{if } s_{[\tau_n(s),\tau_n(s)+k+1]} = upw \\ s_{[1,\tau_n(s)]} p s_{[\tau_n(s)+k+1,\infty)} & \text{if } s_{[\tau_n(s),\tau_n(s)+k+1]} = uqw \end{cases}$$
- Note that ω_n is invertible. So we can define $\psi_n : V^{\mathbb{N}} \to V^{\mathbb{N}}$ by
$$(\psi_n \omega_n(s)) = \begin{cases} \omega_n(s)_{[2k+1,\tau_n(s)+k]} p^* \omega_n(s)_{[\tau_n(s)+1,\infty)} & \text{if } s_{[\tau_n(s),\tau_n(s)+k+1]} = upw \\ \omega_n(s)_{[2k+1,\tau_n(s)+k]} q^* \omega_n(s)_{[\tau_n(s)+1,\infty)} & \text{if } s_{[\tau_n(s),\tau_n(s)+k+1]} = uqw \end{cases}$$

$$= \begin{cases} s_{[2k+1,\tau_n(s)]}qp^*qs_{[\tau_n(s)+k+1,\infty)} & \text{if } s_{[\tau_n(s),\tau_n(s)+k+1]} = upw \\ s_{[2k+1,\tau_n(s)]}pq^*ps_{[\tau_n(s)+k+1,\infty)} & \text{if } s_{[\tau_n(s),\tau_n(s)+k+1]} = uqw \end{cases}$$

- Recall that ν is the Π-stationary measure on V. Let $\tilde{\nu}$ be the associated measure on $V^{\mathbb{N}}$. To be precise, for any $t_1, \ldots, t_n \in V$,
$$\tilde{\nu}(\{s \in V^{\mathbb{N}} :\ s_i = t_i\ \forall 1 \le i \le n\}) = \nu(t_n)\Pi_t = \nu(t_n)\Pi_{t_n,t_{n-1}} \cdots \Pi_{t_2,t_1}.$$
- $C > 0$ be a constant so that almost everywhere holds
$$C^{-1} \le \frac{d(\omega_n^{-1})_*\tilde{\nu}}{d\tilde{\nu}}(s) \le C, \quad C^{-1} \le \frac{d((\psi_n\omega_n)^{-1})_*\tilde{\nu}}{d\tilde{\nu}}(s) \le C$$
The existence of such a constant follows from the finiteness of V (so that there is a uniform bound on the ratio of any two nonzero entries of Π) and explicit computation using the formulae above.

Recall that $\sigma : V^{\mathbb{N}} \to V^{\mathbb{N}}$ is defined by $\sigma(s)_i = s_{i+1}$. Let $d_{V^{\mathbb{N}}}$ denote the distance function on $V^{\mathbb{N}}$ defined by $d_{V^{\mathbb{N}}}\big((s_1, s_2, \ldots), (t_1, t_2, \ldots)\big) = \frac{1}{n}$ where n is the largest natural number such that $s_i = t_i$ for all $i < n$.

Proposition 2.1. *For every $n > 2k + 1$,*
1. $\forall s \in V^{\mathbb{N}}$, $d_{V^{\mathbb{N}}}\big(\psi_n\omega_n(s), \sigma^{2k}\omega_n(s)\big) \le \frac{1}{\tau_n(s)-k}$;
2. $\forall s \in V^{\mathbb{N}}$, $d_{V^{\mathbb{N}}}(s, \omega_n s) \le \frac{1}{\tau_n(s)}$;
3. *the graphs of ω_n and ψ_n are contained in $\mathcal{R}_{sync}$;*
4. $\forall s \in \mathcal{A}^{\mathbb{N}}$, $\alpha(\psi_n\omega_n s, \omega_n s) = \alpha(\sigma^{2k}\omega_n s, s)$.
5. $\forall f \in L^1(\mathcal{A}^{\mathbb{N}})$, $\|f \circ \omega_n\|_1 \le C\,\|f\|_1$ *and* $\|f \circ \psi_n\|_1 \le C^2\,\|f\|_1$.

Proof. Items 1 and 2 are obvious. It is clear that the graph of ω_n is contained in $\mathcal{R}_{sync}$. This implies the graph of $\psi_n\omega_n$ is contained in $\mathcal{R}_{sync}$ and therefore, since ω_n is invertible, the graph of ψ_n is contained in $\mathcal{R}_{sync}$.

For simplicity's sake, we will drop the subscripts n in the following computations. So $\psi = \psi_n, \omega = \omega_n, \tau = \tau_n$.

Suppose that $s \in V^{\mathbb{N}}$ satisfies $s_{[\tau(s),\tau(s)+k+1]} = upw$. Let $N = \tau(s)$. Because $(\psi\omega(s))_i = \omega(s)_i$ for all $i > N$ the definition of α implies

$$\begin{aligned} \alpha(\psi\omega s, \omega s) &= \mathcal{L}(\psi\omega(s)_1) \cdots \mathcal{L}(\psi\omega(s)_N)\Big(\mathcal{L}(\omega(s)_1) \cdots \mathcal{L}(\omega(s)_N)\Big)^{-1} \\ &= \mathcal{L}(s_{1+2k}) \cdots \mathcal{L}(s_N)\mathcal{L}(q_1) \cdots \mathcal{L}(q_k)\mathcal{L}(p_k)^{-1} \cdots \mathcal{L}(p_1)^{-1}\Big(\mathcal{L}(s_1) \cdots \mathcal{L}(s_N)\Big)^{-1} \end{aligned}$$

Because $(\sigma^{2k}\omega s)_{i-2k} = (\omega s)_i = s_i$ for all $i > N + k$ the definition of α implies

$$\begin{aligned} \alpha(\sigma^{2k}\omega s, s) &= \mathcal{L}(\sigma^{2k}\omega(s)_1) \cdots \mathcal{L}(\sigma^{2k}\omega(s)_{N-k})\Big(\mathcal{L}(s_1) \cdots \mathcal{L}(s_{N+k})\Big)^{-1} \\ &= \mathcal{L}((\omega s)_{1+2k}) \cdots \mathcal{L}((\omega s)_{N+k})\Big(\mathcal{L}(s_1) \cdots \mathcal{L}(s_{N+k})\Big)^{-1} \\ &= \mathcal{L}(s_{1+2k}) \cdots \mathcal{L}(s_N)\mathcal{L}(q_1) \cdots \mathcal{L}(q_k)\Big(\mathcal{L}(s_1) \cdots \mathcal{L}(s_N)\mathcal{L}(p_1) \cdots \mathcal{L}(p_k)\Big)^{-1} \\ &= \alpha(\psi\omega s, \omega s). \end{aligned}$$

The case when $s_{[\tau(s),\tau(s)+k+1]} = uqw$ is similar. This proves item 4.

It follows from the choice of $C > 0$ (made right before this proposition) that for every $f \in L^1(V^{\mathbb{N}})$,

$$\|f \circ \omega\|_1 \le C\,\|f\|_1\,, \quad \|f \circ \psi\omega\|_1 \le C\|f\|_1.$$

Since ω is invertible, this implies

$$\|f \circ \psi\|_1 = \|f \circ \psi\omega \circ \omega^{-1}\|_1 \le C\|f \circ \psi\omega\|_1 \le C^2\|f\|_1.$$

Here we used that $\omega = \omega^{-1}$. This establishes the last claim. $\square$

Definition 4. Define $\sigma_X : V^{\mathbb{N}} \times X \to V^{\mathbb{N}} \times X$ by $\sigma_X(s,x) = (\sigma s, \alpha(\sigma s, s)x)$. Note $\alpha(\sigma s, s) = s_1^{-1}$. So we can also write $\sigma_X(s,x) = (\sigma s, s_1^{-1}x)$.

Lemma 2.2. *There exist measurable maps $\Phi_n, \Psi_n, \Omega_n : V^{\mathbb{N}} \times X \to V^{\mathbb{N}} \times X$ (for $n > 2k+1$) such that*

1. *for all $f \in L^1(V^{\mathbb{N}} \times X)$, $\lim_{n\to\infty} \|f \circ \Psi_n \circ \Omega_n - f \circ \sigma_X^{2k} \circ \Phi_n\|_1 = 0$;*
2. *for all $f \in L^1(V^{\mathbb{N}} \times X)$, $\lim_{n\to\infty} \|f \circ \Omega_n - f\|_1 = 0$;*
3. *the graphs of Φ and Ψ are contained in $\mathcal{R}^X_{sync}$.*

Proof. For $n > 2k+1$ an integer, let ψ_n and ω_n be as in Proposition 2.1. Define

$$\begin{aligned} \Omega_n(s,x) &:= (\omega_n s, x)\\ \Phi_n(s,x) &:= (\omega_n s, \alpha(\omega_n s, s)x)\\ \Psi_n(s,x) &:= (\psi_n s, \alpha(\psi_n s, s)x). \end{aligned}$$

Since the graphs of ψ_n and ω_n are contained in $\mathcal{R}_{sync}$, the graphs of Φ_n and Ψ_n are contained in $\mathcal{R}^X_{sync}$. Let d_X be a metric on X that induces its Borel structure and makes X into a compact space. For $(s,x),(s',x') \in V^{\mathbb{N}} \times X$, define $d_*((s,x),(s',x')) = d_X(x,x') + d_{V^{\mathbb{N}}}(s,s')$. By the previous proposition, $d_*(\Omega_n(s,x),(s,x)) = d_{V^{\mathbb{N}}}(\omega_n(s), s) \le 1/\tau_n(s) \le 1/n$. Also by the previous proposition:

$$\begin{aligned} \Psi_n\Omega_n(s,x) &= (\psi_n\omega_n s, \alpha(\psi_n\omega_n s, \omega_n s)x)\\ \sigma_X^{2k}\Phi_n(s,x) &= \sigma_X^{2k}(\omega_n s, \alpha(\omega_n s, s)x) = (\sigma^{2k}\omega_n s, \alpha(\sigma^{2k}\omega_n s, \omega_n s)\alpha(\omega_n s, s)x)\\ &= (\sigma^{2k}\omega_n s, \alpha(\sigma^{2k}\omega_n s, s)x) = (\sigma^{2k}\omega_n s, \alpha(\psi_n\omega_n s, \omega_n s)x). \end{aligned}$$

So the previous proposition implies $d_*(\Psi_n \circ \Omega_n(s,x), \sigma_X^{2k} \circ \Phi_n(s,x)) \le 1/(n-k)$. So if f is a continuous function on $V^{\mathbb{N}} \times X$ then the bounded convergence theorem implies

$$\begin{aligned} \lim_{n\to\infty} \|f \circ \Psi_n \circ \Omega_n - f \circ \sigma_X^{2k} \circ \Phi_n\|_1 &= 0\\ \lim_{n\to\infty} \|f \circ \Omega_n - f\|_1 &= 0. \end{aligned}$$

It follows from the previous proposition that the operators $f \mapsto f \circ \Omega_n$, $f \mapsto f \circ \Phi_n$ and $f \mapsto f \circ \Psi_n$ are all bounded for $f \in L^1(V^{\mathbb{N}} \times X)$ with bound independent of n. It easy to see that $f \mapsto f \circ \sigma_X^{2k}$ is also a bounded operator on $L^1(V^{\mathbb{N}} \times X)$ (because V is finite and $\tilde{\nu}$ is the Markov measure). Since the continuous functions are dense in $L^1(V^{\mathbb{N}} \times X)$, this implies the lemma. $\square$

We can now prove Theorem 1.6.

Proof of Theorem 1.6. Note $\mathcal{F}^X_{2k} \supset \mathcal{F}^X_{sync}$. So it suffices to show that if $f \in L^1(V^{\mathbb{N}} \times X)$ is $\mathcal{R}^X_{sync}$-invariant then it is $\mathcal{R}^X_{2k}$-invariant. Because the map σ_X^{2k} together with $\mathcal{R}^X_{sync}$ generates $\mathcal{R}^X_{2k}$, it suffices to show that if $f \in L^1(V^{\mathbb{N}} \times X)$ is $\mathcal{R}^X_{sync}$-invariant then $f \circ \sigma_X^{2k} = f$.

Let Φ_n, Ψ_n, Ω_n $(n > 2k+1)$ be as in the previous lemma. Because f is $\mathcal{R}^X_{sync}$-invariant and the graph of Ψ_n is contained in $\mathcal{R}^X_{sync}$, it follows that $f \circ \Psi_n = f$ for all n. An easy exercise shows that σ_X preserves the equivalence relation in the sense that

$$\Big((s,x),(t,y)\Big) \in \mathcal{R}_{sync} \Rightarrow \Big(\sigma_X(s,x), \sigma_X(t,y)\Big) \in \mathcal{R}_{sync}.$$

It follows that $f \circ \sigma_X^{2k}$ is $\mathcal{R}^X_{sync}$-invariant. Since the graph of Φ_n is contained in $\mathcal{R}^X_{sync}$, $f \circ \sigma_X^{2k} \circ \Phi_n = f \circ \sigma_X^{2k}$ for all n. We now have

$$\begin{aligned}
\|f - f\circ\sigma_X^{2k}\|_1 &= \|f - f \circ \sigma_X^{2k} \circ \Phi_n\|_1 \\
&\le \|f - f\circ\Psi_n\circ\Omega_n\|_1 + \|f\circ\Psi_n\circ\Omega_n - f\circ\sigma_X^{2k}\circ\Phi_n\|_1 \\
&= \|f - f\circ\Omega_n\|_1 + \|f\circ\Psi_n\circ\Omega_n - f\circ\sigma_X^{2k}\circ\Phi_n\|_1.
\end{aligned}$$

We take the limit as $n \to \infty$ (using the previous lemma) to obtain $f = f \circ \sigma_X^{2k}$ as claimed. □

3. Proof of Theorem 1.1

Proposition 3.1. *Let $\Pi, V, \mathcal{L}$ be as above. For each $v \in V$, let $\Gamma_v \le \mathbb{F}$ be the subgroup generated by all elements of the form $\mathcal{L}(p)$ where pv is a directed path from v to v in G. If $\Gamma_v = \mathbb{F}$ for some $v \in V$ and G is strongly connected then $\mathcal{F}^X_1$ is the σ-algebra generated by all sets of the form $V^{\mathbb{N}} \times A$ where $A \subset X$ is a measurable $\mathbb{F}$-invariant set. In particular, if $\mathbb{F} \curvearrowright (X,\mu)$ is ergodic then $\mathcal{F}^X_1$ is trivial.*

Proof. By decomposing into ergodic components, we may assume that $\mathbb{F} \curvearrowright (X,\mu)$ is ergodic. Because $\mathcal{R}^X_1$ is generated by σ_X, it suffices to prove σ_X is ergodic.

Let $Y \subset V^{\mathbb{N}} \times X$ be the set of all (s,x) such that $s_1 = v$ where $v \in V$ is chosen so that $\Gamma_v = \mathbb{F}$. Let $T : Y \to Y$ be the induced transformation:

$$T(s,v) = \sigma_X^n(s,v)$$

where $n \ge 1$ is the smallest natural number such that $\sigma_X^n(s,v) \in Y$. By Kakutani's random ergodic theorem [24, Theorem 3 (a) $\Rightarrow$ (f)], the ergodicity of $\mathbb{F} \curvearrowright (X,\mu)$ implies T is ergodic.

Now suppose $Z \subset V^{\mathbb{N}} \times X$ is measurable, σ_X-invariant and has positive measure. Then $Y \cap Z$ is T-invariant. Because the graph G is strongly connected, $\tilde{\nu} \times \mu(Y \cap Z) > 0$. Since T is ergodic, this implies $Y \cap Z = Y$ up to measure zero. However, $\cup_{i=0}^{\infty} \sigma_X^i Y = V^{\mathbb{N}} \times X$ (up to measure zero) because G is strongly connected. This implies Z is conull and therefore σ_X is ergodic as claimed. □

Lemma 3.2. *For any $f \in L^1(V^{\mathbb{N}} \times X)$ and any $k \in \mathbb{N}$,*

$$\frac{1}{k}\sum_{i=0}^{k-1} \mathbb{E}[f \circ \sigma_X^i | \mathcal{F}^X_k] = \mathbb{E}[f|\mathcal{F}^X_1].$$

Proof. Because $\mathcal{F}_1^X$ is the sigma-algebra of σ_X-invariant measurable subsets, von Neumann's mean ergodic theorem implies that

$$\frac{1}{nk}\sum_{i=0}^{nk-1} f\circ\sigma_X^i \to \mathbb{E}[f|\mathcal{F}_1^X]$$

in L^1 as $n\to\infty$. By taking conditional expectations on both sides (and remembering that $\mathcal{F}_1^X \subset \mathcal{F}_k^X$), we have

$$\frac{1}{nk}\sum_{i=0}^{nk-1} \mathbb{E}[f\circ\sigma_X^i|\mathcal{F}_k^X] \to \mathbb{E}[f|\mathcal{F}_1^X].$$

Because $\mathcal{F}_k^X$ is σ_X^k-invariant, we have $\mathbb{E}[f\circ\sigma_X^{k+i}|\mathcal{F}_k^X] = \mathbb{E}[f\circ\sigma_X^i|\mathcal{F}_k^X]$ for any i. So for any n

$$\frac{1}{nk}\sum_{i=0}^{nk-1} \mathbb{E}[f\circ\sigma_X^i|\mathcal{F}_k^X] = \frac{1}{k}\sum_{i=0}^{k-1} \mathbb{E}[f\circ\sigma_X^i|\mathcal{F}_k^X].$$

This implies the lemma. □

Proof of Theorem 1.1 from Theorem 1.6. Without loss of generality, we may assume $\mathbb{F}\curvearrowright(X,\mu)$ is ergodic. Let $\pi: V^{\mathbb{N}}\times X \to V\times X$ denote the projection map $\pi(s,x)=(s_1,x)$.

Let $\mathcal{B}_{V\times X}$ denote the Borel sigma-algebra on $V\times X$ and let $\mathcal{F}_{\geq n}^X$ be the smallest sigma-algebra of $V^{\mathbb{N}}\times X$ containing $(\pi\circ\sigma_X^m)^{-1}(\mathcal{B}_{V\times X})$ for every $m\geq n$.

Consider the induced Markov operator $\Pi_X: L^1(V\times X)\to L^1(V\times X)$ given by

$$\Pi_X(\varphi)(x,v) = \sum_{w\in V} \Pi_{w,v}\varphi(w, T_v x).$$

Observe that for $n\geq 2$

$$\Pi_X^n(\varphi)(x,v) \;=\; \sum_{t_1,\ldots,t_n\in V} \Pi_{(t_1,\ldots,t_n,v)}\varphi(t_1, T_{(t_2,\ldots,t_n,v)}x).$$

Thus

$$\begin{aligned}
(\Pi_X^n\varphi)\circ\pi\circ\sigma_X^n)(s,x) &= (\Pi_X^n\varphi)(s_{n+1}, T^{-1}_{(s_1,\ldots,s_n)}x)\\
&= \sum_{t_1,\ldots,t_n\in V} \Pi_{(t_1,\ldots,t_n,s_{n+1})}\varphi(t_1, T_{(t_2,\ldots,t_n,s_{n+1})}T^{-1}_{(s_1,\ldots,s_n)}x)\\
&= \mathbb{E}[\varphi\pi|\mathcal{F}_{\geq n+1}^X](s,x).
\end{aligned}$$

The reverse martingale convergence theorem yields

$$\mathbb{E}[\varphi\pi|\mathcal{F}_{\geq n+1}^X] \to \mathbb{E}[\varphi\pi|\mathcal{F}_{sync}^X]$$

in $L^1(V^{\mathbb{N}}\times X)$ as $n\to\infty$. By Theorem 1.6, $\mathcal{F}_{sync}^X = \mathcal{F}_{2k}^X$. Therefore,

$$(\Pi_X^n\varphi)\circ\pi\circ\sigma_X^n \to \mathbb{E}[\varphi\pi|\mathcal{F}_{2k}^X]$$

in $L^1(V^{\mathbb{N}}\times X)$ as $n\to\infty$. Because conditioning on $\mathcal{F}_{2k}^X$ commutes with σ_X, for any $i\geq 0$

$$(\Pi_X^n\varphi)\circ\pi\circ\sigma_X^{n+i} \to \mathbb{E}[\varphi\pi\circ\sigma_X^i|\mathcal{F}_{2k}^X]$$

in $L^1(V^{\mathbb{N}} \times X)$ as $n \to \infty$. Since $\mathbb{E}[\varphi\pi \circ \sigma_X^i | \mathcal{F}_{2k}^X] = \mathbb{E}[\varphi\pi \circ \sigma_X^{2k+i} | \mathcal{F}_{2k}^X]$ we can also write this as: for any $0 \le i < 2k$,

$$(\Pi_X^n \varphi) \circ \pi \circ \sigma_X^{n-i} \to \mathbb{E}[\varphi\pi \circ \sigma_X^{2k-i} | \mathcal{F}_{2k}^X]$$

in $L^1(V^{\mathbb{N}} \times X)$ as $n \to \infty$. Now Lemma 3.2 and Proposition 3.1 imply

$$\frac{1}{2k}\sum_{i=0}^{2k-1} (\Pi_X^n \varphi) \circ \pi \circ \sigma_X^{n-i} \quad \to \quad \frac{1}{2k}\sum_{i=0}^{2k-1} \mathbb{E}[\varphi\pi \circ \sigma_X^i | \mathcal{F}_{2k}^X] = \mathbb{E}[\varphi\pi | \mathcal{F}_1^X] = \int \varphi \, d\nu \times \mu$$

in L^1 as $n \to \infty$. However,

$$(\Pi_X^n \varphi) \circ \pi \circ \sigma_X^{n-i} = (\Pi_X^{n-i} \Pi_X^i \varphi) \circ \pi \circ \sigma_X^{n-i} \to \mathbb{E}[\Pi_X^i \varphi\pi | \mathcal{F}_{2k}^X]$$

in $L^1(V^{\mathbb{N}} \times X)$ as $n \to \infty$. Similarly,

$$(\Pi_X^{n+i} \varphi) \circ \pi \circ \sigma_X^n \to \Pi_X^i \Big(\mathbb{E}[\varphi\pi | \mathcal{F}_{2k}^X] \Big) = \mathbb{E}[\Pi_X^i \varphi\pi | \mathcal{F}_{2k}^X]$$

in $L^1(V^{\mathbb{N}} \times X)$ as $n \to \infty$. So we have

$$\frac{1}{2k}\sum_{i=0}^{2k-1} (\Pi_X^{n+i} \varphi) \circ \pi \circ \sigma_X^n \to \int \varphi \, d\nu \times \mu$$

in L^1 as $n \to \infty$.

Without loss of generality, we may assume $\int \varphi \, d\nu \times \mu = 0$ in which case the above implies

$$\left\| \frac{1}{2k}\sum_{i=0}^{2k-1} (\Pi_X^{n+i} \varphi) \circ \pi \circ \sigma_X^n \right\| \to 0$$

as $n \to \infty$. However,

$$\left\| \frac{1}{2k}\sum_{i=0}^{2k-1} (\Pi_X^{n+i} \varphi) \circ \pi \circ \sigma_X^n \right\| \quad = \quad \left\| \frac{1}{2k}\sum_{i=0}^{2k-1} \Pi_X^{n+i} \varphi \right\|.$$

So

$$\frac{1}{2k}\sum_{i=0}^{2k-1} \Pi_X^{n+i} \varphi \to 0$$

in L^1 as $n \to \infty$. Next we note that if $\varphi(v,x) = \phi(x)$ for some $\phi \in L^1(X)$ then by a change of variables argument

$$\begin{aligned}
(S_n\phi)(x) &= \sum_{s_1,\ldots,s_n \in V} \nu(s_n) \Pi_{(s_1,\ldots,s_n)} \phi(T_{(s_1,\ldots,s_n)} x) \\
&= \sum_{v \in V} \sum_{s_1,\ldots,s_{n-1} \in V} \nu(v) \Pi_{(s_1,\ldots,s_{n-1},v)} \phi(T_{(s_2,\ldots,s_{n-1},v)} x) \\
&= \sum_{v \in V} \nu(v) (\Pi_X^{n-1} \varphi)(v,x).
\end{aligned}$$

Thus $S_n\phi$ converges to 0 in L^1 as $n \to \infty$.

$\square$

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5

Sternberg linearization theorem for skew products

This Chapter concerns normalization theory : we search for a simpler form of a diffeomoprhism that can be obtained by a change of coordinates. We are interested in a special kind of normalization that works for skew product diffeomorphisms. The normal form is a linear map and what is the most important for us, the conjugacy preserves the fibered structure (more precisely, it changes only the fiber coordinate). It appears, that even in the smooth case, the conjugacy is only Hölder continuous with respect to the base. We calculate explicit bounds for its Hölder exponent and its Hölder constant as well. The Hölder exponent, as often in this subject, reflects the ratio of the dynamics in the fiber and in the base. This normalization theorem may be applied to perturbations of skew products and to the study of new persistent properties of attractors.
This Chapter is reproducing our work with Yulij Ilyashenko. Its results were published in Journal of Dynamical and Control Systems.

STERNBERG LINEARIZATION THEOREM FOR SKEW PRODUCTS

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Abstract. We present a new kind of normalization theorem: linearization theorem for skew products. The normal form is a skew product again, with the fiber maps linear. It appears, that even in the smooth case, the conjugacy is only Hölder continuous with respect to the base. The normalization theorem mentioned above may be applied to perturbations of skew products and to the study of new persistent properties of attractors.

1. Setting and statements

1.1. **Motivation.** This paper is devoted to a normalization theorem for Hölder skew products. We begin with the motivation for the choice of this class of maps.

According to a heuristic principle going back to [1], generic phenomena that occur in random dynamical systems on a compact manifold may also occur for diffeomorphisms of manifolds of higher dimensions. Random dynamical systems are equivalent to skew product homeomorhisms over Bernoulli shift. Some new effects found for these homeomorphisms were transported later to skew product diffeomorphisms over hyperbolic maps with compact fibers. These diffeomorphisms are in no way generic. Their small perturbations are skew products again, whose fiber maps are smooth but only continuous with respect to the base point [8].

Recently it was discovered that these fiber maps are in fact Hölder with respect to the base point [4, 13, 21].

New effects found for skew product diffeomorphisms are then transported to Hölder skew products, and thus proved to be generic. This program was carried on in [3, 9, 10, 11, 16, 17].

This motivates the study of Hölder skew products. We now pass to our main results.

1.2. **Main statements.** Consider a skew product diffeomorphism over an Anosov map in the base with the fiber a segment. In more detail, let $M = \mathbb{T}^d \times I$, $\mathbb{T}^d$ is a a d-dimensional torus, $I = [0, 1]$. Consider a boundary preserving skew product

$$F : M \to M, (b, x) \mapsto (Ab, f_b(x)) \,, \tag{1}$$

where $f_b(0) = 0, f_b(1) = 1$, the fiber map f_b is an orientation preserving diffeomorphism $I \to I$ and the base map A is a linear hyperbolic automorphism of a torus.

Suppose also that f is Hölder continuous in x with respect to the C^k-norm; i.e. that there exist constants $C_k, \beta > 0$ such that for any $b, b' \in \mathbb{T}^d$ the following holds:

$$||f_b - f_{b'}||_{C^k} \leq C_k ||b - b'||^{\beta} \tag{2}$$

This assumption appears in slightly different settings as a statement in a number of articles on partial hyperbolicity: for exmple, in [21] the estimate (2) is true for $k = 0$, in [2] for $k = 1$ and in [4] for any k. Now we will state the main results that we are proving in a hope to apply them to the study of the skew products: for example, to drastically simplify the proofs in [10].

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Theorem 1. *Consider a map F of the form* (1) *with the property* (2) *for some fixed k and C^k – smooth on the x variable, $k \geq 2$. Let $O \in \mathbb{T}^d \times \{0\}$ be its hyperbolic fixed point. Then there exists a neighborhood U of O and a fiber preserving homeomorphism*

$$H : (U, O) \to U, \quad (b, x) \mapsto (b, x + h_b(x)) \quad h_b(0) = \frac{\partial h_b}{\partial x}(0) = 0, \tag{3}$$

such that

1. *H conjugates F in (U, O) with its "fiberwize linearization"*

$$F_0 : (b, x) \mapsto (Ab, \lambda_b x), \tag{4}$$

where

$$\lambda_b = f_b'(0). \tag{5}$$

This means that

$$F \circ H = H \circ F_0. \tag{6}$$

2. *H is smooth on x for b fixed: the degree of smoothness is $k - 2$*
3. *H is fiberwise Hölder: there exist constants $\tilde{C}_l, \alpha > 0$ such that for any $l, 0 \leq l \leq k - 2$ holds*

$$||h_b - h_{b'}||_{C^l} \leq \tilde{C}_l ||b - b'||^\alpha \tag{7}$$

such that

$$\alpha \in \min(\beta, \log_\mu q), \tag{8}$$

where μ is the largest magnitude of eigenvalues of A.

This theorem is local: the conjugacy relation $F \circ H = H \circ F_0$ holds in a neighborhood of O only. We will reduce this theorem to the following two results.

Theorem 2. *Consider the same F as in Theorem 1, with $k = 2$. Let in addition*

$$\lambda_b \leq q < 1 \quad \forall b \in \mathbb{T}^d$$

Then the map H with the properties 1, 2, 3 mentioned in Theorem 1 is defined in a set

$$M_\varepsilon = \left\{(b, x) \in \mathbb{T}^d \times [0, 1] \mid x \in [0, \varepsilon]\right\}$$

for some $\varepsilon > 0$ and is continuous on this set. Moreover, for $l = 0$, the relation (7) *holds for any α as in* (8).

Theorem 3. *Suppose that all assumptions of Theorem 2 hold, except for k is arbitrary now. Then the map H with the same properties as in Theorem 2 exists. Moreover, H is C^{k-2} fiberwise smooth, and satisfies the Hölder condition* (8) *for $l = k - 2$.*

Theorems 2 and 3 are the main results of the paper. They are similar: the first one claims that the fiberwise conjugacy H is continuous in the C-norm with respect to the point of the fiber, and the second one improves this result – by decreasing the neighborhood in the fiber, and replacing the C-norm by the C^l one. The main part of the paper to follow is the proof of Theorem 2. At the end we present a part of the proof of Theorem 3. Namely, we prove that the maps h_b are $(k-2)$-smooth , but we do not prove that the derivatives $\frac{\partial^j h_b}{\partial x^j}, 1 \leq j \leq k-2$ are Hölder in b. This may be proven in the same way as the Hölder property for h_b, but requires more technical details that we skip here.

At the beginning of the next section we deduce Theorem 1 from Theorems 2, 3.

1.3. **Comparison with the theory of "nonstationary normal forms".** This theory is developed in [6] and [5]. The closer to our results is Theorem 1.2 of [6] proved in full detail as Theorem 1 of [5]. For future references we will call it the GK-Theorem. This theorem considers a wide class of maps to which the map (1) belongs, provided that it satisfies the so called *narrow band spectrum condition.* In assumptions of our Theorem 1, this condition has the form:

$$(\max_{b\in\mathbb{T}^d} \lambda_b)^2 < \min_{b\in\mathbb{T}^d} \lambda_b. \tag{9}$$

This is a restrictive condition, not required in Theorem 2. Moreover, GK-Theorem claims properties 1 and 2 of the map H, that is, H is a fiberwise smooth topological conjugacy, and does not claim Hölder continuity of the fiberwise maps of H, see (8). To summarize, Theorems 2, 3 improve the GK-Theorem for the particular class of maps (1), skipping the narrow band spectrum condition and adding Property 3, the Hölder continuity.

Statements 1 and 2 of Theorem 1 may be deduced from the GK-Theorem. Indeed, any continuous function $\lambda < 1$ satisfies condition (9) in a suitable neighborhood of any point. On the other hand, Theorem 1 in its full extent is easily deduced from Theorems 2, 3.

2. The plan of the proof

2.1. **Globalization.** Theorem 1 is proved with the help of standard globalization technics (see, for instance, [14]). Without loss of generality, hyperbolicity of the skew-product F implies that $\lambda_b < 1$ in the neighborhood $U \in \mathbb{T}^d$ of the fixed point O. If not, we pass to the inverse mapping F^{-1}. Let K be a compact subset of U. Let us take a smooth cut function $\varphi : \mathbb{T}^d \to [0,1]$ such that

$$\varphi|_K \equiv 0, \varphi|_{\mathbb{T}^d\setminus U} \equiv 1 \tag{10}$$

Instead of the initial function $f_b(x)$ on the fibers let us consider the function

$$\tilde{f}_b(x) = f_b(x)(1-\varphi) + \frac{x}{2}\varphi \tag{11}$$

Then a map

$$\tilde{F} : X_\varepsilon \to X, (b,x) \mapsto (Ab, \tilde{f}_b(x)) \tag{12}$$

has the following list of properties:

1. $\tilde{F}$ coincides with F in the neighborhood of the fixed point O
2. $\tilde{F}$ is attracting near the zero layer: if $\tilde{\lambda}_b := \tilde{f}_b'(0)$ then

$$\tilde{\lambda}_b < 1 \;\; \forall b \in \mathbb{T}^d \tag{13}$$

So without loss of generality we may assume from the very beginning that $\lambda_b \in (0,1)$ everywhere on the base $\mathbb{T}^d$. Moreover, the conjugacy H is to be found in the whole M_ε: in other words, the equality $F \circ H = H \circ F_0$ will hold on the full neigbourhood M_ε of the base.

All the rest of the article deals with the proof of the global result, i.e. Theorem 2 and Theorem 3.

Now let us prove Theorem 2: the main idea is to use fixed point theorem to prove the existence of the conjugacy H: we should just properly define the functional space and a contraction operator in it. In the following sections we will do all of it, postponing some calculcations as well as the proof of Theorem 3 to the Appendix (Section 6).

2.2. **Homological and functional equations.** Suppose that

$$f_b(x) = \lambda_b x + R_b(x), \quad R_b(x) = O(x^2), x \to 0. \tag{14}$$

Then finding the conjugacy $H : M_\varepsilon \to M_\varepsilon$ of the form (3) satisfying (6) is equivalent to finding the solution $\bar{h}_b(x)$ of a so-called *functional equation*

$$\bar{h}_{Ab}\left(\lambda_b x\right) - \lambda_b \bar{h}_b(x) = R_b\left(x + \bar{h}_b(x)\right) \tag{15}$$

More briefly, this equation can be written in a *compositional form* as

$$\bar{h} \circ F_0 - \lambda \bar{h} = R \circ H \tag{16}$$

Here we denote by λ the operator of multiplication by λ_b, here b is an argument of the considered function.

In the following we will be working not with the quadratic part of the conjugacy itself but with this part divided by x^2. That's why we change the notations in such a way: we write bars for the functions in the space of quadratic parts for possible conjugacy maps and we don't write bars for the same functions divided by x^2, for instance, $h_b(x) := \frac{\bar{h}_b(x)}{x^2}$. In a similar fashion, $Q_b(x) := \frac{R_b(x)}{x^2}$.

The functional equation is a hard one to solve since the function $\bar{h}_b(x)$ is present in both sides of the equation. One may simplify functional equation and consider a gentler form of the equation on $\bar{h}_b(x)$, a *homological equation:*

$$\bar{h} \circ F_0 - \lambda \bar{h} = R \tag{17}$$

The solution of the homological equation doesn't give the conjugacy but is a useful tool in the investigation. Homological eqation can be rewritten equivalently in terms of h and Q as

$$\lambda^2 h \circ F_0 - \lambda h = Q \tag{18}$$

2.3. **Operator approach.** Let us consider a space $\mathcal{M}$ of real-valued functions defined on M which are continuous on $b \in \mathbb{T}^d$ and smooth in $x \in [0,1]$:

$$\mathcal{M} := \left\{\bar{h}_b(x) \in M \mid \bar{h}_\cdot(x) \in C(\mathbb{T}^d), \bar{h}_b(\cdot) \in C^k[0,1]\right\} \tag{19}$$

Let us define an operator $\bar{\Psi} : \mathcal{M} \to \mathcal{M}$ on it which acts on a function $h(b,x)$ by associating to it the left-hand side of the homological equation (17). With the use of this operator, equation (17) can be rewritten in a form $\bar{\Psi}\bar{h} = R$.

Denote $\bar{L}$ an inverse operator to $\bar{\Psi}$. The operator $\bar{L}$ is *solving* homological equation: if the right-hand side is R then $\bar{L}R = \bar{h}$ and $\bar{L}\bar{\Psi} = \mathrm{id}$. From now on, operator $\bar{L}$ will be referred as *homological operator.*

Let us define a *shift operator* $\bar{\Phi} : \mathcal{M} \to \mathcal{M}$ which acts as

$$\bar{\Phi}\bar{h}(b,x) = R_b\left(x + \bar{h}_b(x)\right) \tag{20}$$

Then the functional equation on the function $\bar{h}$ can be rewritten in the form $\bar{\Psi}\bar{h} = \bar{\Phi}\bar{h}$ or, equivalently, $\bar{h} = \bar{L}\bar{\Phi}\bar{h}$. So the search for conjugacy is **equivalent** to the search of a fixed point for the operator $\bar{L}\bar{\Phi}$ in the space $\mathcal{M}$.

Let us note for the future that the operator $\bar{\Psi}$ (as well as its inverse $\bar{L}$) is a linear operator on the space of formal series although operator $\bar{\Phi}$ is not at all linear: for instance, it sends a zero function to $R_b(x)$.

2.4. **Choice of a functional space for the Banach fixed point theorem.** We will be using the simplest of the forms of a contraction mapping principle by considering a contracting mapping defined on a metric space $\mathcal{N}$ and preserving its closed subspace N.

Let us define $\mathcal{N}, d$ and N for our problem: a contraction mapping f will be a slight modification of the composition of operators $\bar{L}\bar{\Phi}$ considered in Section 2.3. Note that operator $\bar{L}$ as well as operator $\bar{\Phi}$ preserve the subspace of $\mathcal{M}$ of functions starting with quadratic terms in x which we will denote by $\mathcal{M}^2$:

$$\mathcal{M}^2 = \left\{\bar{h}_b(x) \in \mathcal{M} | \bar{h}_b(x) = x^2 h_b(x), h_b(x) \in \mathcal{M}\right\} \tag{21}$$

That's why for our comfort we will define the operators L and Φ acting on $\mathcal{M}$ as

$$Lh := \frac{\bar{L}[x^2 h]}{x^2} \qquad \Phi h := \frac{\bar{\Phi}[x^2 h]}{x^2} \tag{22}$$

These operators correspond to the solution of homological equation and to the shift operator but are somewhat normalized.

The linearization theorems we prove will be applicable only in the vicinity of the base, i.e. in $\mathbb{T}^d \times [0, \varepsilon]$. The conditions on the small constant ε will be formulated later. Contraction mapping theorem will be applied to the operator $L\Phi$ acting in the complete metric space $\mathcal{M}_\varepsilon$ of functions from $\mathcal{M}$ restricted to the small neighborhood of a torus $\mathbb{T}^d \times [0, \varepsilon]$. A norm on this space is simply a continuous one, for $h(b, x) \in \mathcal{M}_\varepsilon$ it is defined by

$$||h||_{C,\varepsilon} = \sup_{(b,x)\in\mathbb{T}^d\times[0,\varepsilon]} |h_b(x)|\,. \tag{23}$$

To use contraction mapping principle we define a space

$$\mathcal{N} := \{h \in \mathcal{M}_\varepsilon, ||h||_{C,\varepsilon} \leq A\}\,, \rho(h_1, h_2) := ||h_1 - h_2||_{C,\varepsilon}. \tag{24}$$

with a continuous norm on it.

The constant A will be chosen later. Now we pass to the definition of the set N.

2.5. **Hölder property and a closed subspace** N**.** To prove Theorem 2 we shall work with the three norms: continuous one $||\cdot||_{C,\varepsilon}$ was already defined, now we will define the Lipschitz norm $\mathrm{Lip}_{x,\varepsilon}$ as well as the Hölder one $||\cdot||_{[\alpha],\varepsilon}$. The index ε indicates that these norms are considered for the subspaces of functions in $\mathcal{M}^2_\varepsilon$; but it will be omitted in the case it is matter-of-course.

Definition 1. For a function $h \in \mathcal{M}$ define its *Hölder norm* $||h||_{[\alpha]}$ as

$$||h||_{[\alpha]} := \sup_{b_1,b_2\in\mathbb{T}^d, x\in[0,1]} \frac{|h_{b_1}(x) - h_{b_2}(x)|}{||b_1 - b_2||^\alpha} \tag{25}$$

Hölder norm of a function is sometimes called its Hölder constant.

The subspace of functions $h \in \mathcal{M}$ for which this norm is finite, will be called the space of *Hölder functions with exponent* α and denoted by $\mathcal{H}^\alpha$. In much the same way, the space $\mathcal{H}^\alpha_\varepsilon$ is a subset of functions h in $\mathcal{M}_\varepsilon$ such that $||h||_{[\alpha],\varepsilon} < \infty$ where

$$||h||_{[\alpha],\varepsilon} := \sup_{b_1,b_2\in\mathbb{T}^d, x\in[0,\varepsilon]} \frac{|h_{b_1}(x) - h_{b_2}(x)|}{||b_1 - b_2||^\alpha} \tag{26}$$

Definition 2. For a function $h \in \mathcal{M}$ define its fiberwise *Lipschitz norm* $\mathrm{Lip}_x h$ as

$$\mathrm{Lip}_x h := \sup_{b\in\mathbb{T}^d, x,y\in[0,1]} \frac{|h_b(x) - h_b(y)|}{|x - y|} \tag{27}$$

Analogously, for $h \in \mathcal{M}_\varepsilon$

$$\mathrm{Lip}_{x,\varepsilon} h := \sup_{b\in\mathbb{T}^d, x,y\in[0,\varepsilon]} \frac{|h_b(x) - h_b(y)|}{|x-y|} \tag{28}$$

Once these definitions given, we can say what will be the closed subspace N of the functional space $\mathcal{N}$ (see (24)) for the contraction mapping principle. We will show that there exist constants $\varepsilon > 0$ as well as A_C, A_{Lip} and A_α such that the space

$$N = \left\{h \in \mathcal{M}_\varepsilon, h \in \mathcal{H}^\alpha_\varepsilon \mid ||h||_{C,\varepsilon} \le A_C, \mathrm{Lip}_{x,\varepsilon} h \le A_{\mathrm{Lip}}, ||h||_{[\alpha],\varepsilon} \le A_\alpha\right\} \tag{29}$$

is closed in $(\mathcal{N}, \rho)$ and preserved under $L\Phi$. Note that here we need $A \ge A_C$. In the proof, we will first choose constants A_C, A_{Lip} and A_α, A can be chosen later as $A := A_C$.

2.6. **Three main lemmas and the proof of Theorem 2.** To prove Theorem 2, one needs simply to show that all the conditions of contraction mapping principle hold for $\mathcal{N}$, ρ and N defined correspondingly in (24) and (29). Here we state three main lemmas that will give the result of Theorem 2.

Lemma 1 deals with homological equation and provides an explicit solution of (17) as a formal series. It also states that this series converges exponentially and gives a continuous function on M. Moreover, for α chosen accordingly to (8), the operator L in the space $\mathcal{M}$ preserves the subspace $\mathcal{H}^\alpha$ of Hölder functions with this particular exponent. This is a crucial point that gives us the main claim – Hölder property of a conjugacy.

The two lemmas that are left enable us to apply contraction mapping principle. Lemma 2 deals with composition $L\Phi$: it states that one can choose a closed subspace $N \subset \mathcal{N}$ of the form (29) such that is mapped into itself under the composition $L\Phi$. Lemma 3 proves that $L\Phi$ is indeed a contraction on the space $\mathcal{M}_\varepsilon$ in continuous norm.

Let us state precisely these lemmas.

Lemma 1. *[Solution of a homological equation] Consider a skew product* (1)*. Let us define a sequence of functions on* $\mathbb{T}^d$ *as*

$$\Pi_0(b) := 1, \Pi_n(b) := \lambda_b \lambda_{Ab} \dots \lambda_{A^{n-1}b}, n = 1, 2, \dots \tag{30}$$

Let α *be given by* (8)*, and set*

$$\theta = \theta(\alpha) := \mu^\alpha q < 1 \tag{31}$$

Suppose that conditions (2) *and* (31) *hold, and let* $Q \in \mathcal{H}^\alpha$.

Then the following holds:

1. *There exist a solution* $h_b(x)$ *of the homological equation* (18)*; it can be represented as a formal series*

$$h_b(x) = -\sum_{k=0}^{\infty} \frac{\Pi_k(b) Q \circ F_0^k(b,x)}{\lambda_{A^k b}} \tag{32}$$

2. *The series* (32) *converges uniformly on* M *and its sum is continuous in* b *and as smooth in* x *as* Q.

3. *The solution* h *satisfies Hölder condition with the exponent equal to* $\alpha : h \in \mathcal{H}^\alpha$.

4. *The operator* $L : Q \mapsto h$ *is bounded in* C*–norm on the space* $\mathcal{M}$.

Lemma 2. *[A closed subspace maps inside itself] For a skew product of the form* (1) *there exist constants* $\varepsilon, A_C, A_{\mathrm{Lip}}, A_\alpha > 0$ *such that the operator* $L\Phi$ *acting in the space* $\mathcal{M}$ *maps the closed space* N *defined by* (29) *into itself.*

Lemma 3. *[Contraction property] There exists a constant $A > 0$ such that for any sufficiently small $\varepsilon > 0$ the operator $L\Phi$ acting in the space $\mathcal{N}$ (which depends on A and on ε, see (24)) is contracting in the continuous norm.*

Proof of Theorem 2.

Now the proof follows: first, we take ε defined by Lemma 2 and fix all constants $A_C, A_{\mathrm{Lip}}, A_\alpha$ provided by the same lemma. Then we diminish ε for Lemma 3 to hold. Then the set $\mathcal{N}$ corresponding to such an ε and a constant $A = A_C$ has a C-norm on it defining a complete metric space. Operator $L\Phi$ acts in this space and, by Lemma 3, is a contracting map. Note that the set N defined in (29) is a closed subspace of $\mathcal{N}$ since $A = A_C$. This subspace $N \subset \mathcal{H}^\alpha_\varepsilon$ with a fixed Hölder constant α is preserved by $L\Phi$. Then, by contraction mapping principle, $L\Phi$ has a fixed point $h \in N$ (and hence in $\mathcal{H}^\alpha_\varepsilon$) which gives Hölder conjugacy of the initial skew product with its linearization. Strictly speaking, the Hölder property is proven not for the conjugacy but for its quadratic part divided by x^2 but the Hölder property of the conjugacy follows just because x is bounded. □

3. Proof of Lemma 1: homological equation solution

From the form (18) of the homological equation we deduce that $h(b, x)$ can be represented as

$$h = -\lambda^{-1}Q + \lambda h \circ F_0 \tag{33}$$

Let us take the right composition of this equation with the normalized map F_0 given by (4). And then let us apply the operator of multiplication by λ to this equation. The equality (33) implies

$$\lambda\,(h \circ F_0) = -\lambda\left(\lambda^{-1} \circ A\right) Q \circ F_0 + \lambda\,(\lambda \circ A)\, h \circ F_0^2 \tag{34}$$

Note that the left side of (34) is equal to one of the terms in the right hand side of (33). We continue such a process of taking right composition with F_0 and multiplying by λ. Thus we obtain the infinite sequence of equations that can be all summed up. Let us sum the first $N+1$ of them, then we will have

$$h_b = \Pi_{N+1}(b) h_b \circ F_0^{N+1} - \sum_{k=0}^{N} \frac{\Pi_k(b) Q \circ F_0^k}{\lambda_{A^k b}} \tag{35}$$

Let us now pass to the limit when $N \to \infty$: since $h \in \mathcal{N}$, $||h||_C \leq A$ and λ is bounded by some $q < 1$, we have that the first term on the right-hand side of (35) is bounded by Aq^{N+1} and hence tends to 0. Thus we obtain formula (32) for $h(b, x)$.

Since F is a diffeomorphism, then $\forall b \in \mathbb{T}^d$ we have: $\lambda_b \neq 0$. Then, since λ_b is a continuous function on a compact manifold $\mathbb{T}^d$, there exists a lower bound $D > 0$ such that

$$\lambda_b \geq D > 0 \quad \forall b \in \mathbb{T}^d. \tag{36}$$

Then, since obviously

$$|\Pi_k(b)| \leq q^k, \tag{37}$$

the series (32) is bounded by a converging number series

$$\sum_{k=0}^{\infty} \frac{q^k}{D} ||Q||_C = \frac{||Q||_C}{D(1-q)} \tag{38}$$

So by the Weierstrass majorant theorem, its sum is a continuous function on M, and the normalized homological operator L is bounded in continuous norm. Namely,

$$||L||_C \leq \frac{1}{D(1-q)} \tag{39}$$

Solution $h_b(x)$ is as smooth in x as Q is: that can be verified by differentiation of the series (32) and repetitive applying of Weierstrass majorant theorem. Convergence of the series for the derivative of the solution of homological equation h will be even faster than the convergence of the series for the function itself: indeed, the coefficients in the series (32) will be multiplied by the factors $\Pi_k(b)$ which are rapidly decreasing.

So assertions 1, 2 and 4 of Lemma 1 are proven. What is left to prove is that Hölder property with exponent α is preserved by operator L. We will need the following

Proposition 1. *In the setting of Theorem 1, let the Hölder property* (2) *for* f_b *and some* k *hold. Then, for* λ_b *and* $Q_b(x)$ *given by* $f_b(x) = \lambda_b x + x^2 Q_b(x)$ *we have Hölder properties: for* λ_b *and for the same* k *as in* (2)*; for* $Q_b(x)$ *and for* $k-2$.

Proof. The property for λ_b is obvious since $\lambda_b = \frac{\partial f_b(x)}{\partial x}|_{x=0}$. The property for $Q_b(x)$ follows from an analogue of Hadamard's lemma: for $\varphi \in C^2_{[0,1]}, \varphi(0) = \varphi'(0) = 0$ and $\psi = \frac{\varphi}{x^2}$ we have

$$||\psi||_C \leq ||\varphi||_{C^2} \tag{40}$$

This follows from the well-known formula $\psi(x) = \int_0^x (x-t)\varphi''((t))dt$. The coordinate change $t = xs, s \in [0,1]$ implies $\psi(x) = x^2 \int_0^1 (1-s) f''(xs) ds$. From this (40) follows. The needed corollary that Q_b is Hölder continuous assuming f_b is Hölder as an element of C^2, follows. $\square$

To prove assumption 3 of Lemma 1 let us denote by $C_Q := ||Q||_{[\alpha]}$ and $C_\lambda := ||\lambda||_{[\alpha]}$ the Hölder constants for functions Q and λ respectively. We need to find such $C > 0$ that for all $b_1, b_2 \in \mathbb{T}^d$:

$$|h_{b_1}(x) - h_{b_2}(x)| \leq C||b_1 - b_2||^\alpha \tag{41}$$

Note that even though Hölder exponents for Q and λ can be close to 1, the Hölder exponent for the solution h of normalized homological equation will be close to zero.

For each $k \in \mathbb{Z}_+$ denote

$$P_k(b) := \frac{\Pi_k(b)}{\lambda_{A^k b}} \tag{42}$$

Then, obviously,

$$|P_k(b)| \leq q^k D. \tag{43}$$

Let $Q_k(b,x) := Q \circ F_0^k(b,x)$. Then the solution h can be written in the form

$$h_b(x) = -\sum_{k=0}^{\infty} P_k(b) Q_k(b,x) \tag{44}$$

Take $b_1, b_2 \in \mathbb{T}^d$ and denote $Q_{k,j} := Q \circ F_0^k(b_j, x), j = 1, 2$. Then

$$|h_{b_1}(x) - h_{b_2}(x)| = \sum_{k=0}^{\infty} [(P_k(b_1) - P_k(b_2))Q_{k,1} + P_k(b_2)(Q_{k,1} - Q_{k,2})] \tag{45}$$

So we have the estimate

$$|h_{b_1}(x) - h_{b_2}(x)| \leq \sum_{k=0}^{\infty} \theta_{1,k}(b_1, b_2) + \theta_{2,k}(b_1, b_2) \tag{46}$$

where

$$\theta_{1,k}(b_1, b_2) = |P_k(b_1) - P_k(b_2)|\, ||Q||_C, \;\; \theta_{2,k}(b_1, b_2) = |P_k(b_2)||Q_{k,1} - Q_{k,2}| \tag{47}$$

Let us formulate some propositions that we will need, and postpone their proofs to the Appendix, Section 6.

Proposition 2. *Function $\Pi_n(b)$ defined as a product of λ_b in the first n points of the orbit of a linear diffeomorphism A, see* (30)*, is Hölder continuous with the exponent α, see* (8) *, and*

$$||\Pi_n||_{[\alpha]} \le \frac{C_\lambda \theta^n}{(\mu^\alpha - 1)q} \tag{48}$$

where C_λ is the Hölder constant for λ, θ is defined in (31)*, and μ is the largest magnitude of eigenvalues of A.*

Proposition 3. *Function $P_n(b)$ defined by* (42) *is Hölder with the exponent α, and*

$$||P_n||_{[\alpha]} \le D^2 C_\lambda B \theta^n, \tag{49}$$

where B depends only on the initial map, the precise formula for B is given below, see (71)*.*

Now, using Proposition 3 we can prove that

$$\theta_{1,k}(b_1, b_2) \le ||Q||_C D^2 C_\lambda B \theta^k ||b_1 - b_2||^\alpha \tag{50}$$

The estimate of $\theta_{2,k}$ is somewhat lengthier.

Proposition 4. *Function $\theta_{2,k}(b_1, b_2)$ defined in* (47) *is Hölder with the exponent α, and*

$$||\theta_{2,k}||_{[\alpha]} \le \theta^k D \left(C_Q + q^{k-1} \mathrm{Lip}_x Q \frac{C_\lambda}{\mu^\alpha - 1} \right) \tag{51}$$

The proof of this proposition is using only the triangle inequality and we postpone it till the appendix.

Inserting estimates on $\theta_{1,k}$ and $\theta_{2,k}$ from (50) and (51) into the inequality (46), we can finally use our special choice of α. It is in this place where we crucially use the fact that $\theta < 1$ to establish the convergence of estimating series in the right-hand side of (46). By simple computation of the sum of a geometric progression, we obtain that h is Hölder, and (41) holds for some C_h. The expicit form of C_h is not important for the proof of this lemma, but it will be used in the proof of Lemma 2. That's why we write it out explicitely:

$$C_h = ||Q||_C L_C + C_Q L_{[\alpha]} + \mathrm{Lip}_x Q L_{\mathrm{Lip}}. \tag{52}$$

where

$$L_C = \frac{D^2 C_\lambda B}{1-\theta}, L_{[\alpha]} = \frac{D}{1-\theta}, L_{\mathrm{Lip}} = \frac{C_\lambda}{(\mu^\alpha - 1)\, q} \frac{1}{1-\theta q}. \tag{53}$$

This completes the proof of Lemma 1. □

4. Proof of Lemma 2: the shift operator

Take some $h \in N$ and let us estimate continuous, Lipschitz and Hölder norms of its image under the composition of operators L and Φ.

The plan of the proof is the following: we will first show that there exist constants $\varepsilon_C > 0$ and $A = A_C > 0$ such that the space $\mathcal{N}$ defined by (24) is mapped by $L\Phi$ to itself. So the operator $L\Phi$ doesn't increase too much the continuous norm if we consider it on an appropriate space.

In the following step, we will diminish even more the ε-neighborhood of the base in which the functions are defined, and search for $\varepsilon_{\mathrm{Lip}} < \varepsilon_C$ and we will also search for a good bound A_{Lip} in (24). We will find such $\varepsilon_{\mathrm{Lip}}$ and A_{Lip} that $L\Phi$ won't increase the Lipschitz norm of the function h with conditions $||h||_C \le A_C, ||h||_{\mathrm{Lip}} \le A_{\mathrm{Lip}}$ in the vicinity of the base.

And, in the final step, we will find $\varepsilon_\alpha < \varepsilon_{\mathrm{Lip}}$ and A_α such that the space N defined by (29) is preserved by $L\Phi$.

From the definition (20) of the shift operator $\bar{\Phi}$ we have

$$\bar{\Phi}\bar{h}(b, x) = R_b\left(x + \bar{h}_b(x)\right) = (x + \bar{h}_b(x))^2 Q(b, x + x^2 h_b(x)) = x^2(1 + x h_b(x))^2 Q(b, x + x^2 h_b(x)) \tag{54}$$

hence

$$\Phi h = (1 + xh)^2 Q(b, x + x^2 h) \tag{55}$$

Using the definition (29) of the subspace N as well as the estimate (38) and the expression (55), for any $h \in N$ we have

$$||L\Phi h||_{C,\varepsilon} \leq \frac{1}{D(1-q)} ||\Phi h||_{C,\varepsilon} \leq \frac{||Q||_C}{D(1-q)} (1 + \varepsilon A_C)^2 \tag{56}$$

Hence let us first fix any

$$A_C > \frac{||Q||_C}{D(1-q)} \tag{57}$$

and then choose $\varepsilon = \varepsilon_C$ such that

$$\frac{||Q||_C}{D(1-q)} (1 + \varepsilon A_C)^2 < A_C \tag{58}$$

Note that in the definition of the space $\mathcal{N}$ the constant A bounding the norm should be greater than A_C defined by (57).

For the Lipschitz norm bound, we will need the proposition concerning only the homological operator: it preserves the space of smooth on fiber functions. Since we will deal with derivatives of functions along the fiber let us agree on notations: let us denote the l-th derivative of a function $h(b, x)$ with respect to fiber coordinate x as $h^{(l)}, l \in \mathbb{N}$.

Proposition 5. *The operator L is bounded in the Lipschitz norm: there exists a constant* $\mathrm{Lip}_x L$ *such that for any $h \in \mathcal{M}$ the following holds:*

$$\mathrm{Lip}_x (Lh) \leq \mathrm{Lip}_x L \cdot \mathrm{Lip}_x h.$$

Moreover, if for any $b \in \mathbb{T}^d$, the function $h(b, \cdot) \in C^l$, then Lh has the same smoothness as well and

$$\left|\left|(Lh)^{(l)}\right|\right|_C \leq C_k(L) \left|\left|h^{(l)}\right|\right|_C \tag{59}$$

The proof of this proposition is an easy consequence of the explicit form (32) for the solution of the normalized homological equation, and we give it in the Appendix, Section 6.

Now let us pass to the Lipschitz norm $\mathrm{Lip}_{x,\varepsilon}[L\Phi h] \leq \mathrm{Lip}_x L \times \mathrm{Lip}_{x,\varepsilon} \Phi h$. By using the simple arguments one can prove the following

Proposition 6. *There exist polynomials $T_3(\varepsilon)$ and $T_4^0(\varepsilon)$ of degrees respectively 3 and 4 such that $T_4^0(0) = 0$ and for any $h \in N$ holds*

$$\mathrm{Lip}_{x,\varepsilon}[\Phi h] \leq T_3(\varepsilon) + T_4^0(\varepsilon) A_{\mathrm{Lip}} \tag{60}$$

We postpone the proof to the Appendix.

From here we see that there exists a constant A_{Lip} such that for ε small enough, say $\varepsilon < \varepsilon_{\mathrm{Lip}}$, Lipschitz constant of the image of any function $h \in N$ is bounded by A_{Lip}:

$$\mathrm{Lip}_{x,\varepsilon}[L\Phi h] \leq A_{\mathrm{Lip}}.$$

We can assume that $\varepsilon_{\mathrm{Lip}} < \varepsilon_C$.

What is left is to estimate $||L\Phi h||_{[\alpha],\varepsilon}$: for this, we will need the bounds on how operators L and Φ behave on the space of α-Hölder functions separately.

For the shift operator in Appendix, Section 6 we will prove

Proposition 7. *If $h \in \mathcal{H}^{\alpha}{}_{\varepsilon}$ then $\Phi h \in \mathcal{H}^{\alpha}{}_{\varepsilon}$ as well. And, moreover, for $h \in N$, there exist polynomials $\tilde{T}_2(\varepsilon)$ and $\tilde{T}_4^0(\varepsilon), \tilde{T}_4^0(\varepsilon)(0) = 0$ of degrees 2 and 4 correspondingly such that*

$$|| \; \Phi h||_{[\alpha],\varepsilon} \leq \tilde{T}_4^0(\varepsilon) A_\alpha + \tilde{T}_2(\varepsilon) \tag{61}$$

While proving Lemma 1, we have deduced the bound (52) on Hölder norm of normalized homological operator with $L_C, L_{[\alpha]}, L_{\mathrm{Lip}}$ being some fixed constants defined by (53):

$$||Lh||_{[\alpha],\varepsilon} \leq L_C ||h||_{C,\varepsilon} + L_{[\alpha]} ||h||_{[\alpha],\varepsilon} + L_{\mathrm{Lip}} \mathrm{Lip}_{x,\varepsilon} h \tag{62}$$

Let us now combine (61) and (62) for $h := \Phi h$ to get the bound for $||L\Phi h||_{[\alpha],\varepsilon}$. Here we will be using Propositions 6 and 7 as well as inequality (56) to get the bounds on different norms of Φh in the space $\mathcal{M}_\varepsilon$.

$$\begin{aligned} ||L\Phi h||_{[\alpha],\varepsilon} &\leq L_C \, ||\Phi h||_{C,\varepsilon} + L_{[\alpha]} \, ||\Phi h||_{[\alpha],\varepsilon} + L_{\mathrm{Lip}} \mathrm{Lip}_{x,\varepsilon} \Phi h \leq \\ &\leq L_C ||Q||_C (1 + \varepsilon A_C)^2 + L_{[\alpha]} \left(\tilde{T}_2^0(\varepsilon) A_\alpha + \tilde{T}_2(\varepsilon) \right) + L_{\mathrm{Lip}} \left(T_2(\varepsilon) + T_4^0(\varepsilon) A_{\mathrm{Lip}} \right) \end{aligned} \tag{63}$$

So we see that there exist polynomials $Q_2^0(\varepsilon), Q_4(\varepsilon)$ such that $\deg Q_2^0 = 2, Q_2^0(0) = 0, \deg Q_4(\varepsilon) = 4$ and

$$||L\Phi h||_{[\alpha],\varepsilon} \leq A_\alpha Q_2^0(\varepsilon) + Q_4(\varepsilon) \tag{64}$$

So for ε small enough, $\varepsilon < \varepsilon_{[\alpha]}$, and for some $A_\alpha > 0$ the right-hand side of inequality (64) can be made less than A_α. We can take $\varepsilon_{[\alpha]} < \varepsilon_{\mathrm{Lip}}$. By taking $\varepsilon = \varepsilon_{[\alpha]}$ we obtain the desired preservation of N by operator $L\Phi$. This space is obviously closed in $\mathcal{N}$.

□

5. Proof of Lemma 3: contraction property

Since operator L is linear and uniformly bounded by (38) in the continuous norm, the only thing to prove is that normalized shift operator Φ is strongly contracting in this norm, i.e. for any ε small enough there exists some constant $\nu = \nu(\varepsilon) \in (0, 1)$ such that for any $h, g \in \mathcal{N}$

$$||\Phi h - \Phi g||_{C,\varepsilon} \leq \nu ||h - g||_{C,\varepsilon} \tag{65}$$

Proof.
Suppose $h, g \in \mathcal{M}$ and define $\bar{h}, \bar{g} \in \mathcal{M}^2$ by $\bar{h}_b(x) = x^2 h(b, x), \bar{g}_b(x) = x^2 g_b(x)$. Also denote $Q_h = Q\left(b, x + \bar{h}_b(x)\right)$.

$$\begin{aligned} ||\Phi h - \Phi g||_{C,\varepsilon} &= \left|\left| (1 + x h_b(x))^2 Q_h - (1 + x g_b(x))^2 Q_g \right|\right| \leq \\ &\leq ||Q_h - Q_g||_{C,\varepsilon} + ||2x h_b(x) Q_h - 2x g_b(x) Q_g||_{C,\varepsilon} + \left|\left| x^2 h_b^2(x) Q_h - x^2 g_b^2(x) Q_g \right|\right|_{C,\varepsilon} \leq \\ &\leq \mathrm{Lip}_x Q ||\bar{h} - \bar{g}||_{C,\varepsilon} + 2\varepsilon ||h - g||_{C,\varepsilon} ||Q||_C + 2\varepsilon A ||Q_h - Q_g||_{C,\varepsilon} + \varepsilon^2 \left|\left| (h^2 - g^2) Q_h + g^2 (Q_h - Q_g) \right|\right|_{C,\varepsilon} \leq \\ &||h - g||_{C,\varepsilon} \left(\varepsilon^2 \mathrm{Lip}_x Q + 2\varepsilon ||Q||_C + 2\varepsilon^2 \mathrm{Lip}_x Q A \right) + \varepsilon^2 \left(2A ||h - g||_{C,\varepsilon} ||Q||_C + A^2 \mathrm{Lip}_x Q \varepsilon^2 ||h - g||_{C,\varepsilon} \right) = \\ &= ||h - g||_{C,\varepsilon} \, o(\varepsilon). \end{aligned} \tag{66}$$

Hence operator Φ is strongly contracting. And since from (39) for any function $h \in \mathcal{N}$ the norm $||Lh||_{C,\varepsilon} \leq \frac{D}{1-q} ||h||_{C,\varepsilon}$, applying this to Φh with the fact of the strong contraction property for Φ we get the strong contraction property for $L\Phi$. □

6. Appendix: Proof of Theorem 3 and other calculations

In the appendix we will prove the technical propositions stated above.

6.1. Hölder properties of some auxiliary functions.

First let us prove

Proposition 2. *Function $\Pi_n(b)$ defined as a product of λ_b in the first n points of the orbit of a linear diffeomorphism A, see* (30), *is Hölder with the exponent α and*

$$||\Pi_n||_{[\alpha]} \leq \frac{C_\lambda \theta^n}{(\mu^\alpha - 1)q} \tag{67}$$

where C_λ is Hölder constant for λ, θ is defined in (31). *Here and below α is given by* (8) *and μ* is the largest magnitude of eigenvalues of A.

Proof of Proposition 2:

(68)

$$|\Pi_n(b_1) - \Pi_n(b_2)| = \left|\prod_{k=0}^{n-1} \lambda_{A^k b_1} - \prod_{k=0}^{n-1} \lambda_{A^k b_2}\right| = |\lambda_{b_1} - \lambda_{b_2}| \times \left|\prod_{k=1}^{n-1} \lambda_{A^k b_1}\right| + |\lambda_{b_2}|\,|\Pi_{n-1}(Ab_1) - \Pi_{n-1}(Ab_2)| \leq \ldots$$

$$\leq q^{n-1} C_\lambda \sum_{k=0}^{n-1} \left|\left|A^k b_1 - A^k b_2\right|\right|^\alpha \leq q^{n-1} C_\lambda \frac{\mu^{n\alpha} - 1}{\mu^\alpha - 1} ||b_1 - b_2||^\alpha \leq \frac{C_\lambda \theta^n}{(\mu^\alpha - 1)q} ||b_1 - b_2||^\alpha$$

□

Proposition 3. *Function $P_n(b)$* defined by $P_n(b) := \frac{\Pi_n(b)}{\lambda_{A^n b}}$ *is Hölder with exponent α and*

$$||P_n||_{[\alpha]} \leq D^2 C_\lambda B \theta^n \tag{69}$$

where B depends only on the initial map, the precise formula for B is given below, see (71).

Proof.

$$|P_n(b_1) - P_n(b_2)| = \left|\frac{\Pi_n(b_1)\lambda_{A^n b_2} - \Pi_n(b_2)\lambda_{A^n b_1}}{\lambda_{A^n b_1}\lambda_{A^n b_2}}\right| \leq D^2 \left|\lambda_{A^n b_2} \prod_{k=0}^{n-1} \lambda_{A^k b_1} - \lambda_{A^n b_1} \prod_{k=0}^{n-1} \lambda_{A^k b_2}\right| = \tag{70}$$

$$= D^2 \left|(\lambda_{A^n b_2} - \lambda_{A^n b_1})\, \Pi_n(b_1) + \Pi_{n+1}(b_1) - (\lambda_{A^n b_1} - \lambda_{A^n b_2})\, \Pi_n(b_2) - \Pi_{n+1}(b_2))\right| \leq$$

$$\leq |\Pi_{n+1}(b_1) - \Pi_{n+1}(b_2)| + |\lambda_{A^n b_1} - \lambda_{A^n b_2}||\Pi_n(b_1) - \Pi_n(b_2)| \leq$$

$$\leq D^2 \left[\frac{C_\lambda \theta^{n+1}}{(\mu^\alpha - 1)q} + 2q^n C_\lambda \mu^{n\alpha}\right] ||b_1 - b_2||^\alpha \leq D^2 C_\lambda B \theta^n ||b_1 - b_2||^\alpha$$

where B doesn't depend on anything but initial skew product:

$$B(\theta, \mu, \alpha, q) = \frac{\theta}{(\mu^\alpha - 1)q} + 2 \tag{71}$$

□

Proposition 4. *Function $\theta_{2,k}(b_1, b_2)$ defined as $\theta_{2,k}(b_1, b_2) = |P_k(b_2)||Q_{k,1} - Q_{k,2}|$ is Hölder with α as exponent and*

$$||\theta_{2,k}||_{[\alpha]} \leq \theta^k D \left(C_Q + q^{k-1} \mathrm{Lip}_x Q \frac{C_\lambda}{\mu^\alpha - 1}\right) \tag{72}$$

Here $Q_{k,1} = Q \circ F_0^k(b_1, x)$ and *$Q_{k,2} = Q \circ F_0^k(b_2, x)$, and the definition of $P_k(b)$ was reminded in Proposition 2 above.*

Proof. We use the results of Proposition 2 in the following chain of inequalities.

$$\theta_{2,k} \leq q^k D\,|Q_{A^k b_1}(\Pi_k(b_1)x) - Q_{A^k b_2}(\Pi_k(b_2)x)| \leq q^k D\,|Q_{A^k b_1}(\Pi_k(b_1)x) - Q_{A^k b_2}(\Pi_k(b_1)x)| + \\ + q^k D\,|Q_{A^k b_2}(\Pi_k(b_2)x) - Q_{A^k b_2}(\Pi_k(b_1)x)| \leq q^k D C_Q \mu^{k\alpha} ||b_1 - b_2||^\alpha + q^k D \mathrm{Lip}_x Q ||\Pi_k||_{\mathcal{H}^\alpha} ||b_1 - b_2||^\alpha \leq \\ \theta^k D \left(C_Q + q^{k-1} \mathrm{Lip}_x Q \frac{C_\lambda}{\mu^\alpha - 1} \right) ||b_1 - b_2||^\alpha \tag{73}$$

□

Proposition 5. *Operator L is bounded in the Lipschitz norm: there exists a constant $\mathrm{Lip}_x L$ such that for any $h \in \mathcal{M}$ holds*

$$\mathrm{Lip}_x\,(Lh) \leq \mathrm{Lip}_x L \times \mathrm{Lip}_x h.$$

Moreover, if $h(b,\cdot) \in C^l$ for any $b \in \mathbb{T}^d$, then Lh has the same smoothness as well and

$$\left|\left|(Lh)^{(l)}\right|\right|_C \leq C_k(L)\left|\left|h^{(l)}\right|\right|_C. \tag{74}$$

Proof. Using the explicit formula for the solution (32), as well as bounds (37) and (43), we have:

$$\sup_{x,y\in[0,1]} \left| \frac{Lh(b,x) - Lh(b,y)}{x-y} \right| = \sup_{x,y\in[0,1]} \left| \sum_{k=0}^{\infty} P_k(b) \frac{h \circ F_0^k(b,x) - h \circ F_0^k(b,y)}{x-y} \right| \leq \\ \leq \sup_{x,y\in[0,1]} \sum_{k=0}^{\infty} P_k(b) \frac{\mathrm{Lip}_x h |\Pi_k(b)x - \Pi_k(b)y|}{|x-y|} = \mathrm{Lip}_x h \frac{D}{1-q^2}. \tag{75}$$

The bounds for the derivatives are obtained analgously by differentiating term by term the series (32):

$$(Lh)^{(l)} = -\sum_{k=0}^{\infty} P_k(b) \Pi_k^l(b) h^{(l)} \circ F_0^k. \tag{76}$$

Therefore,

$$\left|\left|(Lh)^{(l)}\right|\right|_C \leq \frac{D}{1-q^{l+1}}\left|\left|h^{(l)}\right|\right|_C. \tag{77}$$

□

Proposition 6. *There exist polynomials $T_3(\varepsilon)$ and $T_4^0(\varepsilon)$ of degrees respectively 3 and 4 such that $T_4^0(0) = 0$ and for any $h \in N$ holds*

$$\mathrm{Lip}_{x,\varepsilon}[\Phi h] \leq T_3(\varepsilon) + T_4^0(\varepsilon) A_{\mathrm{Lip}} \tag{78}$$

Proof of Proposition 6: The proof of this proposition deals with an expression for $\mathrm{Lip}_{x,\varepsilon}\Phi h$ which is given by

$$\sup_{x,y\in[0,\varepsilon]} \frac{\left|(1 + x h_b(x))^2 Q(b, x + \bar{h}_b(x)) - (1 + y h_b(y))^2 Q(b, y + \bar{h}_b(y))\right|}{|x-y|} \tag{79}$$

Since in this proposition the base coordinate b is fixed and x is changing we will permit to ourselves not to write the b index and just suppose that $Q(x) = Q(b, x + \bar{h}_b(x))$ as well as $h(x) = h_b(x)$. The bound is a triangle inequality formula:

$$\text{(80)}\quad \mathrm{Lip}_{x,\varepsilon}\Phi h \le \sup_{x,y\in[0,\varepsilon]}\left|\frac{Q(x)-Q(y)}{x-y}\right| + 2\sup_{x,y\in[0,\varepsilon]}\left|\frac{xh(x)Q(x)-yh(y)Q(y)}{x-y}\right| +$$
$$+\sup_{x,y\in[0,\varepsilon]}\left|\frac{x^2h(x)Q(x)-y^2h(y)Q(y)}{x-y}\right| \le \mathrm{Lip}_xQ(1+\mathrm{Lip}_{x,\varepsilon}\bar{h})+$$
$$+2\sup_{x,y\in[0,\varepsilon]}\left|\frac{xh(x)\left(Q(x)-Q(y)\right)}{x-y}\right| + 2\sup_{x,y\in[0,\varepsilon]}\left|\frac{xQ(y)(h(x)-h(y))}{x-y}\right| + 2\sup_{y\in[0,\varepsilon]}|Q(y)h(y)|+$$
$$+\sup_{x,y\in[0,\varepsilon]}\left|\frac{x^2h(x)\left(Q(x)-Q(y)\right)}{x-y}\right| + \sup_{x,y\in[0,\varepsilon]}\left|\frac{Q(y)\left[x^2\left(h(x)-h(y)\right)+(x^2-y^2)h(y)\right]}{x-y}\right| \le$$
$$\le \mathrm{Lip}_xQ(1+\mathrm{Lip}_{x,\varepsilon}\bar{h}) + 2\varepsilon A_C\mathrm{Lip}_xQ\mathrm{Lip}_{x,\varepsilon}\bar{h} + 2\varepsilon||Q||_CA_{\mathrm{Lip}}+$$
$$+2||Q||_CA_C + \varepsilon^2A_C\mathrm{Lip}_xQ\mathrm{Lip}_{x,\varepsilon}\bar{h} + ||Q||_C(\varepsilon^2A_{\mathrm{Lip}}+2\varepsilon A_C)$$

Let us note that

$$\text{(81)}\quad \mathrm{Lip}_{x,\varepsilon}\bar{h} = \sup_{x,y\in[0,\varepsilon]}\left|\frac{x^2h(x)-y^2h(y)}{x-y}\right| \le \sup_{x,y\in[0,\varepsilon]}\left|\frac{x^2(h(x)-h(y))}{x-y}\right| + \sup_{x,y\in[0,\varepsilon]}\left|\frac{h(y)(x^2-y^2)}{x-y}\right| \le$$
$$\le A_{\mathrm{Lip}}\varepsilon^2 + A_C2\varepsilon$$

After substitution of $\mathrm{Lip}_{x,\varepsilon}\bar{h}$ in (80) by (81) we have the result with

$$T_3(\varepsilon) = 2A_C^2\mathrm{Lip}_xQ\varepsilon^3 + 4A_C^2\mathrm{Lip}_xQ\varepsilon^2 + 2A_C(\mathrm{Lip}_xQ + ||Q||_C)\varepsilon + \mathrm{Lip}_xQ + 2||Q||_CA_C$$
$$T_4^0(\varepsilon) = \mathrm{Lip}_xQA_C\varepsilon^4 + 2A_C\mathrm{Lip}_xQ\varepsilon^3 + \mathrm{Lip}_xQ\varepsilon^2 + 2||Q||_C\varepsilon$$

□

Now let us prove the analogous proposition for the Hölder norm of the operator Φ:

Proposition 7. *If $h \in \mathcal{H}^\alpha{}_\varepsilon$ then $\Phi h \in \mathcal{H}^\alpha{}_\varepsilon$ as well. And, moreover, for $h \in N$, there exist polynomials $\tilde{T}_2(\varepsilon)$ and $\tilde{T}_4^0(\varepsilon), \tilde{T}_4^0(\varepsilon)(0) = 0$ of degrees 2 and 4 correspondingly such that*

$$\text{(82)}\qquad ||\ \Phi h||_{[\alpha],\varepsilon} \le \tilde{T}_4^0(\varepsilon)A_\alpha + \tilde{T}_2(\varepsilon)$$

Proof. To estimate Hölder norm of the shift operator, we need some more triangle inequalities.

$$\text{(83)}\quad |\Phi h(b_1,x) - \Phi h(b_2,x)| = |(1+xh_{b_1})^2Q_{b_1} - (1+xh_{b_2})^2Q_{b_2}| \le$$
$$\le |Q_{b_1}-Q_{b_2}| + 2x\left|h_{b_1}Q_{b_1} - h_{b_2}Q_{b_2}\right| + x^2\left|h_{b_1}^2Q_{b_1} - h_{b_2}^2Q_{b_2}\right| \le$$
$$\le \left|Q(b_1,x+\bar{h}_{b_1}) - Q(b_1,x+\bar{h}_{b_2})\right| + \left|Q(b_1,x+\bar{h}_{b_2}) - Q(b_2,x+\bar{h}_{b_2})\right| +$$
$$+2\varepsilon\left|h_{b_1}\left(Q(b_1,x+\bar{h}_{b_1}) - Q(b_1,x+\bar{h}_{b_2})\right)\right| + 2\varepsilon\left|h_{b_1}\left(Q(b_1,x+\bar{h}_{b_2}) -\right.\right.$$
$$\left.\left.-Q(b_2,x+\bar{h}_{b_2})\right)\right| + 2\varepsilon\left|Q_{b_2}(h_{b_1}-h_{b_2})\right| + \varepsilon^2h_{b_1}^2\left|Q(b_1,x+\bar{h}_{b_1}) - Q(b_1,x+\bar{h}_{b_2})\right| +$$
$$+\varepsilon^2h_{b_1}^2\left|Q(b_1,x+\bar{h}_{b_2}) - Q(b_2,x+\bar{h}_{b_2})\right| + \varepsilon^2|Q_{b_2}|\left|h_{b_1}^2 - h_{b_2}^2\right| \le$$
$$\le ||b_1-b_2||^\alpha\left(\tilde{T}_4^0(\varepsilon)A_\alpha + \tilde{T}_2(\varepsilon)\right)$$

where

$$\text{(84)}\qquad \tilde{T_4}^0(\varepsilon) = \mathrm{Lip}_xQ\varepsilon^2 + 2\varepsilon^3A\mathrm{Lip}_xQ + 2\varepsilon||Q||_C + \varepsilon^4A^2\mathrm{Lip}_Q + 2||Q||_CA\varepsilon^2$$

and

$$\text{(85)}\qquad \tilde{T}_2(\varepsilon) = C_Q + 2\varepsilon AC_Q + \varepsilon^2A^2C_Q$$

□

All the propositions stated above are proven. This completes the proof of our main result – Theorem 2.

Now we are ready to prove that the conjugacy is smooth in fiber variable, and Hölder with its derivatives in base variables.

Proof of Theorem 3:

The proof of a smooth version of Theorem 2 is analogous to the proof of the latter theorem. Here we will give a sketch of the proof: we will only show that the conjugacy H is $(k-2)-$ smooth with respect to the fiber variable. The proof of the fact that its fiber derivatives are now Hölder on b is analogous to the proof of the Hölder property fot the function H itself and we don't give it here.

The idea is to change the space $\mathcal{N}$ in an appropriate way. For some constants $A_0, \dots, A_l > 0$ and $\varkappa_0, \dots, \varkappa_l > 0$ let us define the space

$$\mathcal{N}_l := \left\{ h(\cdot, b) \in C^l([0,\varepsilon]) : ||h||_C \le A_0, \dots, \left|\left|h^{(l)}\right|\right|_C \le A_l \right\} \tag{86}$$

with the norm

$$||h||_* = \varkappa_0 ||h||_C + \ldots + \varkappa_l ||h^{(l)}||_C. \tag{87}$$

We have now to prove the analogues of Lemmas 1, 3 and 2 above, and then follow the argument in Theorem 2. The homological and shift operators will stay the same although the functional spaces in which they act will be smaller, and the metric will be not continuous but a smooth one.

Lemma 1 (smooth case) Operator L is bounded in the norm (87).

Proof.

$$||Lh||_* \;=\; \sum_{j=0}^{l} \varkappa_j ||(Lh)^{(j)}||_C \;\le\; \sum_{j=0}^{l} \frac{D\varkappa_j}{1-q^{j+1}} ||h^{(j)}||_C \;\le\; \frac{D}{1-q} \sum_{j=0}^{\infty} \varkappa_j ||h^{(j)}||_C \;=\; \frac{D}{1-q} ||h||_* \tag{88}$$

□

For the space $\mathcal{N}_l$ to map to itself by $L\Phi$, we should choose constants $A_0, A_1, \dots, A_l$ appropriately. For $L\Phi$ to be contracting in the space, we should appropriately choose $\varkappa_0, \dots, \varkappa_l$. Let us show that these two choices can be made without complications and the analogues of Lemmas 3 and 2 hold.

In what concerns the operator Φ, we will use its presentation (55) and calculate the derivatives for $k = 0, \dots, l$: by the Leibnitz rule:

$$(\Phi h)^{(k)} = \sum_{j=0}^{k} C_k^j ((1 + xh(b,x))^2)^{(j)} Q^{(k-j)}(b, x + \bar{h}) \tag{89}$$

The explicit form of the right-hand side is not as important as a fact that it can be written as a sum of polynomials in derivatives of $h, \bar{h}$ and Q. Indeed, there exist polynomials $\tau_0, \dots, \tau_l$ and $\sigma_0, \dots, \sigma_l$ such that

$$(\Phi h)^{(k)} = \sum_{j=0}^{k} C_k^j \tau_j(x, h, \dots, h^{(j)}) \sigma_j \left(x, \bar{h}, \dots, \bar{h}^{(k-j)}, Q\left(b, x+\bar{h}\right), \dots, \;\; Q^{(k-j)}(b, x+\bar{h}) \right) \tag{90}$$

We will estimate the continuous norm of the right-hand side of (90) in $\mathbb{T}^d \times [0, \varepsilon]$. So we will have that for some polynomials T_j and S_j there is a bound

$$||L\Phi h^{(k)}||_{C,\varepsilon} \le \sum_{j=0}^{k} C_k^j T_j(\varepsilon, A_0, \dots, A_j) S_j \left(\varepsilon, A_0, \dots, A_{k-j}, ||Q||_C, \dots, \;\; ||Q^{(k-j)}||_C \right) \tag{91}$$

Note that the coefficient in front of A_k in this expression is a polynomial that has no free term. Indeed, A_k comes only from T_k or S_0: in both of the cases A_k is multiplied by at least one ε. Hence we need to find $A_0, \dots A_l$ such that $l+1$ equations hold for some polynomials $P_k, P_k^0, P_k^0(0) = 0$:

$$P_k^0(\varepsilon)A_k + P_k(\varepsilon)C(A_0, \dots, A_{k-1}) \le A_k, k = 0, \dots, l \tag{92}$$

First we take ε such that all polynomials $P_k^0(\varepsilon) < 1$. Then we satisfy the equations (92) one by one, starting from $k = 0$ by choosing A_k one by one, starting with A_0 and by increasing the index.

Now we have to prove that operator $L\Phi$ is contracting in the space $\mathcal{N}$ if $\varkappa_0, \dots \varkappa_l$ are properly chosen.

One can show that

$$\begin{aligned} &||L\Phi h - L\Phi g||_* \le \\ &\frac{D}{1-q}\sum_{j=0}^{l}\varkappa_j||\sum_{k=0}^{j} C_j^k \left(\tau_k(x, h, \dots h^{(k)})\sigma_k\left(x, \bar{h}, \dots, \bar{h}^{(j-k)}, Q(b, x+h), \dots,\right.\right. \\ &\left.Q^{(j-k)}(b, x+h)\right) - \tau_k\left(x, g, \dots g^{(k)})\sigma_k(x, \bar{g}, \dots, \bar{g}^{(j-k)}, Q(b, x+g), \dots,\right. \\ &\left.\left.Q^{(j-k)}(b, x+g)\right)\right)||_{C,\varepsilon} \le \sum_{j=0}^{l}||h^{(k)} - g^{(k)}||_{C,\varepsilon}\left(\varkappa_k U_k^0(\varepsilon) + \sum_{j=k+1}^{l} U_j(\varepsilon)\varkappa_j\right) \end{aligned} \tag{93}$$

for some polynomials U_j, U_j^0. For the right-hand side to be less than $\xi||h-g||_*$ for some $\xi < 1$ the following system should be satisfied:

$$\begin{aligned} U_0^0(\varepsilon) + \frac{\varkappa_1}{\varkappa_0}U_0(\varepsilon) + \dots \frac{\varkappa_l}{\varkappa_0}U_0(\varepsilon) &\le \xi \\ U_1^0(\varepsilon) + \frac{\varkappa_2}{\varkappa_1}U_0(\varepsilon) + \dots \frac{\varkappa_l}{\varkappa_1}U_0(\varepsilon) &\le \xi \\ &\dots \\ U_{l-1}^0(\varepsilon) + \frac{\varkappa_l}{\varkappa_{l-1}}U_{l-1}(\varepsilon) &\le \xi \\ U_l^0(\varepsilon) &\le \xi \end{aligned}$$

One can choose ε in such a way that $U_k^0(\varepsilon) < \xi$. Then, the last inequality in the list is true, by taking any $\varkappa_l$ and $\varkappa_{l-1}$ big enough, we satisfy the before-last inequality and we proceed in staisfying these inequalities from the last one till the first one.

So, we obtain a contracting operator. We haven provent that in the space $\mathcal{N}_l$ of functions defined in a neighborhood of the base with a metric chosen appropriately, there is a contracting operator $L\Phi$. Its fixed point is the needed conjugacy which will be sufficiently smooth on the fiber variable x.

□

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Abstract

This thesis deals with the questions of asymptotic behavior of dynamical systems and consists of six independent chapters.

In the first part of this thesis we consider three particular dynamical systems. The first two chapters deal with the models of two physical systems : in the first chapter, we study the geometric structure and limit behavior of Arnold tongues of the equation modeling a Josephson contact ; in the second chapter, we are interested in the Lagrange problem of establishing the asymptotic angular velocity of the swiveling arm on the surface. The third chapter deals with planar geometry of an elliptic billiard.

The forth and fifth chapters are devoted to general methods of studying the asymptotic behavior of dynamical systems. In the forth chapter we prove the convergence of markovian spherical averages for free group actions on a probablility space. In the fifth chapter we provide a normal form for skew-product diffeomorphisms that can be useful in the study of strange attractors of dynamical systems.

Keywords: ergodic theory, Arnold tongues, elliptic billiard, normal forms

Résumé

Cette thèse porte sur le comportement asymptotique des systèmes dynamiques et contient cinq chapitres indépendants.

Nous considérons dans la première partie de la thèse trois systèmes dynamiques concrets. Les deux premiers chapitres présentent deux modèles de systèmes physiques : dans le premier, nous étudions la structure géométrique des langues d'Arnold de l'équation modélisant le contact de Josephson ; dans le deuxième, nous nous intéressons au problème de Lagrange de recherche de la vitesse angulaire asymptotique d'un bras articulé sur une surface. Dans le troisième chapitre nous étudions la géométrie plane du billard elliptique avec des méthodes de la géométrie complexe.

Les quatrième et cinquième chapitres sont dédiés aux méthodes générales d'étude asymptotique des systèmes dynamiques. Dans le quatrième chapitre nous prouvons la convergence des moyennes sphériques pour des actions du groupe libre sur un espace mesuré. Dans le cinquième chapitre nous fournissons une forme normale pour un produit croisé qui peut s'avérer utile dans l'étude des attracteurs étranges de systèmes dynamiques.

Mots-clés: théorie ergodique, langues d'Arnold, billard elliptique, formes normales