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About the shock formation in multi-dimensional scalar conservation laws

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Abstract

We establish a space-time integral estimate of the difference of solutions of non-degenerate scalar conservation laws. It is valid over the maximal domain $(0,T_{\max})\times\mathbb{R}^d$ in which the solutions are shock-free, and it fails beyond $T_{\max}$. When comparing a solution with its shifts, we deduce a Besov-like estimate at the development of its first singularity.

Key words: Divergence-free tensors, Compensated Integrability, Scalar conservation laws, Burgers equation, shock formation.

Mots clés: Tenseurs à Divergence nulle, Intégrabilité par Compensation, lois de conservation scalaires, équation de Burgers, formation des chocs.

MSC2010: 35F55, 35L65.

Notations. The problems under consideration involve functions of time t and space variable $y\in\mathbb{R}^d$. The space-time dimension is thus $n=1+d$. If $1\le p\le+\infty$, the norm in $L^p(U)$ is $\|\cdot\|_p$. The cone of symmetric positive semi-definite $n\times n$ matrices with real entries is $\mathbf{Sym}_n^+$. If $V\in\mathbb{R}^n$, the matrix $V\otimes V\in\mathbf{Sym}_n^+$ has entries v_iv_j. If $h\in\mathbb{R}^d$ and $u:\mathbb{R}^d\to\mathbb{R}$, then τ_hu denotes the function $u(\cdot+h)$. The Lipschitz semi-norm of $s\mapsto g(s)$ is

$$|g|_{\mathrm{Lip}(I)}=\sup_{s\neq s'\in I}\frac{|g(s')-g(s)|}{|s'-s|}.$$

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The sign of a real number is $\mathrm{sgn}(s) \in \{0, \pm 1\}$.

1 Introduction

Let us consider a scalar conservation law in the spacial domain $\mathbb{R}^d$,

$$\partial_t u + \mathrm{div}_y \vec{f}(u) = 0, \qquad t > 0,\, y \in \mathbb{R}^d. \tag{1}$$

The flux $\vec{f} : \mathbb{R} \to \mathbb{R}^d$ is a given smooth vector field. Choosing an appropriate moving frame, we may assume $\vec{f}(0) = \vec{f}'(0) = 0$.

Depending upon $\vec{f}$, equation (1) can be either linear or not, degenerate or not. Recall that if $d = 1$, (1) is said *genuinely non-linear* whenever f'' does not vanish. The corresponding notion when $d \geq 1$, which we call *full non-degeneracy*, is that $\det(\vec{f}'', \dots, \vec{f}^{(n)})$ does not vanish.

According to Kružkov [10], the Cauchy problem is well-posed for L^∞-initial data u_0, in the sense of entropy solutions. It generates a semi-group $S_t : u_0 \mapsto u(t)$, which enjoys several properties:

Comparison. If $u_0 \leq v_0$ almost everywhere, then $S_t u_0 \leq S_t v_0$. In particular

$$\inf u_0 \leq u(t,y) \leq \sup u_0.$$

Contraction. If $v_0 - u_0 \in L^1(\mathbb{R}^d)$, then $S_t v_0 - S_t u_0 \in L^1(\mathbb{R}^d)$, and $t \mapsto \|S_t v_0 - S_t u_0\|_1$ is non-increasing.

Conservation. Under the same assumption as above, $t \mapsto \int_{\mathbb{R}^d} (S_t v_0 - S_t u_0)\, dy$ is constant, equal to $\int_{\mathbb{R}^d} (v_0 - u_0)\, dy$.

We aim at estimating the difference $v - u$ of two solutions associated with initial data u_0, v_0. The L^1-contraction mentionned above holds uniformly in time and is valid for entropy admissible solutions. The contraction fails in L^2-norm, though it has been shown that in one space dimension, the quantity

$$t \mapsto \inf_h \|v(t) - \tau_h u(t)\|_2$$

is non-increasing when u is a pure shock and f is convex ; see N. Leger's analysis [11]. Within the more general context of systems of conservation laws endowed with a strongly convex entropy, C. Dafermos [2] and R. DiPerna [6] established the well-known weak-strong estimate.

This one compares a Lipschitz continuous solution u over $(0,T)\times\mathbb{R}^d$ with a bounded entropy admissible solution v. The relative entropy

$$\eta(v|u) :== \eta(v)-\eta(u)-d\eta(u)\cdot(v-u),$$

satisfies a differential inequality

$$\frac{d}{dt}\int_{\mathbb{R}^d}\eta(v|u)\,dy \le c(\|u\|_\infty)\|\nabla_y u\|_\infty\int_{\mathbb{R}^d}\eta(v|u)\,dy,$$

from which one infers

$$\int_{\mathbb{R}^d}\eta(v(t)|u(t))\,dy \le \exp\left(c(\|u_0\|_\infty)\int_0^t\|\nabla_y u(s)\|_\infty ds\right)\int_{\mathbb{R}^d}\eta(v_0|u_0)\,dy, \tag{2}$$

a kind of L^2-estimate. Other results, dealing with the continuity of the map $u_0\mapsto u$, were obtained by means of Compensated Compactness (CC), see [1, 19].

Returning to the scalar case, we recently [17] established a space-time estimate:

$$\int_0^\infty\int_{\mathbb{R}^d}K(|v(t,y)-u(t,y)|)^{\frac{1}{d}}dy\,dt \le c(\|u_0\|_2,\|v_0\|_2,\|u_0\|_\infty,\|v_0\|_\infty)\,\|v_0-u_0\|_1^{\frac{1}{d}}, \tag{3}$$

where $K:\mathbb{R}_+\to\mathbb{R}_+$ is a function defined in Section 2.1 below. Typically, $K(s)\sim s^{n^2}$ if (1) is fully non-degenerate, thus (3) is a kind of L^p-estimate with $p=n^2/d>2$. However because the solutions are bounded, we may always bound the space integral above, thanks to the L^1-contraction,

$$\int_{\mathbb{R}^d}K(|v(t,y)-u(t,y)|)^{\frac{1}{d}}dy \le c(\|u_0\|_\infty,\|v_0\|_\infty)\,\|v_0-u_0\|_1.$$

Thus the resulting bound of

$$\int_0^T\int_{\mathbb{R}^d}K(|v(t,y)-u(t,y)|)^{\frac{1}{d}}dy\,dt \le c(\|u_0\|_\infty,\|v_0\|_\infty)T\,\|v_0-u_0\|_1 \tag{4}$$

for finite T is way better than (3) as $\|v_0-u_0\|_1\to 0$. The only new feature in (3) is therefore the finiteness of the integral when $T=+\infty$. This kind of dispersion effect is of course intimately related to the Strichartz-like estimates obtained in [18].

Our purpose here is to improve the estimate (3) by restricting the integration to the maximal slab $(0,T)\times\mathbb{R}^d$ in which both u and v are shock-free. Mind that this assumption is more restrictive than that in weak-strong analysis. It is however meaningful since Lipschitz initial data yield shock-free solutions on some non-trivial slab. According to Theorem 6.1.1 of [3], we have

$$-\frac{1}{T}=\min\left\{0,\inf_y \operatorname{div}(\vec{f}'\circ v_0),\inf_y \operatorname{div}(\vec{f}'\circ u_0)\right\}.$$

In order that it be valuable, our improved estimate will not be a naive consequence of the L^1-contraction property: we shall be able to replace the factor $\|v_0-u_0\|_1^{1/d}$ in the right-hand side of (3) by $\|v_0-u_0\|_1^p$, where the exponent $p=n/d$ is now bigger than 1,

$$\int_0^\infty\int_{\mathbb{R}^d} K(|v(t,y)-u(t,y)|)^{\frac{1}{d}}\,dy\,dt\le c(\|u_0\|_\infty,\|v_0\|_\infty)\,\|v_0-u_0\|_1^{\frac{n}{d}}, \tag{5}$$

Likewise, (5) does not follow from (2), because of two reasons. On the one hand the integrand depends upon the amount of nonlinearity of the flux $\vec{f}$; a fully non-degenerate flux yields an estimate in an L^q-space with $q>2$, while (5) is trivial if $\det(\vec{f}'',\dots,\vec{f}^{(n)})\equiv 0$, because then $K\equiv 0$. On the other hand, the estimate is valid when we integrate in t up to the blow-up time T of one solution ; on the contrary, the integral in (2) becomes infinite in general as $t\to T$, because $\|\nabla_y u\|_\infty$ behaves like $(T-t)^{-1}$ generically.

We point out that when taking $v=\tau_h u$ for some small vector h, the estimate (4) is sharp once shocks develop. This is because the left-hand side is proportional to $K([u])^{1/d}|h|$, where $[u]:=u_{\rm right}-u_{\rm left}$ is the jump of u across the shock surface, while $\|\tau_h u_0-u_0\|_1$ in the right-hand side behaves as $|h|\cdot TV(u_0)$. Thus both sides are of order $|h|$ as $h\to 0$. On the contrary (3) writes as the tautology $|h|=O(|h|^{1/d})$. When u is shock-free over $(0,T)\times\mathbb{R}^d$, (5) gives the better estimates.

$$\int_0^T\int_{\mathbb{R}^d} K(|u(t,y+h)-u(t,y)|)^{\frac{1}{d}}\,dy\,dt\le c(\|u_0\|_\infty)\,(|h|\cdot TV(u_0))^{\frac{n}{d}}. \tag{6}$$

We warn that (6) would be false if we integrated beyond the shock formation, since $|h|$ is not an $O(|h|^{n/d})$ as $h\to 0$. The precise statement is given in Theorem 3.2 below.

Inequality (6) is a valuable information about how strong can be the development of a singularity as $t\to T-0$. Since shock-free solutions are reversible, it also tells us something about the "rarefaction waves" ; by this we mean those shock-free solutions whose initial data is $BV\cap L^\infty(\mathbb{R}^d)$, without being continuous. These data are such that the measure $\operatorname{div}(\vec{f}'\circ u_0)$ is bounded below by some negative constant.

Our strategy is a mix between that of our work [18] with L. Silvestre, and that of F. Golse's work [7, 8] in space dimension $d=1$ (improved later on by Golse & Perthame [9]). Because of the multi-dimensional context, we employ the technique of Compensated Integrability (CI), which we developed in [14, 15] and others papers. This tool is slightly less powerful than Compensated Compactness (CC), and our estimate of

$$\int_0^T \int_{\mathbb{R}^d} K(|u(t,y+h)-u(t,y)|)^{\frac{1}{d}} dy\,dt$$

depends upon the regularity of the data (bounded variation in (6)). In one space dimension, where CC is available, Golse could assume only $u_0 \in L^\infty(\mathbb{R})$, so that his result expresses a regularizing effect. Notice that his right-hand side $c(\|u_0\|_\infty, \|v_0\|_\infty)\,|h|$ is weaker than ours $c(\|u_0\|_\infty, \|v_0\|_\infty)\,(|h| \cdot TV(u_0))^2$ as $h \to 0$. Again, this is imposed by its validity over $(0,+\infty) \times \mathbb{R}$, thus beyond the shock formation, where it becomes sharp while ours fail.

Plan of the paper. Section 2 contains the definition of shock-free solutions and of the function K involved in Estimate (6). Section 3 presents the main theorem 3.1 and its consequences. Eventually we discuss the accuracy of our estimate in space dimension $d = 1$. Section 4 presents the remaining steps of the proof, starting with the construction of a Div-BV symmetric positive semi-definite tensor, and then applying CI.

2 Definitions and results

We begin by giving a quantitative notion of non-degeneracy. Then we explain what are shock-free solutions. Eventually we state our main results.

2.1 Non-degeneracy and symmetric matrices

Each Lipschitz function $\eta(s)$ of a real variable can be viewed as an entropy of the conservation law, with entropy flux $\vec{q}$ given by $\vec{q}\,' = \eta' \vec{f}\,'$. We are specially concerned with the entropies f_i, the coordinates of $\vec{f}$! Denoting $\vec{q}_i$ the corresponding fluxes, we have

$$q_{ij}(s) = \int_0^s f_i'(\xi) f_j'(\xi)\, d\xi.$$

We point out that $q_{ij} = q_{ji}$ and thus the $n \times n$ matrix

$$\mathbb{A}(s) := \begin{pmatrix} s & f_1(s) & \dots & f_d(s) \\ f_1(s) & & \vdots & \\ \vdots & \dots & q_{ij}(s) & \dots \\ f_d(s) & & \vdots & \end{pmatrix}$$

is symmetric.

If $s \leq \sigma$, then

$$\mathbb{A}(\sigma;s) := \mathbb{A}(\sigma) - \mathbb{A}(s) = \int_s^\sigma \begin{pmatrix} 1 \\ \vec{f}'(\xi) \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \vec{f}'(\xi) \end{pmatrix} d\xi \tag{7}$$

is positive semi-definite. Following [12], we say that the conservation law is *non-degenerate* if there does not exist a non-trivial interval (s,σ) on which $(1,\vec{f}')$ remains in some hyperplane. This amounts to saying that $\mathbb{A}(\sigma;s)$ is positive definite whenever $s < \sigma$.

Let us assume that (1) is non-degenerate. Then $s < \sigma$ implies $\det \mathbb{A}(\sigma;s) > 0$. Since this quantity is continuous in both arguments, we may define for every bounded interval $I \subset \mathbb{R}$ and any increment $\alpha > 0$,

$$K_I(\alpha) = \min\{\det(\mathbb{A}(s+\alpha) - \mathbb{A}(s)) \,|\, (s,s+\alpha) \subset I\}.$$

The function $\alpha \mapsto K(\alpha) \in (0,+\infty)$ is non-decreasing. Because of the concavity of $\det^{1/n}$ over $\mathbf{Sym}_n^+$, it satisfies

$$K_I(\alpha+\beta)^{\frac{1}{n}} \geq K_I(\alpha)^{\frac{1}{n}} + K_I(\beta)^{\frac{1}{n}}.$$

The paradigm of a fully non-degenerate conservation law is the multi-D Burgers equation:

$$\partial_t u + \partial_1 \frac{u^2}{2} + \cdots + \partial_d \frac{u^n}{n} = 0, \tag{8}$$

for which

$$K_I(\alpha) = H_n \alpha^{n^2}, \qquad H_n := \det \left(\frac{1}{i+j-1} \right)_{1 \leq i,j \leq n}.$$

For more general fluxes, Andreiv's formula

$$\det \mathbb{A}(\sigma;s) = \frac{1}{n!} \int_{[s,\sigma]^n} \begin{vmatrix} 1 & \cdots & 1 \\ \vec{f}'(\xi_0) & \cdots & \vec{f}'(\xi_d) \end{vmatrix}^2 d\xi_0 \cdots d\xi_d$$

implies that $K_I(\alpha)$ is always an $O\left(\alpha^{n^2}\right)$. For a fully non-degenerate flux, we have actually

$$\det(\mathbb{A}(s+\alpha) - \mathbb{A}(s)) \overset{\alpha \to 0+}{\sim} H_n \left(\det(\vec{f}''(s), \ldots, \vec{f^{(n)}}(s)) \right)^2 \alpha^{n^2}.$$

2.2 Shock-free solutions

Let $u_0 \in L^\infty(\mathbb{R}^d)$ be an initial data, and u be the corresponding entropy solution of the Cauchy problem for (1). Its characterization in terms of a kinetic equation has been found by P.-L. Lions & coll. [12] ; see also Theorem 3.21 in [13]. To describe it, we need the step function $\chi(\xi;s)$, which vanishes for $\xi(s-\xi)<0$, and equals $\mathrm{sgn}(\xi)=\mathrm{sgn}(s)$ otherwise. In other terms,

$$\chi(\xi;s)=\frac{\partial}{\partial\xi}\left(\frac{1}{2}(|\xi|-|s-\xi|)\right).$$

Let us define an auxiliary function $h(t,y,,\xi):=\chi(\xi;u(t,y))$. Then h satifies the transport equation

$$\partial_t h+\vec{f}'(\xi)\cdot\nabla_y h=\partial_\xi m, \tag{9}$$

where $m(\cdot;\xi)$ is the non-negative measure which occurs in Kružkov's entropy inequalities:

$$\partial_t|u-\xi|+\mathrm{div}_y[(\mathrm{sgn}(u-\xi))(\vec{f}(u)-\vec{f}(\xi))]=-2m(\xi).$$

We notice that if u_0 takes values in an interval I, then so does u, and thus $m(\xi)\equiv 0$ for $\xi\notin I$.

Definition 2.1 *We say that the solution u of (1) is shock-free in an open domain U if* $m(\xi)|_U\equiv 0$ *for every* $\xi\in\mathbb{R}$.

This amounts to saying that u satisfies every entropy identity (just multiply by $\eta'(\xi)$ and integrate with respect to the kinetic variable))

$$\partial_t\eta(u)+\mathrm{div}\,\vec{q}(u)=0,\qquad \vec{q}\,'=\eta'\vec{f}\,' \tag{10}$$

in U.

A specific property of shock-free solutions is

Proposition 2.1 *Let* $(\eta,\vec{q})$ *be an entropy-flux pair of (1), and let* u,v *be two bounded shock-free solutions in an open domain* U *of* $\mathbb{R}_+\times\mathbb{R}^d$*. Then we have*

$$\partial_t[(\mathrm{sgn}(v-u))(\eta(v)-\eta(u))]+\mathrm{div}_y[(\mathrm{sgn}(v-u))(\vec{q}(v)-\vec{q}(u))]=0 \tag{11}$$

in U.

Proof

Given $\xi \in \mathbb{R}$, the flux of the entropy $s \mapsto (\mathrm{sgn}(s-\xi))(\eta(s)-\eta(\xi))$ is the vector field $s \mapsto (\mathrm{sgn}(s-\xi))(\vec{q}(s)-\vec{q}(\xi))$. Since u and v are shock-free, we thus have the entropy identities

$$\begin{aligned}\partial_t[(\mathrm{sgn}(v-\xi))(\eta(v)-\eta(\xi))]+\mathrm{div}_y[(\mathrm{sgn}(v-\xi))(\vec{q}(v)-\vec{q}(\xi))] &= 0\\ \partial_t[(\mathrm{sgn}(u-\xi))(\eta(u)-\eta(\xi))]+\mathrm{div}_y[(\mathrm{sgn}(u-\xi))(\vec{q}(u)-\vec{q}(\xi))] &= 0.\end{aligned}$$

We conclude by using Kružkov's trick, known as the *doubling of variables* trick. Having started from equalities, we receive an equality, instead of an inequality.

■

Integrating in space and time, we infer a conservation property:

Corollary 2.1 *Let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^1-function, and u_0, v_0 be bounded initial data such that $v_0 - u_0 \in L^1(\mathbb{R})$. Assume that the corresponding solutions u, v are shock-free over $(0,T)\times\mathbb{R}^d$. Then for all $t \in (0,T)$, we have*

$$\int_{\mathbb{R}^d}[(\mathrm{sgn}(v-u))(\eta(v)-\eta(u))](t,y)\,dy = \int_{\mathbb{R}^d}[(\mathrm{sgn}(v_0-u_0))(\eta(v_0)-\eta(u_0))](y)\,dy.$$

Remark 2.1 *Our starting point is somewhat different than in C. Dafermos' analysis [4]. Instead of focusing on the continuity of the solution, we are interested in the full list of entropy identities (10). In one space dimension, Dafermos proved that the former implies the latter, by a clever examination of characteristics.*

3 Statements and discussion

Our most general result writes

Theorem 3.1 *Let $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ be given initial data, such that $v_0 - u_0 \in L^1(\mathbb{R}^d)$. Denote*

$$I = [\min\{\inf u_0, \inf v_0\}, \max\{\sup u_0, \sup v_0\}].$$

Let us assume that the corresponding solutions u, v of (1) are shock-free over $(0,T)\times\mathbb{R}^d$. Then we have an inequality

$$\int_0^T\int_{\mathbb{R}^d} K_I(|v(t,y)-u(t,y)|)^{\frac{1}{d}}\,dy\,dt \leq c\left(|\vec{f}|_{\mathrm{Lip}(I)}\right)\|v_0-u_0\|_1^{\frac{n}{d}}. \tag{12}$$

We may take

$$c\left(|\vec{f}|_{\mathrm{Lip}(I)}\right) = c_d\left(\prod_{j=1}^{d} |f_j|_{\mathrm{Lip}(I)}\right)^{\frac{1}{d}}$$

for some absolute constant c_d.

When choosing $v = \tau_h u$, this yields

Corollary 3.1 *Let* $u_0 \in L^\infty(\mathbb{R}^d)$ *be a given initial data, and* $h \in \mathbb{R}^d$ *be such that* $u_0(\cdot + h) - u_0 \in L^1(\mathbb{R}^d)$. *Denote*

$$I = [\inf u_0, \sup u_0].$$

Let us assume that the corresponding solution u of (1) is shock-free over $(0,T) \times \mathbb{R}^d$. *Then we have an inequality*

$$\int_0^T \int_{\mathbb{R}^d} K_I(|u(t,y+h) - u(t,y)|)^{\frac{1}{d}} dy\, dt \leq c\left(|\vec{f}|_{\mathrm{Lip}(I)}\right) \omega_1(h;u_0)^{\frac{n}{d}}, \tag{13}$$

where $\omega_1(h;u_0) := \|\tau_h u_0 - u_0\|_1$.

The best regularity estimate is thus

Theorem 3.2 *Let* $u_0 \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ *be a given initial data, and*

$$I = [\inf u_0, \sup u_0].$$

Let us assume that the corresponding solution u of (1) is shock-free over $(0,T) \times \mathbb{R}^d$. *Then we have an inequality*

$$\int_0^T \int_{\mathbb{R}^d} K_I(|u(t,y+h) - u(t,y)|)^{\frac{1}{d}} dy\, dt \leq c\left(|\vec{f}|_{\mathrm{Lip}(I)}\right) (|h| \cdot TV(u_0))^{\frac{n}{d}}. \tag{14}$$

Likewise, we may choose $v(t,y) = u(t+\tau,y)$. The conservation law tells us that $|\partial_t u| \leq |\vec{f}|_{\mathrm{Lip}(I)} |\nabla_y u|$, so that $\|u(\tau) - u_0\|_1 \leq |\vec{f}|_{\mathrm{Lip}(I)} \tau \cdot TV(u_0)$:

Theorem 3.3 *Under the same assumptions as in Theorem 3.2, and for* $\tau \in (0,T)$, *we have an inequality*

$$\int_0^{T-\tau} \int_{\mathbb{R}^d} K_I(|u(t+\tau,y) - u(t,y)|)^{\frac{1}{d}} dy\, dt \leq c_1\left(|\vec{f}|_{\mathrm{Lip}(I)}\right) (\tau \cdot TV(u_0))^{\frac{n}{d}}. \tag{15}$$

For a fully non-degenerate flux, where $K(s) \sim s^{n^2}$, (14,15) mean together that the restriction of u to $(0,T)\times\mathbb{R}^d$ belongs to the homogeneous Besov space $\dot{B}_\infty^{1/n,n^2/d}$. We point out that this property is not implied, even locally, by $u \in BV(\mathbb{R}_+\times\mathbb{R}^d)$ and embeddings, because of

$$\frac{d}{n^2}-\frac{1}{n^2}=\frac{d-1}{n^2}<1-\frac{1}{n}=\frac{d}{n}.$$

Notice also that the one-dimensional case $u \in \dot{B}_\infty^{1/2,4}((0,T)\times\mathbb{R})$ is better than the optimal result $B_\infty^{1/3,3}$ (see [9, 5]), of course because we avoid the shocks.

3.1 Accuracy of Theorem 3.2 in dimension $d=1$

In one space dimension, Estimate (14) turns out to be sharp in the following sense. Consider a genuinely non-linear conservation law, say Burgers,

$$\partial_t u + \partial_y \frac{u^2}{2} = 0.$$

Then (14) writes

$$\int_0^T \int_{\mathbb{R}^d} |u(t,y+h)-u(t,y)|^4 dy\,dt \le c\|u_0\|_\infty (h\cdot TV(u_0))^2.$$

Rarefaction wave. Assume first that $u_0(y)=u_\pm$ where $u_- < u_+$ are constants. Then the solution, a rarefaction wave, is shock-free over $\mathbb{R}_+\times\mathbb{R}$, and the estimate above is valid for $T=+\infty$. Because $u\equiv u_-$ for $y<tu_-$ and $u\equiv u_+$ for $y>tu_+$, the difference $u(t,y+h)-u(t,y)$ equals $[u]:=u_+-u_-$ on a small triangle, whose basis is the segment $\{0\}\times(-h,0)$ and upper vertex is at $t=h/[u]$. Therefore

$$\int_0^T \int_{\mathbb{R}^d} |u(t,y+h)-u(t,y)|^4 dy\,dt \ge \frac{[u]^3 h^2}{2},$$

and both sides of the estimate have the same order h^2 as $h\to 0$.

Of course, this can be reversed by $\hat{u}(t,y)=u(T-t,-y)$, to transform the initial singularity into a final one, a shock formation.

Generic singularities. The above example is non-generic. What happens usually is that $u(t,\cdot)$ is locally decreasing, with a non-degenerate inflexion point. As t increases, the derivative $\partial_y u$ at the inflexion point decreases and tends to $-\infty$ as $t \to T-0$. At time T, the solution is still continuous, but experiences a cubic root singularity in the space variable. Reversing time and space to make it an initial singularity, we are lead to consider the example of a smooth bounded increasing initial data $u_0(y)$ which coincides with $y^{1/3}$ in some neighbouhood of the origin. Say that $u_0 \equiv u_\pm$ takes constant values away from a compact interval.

The corresponding solution of the Burgers equation is globally shock-free. Thanks to the finite velocity of waves, it coincides locally with a self-similar solution:

$$u(t,y) = \sqrt{t}\, U\left(\frac{y}{t^{3/2}}\right), \qquad U(z)^3 + U(z) = z, \qquad \text{say over } (0,1)\times(-2,2).$$

Let us split the integral

$$\int_0^1 \int_{\mathbb{R}} |u(t,y+h) - u(t,y)|^4 dy\,dt = \int_0^1 \int_{-1}^{+1} (\cdots) + \int_0^1 \int_{|y|>1} (\cdots).$$

Since u is smooth away from $(0,0)$, and is constant $\equiv u_\pm$ for $|y| > L$ for a suitable $L < +\infty$, the second integral above is an $O(h^4)$. As for the first one, it equals

$$\begin{aligned}
\int_0^1 \int_{-1}^{+1} |u(t,y+h) - u(t,y)|^4 dy\,dt &= \int_0^1 \int_{-1}^{+1} t^2 \left| U\left(\frac{y+h}{t^{3/2}}\right) - U\left(\frac{y}{t^{3/2}}\right)\right|^4 dy\,dt \\
&= \frac{2h^3}{3} \iint_{b-a>h\max\{1,|a|\}} \left(\frac{U(b)-U(a)}{b-a}\right)^4 da\,db \\
&= \frac{2h^3}{3} \iint_{b-a>h\max\{1,|a|\}} \frac{da\,db}{(1+U(a)^2+U(a)U(b)+U(b)^2)^4},
\end{aligned}$$

where we first made the change of variables $(a,b) = (yt^{-3/2}, (y+h)t^{-3/2})$, and then used the cubic equation that U satisfies. Since

$$1 + U(a)^2 + U(a)U(b) + U(b)^2 \geq 1 + \frac{1}{2}(U(a)^2 + U(b)^2) \geq C(1 + r^{2/3}), \qquad r := \sqrt{a^2+b^2}$$

for some constant $C > 0$, and because

$$\int_{\mathbb{R}^2} \frac{dx}{1+r^{8/3}} < +\infty,$$

the quantity $(1+U(a)^2+U(a)U(b)+U(b)^2)^{-4}$ is integrable over $\mathbb{R}^2$. Thus

$$\int_0^1 \int_{-1}^{+1} |u(t,y+h)-u(t,y)|^4 dy\,dt \sim \kappa h^3,$$

where (mind that U is odd)

$$\kappa := \frac{1}{3} \iint_{\mathbb{R}^2} \frac{da\,db}{(1+U(a)^2+U(a)U(b)+U(b)^2)^4} < \infty.$$

Summing up the calculations above, we deduce that

$$\int_0^1 \int_{\mathbb{R}} |u(t,y+h)-u(t,y)|^4 dy\,dt \sim \kappa h^3, \qquad \text{as } h \to 0.$$

Conclusion. Theorem 3.2 is sharp in space dimension 1, but only for non-generic initial data. For the Burgers equation and generic data, there is a gap between the order h^2 of our upper bound, and the equivalent $\mathrm{cst} \cdot h^3$ of the left-hand side of (14). We leave open the question whether (14) is sharp in higher dimension, or not. That is, whether there exist shock-free solutions for which the left-hand side is bounded below by a constant times $|h|^{n/d}$ as $h \to 0$, at least in some directions. An answer seems to need a good understanding of the worst singularities that are consistent with the conservation law.

4 Proof of Theorem 3.1

4.1 A positive Div-BV tensor

From the solutions u and v, we build the symmetric tensor

$$A := (\mathrm{sgn}(v-u))(\mathbb{A} \circ v - \mathbb{A} \circ u).$$

Thanks to formula (7), A is positive semi-definite. We have by definition

$$\det A(t,y) \geq K_I(|v(t,y)-u(t,y)|).$$

Since u and v are shock-free over $(0,T) \times \mathbb{R}^d$, Proposition 2.1 tells us that A is Divergence-free.

Finally we define a tensor B over $\mathbb{R}^{1+d}$ by extension:

$$B(t,y) = \begin{cases} A(t,y) & \text{if} \quad t \in (0,T), \\ 0_n & \text{if not.} \end{cases}$$

Let us recall that the Divergence of a symmetric tensor $x \mapsto S$ is defined row-wise:

$$(\mathrm{Div}\, S)_i := \sum_j \frac{\partial s_{ij}}{\partial s_j},$$

where the derivatives are understood in the distributional sense. The tensor S is said Div-BV over U if its entries *and* the coordinates of $\mathrm{Div}\, S$ are bounded measures over U. It is Div-free if $\mathrm{Div}\, S \equiv 0$.

Taking $x = (t, y)$, we have

$$\begin{aligned} \mathrm{Div}\, B &= \begin{pmatrix} |v_0 - u_0| \\ (\mathrm{sgn}(v_0 - u_0))(\vec{f}(v_0) - \vec{f}(u_0)) \end{pmatrix} \mathcal{L}^d|_{t=0} \\ &\quad - \begin{pmatrix} |v(T) - u(T)| \\ (\mathrm{sgn}(v(T) - u(T)))(\vec{f}(v(T)) - \vec{f}(u(T))) \end{pmatrix} \mathcal{L}^d|_{t=T} \end{aligned}$$

where $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure. Therefore B is Div-BV and we have

$$\|(\mathrm{Div}\, B)_0\|_{\mathcal{M}} = \|v_0 - u_0\|_1 + \|v(T) - u(T)\|_1 \leq 2\|v_0 - u_0\|_1.$$

Likewise, for $1 \leq j \leq d$,

$$\begin{aligned} \|(\mathrm{Div}\, B)_j\|_{\mathcal{M}} &= \|f_j \circ v_0 - f_j \circ u_0\|_1 + \|f_j \circ v(T) - f_j \circ u(T)\|_1 \\ &\leq |f_j|_{\mathrm{Lip}(I)}(\|v_0 - u_0\|_1 + \|v(T) - u(T)\|_1) \leq 2|f_j|_{\mathrm{Lip}(I)}\|v_0 - u_0\|_1. \end{aligned}$$

4.2 Applying Compensated Integrabilty

Let us recall a version of CI, taken from [16] (Theorem 2.1):

Theorem 4.1 *Let S be a symmetric positive semi-definite, Div-BV tensor over $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} (\det S)^{\frac{1}{n-1}} dx \leq c_n \left(\prod_{i=1}^n \|(\mathrm{Div}\, S)_i\|_{\mathcal{M}} \right)^{\frac{1}{n-1}}. \tag{16}$$

Applying (16) to our tensor B, and using the estimates established in the previous paragraph, we infer

$$\int_0^T \int_{\mathbb{R}^d} K(|v(t,y) - u(t,y)|)^{\frac{1}{d}} dy\, dt \leq 2c_n \left(2 \prod_{i=1}^d |f_j|_{\mathrm{Lip}(I)} \right)^{\frac{1}{d}} \|v_0 - u_0\|_1^{\frac{n}{d}},$$

which is (12). This ends the proof.

References

[1] G.-Q. Chen, Y.-G. Liu. The study on application way of the Compensated Compactness (in Chinese). *Chinese Sci. Bull.*, **34** (1989), pp 15–19.

[2] C. Dafermos. Entropy and the stability of classical solutions of hyperbolic systems of conservation laws. In *Recent mathematical methods in nonlinear wave propagation (Montecatini Terme, 1994)*, vol. 1640 of *Lecture Notes in Math.*, Springer-Verlag, Berlin (1996), pp 48–69.

[3] C. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin (2000).

[4] C. Dafermos. Continuous solutions for balance laws. *Ricerche di Matematica*, **55** (2006), pp 79–91.

[5] C. De Lellis, M. Westdickenberg. On the optimality of velocity averaging lemmas. *Ann. Inst. H. Poincaré Anal. non Linéaire*, **20** (2003), pp 1075–1085.

[6] R. DiPerna. Uniqueness of solutions to hyperbolic conservation laws. *Indiana Univ. Math. J.*, **28** (1979), pp 137–188.

[7] F. Golse. Nonlinear regularizing effect for conservation laws. *Nonlinear Hyperbolic Problems; theory, numerics and applications*, 73–92, Proc. Sympos. App. Math., **67**, Part 1, *Amer. Math. Soc., Providence, RI* (2009).

[8] F. Golse. Nonlinear regularizing effect for hyperbolic partial differential equations. *International Congress on Mathematical Physics*, 433–437, *World Sci. Publ., Hackensack, NJ* (2010).

[9] F. Golse, B. Perthame. Optimal regularizing effect for scalar conservation laws. *Rev. Mat. Iberoam.*, **29** (2013), pp 1477–1504.

[10] S. Kružkov. First order quasilinear equations with several independent variables (in Russian). *Mat. Sbornik (N.S.)*, **81 (123)** (1970), pp 228–255.

[11] N. Leger. L^2-stability estimates for shock solutions of scalar conservation laws using the relative entropy method. *Arch Rat. Mech. & Anal.*, **199** (2011), pp 761–778.

[12] P.-L. Lions, B. Perthame, E. Tadmor. A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Amer. Math. Soc.*, **7** (1994), pp 169–191.

[13] B. Perthame. *Kinetic formulation of conservation laws*, Oxford lecture series in Math. & its Appl. **21**. Oxford (2002).

[14] D. Serre. Divergence-free positive symmetric tensors and fluid dynamics. *Annales de l'Institut Henri Poincaré (analyse non linéaire)*, **35** (2018), pp 1209–1234.

[15] D. Serre. Compensated integrability. Applications to the Vlasov–Poisson equation and other models of mathematical physics. *J. Math. Pures & Appl.*, **127** (2019), pp 67–88.

[16] D. Serre. Mixed determinants and new *a priori* estimates in gas dynamics. *Quart. Appl. Math*, to appear (2023).

[17] D. Serre. Compensated Integrability and conservation laws. In: Castro, M. J., Morales de Luna, T.; Muñoz, M. L., Parés, C. (eds) *Int. Conf. Hyperbolic Problems: Theory, Numerics and Applications (Malaga 2022)*. SEMA SIMAI Springer Series (to appear).

[18] D. Serre, L. Silvestre. Multi-dimensional scalar conservation laws with unbounded initial data: well-posedness and dispersive estimates. *Arch Rat. Mech. & Anal.*, **234** (2019), pp 1391–1411.

[19] L. Tartar. Compensated compactness and applications to partial differential equations. In: R. J. Knops ed., *Nonlinear Analysis and Mechanics*, Heriott-Watt Symposium IV, Pitman Res. Notes in Math. **39**, Pitman, Boston (1979), pp 136–212.