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Improvement and generalization of Papasoglu's lemma

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Abstract

We improve an isoperimetric inequality due to Panos Papasoglu. We also generalize this inequality to the Finsler case by proving an optimal Finsler version of the Besicovitch's lemma which holds for any notion of Finsler volume.

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1 Introduction

In [Pap09, Prop. 2.3], Panos Papasoglu shows the

Lemma 1.1. *Let $(\mathbb{S}^2, g)$ be a Riemannian 2-sphere and denote by $\mathcal{A}$ its Riemannian area. Then for any $\varepsilon > 0$ there exists a closed curve γ dividing $(\mathbb{S}^2, g)$ into two disks D_1 and D_2 of area at least $\frac{\mathcal{A}(\mathbb{S}^2,g)}{4}$ and whose length satisfies*

$$\text{length}(\gamma) \leq 2\sqrt{3\mathcal{A}(\mathbb{S}^2, g)} + \varepsilon.$$

This lemma has several deep consequences in metric geometry: using it, Papasoglu gives estimates of the Cheeger constant of surfaces, Liokumovich, Nabutovsky and Rotman use it to answer a question asked by Frankel, Katz and Gromov in [LNR15] whereas Balacheff uses it to estimate 2-spheres width in [Bal15]. In this article, we give two different ways to improve Papasoglu's estimate. First by a factor of $\sqrt{2}$ by using directly the coarea formula, then by a factor of $2\sqrt{\frac{2}{\pi}}$ by using Pu's inequality (an argument suggested by an anonymous reviewer and already used by Gromov to give the filling radius of $\mathbb{S}^1$ in the simply connected case). It gives automatically better estimates: for instance, in [Lio14], the constants 52 and 26, given by Liokumovich in the abstract, could be divided by $2\sqrt{\frac{2}{\pi}}$, thus

Corollary 1.2. *There exists a Morse function $f : M \to \mathbb{R}$, which is constant on each connected component of a Riemannian 2-sphere with $k \geq 0$ holes M and has fibers of length no more than $26\sqrt{\frac{\pi}{2}\mathcal{A}(M)}$.*

Corollary 1.3. *On every 2-sphere there exists a simple closed curve of length $\leq 13\sqrt{\frac{\pi}{2}\mathcal{A}(\mathbb{S}^2)}$ subdividing the sphere into two discs of area $\geq \frac{1}{3}\mathcal{A}(\mathbb{S}^2)$.*

In his original proof, Papasoglu used the so-called Besicovitch's lemma. This lemma asserts that, given a parallelotope $P \subset \mathbb{R}^n$ endowed with a Riemannian metric g, one has

$$v(P, g) \geq \prod_{i=1}^{n} d_i,$$

where v denotes the Riemannian volume of (P, g) and the d_i denote the Riemannian distances between two opposite sides of P (see for instance [Gro01, Section 4.28]). In this article, we give a natural generalization of Besicovitch's lemma extending it to Finsler parallelotopes – that is parallelotopes continuously endowed with a norm at each of their points. As for such a manifold, there is not one good definition of volume, we prove an optimal inequality satisfied by any Finsler volume in the sense of [BBI01, §5.5.3] such as the Busemann-Hausdorff and Holmes-Thompson ones. Our proof is based on the Gromov one given in [Gro01]. We then use it in order to extend the Papasoglu lemma to Finsler 2-spheres.

Acknowledgments

I am very grateful to my internship advisor Florent Balacheff who introduced me to systolic geometry. He encouraged me to write this article and helped me during the preparation of the paper.

2 Improvements of Papasoglu's lemma

We remind the reader of the following simple version of the coarea formula:

Proposition 2.1 (Coarea formula)**.** *Let (M, g) be a Riemannian surface with boundaries and denote by $\mathcal{A}$*

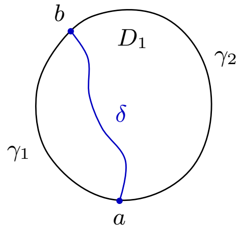

Fig. 1: δ cannot be an ε-shortcut

its Riemannian area. Let $x_0 \in M$, then $\partial B(x_0, r)$ is a smooth curve (not necessarily connected) for almost every $r > 0$ and

$$\mathcal{A}(B(x_0, R)) = \int_0^R \mathrm{length}(\partial B(x_0, r))\, \mathrm{d}r.$$

This is the simplest case of a much more general formula that can be found in [Fed96, Section 3.2]. The coarea formula gives us a way to improve Papasoglu's isoperimetric inequality:

Proposition 2.2. *Let $(\mathbb{S}^2, g)$ be a Riemannian 2-sphere and denote by $\mathcal{A}$ its Riemannian area. Then for any $\varepsilon > 0$ there exists a closed curve γ dividing $(\mathbb{S}^2, g)$ into two disks D_1 and D_2 of area at least $\frac{\mathcal{A}(\mathbb{S}^2,g)}{4}$ and whose length satisfies*

$$\mathrm{length}(\gamma) \leq \sqrt{6\mathcal{A}(\mathbb{S}^2, g)} + \varepsilon.$$

Proof. Let Γ be the set of simple closed curves dividing $(\mathbb{S}^2, g)$ into two disks of area $\geq \frac{\mathcal{A}(\mathbb{S}^2,g)}{4}$. Let $L = \inf_{\gamma \in \Gamma} \mathrm{length}(\gamma)$. Now if we fix an $\varepsilon > 0$, we can take $\gamma \in U$ such as $\mathrm{length}(\gamma) < L + \varepsilon$ and denote by D_1 and D_2 the two disks bounded by γ with $\mathcal{A}(D_1) \geq \mathcal{A}(D_2)$ (which implies $\mathcal{A}(D_1) \geq \frac{\mathcal{A}(\mathbb{S}^2,g)}{2}$).

Then γ cannot be ε-shortcut on D_1 – that is there does not exist any $\delta \subset D_1$ joining two points a and b of γ of length $\mathrm{length}(\delta) < \mathrm{length}(\gamma_1) - \varepsilon$ where $\gamma_1 \subset \gamma$ is the shortest curve between a and b on γ. In the contrary either $\delta \cup \gamma_1$ or $\delta \cup \gamma_2$ would bound a disk of area $\geq \frac{\mathcal{A}(\mathbb{S}^2,g)}{4}$ with a length $< L$ (calling $\gamma_2 = \gamma \setminus \gamma_1$), a contradiction.

Now fix $\varepsilon > 0$ and rather take $\gamma \in U$ a curve of length $\mathrm{length}(\gamma) < L + \frac{\varepsilon}{L}$ (taking ε small enough to have $\mathrm{length}(\gamma) < 2L$). On D_1 the disk of greatest area, there is not any $\frac{\varepsilon}{L}$-shortcut between two points of γ. Fix any point A of γ and denote for every $r \geq 0$ $F_r := \{m \in \overline{D_1} \mid d(A, m) = r\}$. Since $d(A, \cdot)$ is a Lipschitz continuous function, it is differentiable almost everywhere and, according to Sard's lemma, F_r is a submanifold for almost every r; we will restrict ourselves to such r. Let $u(r)$ and $v(r)$ be the two points of γ away from r from A when $r < \frac{\mathrm{length}(\gamma)}{2}$. Since F_r is a submanifold on D_1 which is a submanifold with boundary on $D_1 \cup \gamma$, there is a path δ_r of F_r connecting $u(r)$ to $v(r)$. Then, since $\frac{\varepsilon}{L}$-shortcuts do not exist,

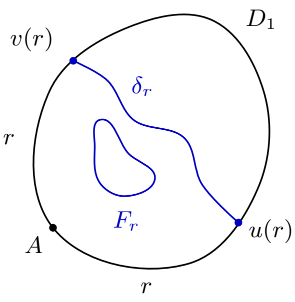

Fig. 2: The loci F_r and δ_r

$$\mathrm{length}(\delta_r) \geq \begin{cases} 2r - \frac{\varepsilon}{L} & \text{if} \quad r \leq \frac{\mathrm{length}(\gamma)}{4} \\ \mathrm{length}(\gamma) - 2r - \frac{\varepsilon}{L} & \text{if} \quad \frac{\mathrm{length}(\gamma)}{4} \leq r \leq \frac{\mathrm{length}(\gamma)}{2}. \end{cases}$$

Using the coarea formula (Proposition 2.1), we obtain:

$$\begin{aligned} \mathcal{A}(D_1) &= \int_0^{+\infty} \mathrm{length}(F_r)\, \mathrm{d}r \\ &\geq \int_0^{\frac{\mathrm{length}(\gamma)}{2}} \mathrm{length}(\delta_r)\, \mathrm{d}r \\ &\geq 2\int_0^{\frac{\mathrm{length}(\gamma)}{4}} \left(2r - \frac{\varepsilon}{L}\right) \mathrm{d}r \\ &\geq \frac{\mathrm{length}(\gamma)^2}{8} - \varepsilon. \end{aligned}$$

In addition to the fact that $\frac{3}{4}\mathcal{A}(\mathbb{S}^2) \geq \mathcal{A}(D_1)$ and that this inequality holds for any $\varepsilon > 0$ short enough, we can conclude. □

As it turns out, we can obtain an even better improvement of Papasoglu's isoperimetric inequality based on Pu's systolic inequality. This improvement was suggested by an anonymous reviewer and is discussed next.

We recall that the *systole* of a closed Riemannian manifold (M, g) is defined by:

$$\mathrm{sys}(M, g) := \inf\left\{\mathrm{length}_g(\gamma) \mid \gamma : \mathbb{S}^1 \to M \text{is a non-contractible loop}\right\}.$$

A systolic inequality is an inequality of the form $v(M, g) \geq C\mathrm{sys}(M, g)^n$ where v is the Riemannian volume of (M, g), n is the dimension of M and $C > 0$ does not depend on the metric g. Pu's systolic inequality is the following

Theorem 2.3 (Pu's systolic inequality [Pu52, Theorem 1])**.** *Let $(\mathbb{R}P^2, g)$ be a Riemannian real projective plane and denote by $\mathcal{A}$ its Riemannian area, then*

$$\mathcal{A}(\mathbb{R}P^2, g) \geq \frac{2}{\pi}\mathrm{sys}(\mathbb{R}P^2, g)^2,$$

with equality if and only if g is a Riemannian metric of constant curvature.

For more information about systolic inequalities, we refer the reader to the article survey [CK03]. Our second improvement of Paposoglu's inequality takes the following form.

Proposition 2.4. *Let $(\mathbb{S}^2, g)$ be a Riemannian 2-sphere and denote by $\mathcal{A}$ its Riemannian area. Then for any $\varepsilon > 0$ there exists a closed curve γ dividing $(\mathbb{S}^2, g)$ into two disks D_1 and D_2 of area at least $\frac{\mathcal{A}(\mathbb{S}^2,g)}{4}$ and whose length satisfies*

$$\text{length}(\gamma) \leq \sqrt{\frac{3\pi}{2}\mathcal{A}(\mathbb{S}^2, g)} + \varepsilon.$$

Proof. We will use an argument given by Gromov in [Gro83, Section 5.5.B' (e)]. Let us have the same approach as the previous proof, taking $\gamma \in \Gamma$ a curve of length $\text{length}(\gamma) < L + \varepsilon$ dividing $\mathbb{S}^2$ into two disks D_1 and D_2 with the same conditions.

Since there is no ε-shortcut, any curve joining two antipodal points of ∂D_1 is longer than $\frac{\text{length}(\gamma)}{2} - \varepsilon$. By identification of these antipodal points, D_1 gives a projective plane of systole greater than $\frac{\text{length}(\gamma)}{2} - \varepsilon$, thus, applying Pu's systolic inequality (Theorem 2.3),

$$\mathcal{A}(D_1) \geq \frac{2}{\pi}\left(\frac{\text{length}(\gamma)}{2} - \varepsilon\right)^2.$$

Since $\frac{3}{4}\mathcal{A}(\mathbb{S}^2) \geq \mathcal{A}(D_1)$, we then conclude. □

Remark 2.5. The equality case of Pu's theorem tells us about the unoptimality of this inequality. Precisely, there is no Riemannian 2-sphere $(\mathbb{S}^2, g)$ whose minimal closed curve γ satisfying Papasoglu's hypothesis has length:

$$\text{length}(\gamma) = \sqrt{\frac{3\pi}{2}\mathcal{A}(\mathbb{S}^2, g)}.$$

As a matter of fact, this would imply the equality cases $\frac{3}{4}\mathcal{A}(\mathbb{S}^2, g) = \mathcal{A}(D_1)$ and $\mathcal{A}(D_1) = \frac{2}{\pi}\left(\frac{\text{length}(\gamma)}{2}\right)^2$. By Pu's theorem, D_1 is then a hemisphere of the round sphere of radius $\frac{\text{length}(\gamma)}{2\pi}$. Let us see that D_1 is a hemisphere of 2-sphere of radius r implies that $(\mathbb{S}^2, g)$ is the round sphere of radius r. This will complete the proof because γ would be an equator which is obviously not minimal for Papasoglu's lemma.

In order to prove it, we will apply Pu's theorem to D_2, thus D_2 would be a round hemisphere of radius r. Let us see that any curve δ_2 joining two antipodal points N and S of D_2 is longer than $\frac{\text{length}(\gamma)}{2} = \pi r$. Suppose the contrary for some δ_2 in D_2 joining N and S, then, gluing this curve with any meridian δ_1 of the hemisphere D_1 joining N and S, we obtain a close simple curve $\delta = \delta_1 \cdot \delta_2$. Since meridians of a round hemisphere of radius r have length πr, $\text{length}(\delta) < \text{length}(\gamma)$. But, according to the intermediate value theorem, there exists a meridian δ_1 such as δ divides $(\mathbb{S}^2, g)$ into disks of same area, a contradiction with γ minimality.

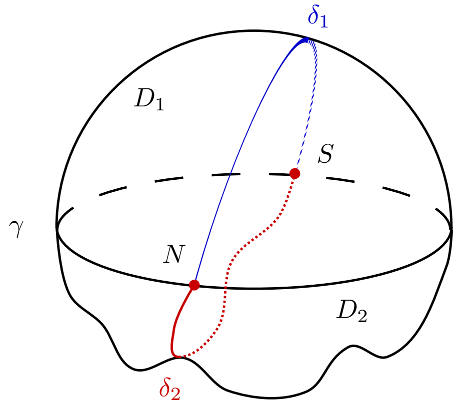

Fig. 3: δ_2 glued with a meridian δ_1

3 Besicovitch's lemma for Finsler manifolds

In this section, we extend Papasoglu's lemma to Finsler manifolds for any good notion of Finsler area. For this, we first give a natural generalization of the Besicovitch lemma.

3.1 Length metric and volume on Finsler manifolds

The manifolds used here will be closed and connected. See [BBI01] for details about the results of this section.

Recall that a *continuous Finsler metric* on a manifold M is a continuous function $\Phi : TM \to [0, +\infty[$ whose restriction to every tangent space is an asymmetric norm. Such a manifold M is said to be a *Finsler manifold* (M, Φ). If $\Phi(-v_x) = \Phi(v_x)$ for all tangent vectors, we shall say that Φ is a *reversible* continuous Finsler metric. We can then define a *length metric* d_Φ on M by:

$$\forall x, y \in M, \quad d_\Phi(x, y) = \inf_{\gamma : x \rightsquigarrow y} \text{length}_\Phi(\gamma)$$

where the infimum is taken on the piecewise-$\mathcal{C}^1$ curves $\gamma : [0, 1] \to M$ joining x to y and

$$\text{length}_\Phi(\gamma) := \int_0^1 \Phi(\gamma'(t))\, dt.$$

We will restrict ourselves to the case of reversible continuous Finsler metrics.

Contrary to the Riemannian case, there is no natural way to define a volume on Finsler manifolds. We will give two classical examples of Finsler volume. The Busemann-Hausdorff volume could be defined, for all open subsets $U \subset M$, as:

$$v_{BH}(U) := \int_U \frac{|B_g|}{|B_\Phi|} v_g,$$

where v_g is the volume associated to g which is a Riemannian auxiliary metric, for every $p \in M$, $|A|$ designates the g_p-normalized Lebesgue measure of $A \subset T_pM$

and B_g and B_Φ are unit balls of T_pM endowed with the normed metrics g_p and Φ_p respectively. This definition does not depend on g and boils down to normalizing the volume of the unit ball of each tangent space (T_pM, Φ_p).

On the other hand, the Holmes-Thompson volume is defined, for all open subsets $U \subset M$ as:

$$v_{HT}(U) := \int_U \frac{|B_\Phi^*|}{|B_g|} v_g,$$

where, for all $p \in M$ and all convex K of the Euclidean space (T_pM, g_p),

$$K^* := \{u \in T_pM \mid \forall w \in K,\ g_p(u,w) \leq 1\}$$

is the dual convex of K. Compared to the Busemann-Hausdorff volume, here we normalize the unit dual ball. In the case of a Riemannian manifold (M, g), Busemann-Hausdorff and Holmes-Thompson volumes are equal and $v_{BH} = v_{HT} = v_g$. Both Busemann-Hausdorff and Holmes-Thompson volumes are examples of Finsler volumes. We refer to [BBI01, §5.5.3] for an in-depth analysis of this general notion. Let's just underline that any Finsler volume v is *monotonous* in the sense that for all short applications between Finsler manifolds $f : (M, \Phi) \to (N, \Psi)$,

$$v(f(M), \Psi) \leq v(M, \Phi).$$

3.2 Finsler Besicovitch's Lemma

We show that we can deduce a more general statement of Besicovitch's lemma from the proof given in [Gro01, Section 4.28]:

Proposition 3.1 (Finsler Besicovitch's lemma). *Let $P \subset \mathbb{R}^n$ be a n-dimensional parallelotope endowed with a reversible continuous Finsler metric Φ. If (F_i, G_i) (with $1 \leq i \leq n$) denote its pairs of opposite faces and $d_i := d_\Phi(F_i, G_i)$, then, for any Finsler volume v,*

$$v(P, \Phi) \geq v\left(\prod_{i=1}^n [0, d_i], \|\cdot\|_\infty\right).$$

Proof. Let f be the continuous function

$$f : \left\{ \begin{array}{ccc} P & \to & \mathbb{R}^n \\ x & \mapsto & (d_\Phi(x, F_i))_{1 \leq i \leq n} \end{array} \right. .$$

Since for all points x and y in P,

$$|d_\Phi(x, F_i) - d_\Phi(y, F_i)| \leq d_\Phi(x, y) \qquad (3.1)$$

for all i, considering the maximum among i, one has that $f : (P, \Phi) \to (\mathbb{R}^n, \|\cdot\|_\infty)$ is short. Thus, proving $f(P) \supset \prod_{i=1}^n [0, d_i] =: C$ is enough to obtain the inequality.

Note that the boundary of P is mapped outside the interior of C, more precisely, writing $f = (f^1, \ldots, f^n)$, $f^i(F_i) = 0$ whereas $f^i(G_i) \subset [d_i, +\infty[$. From the definition of P, there exists an homeomorphism $h : P \to C$

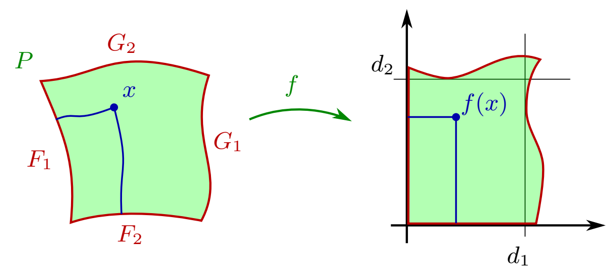

Fig. 4: Scheme of the proof

mapping each face onto a face (with the obvious choice). So $f_t = t f_{|\partial P} + (1-t) h_{|\partial P}$ defines a homotopy from $h_{|\partial P}$ to $f_{|\partial P}$ with values in $\mathbb{R}^n \setminus \mathring{C}$. If there exists $y \in \mathring{C} \setminus f(P)$, then f should be homotopic to 0 in $\mathbb{R}^n \setminus y \supset \mathbb{R}^n \setminus \mathring{C}$, so $h_{|\partial P}$ should also be homotopic to 0 in $\mathbb{R}^n \setminus y$, a contradiction ($h(P) \ni y$).

Since v is monotonous and f is short, one has the chain of inequalities

$$v(P, \Phi) \geq v\left(f(P), \|\cdot\|_\infty\right) \geq v(C, \|\cdot\|_\infty).$$

□

Remark 3.2. The proof provides us with some information about the equality case. Since $f(P) \supset C$, in order to have $v\left(f(P), \|\cdot\|_\infty\right) = v(C, \|\cdot\|_\infty)$, $f(P)$ and C must only differ from a negligible set of $\mathbb{R}^n$. Since $f : (P, \Phi) \to (\mathbb{R}^n, \|\cdot\|_\infty)$ is short, in order to have $v(P, \Phi) = v(f(P), \|\cdot\|_\infty)$, f needs to be locally isometric almost everywhere – meaning that df_x, which is defined for almost every x, has norm 1 almost everywhere. Finally, $v(P, \Phi) = v(C, \|\cdot\|_\infty)$ implies (P, Φ) to be locally isometric almost everywhere to

$$\left(\widetilde{C}, \|\cdot\|_\infty\right) \supset (C, \|\cdot\|_\infty)$$

with $\widetilde{C} \setminus C$ negligible in $\mathbb{R}^n$.

Examples 3.3.

- For $v = v_{BH}$, it gives the sharp inequality:

  $$v_{BH}(P, \Phi) \geq \frac{b_n}{2^n} \prod_{i=1}^n d_i$$

  where $b_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(n + \frac{1}{2}\right)}$ designates the volume of the standard Euclidean unit ball.

- For $v = v_{HT}$, it gives the sharp inequality:

  $$v_{HT}(P, \Phi) \geq \frac{2^n}{n! b_n} \prod_{i=1}^n d_i.$$

The symmetry of d_Φ is key to get the inequality (3.1), thus we cannot directly extend this proof to the asymmetric Finsler case. Nevertheless, in the case of the Holmes-Thompson volume, the Roger-Shepard inequality allows us to assert the following

Proposition 3.4. *Let $P \subset \mathbb{R}^n$ be a n-dimensional parallelotope endowed with an asymmetric continuous Finsler metric Φ. If (F_i, G_i) (with $1 \leq i \leq n$) denote its pairs of opposite faces, then,*

$$v_{HT}(P,\Phi) \geq \frac{n!}{(2n)!}\frac{2^n}{b_n}\prod_{i=1}^{n}\left(d_\Phi(F_i,G_i)+d_\Phi(G_i,F_i)\right).$$

Proof. Following the proof of [APBT16, Theorem 4.13], we consider the symmetrized Finsler metric Ψ defined by

$$\forall u \in TP, \quad \Psi(u) := \Phi(u) + \Phi(-u)$$

so that, for all curves γ,

$$\mathrm{length}_\Psi(\gamma) = \mathrm{length}_\Phi(\gamma) + \mathrm{length}_\Phi(\check{\gamma}),$$

where $\check{\gamma}$ designates the time-reversed curve. Thus, for all $x, y \in P$, $d_\Psi(x,y) \geq d_\Phi(x,y) + d_\Phi(y,x)$, hence

$$\forall i, \quad d_\Psi(F_i,G_i) \geq d_\Phi(F_i,G_i) + d_\Phi(G_i,F_i).$$

On the other hand, at every $p \in P$, $B_{\Psi_p} = B_{\Phi_p} - B_{\Phi_p}$, thus, applying the Rogers-Shepard inequality at every cotangent space we have that

$$v_{HT}(P,\Phi) \geq \frac{(n!)^2}{(2n)!}v_{HT}(P,\Psi).$$

The inequality then follows from Proposition 3.1 applied to (P,Ψ). □

Remark 3.5. We cannot hope such an inequality for the Busemann-Hausdorff volume in the asymmetric case. Here d_i will designate $\min(d(F_i,G_i), d(G_i,F_i))$. To see it in $\mathbb{R}^2$, let us take $P = [0,1]^2$ and let us define an asymmetric norm Φ on $\mathbb{R}^2$ by its unit ball B_Φ. Let $a = \left(-\frac{1}{2}, 0\right)$, $b = a + h\left(\frac{2}{3}, 1\right)$ and $c = a + h\left(\frac{2}{3}, -1\right)$ where $h > \frac{3}{2}$; we define B_Φ as the triangle abc. As h tends to infinity, $d_1 \sim \frac{1}{h}$, $d_2 = 1$ and $v_{BH}(P,\Phi) = \frac{3\pi}{2h^2}$, thus

$$\frac{v_{BH}(P,\Phi)}{d_1 d_2} \underset{h\to+\infty}{\longrightarrow} 0.$$

However, in the asymmetric *flat* case (that is $x \mapsto \Phi_x$ is identically equal to some norm on $\mathbb{R}^n$), we still have the weaker (sharp) inequality:

$$v_{BH}(P,\Phi) \geq \frac{b_n}{2^n}\left(\min_{1\leq i\leq n} d_i\right)^n. \tag{3.2}$$

As a matter of fact, taking $P = [0,1]^n$ without loss of generality,

$$d := \min_{1\leq i\leq n} d_i = \inf\{\alpha > 0,\ \alpha B_\Phi \cap \partial[-1,1]^n \neq \emptyset\},$$

thus for all $\alpha < d$, $\alpha B_\Phi \subset [-1,1]^n$, so $|dB_\Phi| \leq 2^n$ (where $|\cdot|$ designates the standard Lebesgue measure of $\mathbb{R}^n$) which is equivalent to (3.2).

We can also show, with some duality, the Holmes-Thompson analogous of this last inequality: for all flat metrics

$$v_{HT}(P,\Phi) \geq \frac{2^n}{n!b_n}\left(\min_{1\leq i\leq n} d_i\right)^n.$$

As a matter of fact, with the last notations, for all $\alpha > 0$,

$$\alpha B_\Phi \cap \partial[-1,1]^n \neq \emptyset \Leftrightarrow \exists i,\ \alpha B_\Phi \cap \{x,\ \langle e_i, x\rangle = \pm 1\} \neq \emptyset$$
$$\Leftrightarrow \exists i,\ e_i \notin (\alpha B_\Phi)^* \ \text{ or } \ -e_i \notin (\alpha B_\Phi)^*$$

where the e_j form the canonical base of $\mathbb{R}^n$. Thus for all $\alpha < d$, $(\alpha B_\Phi)^* \supset B_{\|\cdot\|_1}$ the convex hull of the $\pm e_j$, so $|(dB_\Phi)^*| \geq \frac{2^n}{n!}$.

3.3 Finsler Papasoglu's lemma

We can now extend the original proof of Papasoglu to Finsler 2-spheres.

Proposition 3.6. *Let $(\mathbb{S}^2, \Phi)$ be a reversible Finsler 2-sphere. Fix a Finsler volume $\mathcal{A}$ and denote by $c > 0$ the constant such that $\mathcal{A}([0,d_1]\times[0,d_2], \|\cdot\|_\infty) = cd_1d_2$ for all $d_i > 0$. Then for any $\varepsilon > 0$ there exists a closed curve γ dividing $(\mathbb{S}^2,\Phi)$ into two disks D_1 and D_2 of area at least $\frac{\mathcal{A}(\mathbb{S}^2,\Phi)}{4}$ and whose length satisfies*

$$\mathrm{length}(\gamma) \leq 2\sqrt{\frac{3}{c}\mathcal{A}(\mathbb{S}^2,\Phi)} + \varepsilon. \tag{3.3}$$

Proof. Let us have the same approach as the Riemannian proof, taking $\gamma \in \Gamma$ a curve of length

$$\mathrm{length}(\gamma) < L + \varepsilon$$

dividing $\mathbb{S}^2$ into two disk D_1 and D_2 with the same conditions.

Let us divide γ into 4 curves $\gamma = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ of the same length $\frac{\mathrm{length}(\gamma)}{4}$. Since there is no ε-shortcut, we have got that

$$d_\Phi(\alpha_1,\alpha_3), d_\Phi(\alpha_2,\alpha_4) \geq \frac{\mathrm{length}(\gamma)}{4} - \varepsilon.$$

Hence, by Besicovitch's lemma and the Example 3.3,

$$\mathcal{A}(D_1) \geq c\frac{(\mathrm{length}(\gamma) - 4\varepsilon)^2}{16}.$$

But $\mathcal{A}(D_1) \leq \frac{3}{4}\mathcal{A}(\mathbb{S}^2,\Phi)$, thus

$$\mathrm{length}(\gamma) \leq 2\sqrt{\frac{3}{c}\mathcal{A}(\mathbb{S}^2,\Phi)} + 4\varepsilon.$$

□

Examples 3.7.

- For $\mathcal{A} = v_{BH}$, one has $c = \frac{\pi}{4}$ and (3.3) becomes:

$$\text{length}(\gamma) \leq 4\sqrt{\frac{3}{\pi}\mathcal{A}(\mathbb{S}^2, \Phi)} + \varepsilon.$$

- For $\mathcal{A} = v_{HT}$, one has $c = \frac{2}{\pi}$ and (3.3) becomes:

$$\text{length}(\gamma) \leq \sqrt{6\pi\mathcal{A}(\mathbb{S}^2, \Phi)} + \varepsilon.$$

Nevertheless, the proof of Proposition 2.4 gives a better estimate in these two special cases:

Proposition 3.8. *Let* $(\mathbb{S}^2, \Phi)$ *be a reversible Finsler 2-sphere. Then for* $\mathcal{A} = v_{HT}$ *or* v_{BH} *and for any* $\varepsilon > 0$*, there exists a closed curve* γ *dividing* $(\mathbb{S}^2, \Phi)$ *into two disks* D_1 *and* D_2 *of area at least* $\frac{\mathcal{A}(\mathbb{S}^2,\Phi)}{4}$ *and whose length satisfies*

$$\text{length}(\gamma) \leq \sqrt{\frac{3\pi}{2}\mathcal{A}(\mathbb{S}^2, g)} + \varepsilon,$$

with the same hypothesis on γ *and* ε *as in the previous proposition.*

Proof. Let $(\mathbb{S}^2, \Phi)$ be a reversible Finsler 2-sphere. According to [Iva11], Pu's systolic inequality:

$$\mathcal{A}(\mathbb{R}P^2, \Phi) \geq \frac{2}{\pi}\text{sys}(\mathbb{R}P^2, \Phi)^2$$

remains true in this Finsler setting for $\mathcal{A} = v_{HT}$ (Ivanov's Theorem 3) and $\mathcal{A} = v_{BH}$ (Ivanov's Theorem 4). Thus, proof of Proposition 2.4 remains valid in this case. □

Remark 3.9. the optimality issue discussed in Remark 2.5 still applies in the Busemann-Hausdorff case, according to the optimality stated in [Iva11, Theorem 4].

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