On the K_4 group of modular curves

François Brunault

ON THE K_4 GROUP OF MODULAR CURVES

FRANÇOIS BRUNAULT

Abstract. We construct elements in the group K_4 of modular curves using the polylogarithmic complexes of weight 3 defined by Goncharov and de Jeu. The construction is uniform in the level and makes use of new modular units obtained as cross-ratios of division values of the Weierstraß $\wp$ function. These units provide explicit triangulations of the Manin 3-term relations in K_2 of modular curves, which in turn gives rise to elements in K_4. Based on numerical computations and on recent results of Weijia Wang, we conjecture that these elements are proportional to the Beilinson elements defined using the Eisenstein symbol.

1. Introduction

The motivic cohomology of algebraic varieties is a fundamental invariant which appears, for example, in the statement of Beilinson's general conjectures on special values of L-functions. However this invariant is very difficult to handle in general: no universal recipe is known to produce non-trivial elements in motivic cohomology. At the same time, finite generation results (for varieties defined over number fields, and in the right degree and twist) seem to be completely out of reach in general.

Let us describe in more detail the situation for fields. Recall that for a field F, the motivic cohomology group $H^i_{\mathcal{M}}(F, \mathbb{Q}(n))$ is isomorphic to the Adams eigenspace $K^{(n)}_{2n-i}(F)$ of Quillen's K-group $K_{2n-i}(F) \otimes \mathbb{Q}$. The groups $K_0(F)$ and $K_1(F)$ are isomorphic to $\mathbb{Z}$ and $F^\times$ respectively. The group $K_2(F)$ is described by Matsumoto's theorem, which gives generators and relations for this group:

$$K_2(F) = \frac{F^\times \otimes_{\mathbb{Z}} F^\times}{\langle x \otimes (1-x) : x \in F\backslash\{0,1\}\rangle}.$$

The class of $x \otimes y$ in $K_2(F)$ is denoted by $\{x, y\}$ and is called a Milnor symbol. The relations $\{x, 1-x\} = 0$ are called the Steinberg relations. The group $K_3(F)$ has a Milnor part, generated by symbols $\{x, y, z\}$, and an indecomposable part which is isomorphic, after tensoring with $\mathbb{Q}$, to the Bloch group of F (Suslin's theorem; see [11, Theorem 1.13]).

It turns out that the higher K-groups of F are much more difficult to deal with, as they are not described by generators and relations. However, for any weight $n \geq 1$, Goncharov has defined in [11] a polylogarithmic motivic complex $\Gamma(F; n)$ whose cohomology in degree $1 \leq i \leq n$ is expected to compute $H^i_{\mathcal{M}}(F, \mathbb{Q}(n))$. In this direction, Goncharov has constructed a map from motivic cohomology to the cohomology of $\Gamma(F; n) \otimes \mathbb{Q}$ when $n \leq 4$ (see [11] for $n = 3$ and [15] for $n = 4$).

These motivic complexes are quite explicit (at least in small weight) and can be used to construct (potential) elements in motivic cohomology. In weight $n = 3$, de Jeu [8, Theorem 5.4] constructed a map from $H^2\Gamma(F; 3) \otimes \mathbb{Q}$ to $H^2_{\mathcal{M}}(F, \mathbb{Q}(3))$ for a field F of characteristic 0. As a consequence, he was able to construct elements in $H^2_{\mathcal{M}}(E, \mathbb{Q}(3)) \cong K_4^{(3)}(E)$ for certain

Date: November 8, 2020.

elliptic curves E over $\mathbb{Q}$. He also checked that their images under Beilinson's regulator map are numerically related to $L(E,3)$, as predicted by Beilinson's conjecture.
In this article, we construct elements in K_4 of modular curves using Goncharov's motivic complex in weight 3. One key ingredient is a new construction of modular units which are solutions to the S-unit equation for the function field of a modular curve, where S is the set of cusps. Another ingredient is given by the Manin 3-term relations in K_2 of modular curves, proved in [4] and [14]. Here we make explicit the Steinberg relations underlying them, and explain how to use this to build elements in K_4.
We also devise a method to compute numerically using PARI/GP the image of our K_4 elements under Beilinson's regulator map. The approach is different from [7, 8] in that we integrate the regulator 1-form along Manin symbols, instead of integrating it against a cusp form. This enables us to check numerically Beilinson's conjecture on $L(E,3)$ for elliptic cuves E of small conductor.
Special elements in K_4 of modular curves have already been defined by Beilinson [1] through a different method, using his theory of the Eisenstein symbol. Our computations lead us to conjecture that our elements coincide (up to a simple rational factor) with the Beilinson elements. Since the regulators of the Beilinson elements are known to be related to L-values, proving this conjecture would have interesting consequences, concerning for example the Mahler measure of certain 3-variable polynomials like $(1+x)(1+y)+z$; see [6, Chapter 8] and [18].
The outline of this paper is as follows. In Section 2, we recall Mason's theorem giving a bound on the degree of solutions to the S-unit equation for curves. Section 3 introduces the modular units $u(a,b,c,d)$ as cross-ratios of division values of the Weierstraß $\wp$ function. Section 4 contains an "effective" proof of the Manin relations in $K_2(Y(N)) \otimes \mathbb{Z}[\frac{1}{6N}]$, which are then used in Sections 5 and 6 to construct the elements in $K_4(Y(N)) \otimes \mathbb{Q}$. In Sections 7 and 8, we explain how to compute numerically the regulators of these elements. Finally, we formulate in Section 9 the conjecture relating our elements and the Beilinson elements.

Acknowledgements. I would like to thank Emmanuel Lecouturier for asking me for a more natural proof of the Manin 3-term relation in K_2, and for his help in the last step of the proof of Theorem 4.1. I am also grateful to Spencer Bloch for asking me about an explicit triangulation of the 3-term relation, and to Andrea Surroca for interesting discussions around the S-unit equation. Finally, I would like to thank my colleagues for fruitful exchanges, among them Vasily Golyshev and the International GdT in Paris, Matilde Lalín, Riccardo Pengo, Jun Wang, Weijia Wang and Wadim Zudilin.

2. The S-unit equation for curves

Let X be a smooth connected projective curve over $\mathbb{C}$. Let S be a finite set of closed points of X. The S-unit equation for X is the equation $f+g=1$, where f,g are non-constant rational functions on X whose zeros and poles are contained in S. Geometrically, this amounts to find the non-constant morphisms $f\colon X\backslash S \to \mathbb{P}^1\backslash\{0,1,\infty\}$.
Solving the S-unit equation for curves has two potential applications:

(1) Prove relations in K_2 of curves;
(2) Construct elements in K_4 of curves.

Namely, each solution (f, g) to the S-unit equation provides the Steinberg relation $\{f, g\} = 0$ in the group $K_2(X \backslash S)$. Moreover, as we shall see in the case of modular curves, relations in K_2 can be used to construct elements in K_4; see Section 6.
We first recall the following bound on the degrees of the solutions to the S-unit equation, due to Mason [19, p. 222].

Theorem 2.1. *If (f, g) is a solution to the S-unit equation for X, then* $\deg(f) \leq 2g_X - 2 + |S|$, *where g_X is the genus of X.*

Corollary 2.2. *The set of solutions to the S-unit equation for X is finite.*

Proof of Corollary 2.2. By Theorem 2.1, there are only finitely many possibilities for the divisors of f and g. Moreover, if (f, g) is a solution, then g must vanish at some point $p \in S$, which implies $f(p) = 1$. This shows that for a given divisor D, there are only finitely many solutions (f, g) such that $\mathrm{div}(f) = D$. □

As I learnt from A. Javanpeykar [16], the finiteness of solutions to the S-unit equation for curves can also be proved using the De Franchis-Severi theorem for hyperbolic curves.
The proof of Corollary 2.2 above actually provides an algorithm to find all the solutions to the S-unit equation. I implemented this algorithm in Magma [3]. In the case of elliptic curves, one may view this algorithm as an extension of Mellit's technique of parallel lines [20]. Namely, the rational functions appearing in [20] have degree at most 3, while here the degree is arbitrary. Of course, looping over the possible divisors becomes impracticable when the cardinality of S or the Mason bound is large.
Regarding the S-unit equation, here are some interesting situations:

- $X = \mathbb{P}^1$ and $S = \{0, \infty\} \cup \mu_N$, where μ_N denotes the Nth roots of unity;
- $X = E$ is an elliptic curve, and S is a finite subgroup of E;
- X is the Fermat curve with projective equation $x^N + y^N = z^N$, and S is the set of points with one coordinate equal to 0;
- X is a modular curve, and S is the set of cusps of X.

In this article, we will concentrate on the case of modular curves.

3. The modular units $u(a, b, c, d)$

We denote by $\mathcal{H}$ the Poincaré upper half-plane. Let $N \geq 1$ be an integer. For any $a = (a_1, a_2) \in (\mathbb{Z}/N\mathbb{Z})^2$, $a \neq (0, 0)$, we define

$$\wp_a(\tau) = \wp\Big(\tau, \frac{a_1\tau + a_2}{N}\Big) \qquad (\tau \in \mathcal{H}),$$

where $\wp$ is the Weierstraß function. We have the transformation formula $\wp_a|_2\gamma = \wp_{a\gamma}$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where $|_2$ denotes the slash action in weight 2. In particular $\wp_a$ is a modular form of weight 2 on the principal congruence subgroup $\Gamma(N)$.
Since the Weierstraß $\wp$-function has a double pole at the origin, we also set $\wp_0 = \infty$. Note that since $\wp$ is even, we have $\wp_{-a} = \wp_a$ for every $a \in (\mathbb{Z}/N\mathbb{Z})^2$.

Definition 3.1. Let a, b, c, d be distinct elements of $(\mathbb{Z}/N\mathbb{Z})^2/\pm 1$. We define

$$u(a, b, c, d) = [\wp_a, \wp_b, \wp_c, \wp_d] = \frac{\wp_c - \wp_a}{\wp_c - \wp_b} \div \frac{\wp_d - \wp_a}{\wp_d - \wp_b}.$$

Recall that a modular unit for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a non-zero meromorphic function on $\overline{\Gamma\backslash\mathcal{H}}$ whose zeros and poles are concentrated at the cusps.

Lemma 3.2. *The function $u(a,b,c,d)$ is a modular unit for $\Gamma(N)$.*

Proof. It is a well-known fact from the theory of elliptic functions that $\wp(\tau,z) = \wp(\tau,z')$ if and only if $z' = \pm z \bmod \mathbb{Z}+\tau\mathbb{Z}$. It follows that $u(a,b,c,d)$ is holomorphic and non-vanishing on $\mathcal{H}$. The modularity of $u(a,b,c,d)$ follows from that of the $\wp_a$'s. □

Modular units of the form $(\wp_a - \wp_b)/(\wp_c - \wp_d)$ are called Weierstraß units and have been investigated in the literature [17]. Here $u(a,b,c,d)$ is a quotient of two Weierstraß units but is not a priori a Weierstraß unit.
Note the following transformation formula:

$$u(a,b,c,d)|\gamma = u(a\gamma, b\gamma, c\gamma, d\gamma) \qquad (\gamma \in \mathrm{SL}_2(\mathbb{Z})).$$

Notation 3.3. For any distinct elements a,b,c,d in $(\mathbb{Z}/N\mathbb{Z})/\pm 1$, we write

$$u_1(a,b,c,d) = u((0,a),(0,b),(0,c),(0,d)).$$

Since row vectors of the form $(0,a)$ are fixed by the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, the function $u_1(a,b,c,d)$ is a modular unit for the larger group $\Gamma_1(N)$.
The definition of $u(a,b,c,d)$ as a cross-ratio makes it clear that

$$u(a,b,c,d) + u(a,c,b,d) = 1.$$

It follows that $u(a,b,c,d)$ is a solution to the S-unit equation for the modular curve $X(N)$, where S is the set of cusps. Since a,b,c,d are arbitrary, this provides us with plenty of solutions, of the order of $N^8/64$ (taking into account that permuting a,b,c,d gives rise to 6 distinct units).
It turns out that these modular units have remarkably low degree. Here are a few facts in the case of the modular curve $X_1(N)$.

(1) For $N \geq 5$ prime, the Mason bound is equal to $2g_{X_1(N)} - 2 + |\text{cusps}| = (N^2-1)/12$.
(2) For N prime, $7 < N < 300$, $N \neq 31$, the lowest degree among the units $u_1(a,b,c,d)$ is attained for the quadruplet $(a,b,c,d) = (1,2,3,5)$.
(3) Up to composing by an homography, the unit $u_1(1,2,3,5)$ is equal to the unit F_7/F_8 studied by van Hoeij and Smith in [25]; they prove that $\deg(F_7/F_8) = [11N^2/840]$ for $N > 7$ prime, where $[\cdot]$ denotes the nearest integer.
(4) In fact, for $N > 7$ prime, the unit $u_1(1,2,3,5)$ yields the lowest known degree for a non-constant map $X_1(N) \to \mathbb{P}^1$ except for $N \in \{31, 67, 101\}$, where the lowest known degree is one less [9, Table 1].
(5) For N prime, $7 < N < 300$, the highest degree among the $u_1(a,b,c,d)$ is attained for the quadruplet $(0,1,3,4)$, the degree being $[N^2/35]$.

We now return to the general case, namely the curve $X(N)$, and explain how to express $u(a,b,c,d)$ in terms of Siegel units. For $a = (a_1,a_2) \in \mathbb{Z}^2$, $a \not\equiv (0,0) \bmod N$, consider the following infinite product

$$\gamma_a(\tau) = \prod_{n\geq 0}\Big(1 - e\big(n\tau + \frac{a_1\tau + a_2}{N}\big)\Big)\prod_{n\geq 1}\Big(1 - e\big(n\tau - \frac{a_1\tau+a_2}{N}\big)\Big) \qquad (\tau \in \mathcal{H}), \tag{1}$$

with $e(z) = \exp(2\pi i z)$. To ease notations, we write $q^\alpha = e(\alpha\tau)$ for any $\tau \in \mathcal{H}$ and $\alpha \in \mathbb{Q}$. We also denote $\zeta_N = e^{2\pi i/N}$. Yang considered in [27] the following function

$$E_a(\tau) = q^{B_2(a_1/N)/2}\gamma_a(\tau), \tag{2}$$

where $B_2(x) = x^2 - x + \frac{1}{6}$ is the Bernoulli polynomial. One can show that

$$E_{a_1+N,a_2} = -\zeta_N^{-a_2} E_{a_1,a_2}, \qquad E_{a_1,a_2+N} = E_{a_1,a_2}. \tag{3}$$

In the case $0 \leq a_1 < N$, the function E_a is, by definition, nothing else but the Siegel unit $g_{\overline{a}}$, where $\overline{a}$ is the image of a in $(\mathbb{Z}/N\mathbb{Z})^2$. However, it will be convenient to allow arbitrary values of a_1. The relation (3) gives immediately

$$E_{a_1,a_2} = (-\zeta_N^{-a_2})^{\lfloor a_1/N \rfloor} \cdot g_{\overline{a_1},\overline{a_2}}. \tag{4}$$

For any $x \in (\mathbb{Z}/N\mathbb{Z})^2$, $x \neq (0,0)$, it is known that g_x^{12N} is a modular unit on $X(N)$.

Proposition 3.4. *Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be distinct elements of $(\mathbb{Z}/N\mathbb{Z})^2/\pm 1$. We have*

$$u(\bar{a}, \bar{b}, \bar{c}, \bar{d}) = \frac{E_{c+a}E_{c-a}E_{d+b}E_{d-b}}{E_{c+b}E_{c-b}E_{d+a}E_{d-a}} = \zeta \cdot \frac{g_{\bar{c}+\bar{a}}g_{\bar{c}-\bar{a}}g_{\bar{d}+\bar{b}}g_{\bar{d}-\bar{b}}}{g_{\bar{c}+\bar{b}}g_{\bar{c}-\bar{b}}g_{\bar{d}+\bar{a}}g_{\bar{d}-\bar{a}}},$$

where a, b, c, d are arbitrary representatives of $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ in $\mathbb{Z}^2$, and the root of unity ζ is determined by (4).

Proof. By [23, Corollary I.5.6(a) and Theorem I.6.4], we have

$$\wp_a(\tau) - \wp_b(\tau) = -(2\pi i)^2 q^{b_1/N}\zeta_N^{b_2} \prod_{n \geq 1}(1-q^n)^4 \cdot \frac{\gamma_{a+b}(\tau)\gamma_{a-b}(\tau)}{\gamma_a(\tau)^2\gamma_b(\tau)^2}.$$

It follows that the Weierstraß units can be expressed as

$$\frac{\wp_a - \wp_b}{\wp_c - \wp_d} = \zeta_N^{b_2-d_2} \frac{E_{a+b}E_{a-b}}{E_a^2 E_b^2} \frac{E_c^2 E_d^2}{E_{c+d}E_{c-d}}. \tag{5}$$

The definition of $u(a,b,c,d)$ as a quotient of two Weierstraß units gives the result. □

Remark 3.5. We take this opportunity to point out a mistake in [4]: the first equation of p. 288 is off by a root of unity. This root of unity can be determined from (4) and (5).

From now on, we will consider the modular curves $Y(N)$ and $Y_1(N)$ as algebraic curves defined over $\mathbb{Q}$. We recall that the field of constants of $Y(N)$ is $\mathbb{Q}(\zeta_N)$, and that $Y(N)(\mathbb{C})$ is a disjoint union of copies of $\Gamma(N)\backslash\mathcal{H}$ indexed by $(\mathbb{Z}/N\mathbb{Z})^\times$.

Proposition 3.6. *Let a, b, c, d be distinct elements in $(\mathbb{Z}/N\mathbb{Z})^2/\pm 1$. The unit $u(a,b,c,d)$ belongs to $\mathcal{O}(Y(N))^\times$.*

Proof. The group $\mathcal{O}(Y(N))^\times$ can be identified with the group of modular units for $\Gamma(N)$ whose Fourier expansion at ∞ has coefficients in $\mathbb{Q}(\zeta_N)$. Proposition 3.4 shows that this holds for $u(a,b,c,d)$. □

Proposition 3.7. *Let a, b, c, d be distinct elements in $(\mathbb{Z}/N\mathbb{Z})/\pm 1$. The unit $u_1(a,b,c,d)$ belongs to $\mathcal{O}(Y_1(N))^\times$.*

Proof. Since $E_{0,a} = g_{0,a}$, Proposition 3.4 shows that

$$u_1(a,b,c,d) = \frac{g_{0,c+a}g_{0,c-a}g_{0,d+b}g_{0,d-b}}{g_{0,c+b}g_{0,c-b}g_{0,d+a}g_{0,d-a}}. \tag{6}$$

Since the cusp $0 \in X_1(N)$ is defined over $\mathbb{Q}$, it suffices to show that the Fourier expansion of $u_1(a,b,c,d)$ at the cusp 0 is $\mathbb{Q}$-rational. By [5, Lemma 4], we have

$$g_{0,x}\big(-\frac{1}{\tau}\big) = -ie^{\pi i\tilde{x}/N} g_{x,0}(\tau),$$

where $\tilde{x}$ is the representative of x satisfying $0 < \tilde{x} < N$. Replacing in (6) gives

$$u_1(a,b,c,d)\big(-\frac{1}{\tau}\big) = e^{\pi i\alpha/N} \frac{g_{c+a,0}g_{c-a,0}g_{d+b,0}g_{d-b,0}}{g_{c+b,0}g_{c-b,0}g_{d+a,0}g_{d-a,0}}(\tau),$$

where $\alpha \in \mathbb{Z}$ is congruent to 0 modulo N, hence $e^{\pi i\alpha/N} = \pm 1$. We conclude by noting that $g_{x,0}$ has rational Fourier coefficients. □

4. The Manin relations in K_2

We begin by recalling the classical Manin relations in the homology of modular curves. For any two cusps $\alpha \neq \beta$ in $\mathbb{P}^1(\mathbb{Q})$, the modular symbol $\{\alpha,\beta\}$ is the hyperbolic geodesic from α to β in $\mathcal{H}$. For any congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, the symbol $\{\alpha,\beta\}$ defines an element of the first homology group of $\overline{\Gamma\backslash\mathcal{H}}$ relative to the cusps. It is known that this group is generated by the Manin symbols $\xi(g) = \{g0, g\infty\}$ with $g \in \Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$. They satisfy the following relations:

$$\xi(g) + \xi(g\sigma) = 0, \qquad \xi(g) + \xi(g\tau) + \xi(g\tau^2) = 0,$$

where the matrices $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\tau = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ have order 4 and 3, respectively.

We now state the main result of this section. Consider the modular curve $Y(N)$. Recall that the Siegel units g_a with $a \in (\mathbb{Z}/N\mathbb{Z})^2$ belong to $\mathcal{O}(Y(N))^\times \otimes \mathbb{Z}[\frac{1}{6N}]$ (by convention, we put $g_{0,0} = 1$).

Theorem 4.1. *For any $a,b,c \in (\mathbb{Z}/N\mathbb{Z})^2$ such that $a+b+c = 0$, we have*

$$\{g_a, g_b\} + \{g_b, g_c\} + \{g_c, g_a\} = 0 \qquad \text{in } K_2(Y(N)) \otimes \mathbb{Z}\Big[\frac{1}{6N}\Big]. \tag{7}$$

Theorem 4.1 was previously known with $\mathbb{Q}$-coefficients; see [4] when $(N,3) = 1$, and [14] in general. The analogy with the Manin 3-term relations goes as follows: for any two row vectors x, y in $(\mathbb{Z}/N\mathbb{Z})^2$, consider the matrix $M = \begin{pmatrix} x \\ y \end{pmatrix}$ with rows x and y, and let $\rho(M) = \{g_x, g_y\}$. Then the relation (7) is equivalent to $\rho(M) + \rho(\tau M) + \rho(\tau^2 M) = 0$ with $M = \begin{pmatrix} a \\ -b \end{pmatrix}$ (here we use the relation $g_{-x} = g_x$).

We will actually need a more precise version of Theorem 4.1, where the Steinberg relations underlying (7) are made explicit. This will be a key ingredient in the construction of elements in $K_4(Y(N))$.

Notation 4.2. For any distinct elements a,b,c,d in $(\mathbb{Z}/N\mathbb{Z})^2/\pm 1$, define

$$\delta(a,b,c,d) = u(a,b,c,d) \wedge u(a,c,b,d) \in \Lambda^2 \mathcal{O}(Y(N))^\times.$$

Since $u(a,c,b,d) = 1 - u(a,b,c,d)$, the image of $\delta(a,b,c,d)$ in $K_2(Y(N))$ is trivial. If a,b,c,d are not distinct, we put $\delta(a,b,c,d) = 0$.
Theorem 4.1 will be a consequence of the following theorem.

Theorem 4.3. *Let G be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^2$, and let $a,b,c \in G$ with $a+b+c=0$. We have the following equality in $\Lambda^2 \mathcal{O}(Y(N))^\times \otimes \mathbb{Z}[\frac{1}{6N}]$:*

(8)

$$g_a \wedge g_b + g_b \wedge g_c + g_c \wedge g_a = \frac{1}{|G|} \sum_{x \in G} \delta(0, x, a-x, b+x) + \frac{1}{4|G|^2} \sum_{x,y \in G} \Big(\delta(0,a,b+2x,y) - \delta(0,b,a+2x,y) - \delta(0,a+b,b+2x,y) \Big).$$

In the case $|G|$ is odd, this simplifies to

$$g_a \wedge g_b + g_b \wedge g_c + g_c \wedge g_a = \frac{1}{|G|} \sum_{x \in G} \delta(0, x, a-x, b+x). \tag{9}$$

We call (8) and (9) *triangulations* of the Manin 3-term relation in K_2. Note that the group G is arbitrary. For example, we may take $G = \{0\} \times \mathbb{Z}/N\mathbb{Z}$ when working with the modular curve $Y_1(N)$.

Proof. Using Proposition 3.4 and expanding, we get

$$\begin{aligned} \delta(a,b,c,d) &= (g_{c+a} g_{c-a} \cdot g_{d+b} g_{d-b}) \wedge (g_{b+a} g_{b-a} \cdot g_{d+c} g_{d-c}) \\ &\quad + (g_{b+a} g_{b-a} \cdot g_{d+c} g_{d-c}) \wedge (g_{c+b} g_{c-b} \cdot g_{d+a} g_{d-a}) \\ &\quad + (g_{c+b} g_{c-b} \cdot g_{d+a} g_{d-a}) \wedge (g_{c+a} g_{c-a} \cdot g_{d+b} g_{d-b}) \\ &= \varphi(a,b,c) + \varphi(c,d,a) + \varphi(b,a,d) + \varphi(d,c,b) \end{aligned} \tag{10}$$

where we have set

$$\varphi(x,y,z) = g_{z+x} g_{z-x} \wedge g_{y+x} g_{y-x} + g_{y+x} g_{y-x} \wedge g_{z+y} g_{z-y} + g_{z+y} g_{z-y} \wedge g_{z+x} g_{z-x}.$$

We can check that $\varphi(\sigma(x,y,z)) = \varepsilon(\sigma) \varphi(x,y,z)$ for every permutation σ of (x,y,z). It follows that $\delta(a,b,c,d)$ is also antisymmetric with respect to (a,b,c,d).

Lemma 4.4. *For any $y,z \in G$, we have $\sum_{x \in G} \varphi(x,y,z) = 0$.*

Proof of Lemma 4.4. A simple computation shows that the sum simplifies to

$$\sum_{x \in G} \varphi(x,y,z) = \sum_{x \in G} g_{z+x} g_{z-x} \wedge g_{y+x} g_{y-x} = 2 \sum_{x \in G} g_{z+x} \wedge g_{y+x} + g_{z+x} \wedge g_{y-x}.$$

Note that the expression $S(a,b) = \sum_{x \in G} g_{x+a} \wedge g_{x+b}$ is antisymmetric in a and b. But changing variables $x' = x+a+b$, we have

$$S(a,b) = \sum_{x' \in G} g_{x'-b} \wedge g_{x'-a} = \sum_{x' \in G} g_{-x'-b} \wedge g_{-x'-a} = S(b,a).$$

Since S is both symmetric and antisymmetric, we conclude that $S = 0$. □

Lemma 4.5. *For any $a,b,c \in G$, we have*

$$\varphi(a,b,c) = \frac{1}{|G|} \sum_{d \in G} \delta(a,b,c,d).$$

Proof of Lemma 4.5. It follows from summing (10) over $d \in G$ and using Lemma 4.4. $\square$

Let $\psi(a,b) = g_a \wedge g_b + g_b \wedge g_c + g_c \wedge g_a$, where c is chosen so that $a+b+c=0$. Our next task is to show that $\psi(a,b)$ is a linear combination of values of φ. The definition of φ gives us

$$\varphi(x,y,z) = \psi(z+x,-y-x) + \psi(z+x,y-x) + \psi(z-x,-y+x) + \psi(z-x,y+x). \tag{11}$$

Changing variables and putting $a = z+x$ and $b = -y-x$, this becomes

$$\begin{aligned}\varphi(x,-b-x,a-x) &= \psi(a,b) + \psi(a,-b-2x) + \psi(a-2x,b+2x) + \psi(a-2x,-b)\\ &= \psi(a,b) - \psi(-b-2x,a) + \psi(b+2x,-a-b) + \psi(a-2x,-b).\end{aligned}$$

Here we used $\psi(u,v) = \psi(v,-u-v) = -\psi(v,u)$. Summing over $x \in G$, we get

$$\sum_{x\in G} \varphi(x,-b-x,a-x) = |G| \cdot \psi(a,b) - R_{-b}(a) + R_b(-a-b) + R_a(-b), \tag{12}$$

where $R_u(v) = \sum_{x\in G} \psi(u+2x,v)$.

Lemma 4.6. *For any $u,v \in G$, we have $R_u(v) = \frac{1}{4}\sum_{x\in G} \varphi(0,v,u+2x)$.*

Proof of Lemma 4.6. Taking $x=0$ in (11), we obtain

$$\varphi(0,y,z) = 2(\psi(z,y) + \psi(z,-y)) = 2(\psi(z,y) + \psi(-z,y)).$$

Specialising to $z = u+2x$, $y = v$, and summing over $x \in G$ gives

$$\begin{aligned}\sum_{x\in G} \varphi(0,v,u+2x) &= 2\sum_{x\in G} \psi(u+2x,v) + \psi(-u-2x,v)\\ &= 2\sum_{x\in G} \psi(u+2x,v) + \psi(u+2(-x-u),v) = 4R_u(v).\end{aligned}$$

$\square$

Using Lemmas 4.5 and 4.6, the equation (12) becomes

$$\begin{aligned}\psi(a,b) = \frac{1}{|G|^2} \sum_{x,y\in G} \Big(&\delta(x,-b-x,a-x,y) + \frac{1}{4}\delta(0,a,-b+2x,y)\\ &- \frac{1}{4}\delta(0,-a-b,b+2x,y) - \frac{1}{4}\delta(0,-b,a+2x,y)\Big)\end{aligned}$$

Since $\delta(\pm a,\pm b,\pm c,\pm d) = \delta(a,b,c,d)$, this can be rewritten

$$\begin{aligned}\psi(a,b) &= \frac{1}{|G|^2} \sum_{x,y\in G} \delta(x,b+x,a-x,y)\\ &\quad + \frac{1}{4|G|^2} \sum_{x,y\in G} \delta(0,a,b+2x,y) - \delta(0,a+b,b+2x,y) - \delta(0,b,a+2x,y).\end{aligned} \tag{13}$$

We now wish to simplify the first sum. We will need some facts about the (pre-)Bloch group [28, Section 2]. Let F be a field, and let $\mathbb{Z}[\mathbb{P}^1(F)]$ be the free abelian group generated by the symbols $\{x\}_2$ with $x \in \mathbb{P}^1(F)$.

Definition 4.7. The Bloch group $B_2(F)$ is the quotient of $\mathbb{Z}[\mathbb{P}^1(F)]$ by the subgroup $R_2(F)$ generated by $\{0\}_2$, $\{1\}_2$, $\{\infty\}_2$ and the 5-term relations

$$\sum_{i\in\mathbb{Z}/5\mathbb{Z}} \{[a_i, a_{i+1}, a_{i+2}, a_{i+3}]\}_2, \qquad (a_i) \in \mathbb{P}^1(F)^{\mathbb{Z}/5\mathbb{Z}},$$

where $[\cdot]$ is the cross-ratio, with the convention $\{[a,b,c,d]\}_2 = 0$ is a,b,c,d are not distinct.

Consider the Bloch–Suslin map

$$\begin{aligned} \delta_2 \colon \mathbb{Z}[\mathbb{P}^1(F)] &\to \Lambda^2 F^\times \\ \{x\}_2 &\mapsto \begin{cases} x \wedge (1-x) & \text{if } x \neq 0, 1, \infty \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{14}$$

It is known that $\delta_2(R_2(F)) = 0$, so that δ_2 induces a map $B_2(F) \to \Lambda^2 F^\times$. Now let's take F to be the field generated by the modular forms $\wp_a$, $a \in (\mathbb{Z}/N\mathbb{Z})^2$. Recall that the modular units $u(a,b,c,d)$ are defined as cross-ratios of the $\wp_a$'s. Considering the associated 5-term relations and applying the map δ_2, we get

$$\sum_{i\in\mathbb{Z}/5\mathbb{Z}} \delta(a_i, a_{i+1}, a_{i+2}, a_{i+3}) = 0$$

for any family $(a_i)_{i\in\mathbb{Z}/5\mathbb{Z}}$ of elements of $(\mathbb{Z}/N\mathbb{Z})^2/\pm 1$. In particular:

$$\begin{aligned} &\delta(x, b+x, a-x, y) + \delta(b+x, a-x, y, 0) + \delta(a-x, y, 0, x) \\ &\qquad + \delta(y, 0, x, b+x) + \delta(0, x, b+x, a-x) = 0. \end{aligned} \tag{15}$$

Lemma 4.8. *For any $\alpha, \beta, z, t \in G$, we have $\sum_{x\in G} \delta(\alpha+x, \beta+x, z, t) = 0$.*

Proof. Denote this sum by S. The change of variables $x \to -\alpha-\beta-x$ gives

$$S = \sum_{x\in G} \delta(-\beta-x, -\alpha-x, z, t) = \sum_{x\in G} \delta(\beta+x, \alpha+x, z, t) = -S. \qquad \square$$

Summing (15) over $x \in G$ and using Lemma 4.8, we obtain

$$\sum_{x\in G} \delta(x, b+x, a-x, y) = -\sum_{x\in G} \delta(0, x, b+x, a-x) = \sum_{x\in G} \delta(0, x, a-x, b+x).$$

Together with (13), this proves (8). Finally, let us suppose that $|G|$ is odd. For any $\alpha \in G$, the map $x \mapsto \alpha + 2x$ is a bijection of G. Therefore

$$\sum_{x,y\in G} \delta(*, *, \alpha+2x, y) = \sum_{x,y\in G} \delta(*, *, x, y) = 0$$

by antisymmetry with respect to (x,y). Therefore the second line of (8) vanishes. This finishes the proof of Theorem 4.3. $\square$

Remark 4.9. Thanks to the 5-term relation, every symbol $\{u(a,b,c,d)\}_2$ is a linear combination of symbols of the form $\{u(0,x,y,z)\}_2$. However, this breaks the symmetry.

5. The Goncharov complex

We begin by recalling Goncharov's theory of polylogarithmic complexes [10, Section 4]. Let F be a field, and let $n \geq 1$ be an integer. Goncharov has constructed a *weight n polylogarithmic complex* $\Gamma(F;n)$ of the following shape:

$$B_n(F) \to B_{n-1}(F) \otimes F^\times \to B_{n-2}(F) \otimes \Lambda^2 F^\times \to \cdots \to B_2(F) \otimes \Lambda^{n-2} F \to \Lambda^n F^\times,$$

where $B_n(F)$ is defined as the quotient $\mathbb{Z}[\mathbb{P}^1(F)]$ by a certain subgroup $R_n(F)$ related to the functional equations of the n-th polylogarithm. For any $x \in \mathbb{P}^1(F)$, we denote by $\{x\}$ the associated basis element in $\mathbb{Z}[\mathbb{P}^1(F)]$, and by $\{x\}_n$ its image in $B_n(F)$.
The complex $\Gamma(F;n)$ sits in degrees 1 to n and, after tensoring with $\mathbb{Q}$, it is expected to compute the weight n motivic cohomology of $\operatorname{Spec} F$. More precisely, Goncharov conjectures that $H^i\Gamma(F;n) \otimes \mathbb{Q} \cong H^i_{\mathcal{M}}(F, \mathbb{Q}(n))$ for every $1 \leq i \leq n$; see [10, Conjecture A].

Examples 5.1.
- $n = 1$. We have $B_1(F) \cong F^\times \cong H^1_{\mathcal{M}}(F, \mathbb{Z}(1))$.
- $n = 2$. The complex $\Gamma(F;2)$ is the Bloch–Suslin map $\delta_2 : B_2(F) \to \Lambda^2 F^\times$ defined in (14). The kernel of $\delta_2 \otimes \mathbb{Q}$ is isomorphic to $K_3^{\mathrm{ind}}(F) \otimes \mathbb{Q} \cong H^1_{\mathcal{M}}(F, \mathbb{Q}(2))$ (Suslin's theorem), and the cokernel of δ_2 is isomorphic to $K_2(F)$ up to 2-torsion (Matsumoto's theorem).
- $n = 3$. The complex $\Gamma(F;3)$ is
$$B_3(F) \to B_2(F) \otimes F^\times \to \Lambda^3 F^\times.$$
Here we define $B_2(F) = \mathbb{Z}[\mathbb{P}^1(F)]/R_2(F)$ as in Definition 4.7. The differentials are given by $\delta(\{x\}_3) = \{x\}_2 \otimes x$ and $\delta(\{x\}_2 \otimes y) = x \wedge (1-x) \wedge y$ for any x, y in $F^\times$ (by convention δ maps any expression containing $\{0\}$, $\{1\}$ or $\{\infty\}$ to 0). We will be interested in the cohomology in degree 2, which is expected to compute $H^2_{\mathcal{M}}(F, \mathbb{Q}(3)) \cong K_4^{(3)}(F)$.
- $n \geq 4$. In this case Goncharov's conjecture says that $H^2_{\mathcal{M}}(F, \mathbb{Q}(n)) \cong K^{(n)}_{2n-2}(F)$ should be isomorphic to a certain subquotient of $B_{n-1}(F) \otimes F^\times \otimes \mathbb{Q}$.

In this article, we will be mainly concerned with the case $n = 3$, and F is the function field of a smooth connected curve Y defined over a number field k. For every $x \in Y$, there is a residue map from the complex $\Gamma(F;3)$ to the complex $\Gamma(k(x);2)$ shifted by -1:

$$\begin{array}{ccccc} B_3(F) & \longrightarrow & B_2(F) \otimes F^\times & \longrightarrow & \Lambda^3 F^\times \\ & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\ & & \bigoplus_{x \in Y} B_2(k(x)) & \longrightarrow & \bigoplus_{x \in Y} \Lambda^2 k(x)^\times. \end{array} \tag{16}$$

In degree 2, the residue map is given by the formula $\mathrm{Res}_x(\{f\}_2 \otimes g) = \mathrm{ord}_x(g)\{f(x)\}_2$ for any rational functions f, g on Y. Then Goncharov [11, 1.15] defines the motivic complex $\Gamma(Y;3)$ as the simple complex associated to the double complex (16). In particular, we have

$$H^2\Gamma(Y;3) \cong \ker\Big(H^2\Gamma(F;3) \xrightarrow{\mathrm{Res}} \bigoplus_{x \in Y} B_2(k(x))\Big).$$

It is expected that the residue maps above are compatible with the localisation sequence in motivic cohomology. In particular, we expect $H^2\Gamma(Y;3) \otimes \mathbb{Q} \overset{?}{\cong} H^2_{\mathcal{M}}(Y, \mathbb{Q}(3))$. In fact, de

Jeu [7, 8] has constructed a map

$$H^2\Gamma(Y;3)\otimes\mathbb{Q} \longrightarrow H^2_{\mathcal{M}}(Y,\mathbb{Q}(3))$$

and Goncharov [12] has constructed a map in the other direction. Moreover, there exists a regulator map for the complex $\Gamma(Y;3)$ which is compatible with the Beilinson regulator map on motivic cohomology; see [8] and [13].

We may now describe our strategy to construct elements in $H^2_{\mathcal{M}}(Y,\mathbb{Q}(3))$. We assume that $Y = X\backslash S$ where X is smooth projective, and S is finite. The idea is to approximate the Goncharov complex $\Gamma(Y;3)$ by a certain subcomplex where all the functions involved are supported in S (that is, their zeros and poles are contained in S). In order for the differential of the symbol $\{f\}_2\otimes g$ to make sense, we need the condition that both f and $1-f$ are supported in S. That is, the function f must be a solution to the S-unit equation for X. The important point is that the subcomplex will consist of groups which are (essentially) of finite rank, hence we may do the linear algebra.

In more detail, let F be the function field of X. We denote by $\mathcal{O}(Y)^\times$ the subgroup of $F^\times$ consisting of the functions that are supported in S. Let U be the set of rational functions f on X such that both f and $1-f$ belong to $\mathcal{O}(Y)^\times$. Consider the following complex:

$$\Gamma_S(Y;3)\colon \qquad \mathbb{Z}[U] \longrightarrow \mathbb{Z}[U]\otimes\mathcal{O}(Y)^\times \longrightarrow \Lambda^3\mathcal{O}(Y)^\times$$

where, as above, the differentials are given by $\delta(\{f\}_3) = \{f\}_2\otimes f$ and $\delta(\{f\}_2\otimes g) = f\wedge(1-f)\wedge g$. Note that the set U is finite by Corollary 2.2. Moreover, after modding out by the non-zero constants, the group $\mathcal{O}(Y)^\times$ is of finite rank. Therefore all the groups occurring in $\Gamma_S(Y;3)$ are essentially of finite rank.

There is a natural morphism of complexes $\Gamma_S(Y;3)\to\Gamma(F;3)$ and since all the functions involved are supported in S, this morphism takes values in the subcomplex $\Gamma(Y;3)$. Therefore, we get natural maps

$$H^2\Gamma_S(Y;3)\otimes\mathbb{Q} \longrightarrow H^2\Gamma(Y;3)\otimes\mathbb{Q} \longrightarrow H^2_{\mathcal{M}}(Y,\mathbb{Q}(3)). \tag{17}$$

This strategy can already be implemented to construct elements in K_4 of a given curve. In the rest of the article, we will consider only the case of modular curves.

6. Construction of the elements in K_4

In this section, we explain how to construct elements in $K_4^{(3)}(Y(N)) \cong H^2_{\mathcal{M}}(Y(N),\mathbb{Q}(3))$. We actually work in the following more general setting. Let G be any subgroup of $(\mathbb{Z}/N\mathbb{Z})^2$. The group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts by right multiplication on the set of row vectors $(\mathbb{Z}/N\mathbb{Z})^2$. Let

$$\Gamma_G = \{\gamma\in\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) : \forall g\in G,\ g\gamma = g\}.$$

We may then consider the modular curve $Y(\Gamma_G) := \Gamma_G\backslash Y(N)$. For example, the group $G = \{0\}\times(\mathbb{Z}/N\mathbb{Z})$ gives rise to the usual modular curve $Y_1(N)$, since we have $\Gamma_G = \{\left(\begin{smallmatrix} * & * \\ 0 & 1\end{smallmatrix}\right)\}$. Let S be the set of cusps of $Y(\Gamma_G)$.

Construction 6.1. Let $a,b,c\in G$ with $a+b+c=0$. Write the triangulation (8) (or (9) when $|G|$ is odd), as follows:

$$g_a\wedge g_b + g_b\wedge g_c + g_c\wedge g_a = \sum_i m_i\cdot u_i\wedge(1-u_i)$$

with coefficients $m_i \in \mathbb{Q}$, and modular units $u_i \in \mathcal{O}(Y(\Gamma_G))^\times$. Then the element

$$\xi_G(a,b) = \sum_i m_i \{u_i\}_2 \otimes \left(\frac{g_b}{g_a}\right)$$

is a cocycle in the Goncharov complex $\Gamma_S(Y(\Gamma_G);3) \otimes \mathbb{Q}$. Indeed, we have

$$\delta\xi_G(a,b) = \sum_i m_i \cdot u_i \wedge (1-u_i) \wedge \left(\frac{g_b}{g_a}\right) = \big(g_a \wedge g_b + g_b \wedge g_c + g_c \wedge g_a\big) \wedge \left(\frac{g_b}{g_a}\right)$$
$$= g_c \wedge g_a \wedge g_b - g_b \wedge g_c \wedge g_a = 0.$$

Hence, using the map (17), $\xi_G(a,b)$ defines an element in $H^2_{\mathcal{M}}(Y(\Gamma_G), \mathbb{Q}(3))$.

Note that the cohomology class $\xi_G(a,b)$ depends a priori on the triangulation.

Notation 6.2. For $a, b \in (\mathbb{Z}/N\mathbb{Z})^2$, we write

$$\xi(a,b) := \xi_{(\mathbb{Z}/N\mathbb{Z})^2}(a,b) \in H^2_{\mathcal{M}}(Y(N), \mathbb{Q}(3)).$$

For $a, b \in \mathbb{Z}/N\mathbb{Z}$, we write

$$\xi_1(a,b) := \xi_{\{0\}\times(\mathbb{Z}/N\mathbb{Z})}((0,a),(0,b)) \in H^2_{\mathcal{M}}(Y_1(N), \mathbb{Q}(3)).$$

For example, if N is odd, we have

$$\xi_1(a,b) = \frac{1}{N} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} \{u(0,x,a-x,b+x)\}_2 \otimes \left(\frac{g_b}{g_a}\right),$$

with the convention $\{u(x,y,z,t)\}_2 = 0$ if x,y,z,t are not distinct in $(\mathbb{Z}/N\mathbb{Z})/\pm 1$.
We will show in the next sections that the elements $\xi_1(a,b)$ are non-trivial by computing numerically their regulators.
We say that $\xi_1(a,b)$ extends to $X_1(N)$ if the residues of $\xi_1(a,b)$ at the cusps are trivial, in other words $\xi_1(a,b)$ defines an element of $H^2_{\mathcal{M}}(X_1(N), \mathbb{Q}(3))$. We now give a sufficient condition for $\xi_1(a,b)$ to extend to $X_1(N)$.

Lemma 6.3. *If $N = p$ or $N = 2p$ with p prime, then $\xi_1(a,b)$ defines an element of $H^2_{\mathcal{M}}(X_1(N), \mathbb{Q}(3))$ for every $a, b \in \mathbb{Z}/N\mathbb{Z}$.*

Proof. Let N be arbitrary. A set of representatives of the Galois orbits of the cusps of $X_1(N)$ is given by the cusps $1/v$ with $0 \le v \le N/2$. Among them, the real cusps are given by $v = 0$, $v = N/2$ (for even N), and the integers $0 < v < N/2$ such that $\gcd(v,N) \in \{1,2\}$. It follows that in the cases $N = p$ and $N = 2p$ with p prime, all the cusps are totally real. But for a totally real number field k, we have $H^1_{\mathcal{M}}(k, \mathbb{Q}(2)) = 0$ by Borel's theorem, hence the residues are automatically trivial. □

In general, the elements $\xi_1(a,b)$ do not always extend to $X_1(N)$, as can be shown in the particular cases $N = 15$ and $N = 21$.
Nevertheless, we can modify the elements as follows. Let us return to the general situation where G is any subgroup of $(\mathbb{Z}/N\mathbb{Z})^2$. To ease notation, write $Y = Y(\Gamma_G)$ and $X = X(\Gamma_G)$. Denote by S the set of cusps, seen as a closed subscheme of X. We have the following localisation exact sequence in motivic cohomology:

$$H^0_{\mathcal{M}}(S, \mathbb{Q}(2)) \to H^2_{\mathcal{M}}(X, \mathbb{Q}(3)) \to H^2_{\mathcal{M}}(Y, \mathbb{Q}(3)) \to H^1_{\mathcal{M}}(S, \mathbb{Q}(2)) \to H^3_{\mathcal{M}}(X, \mathbb{Q}(3))$$

The first group is zero by Borel's theorem, hence we may view $H^2_{\mathcal{M}}(X, \mathbb{Q}(3))$ as a subspace of $H^2_{\mathcal{M}}(Y, \mathbb{Q}(3))$.

Proposition 6.4. *The map $H^2_{\mathcal{M}}(X,\mathbb{Q}(3)) \to H^2_{\mathcal{M}}(Y,\mathbb{Q}(3))$ admits a natural retraction.*

Proof. Let k be the field of constants of X, and let k'/k be the splitting field of S. Since motivic cohomology with $\mathbb{Q}$-coefficients satisfies Galois descent, it suffices to prove that $H^2_{\mathcal{M}}(X_{k'},\mathbb{Q}(3)) \to H^2_{\mathcal{M}}(Y_{k'},\mathbb{Q}(3))$ has a retraction that is $\mathrm{Gal}(k'/k)$-equivariant. Write $i\colon S_{k'} \hookrightarrow X_{k'}$ and $\pi\colon X_{k'} \to \operatorname{Spec} k'$. We have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^2_{\mathcal{M}}(X_{k'},\mathbb{Q}(3)) & \longrightarrow & H^2_{\mathcal{M}}(Y_{k'},\mathbb{Q}(3)) & \xrightarrow{\mathrm{Res}} & \bigoplus_{x\in S(k')} H^1_{\mathcal{M}}(k',\mathbb{Q}(2)) & \xrightarrow{i_*} & H^3_{\mathcal{M}}(X_{k'},\mathbb{Q}(3)) \\
 & & & & & & & \searrow^{\Sigma} & \downarrow{\scriptstyle \pi_*} \\
 & & & & & & & & H^1_{\mathcal{M}}(k',\mathbb{Q}(2))
\end{array}$$

where the first row is exact, and the diagonal arrow Σ is the sum map. Define

$$T = \langle H^1_{\mathcal{M}}(k',\mathbb{Q}(2)) \cup \mathcal{O}(Y_{k'})^\times \rangle \subset H^2_{\mathcal{M}}(Y_{k'},\mathbb{Q}(3)).$$

We have $\mathrm{Res}(\lambda \cup u) = \lambda \otimes \mathrm{div}(u)$ for any $\lambda \in H^1_{\mathcal{M}}(k',\mathbb{Q}(2))$ and any modular unit $u \in \mathcal{O}(Y_{k'})^\times$. By the Manin–Drinfeld theorem, the cusps of X are torsion in the Jacobian of X. This implies that $\mathrm{Res}(T) = \ker(\Sigma)$. We then claim that

$$H^2_{\mathcal{M}}(Y_{k'},\mathbb{Q}(3)) = H^2_{\mathcal{M}}(X_{k'},\mathbb{Q}(3)) \oplus T.$$

The fact that $H^2_{\mathcal{M}}(Y_{k'},\mathbb{Q}(3))$ is generated by $H^2_{\mathcal{M}}(X_{k'},\mathbb{Q}(3))$ and T follows from the localisation sequence above. Now consider the composite map

$$H^1_{\mathcal{M}}(k',\mathbb{Q}(2)) \otimes \mathcal{O}(Y_{k'})^\times \xrightarrow{\cup} T \xrightarrow{\mathrm{Res}} \bigoplus_{x\in S(k')} H^1_{\mathcal{M}}(k',\mathbb{Q}(2)).$$

The kernel of this map is $H^1_{\mathcal{M}}(k',\mathbb{Q}(2)) \otimes k'^\times$. Therefore the intersection of $H^2_{\mathcal{M}}(X_{k'},\mathbb{Q}(3))$ and T is contained in $H^2_{\mathcal{M}}(k',\mathbb{Q}(3))$, which is zero by Borel's theorem. Finally, we note that T is stable under $\mathrm{Gal}(k'/k)$, hence the retraction descends. ☐

Notation 6.5. For $a,b \in G$, we denote by $\xi'_G(a,b)$ the image of $\xi_G(a,b)$ under the retraction $H^2_{\mathcal{M}}(Y(\Gamma_G),\mathbb{Q}(3)) \to H^2_{\mathcal{M}}(X(\Gamma_G),\mathbb{Q}(3))$.

We now study how the elements $\xi_G(a,b)$ and $\xi'_G(a,b)$ depend on the triangulation. Say we have two triangulations

$$g_a \wedge g_b + g_b \wedge g_c + g_c \wedge g_a = \sum_i m_i \cdot u_i \wedge (1-u_i) = \sum_j n_j \cdot v_j \wedge (1-v_j).$$

Let F be the function field of $Y(\Gamma_G)$. Then

$$\lambda := \sum_i m_i \{u_i\}_2 - \sum_j n_j \{v_j\}_2 \in H^1_{\mathcal{M}}(F,\mathbb{Q}(2)).$$

Merkurjev and Suslin formulated a rigidity conjecture [21, Conjecture 4.10] which implies the following statement.

Conjecture 6.6. *We have an isomorphism $H^1_{\mathcal{M}}(F,\mathbb{Q}(2)) \cong H^1_{\mathcal{M}}(k,\mathbb{Q}(2))$, where $k \subset F$ is the field of constants of F.*

Remark 6.7. This statement is also implied by Beilinson's conjecture for the L-value of the motive $h^0(X(\Gamma_G))$ at $s=2$, together with the localisation sequence.

Assuming $\lambda \in H^1_{\mathcal{M}}(k, \mathbb{Q}(2))$, we see that the element $\lambda \cup (g_b/g_a)$ is killed by the retraction of Proposition 6.4. Therefore we have shown:

Proposition 6.8. *Assuming Conjecture 6.6, the element $\xi'_G(a,b) \in H^2_{\mathcal{M}}(X(\Gamma_G), \mathbb{Q}(3))$ does not depend on the triangulation.*

Finally, we investigate how our element $\xi'_G(a,b)$ depend on G. It will be convenient to work with the tower of modular curves $Y(N)$. First note that for $(\alpha, \beta) \in \mathbb{Z}^2$, the Siegel unit $g_{(\alpha,\beta) \bmod N}$ depends only on the class of $(\frac{\alpha}{N}, \frac{\beta}{N})$ in $(\mathbb{Q}/\mathbb{Z})^2$. In this way, we may define the Siegel unit g_x for any $x \in (\mathbb{Q}/\mathbb{Z})^2$, which lives in the direct limit

$$\mathcal{O}(Y(\infty))^\times \otimes \mathbb{Q} := \varinjlim_{N \geq 1} \mathcal{O}(Y(N))^\times \otimes \mathbb{Q},$$

where the transition maps are the pull-backs associated to the canonical projection maps $Y(N') \to Y(N)$ for N dividing N'.
Now, let a, b be two elements of $(\mathbb{Q}/\mathbb{Z})^2$. Choose a finite subgroup G of $(\mathbb{Q}/\mathbb{Z})^2$ containing a and b, and choose an integer $N \geq 1$ such that G is killed by N. Then G identifies with a subgroup of $(\mathbb{Z}/N\mathbb{Z})^2$, and we have defined above elements $\xi_G(a,b) \in H^2_{\mathcal{M}}(Y(\Gamma_G), \mathbb{Q}(3))$ and $\xi'_G(a,b) \in H^2_{\mathcal{M}}(X(\Gamma_G), \mathbb{Q}(3))$. Since $X(\Gamma_G)$ is a quotient of $X(N)$, this gives rise to an element of

$$H^2_{\mathcal{M}}(X(\infty), \mathbb{Q}(3)) := \varinjlim_{N \geq 1} H^2_{\mathcal{M}}(X(N), \mathbb{Q}(3)),$$

where the transition maps are defined as above.

Proposition 6.9. *Assuming Conjecture 6.6, the image of $\xi'_G(a,b)$ in $H^2_{\mathcal{M}}(X(\infty), \mathbb{Q}(3))$ does not depend on G and N.*

Proof. This follows from the independence of the triangulation proved in Proposition 6.8. □

The idea of taking the cohomology at infinite level will be used in Section 9, when trying to compare with the elements in K_4 constructed by Beilinson.
To conclude this section, let us mention that the elements $\xi'_G(a,b)$ also satisfy the analogue of the Manin relations.

Proposition 6.10. *Let G be a finite subgroup of $(\mathbb{Q}/\mathbb{Z})^2$. Assuming Conjecture 6.6, we have*

$$\begin{aligned} &\xi'_G(a,b) = \xi'_G(\pm a, \pm b) = \xi'_G(b,a) && (a, b \in G) \\ &\xi'_G(a,b) + \xi'_G(b,c) + \xi'_G(c,a) = 0 && (a,b,c \in G,\ a+b+c=0). \end{aligned}$$

Proof. We give the proof for the 3-term relation, as the others are treated similarly. Write

$$T(a,b) := g_a \wedge g_b + g_b \wedge g_c + g_c \wedge g_a = \sum_i m_i \cdot u_i \wedge (1-u_i).$$

Note that $T(a,b) = T(b,c) = T(c,a)$. Therefore

$$\xi_G(a,b) + \xi_G(b,c) + \xi_G(c,a) = \sum_i m_i \cdot \{u_i\}_2 \otimes \left(\frac{g_b}{g_a}\frac{g_c}{g_b}\frac{g_a}{g_c}\right) = 0.$$

Projecting onto $H^2_{\mathcal{M}}(X(\infty), \mathbb{Q}(3))$ gives the result. □

Proposition 6.10 gives some hope to find an inductive procedure to construct motivic elements in the higher K-groups $K^{(n)}_{2n-2}(X(\infty)) \cong H^2_{\mathcal{M}}(X(\infty), \mathbb{Q}(n))$.

7. Computation of the regulator

Goncharov has defined in [13] explicit regulator maps on the polylogarithmic complexes, which are compatible with the regulator maps on motivic cohomology defined by Beilinson. In this section we explain how to compute numerically Goncharov's regulator maps in the case of modular curves. We implemented this computation in Pari/GP [22].
We first show that Goncharov's regulator integrals are absolutely convergent in the case of curves (Proposition 7.2 and Corollary 7.3). Then, we explain how to compute them in the case of modular curves, using generalised Mellin transforms.

7.1. **Convergence of Goncharov's integrals.** Let X be a smooth connected projective curve over $\mathbb{C}$, and let F be the function field of X. Let $n \geq 3$. According to Goncharov's theory, the motivic cohomology group $H^2_{\mathcal{M}}(F, \mathbb{Q}(n))$ should be isomorphic to a certain subquotient of $B_{n-1}(F) \otimes F^\times$. Goncharov constructs in [13, Theorem 2.2] a regulator map

$$r_n(2) \colon B_{n-1}(F) \otimes F^\times \longrightarrow \mathcal{A}^1(\eta_X)(n-1)$$

where $\mathcal{A}^1(\eta_X)(n-1)$ is the space of $(2\pi i)^{n-1}\mathbb{R}$-valued differential 1-forms on X which are regular outside a finite subset of X. Concretely, for $f \in F \backslash \{0,1\}$ and $g \in F^\times$, we have

$$r_n(2)(\{f\}_{n-1} \otimes g) = i\widehat{\mathcal{L}}_{n-1}(f) \operatorname{darg} g - \frac{2^{n-1} B_{n-1}}{(n-1)!} \alpha(1-f, f) \cdot \log^{n-3}|f| \log|g| - \sum_{k=2}^{n-2} \frac{2^k B_k}{k!} \widehat{\mathcal{L}}_{n-k}(f) \log^{k-2}|f| \operatorname{dlog}|f| \cdot \log|g|, \tag{18}$$

where $\widehat{\mathcal{L}}_m \colon \mathbb{P}^1(\mathbb{C}) \to (2\pi i)^{m-1}\mathbb{R}$ is the single-valued polylogarithm defined in [13, 2.1], $B_2 = \frac{1}{6}$, $B_3 = 0 \ldots$ are the Bernoulli numbers, and

$$\alpha(f, g) = -\log|f| \operatorname{dlog}|g| + \log|g| \operatorname{dlog}|f|.$$

For $m \geq 2$, the function $\widehat{\mathcal{L}}_m$ is real-analytic outside $\{0, 1, \infty\}$ and is continuous on $\mathbb{P}^1(\mathbb{C})$ with $\widehat{\mathcal{L}}_m(0) = \widehat{\mathcal{L}}_m(\infty) = 0$ (see [28, p. 413-414], where $\widehat{\mathcal{L}}_m = P_m$ for odd m, and $\widehat{\mathcal{L}}_m = iP_m$ for even m). It follows that the 1-form $r_n(2)(\{f\}_{n-1} \otimes g)$ is defined and real-analytic outside the set of zeros and poles of f, $1-f$ and g.

Lemma 7.1. *Let f, g be non-zero rational functions on X. Let $z = re^{i\theta}$ be a local coordinate at $p \in X$. In a neighbourhood of the point p, we have*

$$\alpha(f, g) = \Big(-\frac{\log|\partial_p(f,g)|}{r} + O(\log r)\Big) \cdot dr + O(r \log r) d\theta$$

where $\partial_p(f,g) = (-1)^{\operatorname{ord}_p(f)\operatorname{ord}_p(g)} \big(f^{\operatorname{ord}_p(g)}/g^{\operatorname{ord}_p(f)}\big)(p)$ is the tame symbol of (f,g) at p.

Proof. Write $f(z) \sim az^m$, $g(z) \sim bz^n$ with $a, b \in \mathbb{C}^\times$ and $m, n \in \mathbb{Z}$. A direct computation gives

$$\operatorname{dlog} f = (m + O(z))\frac{dz}{z} = (m + O(z))\frac{dr}{r} + (im + O(z))d\theta.$$

Taking the real and imaginary parts, we get

$$\operatorname{dlog}|f| = (\frac{m}{r} + O(1))dr + O(r)d\theta, \tag{19}$$

$$\operatorname{darg} f = O(1)dr + (m + O(r))d\theta. \tag{20}$$

On the other hand $\log|f| = \log|a| + m\log r + O(r)$. Putting things together, we arrive at

$$\alpha(f,g) = \Big(-\frac{1}{r}\log\Big|\frac{a^n}{b^m}\Big| + O(\log r)\Big)dr + O(r\log r)d\theta. \qquad \square$$

Proposition 7.2. *Let $f \in F\backslash\{0,1\}$ and $g \in F^\times$. Let S be the set of zeros and poles of the functions f, $1-f$ and g. Let $\gamma : [0,1] \to X$ be a C^∞ path such that*

(a) *γ avoids S except possibly at the endpoints;*
(b) *If an endpoint p of γ belongs to S, then the argument of $\gamma(t)$ with respect to a local coordinate at p is of bounded variation when $\gamma(t)$ approaches p.*

Then for every $n \geq 3$, the integral $\int_\gamma r_n(2)(\{f\}_{n-1} \otimes g)$ converges absolutely.

Proof. As noted above, the integrand is C^∞ outside S. We are going to show the convergence of the integral at the endpoint $t = 0$ (the case $t = 1$ is identical). Let $z = re^{i\theta}$ be a local coordinate at $p = \gamma(0)$. Assumption (b) means that the form $d\theta$ is (absolutely) integrable along γ near $t = 0$. Moreover $dr = e^{-i\theta}dz - ird\theta$, so that dr is also integrable along γ. Using (20), we deduce that $\operatorname{darg} g$ and the first term of (18) are integrable. Regarding the second term, Lemma 7.1 and the fact that $\partial_p(1-f, f) = 1$ give

$$\alpha(1-f, f) = O(\log r)dr + O(r\log r)d\theta.$$

It follows that the second term in (18) has at worst logarithmic singularities, hence is integrable. Finally, the integrability of the third term in (18) can be proved similarly, noting that $\widehat{\mathcal{L}}_m(z) = O(|z|\log^{m-1}|z|)$ when $z \to 0$ for any $m \geq 2$, and using the functional equation $\widehat{\mathcal{L}}_m(1/z) = (-1)^{m-1}\widehat{\mathcal{L}}_m(z)$ to get the asymptotic behaviour when $z \to \infty$. $\square$

Corollary 7.3. *Let $X = \overline{\Gamma\backslash\mathcal{H}}$ be a modular curve, and let u, v be modular units such that $1-u$ is also a modular unit. Then for any $n \geq 3$ and any two cusps $\alpha \neq \beta$ in $\mathbb{P}^1(\mathbb{Q})$, the integral of $r_n(2)(\{u\}_{n-1} \otimes v)$ along the modular symbol $\{\alpha, \beta\}$ converges absolutely.*

Proposition 7.2 also holds for $n = 2$; in fact in this case we don't need to include the function $1-f$ in the definition of S. This follows from a similar computation, the regulator being defined by $(f,g) \mapsto \log|f|\operatorname{darg} g - \log|g|\operatorname{darg} f$. As a consequence, Corollary 7.3 also holds in the case $n = 2$, without assuming that $1-u$ is a modular unit.

Remark 7.4. We emphasize that the integral considered in Proposition 7.2 depends on the direction from which γ approaches the endpoints. This is because the differential 1-form $r_n(2)(\{f\}_{n-1} \otimes g)$ may have non-trivial residues at the points of S. As a consequence, in the setting of Corollary 7.3, the formula $\int_\alpha^\gamma = \int_\alpha^\beta + \int_\beta^\gamma$ does not always hold. A convenient framework to deal with this issue is Stevens's theory of extended modular symbols [24].

7.2. **Generalised Mellin transforms.** Let u, v be modular units for $\Gamma(N)$ such that $1-u$ is also a modular unit. Let $n \geq 3$. For any two cusps $\alpha \neq \beta$ in $\mathbb{P}^1(\mathbb{Q})$, we would like to compute the integral

$$\int_\alpha^\beta r_n(2)(\{u\}_{n-1} \otimes v). \tag{21}$$

The modular symbol $\{\alpha, \beta\}$ may be written as a linear combination $\sum_i \{\gamma_i 0, \gamma_i\infty\}$ for some elements $\gamma_i \in \mathrm{SL}_2(\mathbb{Z})$. Therefore the computation of (21) reduces to the case $\alpha = \gamma 0$ and

$\beta = \gamma\infty$ with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, together with the computation of some residues at the cusps. Moreover, we have

$$\int_{\gamma 0}^{\gamma\infty} r_n(2)(\{u\}_{n-1} \otimes v) = \int_0^\infty r_n(2)(\{u|\gamma\}_{n-1} \otimes v|\gamma)$$

and the functions $u|\gamma$, $v|\gamma$ are also modular units. We are thus reduced to the case $\alpha = 0$ and $\beta = \infty$. In this case, let us write

$$\int_0^\infty r_n(2)(\{u\}_{n-1} \otimes v) = \int_0^\infty \phi(y)dy,$$

where $\phi\colon (0,+\infty) \to \mathbb{C}$ is a C^∞ function. We have seen in Corollary 7.3 that ϕ is absolutely integrable. We are going to show that ϕ belongs to a specific class of functions, for which the (generalised) Mellin transform can be computed rapidly.

Definition 7.5. Let $\mathcal{P}$ be the class of functions $\phi\colon (0,+\infty) \to \mathbb{C}$ such that

$$\phi(y) = \sum_{j=0}^{j_\infty} y^j \sum_{n=0}^{\infty} a_n^{(j)} e^{-2\pi n y/N}, \quad \text{and} \quad \phi\Big(\frac{1}{y}\Big) = \sum_{j=0}^{j_0} y^j \sum_{n=0}^{\infty} b_n^{(j)} e^{-2\pi n y/N}, \tag{22}$$

for some integers $j_\infty, j_0 \geq 0$, and where the sequences $(a_n^{(j)})_{n\geq 0}$ and $(b_n^{(j)})_{n\geq 0}$ have polynomial growth when $n \to \infty$.

By considering the asymptotic expansion, it is easy to see that the coefficients $a_n^{(j)}$ and $b_n^{(j)}$ are uniquely determined by ϕ. Moreover, the function ϕ is absolutely integrable on $(0,+\infty)$ if and only if $a_0^{(j)} = 0$ for $j \geq 0$ and $b_0^{(j)} = 0$ for $j \geq 1$.
Recall that the generalised Mellin transform of a function $\phi \in \mathcal{P}$ is defined as follows

$$\begin{aligned}\mathcal{M}(\phi, s) = \int_0^\infty \phi(y) y^s \frac{dy}{y} &:= \text{a.c.}\Big(\int_1^\infty \phi(y) y^s \frac{dy}{y}\Big) + \text{a.c.}\Big(\int_0^1 \phi(y) y^s \frac{dy}{y}\Big) \\ &= \text{a.c.}\Big(\int_1^\infty \phi(y) y^s \frac{dy}{y}\Big) + \text{a.c.}\Big(\int_1^\infty \phi\Big(\frac{1}{y}\Big) y^{-s} \frac{dy}{y}\Big),\end{aligned}$$

where $s \in \mathbb{C}$ and "a.c." means analytic continuation with respect to s. Note that the first integral converges for $\mathrm{Re}\,(s) \ll 0$ while the second integral converges for $\mathrm{Re}\,(s) \gg 0$; both have a meromorphic continuation to $s \in \mathbb{C}$ thanks to (22).
In the case ϕ is absolutely integrable, the integral $\int_0^\infty \phi = \mathcal{M}(\phi, 1)$ is given by the following series with exponential decay:
(23)

$$\int_0^\infty \phi = \sum_{n=1}^{\infty} \sum_{j=0}^{j_\infty} a_n^{(j)} \Big(\frac{N}{2\pi n}\Big)^{j+1} \Gamma\Big(j+1, \frac{2\pi n}{N}\Big) + \sum_{n=1}^{\infty} \sum_{j=0}^{j_0} b_n^{(j)} \Big(\frac{N}{2\pi n}\Big)^{j-1} \Gamma\Big(j-1, \frac{2\pi n}{N}\Big) + b_0^{(0)},$$

where $\Gamma(s,x)$ is the incomplete gamma function. So the integral of ϕ over $(0,+\infty)$ can be computed efficiently, provided sufficiently many coefficients $a_n^{(j)}$ and $b_n^{(j)}$ are known.
The class $\mathcal{P}$ is a $\mathbb{C}$-algebra stable under the differentiation $\frac{d}{dy}$. However it is not stable under taking primitive (e.g. consider a constant function). In fact, given a function $\phi \in \mathcal{P}$, we have the following equivalence:

$$\phi \text{ has a primitive in } \mathcal{P} \quad \Leftrightarrow \quad \forall n \geq 0,\ b_n^{(0)} = b_n^{(1)} = 0.$$

We denote by $\mathcal{P}'$ the image of the operator $\frac{d}{dy}$. The criterion above shows that $\mathcal{P}'$ is an ideal of $\mathcal{P}$.

7.3. **Modular regulators.** We are now going to show, in the case of modular curves, that the regulators defined by Goncharov belong to $\mathcal{P}$.

Lemma 7.6. *For any modular unit u for $\Gamma(N)$, we have $\log u \in \mathcal{P}$. In particular $\log|u| \in \mathcal{P}$, and the forms $\operatorname{dlog}|u|$ and $\operatorname{darg} u$ belong to $d\mathcal{P} = \mathcal{P}'dy$.*

Proof. Here $\log u$ is any determination of the logarithm of u on $\mathcal{H}$. Since the group of modular units is generated by the Siegel units g_a modulo the constants, it suffices to prove the result for them. For the asymptotic expansion of $\log g_a(iy)$ when $y \to +\infty$, this follows from taking the logarithm of (1) and (2), and expanding as a power series in $e^{-2\pi y/N}$. The expansion when $y \to 0$ also has the correct shape since $g_a(-1/\tau)$ is (a root of) a modular unit for $\Gamma(N)$. □

Proposition 7.7. *Let u be a modular unit for $\Gamma(N)$ such that $1-u$ is also a modular unit. For every $n \geq 2$, we have $\widehat{\mathcal{L}}_n(u) \in \mathcal{P}$.*

Proof. We will prove this by complete induction on n. For $n = 2$, we have $\widehat{\mathcal{L}}_2 = iD$, where D is the Bloch-Wigner dilogarithm [28, Section 2]. We have

$$dD(u) = \log|u| \operatorname{darg}(1-u) - \log|1-u| \operatorname{darg}(u).$$

From Lemma 7.6, it follows that $dD(u)(iy) \in d\mathcal{P}$, hence $D(u) \in \mathcal{P}$.
Now let $n \geq 3$. By the commutative diagram in [13, Theorem 2.2], we have

$$\begin{aligned}
d\widehat{\mathcal{L}}_n(u) &= r_n(2)(\{u\}_{n-1} \otimes u) \\
&= i\widehat{\mathcal{L}}_{n-1}(u) \operatorname{darg} u - \frac{2^{n-1}B_{n-1}}{(n-1)!}\alpha(1-u,u) \cdot \log^{n-2}|u| \\
&\quad - \sum_{k=2}^{n-2} \frac{2^k B_k}{k!} \widehat{\mathcal{L}}_{n-k}(u) \log^{k-1}|u| \operatorname{dlog}|u|
\end{aligned}$$

By the induction hypothesis $\widehat{\mathcal{L}}_m(u)$ belongs to $\mathcal{P}$ for $m < n$. The result now follows from Lemma 7.6 and the fact that $\mathcal{P}'$ is an ideal of $\mathcal{P}$. □

Note that the proof of Proposition 7.7 provides a way to compute the Fourier coefficients of $\widehat{\mathcal{L}}_n(u)$ (inductively on n): we first compute the Fourier expansions of $d\widehat{\mathcal{L}}_n(u)$ at 0 and ∞, and then integrate term by term. The constant of integration is determined by computing the value of $\widehat{\mathcal{L}}_n(u)$ at ∞ (note that this value is always finite), which should be equal to the coefficient $a_0^{(0)}$ of the expansion.

Theorem 7.8. *Let $n \geq 3$. Let u, v be two modular units for $\Gamma(N)$ such that $1-u$ is also a modular unit. Write*

$$r_n(2)(\{u\}_{n-1} \otimes v)|_{\{0,\infty\}} = \phi(y)dy.$$

Then ϕ belongs to $\mathcal{P}$.

Proof. This follows from (18), Lemma 7.6 and Proposition 7.7. □

In the case $u = u(a,b,c,d)$ and $v = g_e$, Proposition 3.4, the equation (18) and the proof of Proposition 7.7 actually provide an algorithm to compute the asymptotic expansion of $r_n(2)(\{u\}_{n-1} \otimes v)$ at 0 and ∞, and thus the associated regulator integral by (23).

8. Numerical verification of Beilinson's conjecture

Let E be an elliptic curve of conductor N. Using his theory of Eisenstein symbols, Beilinson has constructed elements in $H^2_{\mathcal{M}}(E, \mathbb{Q}(3))$ whose regulators are proportional to $L(E,3)$, confirming his conjecture for this L-value.
In this section, we investigate the regulators of our elements $\xi_1(a,b)$ with $a,b \in \mathbb{Z}/N\mathbb{Z}$. We explain how to compute them numerically and show that they are numerically related to $L(E,3)$. According to Goncharov's theory, we should look at the following integrals

$$\int_\gamma r_3(2)(\xi_1(a,b)) \tag{24}$$

with $\gamma \in H_1(Y_1(N)(\mathbb{C}), \mathbb{Z})$.

Lemma 8.1. *The differential* 1*-form* $r_3(2)(\xi_1(a,b))$ *is invariant under complex conjugation acting on* $Y_1(N)(\mathbb{C})$*.*

Proof. Let $u, v \in \mathcal{O}(Y_1(N))^\times$. We have

$$r_3(2)(\{u\}_2 \otimes v) = -D(u)\,\mathrm{d}\arg v - \frac{1}{3}\alpha(1-u,u) \cdot \log|v|. \tag{25}$$

Let c denote the complex conjugation on $Y_1(N)(\mathbb{C})$. Since u and v are defined over $\mathbb{Q}$, we have $c^*D(u) = D(u \circ c) = D(\overline{u}) = -D(u)$ and $c^*\,\mathrm{d}\arg v = \mathrm{d}\arg \overline{v} = -\,\mathrm{d}\arg v$. The other term involving $\alpha(1-u,u)$ is dealt with similarly. $\square$

By Lemma 8.1, we should restrict in (24) to cycles γ that are invariant under complex conjugation.
Also, we will not integrate over closed cycles in $Y_1(N)(\mathbb{C})$, but rather over Manin symbols $\{\gamma 0, \gamma\infty\}$ with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. This means that we need to choose a representative of the regulator of $\xi_1(a,b)$. We do this by writing $\xi_1(a,b)$ as the class of a cocycle $\sum_i \{u_i\}_2 \otimes v_i$, and choosing (25) as representative for the regulator of each term. Once the differential 1-form has been fixed, we integrate it over a cycle γ_E^+ which lies in the eigenspace corresponding to E. Let us write

$$\gamma_E^+ = \sum_j n_j \{\gamma_j 0, \gamma_j \infty\} \in H_1(X_1(N)(\mathbb{C}), \mathbb{Z})^+.$$

We obtain γ_E^+ using Magma [3]. Here is a sample code for the elliptic curve $E = 11a3$:

```
N := 11;
M := ModularSymbols(Gamma1(N), 2, +1);
Snew := NewSubspace(CuspidalSubspace(M));
Mf := NewformDecomposition(Snew);
IntegralBasis(Mf[1])[1];
```

This returns

```
-1*{-1/2, 0} + {-1/4, 0} + -1*{7/15, 1/2}
```

which means that

$$\gamma_E^+ = -\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\{0,\infty\} + \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}\{0,\infty\} - \begin{pmatrix} 1 & 7 \\ 2 & 15 \end{pmatrix}\{0,\infty\}.$$

Returning to the general situation, we then compute the regulator integral (24) as follows:

$$\int_{\gamma_E^+} r_3(2)(\xi_1(a,b)) = \sum_{i,j} n_j \int_0^\infty r_3(2)\big(\{u_i|\gamma_j\}_2 \otimes v_i|\gamma_j\big)$$

We know that the integral $\int_0^\infty$ is absolutely convergent, and we can compute it numerically thanks to Section 7.3. Note that $u_i|\gamma_j$ and $v_i|\gamma_j$ are modular units for $\Gamma(N)$ but in general not for $\Gamma_1(N)$.
The results of our computations can be summarised in the following theorem.

Theorem 8.2. *For every elliptic curve E of conductor $N \leq 50$, there exists $a, b \in \mathbb{Z}/N\mathbb{Z}$ and $r \in \mathbb{Z}\backslash\{0\}$ such that $\xi_1(a,b)$ extends to $X_1(N)$ and*

$$\int_{\gamma_E^+} r_3(2)\big(\xi_1(a,b)\big) \stackrel{?}{=} \frac{r\pi^2}{N} \cdot L'(E,-1)$$

to at least 40 decimal places. In particular $\xi_1(a,b)$ is a non-zero element of $H^2_{\mathcal{M}}(X_1(N), \mathbb{Q}(3))$.

The integer r in Theorem 8.2 is always equal to ± 3, except for the curves $38a1$ ($r = 9$) and the curves $42a1, 43a1$ ($r = \pm 6$). This property depends of course on how Magma normalises the cycle γ_E^+. In any case, we have no explanation to offer.
Theorem 8.2 provides non-trivial elements in $H^2_{\mathcal{M}}(E, \mathbb{Q}(3))$ for every elliptic curve E of conductor ≤ 50. Moreover, their regulators are numerically related to $L(E,3)$, as predicted by Beilinson.

9. Comparison with Beilinson's elements and applications

In this section, we will see that our elements $\xi_1(a,b)$ seem to be related to the Beilinson elements defined using Eisenstein symbols.
Let us recall the definition of the Beilinson elements. Let $p \colon E_1(N) \to Y_1(N)$ denote the universal elliptic curve. In [1, Section 3], Beilinson constructs Eisenstein symbols

$$\mathrm{Eis}^1(0,a) \in H^2_{\mathcal{M}}(E_1(N), \mathbb{Q}(2)) \qquad (a \in \mathbb{Z}/N\mathbb{Z}).$$

Taking cup-product and applying push-forward, one gets

$$\mathrm{Eis}^{0,0,1}(a,b) := p_*\big(\mathrm{Eis}^1(0,a) \cup \mathrm{Eis}^1(0,b)\big) \in H^2_{\mathcal{M}}(Y_1(N), \mathbb{Q}(3)).$$

for any $a, b \in \mathbb{Z}/N\mathbb{Z}$.
We may compare the elements $\xi_1(a,b)$ and $\mathrm{Eis}^{0,0,1}(a,b)$ by computing numerically their regulators. More precisely, we use Magma to find a basis of $H_1(X_1(N)(\mathbb{C}), \mathbb{Q})^+$, and we integrate the regulators over this basis. The integral of the regulator of $\xi_1(a,b)$ is computed as in Section 8. The integral of the regulator of $\mathrm{Eis}^{0,0,1}(a,b)$ is computed thanks to the following recent result of W. Wang [26, Theorem 0.1.3].

Theorem 9.1. *For any integer N and any $a, b \in (\mathbb{Z}/N\mathbb{Z})\backslash\{0\}$, we have*

$$\int_0^\infty r_3(2)\big(\mathrm{Eis}^{0,0,1}(a,b)\big) = -\frac{36\pi^2}{N^3} L'(\tilde{s}_a \tilde{s}_b, -1),$$

where the $\tilde{s}_x$ are Eisenstein series of weight 1 and level $\Gamma_1(N)$ defined by

$$\tilde{s}_x(\tau) = \frac{1}{2} - \left\{\frac{x}{N}\right\} + \sum_{\substack{m,n\geq 1 \\ n\equiv x \bmod N}} q^{mn} - \sum_{\substack{m,n\geq 1 \\ n\equiv -x \bmod N}} q^{mn} \qquad (q = e^{2\pi i\tau}),$$

where $\{\cdot\}$ denotes the fractional part.

The proof of Theorem 9.1 uses the Rogers–Zudilin method. In fact, Wang proves a much more general statement concerning the regulators of the motivic classes $\mathrm{Eis}^{k_1,k_2,j}(u_1, u_2)$ for every $k_1, k_2, j \geq 0$; the formula involves the completed L-function of a modular form of weight $k_1 + k_2 + 2$ evaluated at $s = -j$ [26, Chapter 6].

The Eisenstein series $\tilde{s}_x$ appear in the work of Borisov and Gunnells [2, 3.18]. Moreover, the Fricke involution $W_N(\tilde{s}_x)$ is a multiple of the Eisenstein series denoted by s_x in [2, 3.5]. From this, we can use the standard W_N-trick to compute numerically the L-value $L'(\tilde{s}_a\tilde{s}_b, -1)$, and thus the regulator of $\mathrm{Eis}^{0,0,1}(a, b)$.

The result of our computation is as follows.

Theorem 9.2. *For every integer $N \leq 28$ and every $a, b \in \mathbb{Z}/N\mathbb{Z}$, we have*

$$\int_{\gamma_i} r_3(2)\bigl(\xi_1(a, b)\bigr) \stackrel{?}{=} \frac{N^2}{6} \int_{\gamma_i} r_3(2)\bigl(\mathrm{Eis}^{0,0,1}(a, b)\bigr) \qquad (1 \leq i \leq g(X_1(N)))$$

to at least 40 decimal places, where $(\gamma_i)_i$ is the basis of $H_1(X_1(N)(\mathbb{C}), \mathbb{Q})^+$ computed by Magma.

Based on Theorem 9.2, we formulate the following conjecture.

Conjecture 9.3. *For every integer $N \geq 1$ and every $a, b \in \mathbb{Z}/N\mathbb{Z}$, we have*

$$\xi_1(a, b) = \frac{N^2}{6}\mathrm{Eis}^{0,0,1}(a, b).$$

Conjecture 9.3 relates two motivic elements $\mathrm{Eis}^{0,0,1}(a, b)$ and $\xi_1(a, b)$ whose constructions are technically quite different. However, both constructions are of modular nature, so we expect a modular proof of the relation between them.

Here is one way to approach Conjecture 9.3. Let $a, b \in \mathbb{Z}/N\mathbb{Z}$. We saw in Section 6 that conjecturally, the image of $\xi_1'(a, b)$ in the cohomology at infinite level $H^2_{\mathcal{M}}(X(\infty), \mathbb{Q}(3))$ depends only of the image of a and b in $\mathbb{Q}/\mathbb{Z}$ (where the image of $\overline{x} \in \mathbb{Z}/N\mathbb{Z}$ is defined as the class of x/N). In fact, we may choose any level N' divisible by N, and consider the element

$$\xi_G'((0, a), (0, b)) \in H^2_{\mathcal{M}}(X(\infty), \mathbb{Q}(3))$$

with $G = (\mathbb{Z}/N'\mathbb{Z})^2$. This class should not depend on N'. It is defined as a sum over G; assuming N' odd for simplicity, we have

$$\xi_G'(a, b) = \sum_{x\in G} \{u(0, x, a - x, b + x)\}_2 \otimes \left(\frac{g_b}{g_a}\right), \tag{26}$$

where a, b, x are seen as elements of $(\mathbb{Q}/\mathbb{Z})^2$. Note that one may view G as the full N'-torsion subgroup of the universal elliptic curve $E_1(N)$ over $Y_1(N)$. Applying the regulator map and taking the limit when $N' \to \infty$, this transforms the sum (26) into an integral along the fibres of $E_1(N) \to Y_1(N)$. This is reminiscent of the definition of $\mathrm{Eis}^{0,0,1}(a, b)$, which is also obtained by integrating along the fibres of the universal elliptic curve. However, it remains to relate these two integrals.

We finally come to one possible application of the motivic elements $\xi_1(a,b)$, namely for the Mahler measure of the polynomial $(1+x)(1+y)+z$. Recall that the (logarithmic) Mahler measure of a polynomial $P \in \mathbb{C}[x_1,\ldots,x_n]$ is defined by

$$m(P) = \frac{1}{(2\pi i)^n} \int_{T^n} \log|P(x_1,\ldots,x_n)| \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n},$$

where T_n is the n-torus $|x_1| = \cdots = |x_n| = 1$. The following conjecture was formulated by Boyd and Rodriguez Villegas; see [6, Section 8.4] and [18].

Conjecture 9.4. *We have* $m\big((1+x)(1+y)+z\big) = -2L'(E,-1)$*, where* $E = 15a8$ *is the elliptic curve of conductor 15 defined by*

$$E : (1+x)\Big(1+\frac{1}{x}\Big)(1+y)\Big(1+\frac{1}{y}\Big) = 1.$$

Lalín has shown in [18] that

$$m\big((1+x)(1+y)+z\big) = \frac{1}{4\pi^2} \int_{\gamma_E^+} r_3(2)\big(\{-x\}_2 \otimes y - \{-y\}_2 \otimes x\big),$$

where γ_E^+ is a generator of $H_1(E(\mathbb{C}),\mathbb{Z})^+$. The symbol $\xi := \{-x\}_2 \otimes y - \{-y\}_2 \otimes x$ defines an element of $H^2_{\mathcal{M}}(E,\mathbb{Q}(3))$. Moreover, the curve E is isomorphic to $X_1(15)$ and one can show that

$$-x = u(1,2,3,7), \qquad -y = u(2,4,6,1).$$

It would be interesting to express ξ in terms of the symbols $\xi_1(a,b)$ with $a,b \in \mathbb{Z}/15\mathbb{Z}$. Conjecture 9.4 would then follow from Theorem 9.1 and Conjecture 9.3.

References

[1] A. A. Beilinson, Higher regulators of modular curves, in *Applications of algebraic K-theory to algebraic geometry and number theory*, Part I, II (Boulder, Colo., 1983), Contemp. Math., 55 (Amer. Math. Soc., Providence, RI, 1986), 1–34.

[2] L. A. Borisov and P. E. Gunnells, Toric modular forms and nonvanishing of L-functions, *J. Reine Angew. Math.* **539** (2001), 149–165.

[3] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system. I. The user language. Computational algebra and number theory (London, 1993), *J. Symbolic Comput.* **24** (1997), no. 3-4, 235–265.

[4] F. Brunault, Beilinson-Kato elements in K_2 of modular curves, *Acta Arith.* **134** (2008), no. 3, 283–298.

[5] F. Brunault, Regulators of Siegel units and applications, *J. Number Theory* **163** (2016), 542–569.

[6] F. Brunault and W. Zudilin, *Many Variations of Mahler Measures: A Lasting Symphony*, Australian Mathematical Society Lecture Series **28** (Cambridge University Press, Cambridge, 2020).

[7] R. de Jeu, On $K_4^{(3)}$ of curves over number fields, *Invent. Math.* **125** (1996), no. 3, 523–556.

[8] R. de Jeu, Towards regulator formulae for the K-theory of curves over number fields, *Compositio Math.* **124** (2000), no. 2, 137–194.

[9] M. Derickx and M. van Hoeij, Gonality of the modular curve $X_1(N)$, *J. Algebra* **417** (2014), 52–71.

[10] A. B. Goncharov, Polylogarithms in arithmetic and geometry, in *Proceedings of the International Congress of Mathematicians*, Vol. 1, 2 (Zürich, 1994) (Birkhäuser, Basel, 1995), 374–387.

[11] A. B. Goncharov, Geometry of configurations, polylogarithms, and motivic cohomology, *Adv. Math.* **114** (1995), no. 2, 197–318.

[12] A. B. Goncharov, Deninger's conjecture on L-functions of elliptic curves at $s = 3$, Algebraic geometry, 4. *J. Math. Sci.* **81** (1996), no. 3, 2631–2656.

[13] A. B. Goncharov, Explicit regulator maps on polylogarithmic motivic complexes, in *Motives, polylogarithms and Hodge theory, Part I* (Irvine, CA, 1998), Int. Press Lect. Ser. **3**, I (Int. Press, Somerville, MA, 2002), 245–276.
[14] A. B. Goncharov, Euler complexes and geometry of modular varieties, *Geom. Funct. Anal.* **17** (2008), no. 6, pp. 1872–1914.
[15] A. B. Goncharov and D. Rudenko, Motivic correlators, cluster varieties and Zagier's conjecture on $\zeta_F(4)$, *Preprint* arXiv:1803.08585v1 [math.NT].
[16] A. Javanpeykar, The S-unit equation for functions on curves, MathOverflow, URL: `https://mathoverflow.net/q/355779` (version: 2020-03-26).
[17] D. S. Kubert and S. Lang, *Modular units*, Grundlehren der Mathematischen Wissenschaften **244** (Springer-Verlag, New York, 1981).
[18] M. Lalín, Mahler measure and elliptic curve L-functions at $s = 3$, *J. Reine Angew. Math.* **709** (2015), 201–218.
[19] R. C. Mason, The hyperelliptic equation over function fields. *Math. Proc. Cambridge Philos. Soc.* **93** (1983), no. 2, pp. 219–230.
[20] A. Mellit, Elliptic dilogarithms and parallel lines, *J. Number Theory* **204** (2019), 1–24.
[21] A. S. Merkurjev and A. A. Suslin, The group K_3 for a field, *Math. USSR-Izv.* **36** (1991), no. 3, 541–565.
[22] The PARI Group, PARI/GP version `2.11.4`, Univ. Bordeaux, 2020, `http://pari.math.u-bordeaux.fr/`.
[23] J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Math. **151** (Springer-Verlag, New York, 1994).
[24] G. Stevens, The Eisenstein measure and real quadratic fields, in *Théorie des nombres* (Quebec, PQ, 1987), de Gruyter, Berlin, 1989, 887–927.
[25] M. van Hoeij and H. Smith, A divisor formula and a bound on the $\mathbb{Q}$-gonality of the modular curve $X_1(N)$, *Preprint* arXiv:2004.13644v2 [math.NT].
[26] W. Wang, *Regularized integrals and L-functions of modular forms via the Rogers–Zudilin method*, PhD thesis (École normale supérieure de Lyon, 2020).
[27] Y. Yang, Transformation formulas for generalized Dedekind eta functions, *Bull. London Math. Soc.* **36** (2004), no. 5, 671–682.
[28] D. Zagier, Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields, in *Arithmetic algebraic geometry* (Texel, 1989), *Progr. Math.* **89** (Birkhäuser Boston, Boston, MA, 1991), 391–430.

François Brunault, francois.brunault@ens-lyon.fr
UMPA, ÉNS Lyon, 46 allée d'Italie, 69007 Lyon, France