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Alexey Glutsyuk. On odd-periodic orbits in complex planar billiards. Journal of Dynamical and Control Systems, Springer Verlag, 2014, 20 (3), pp.293-306. 10.1007/s10883-014-9236-5 . ensl-02443723

HAL Id: ensl-02443723

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Submitted on 17 Jan 2020

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On odd-periodic orbits in complex planar billiards

Alexey Glutsyuk [*†‡§]

April 8, 2014

Abstract

The famous conjecture of V.Ya.Ivrii (1978) says that *in every billiard with infinitely-smooth boundary in a Euclidean space the set of periodic orbits has measure zero.* In the present paper we study the complex version of Ivrii's conjecture for odd-periodic orbits in planar billiards, with reflections from complex analytic curves. We prove positive answer in the following cases: 1) triangular orbits; 2) odd-periodic orbits in the case, when the mirrors are algebraic curves avoiding two special points at infinity, the so-called isotropic points. We provide immediate applications to the partial classification of k-reflective real analytic pseudo-billiards with odd k, the real piecewise-algebraic Ivrii's conjecture and its analogue in the invisibility theory: Plakhov's invisibility conjecture.

1 Introduction

The famous V.Ya.Ivrii's conjecture [7] says that *in every billiard with infinitely-smooth boundary in a Euclidean space of any dimension the set of periodic orbits has measure zero.* As it was shown by V.Ya.Ivrii [7], it implies the famous H.Weyl's conjecture on the two-term asymptotics of the spectrum of Laplacian [17]. A brief historical survey of both conjectures with references

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[§]Supported by part by RFBR grants 10-01-00739-a, 13-01-00969-a, NTsNIL_a (RFBR-CNRS) grant 10-01-93115 and by ANR grants ANR-08-JCJC-0130-01, ANR-13-JS01-0010.

is presented in [6]. For triangular orbits Ivrii's conjecture was proved in [2, 11, 13, 16, 18]. For quadrilateral orbits in dimension two it was proved in [5, 6].

Remark 1.1 Ivrii's conjecture is open already for piecewise-analytic billiards, and we believe that this is its principal case. In the latter case Ivrii's conjecture is equivalent to the statement saying that for every $k \in \mathbb{N}$ the set of k-periodic orbits has empty interior. In the case, when the boundary is analytic, regular and convex, this was proved for arbitrary period in [15].

In the present paper we study a complexified version of Ivrii's conjecture in complex dimension two for odd periods. More precisely, we consider the complex plane $\mathbb{C}^2$ equipped with the complexified Euclidean metric, which is the standard complex-bilinear quadratic form. This defines notion of symmetry with respect to a complex line. Reflections of complex lines with respect to complex analytic curves are defined by the same formula, as in the real case. See [3, subsection 2.1] and Subsection 2.2 below for more detail.

Remark 1.2 Ivrii's conjecture has an analogue in the invisibility theory: Plakhov's invisibility conjecture. It appears that both conjectures have the same complexification. Thus, results on the complexified Ivrii's conjecture have applications to both Ivrii's and Plakhov's conjectures. See Section 5 for more details.

Main results and an application to the real Ivrii's conjecture and the plan of the paper are presented in Subsection 1.1.

1.1 Complex billiards, main results and plan of the paper.

Definition 1.3 A complex projective line $l \subset \mathbb{CP}^2 \supset \mathbb{C}^2$ is *isotropic*, if either it coincides with the infinity line, or the complexified Euclidean quadratic form $dz_1^2 + dz_2^2$ on $\mathbb{C}^2$ vanishes on l. Or equivalently, a line is isotropic, if it passes through some of two points at infinity with homogeneous coordinates $(1 : \pm i : 0)$: the *isotropic points at infinity.* In what follows we denote the latter points by

$$I_1 = (1 : i : 0), \quad I_2 = (1 : -i : 0).$$

Definition 1.4 The *symmetry* $\mathbb{C}^2 \to \mathbb{C}^2$ with respect to a non-isotropic complex line $L \subset \mathbb{CP}^2$ is the unique non-trivial complex-isometric involution fixing the points of L. It extends to a projective transformation of the ambient plane $\mathbb{CP}^2$.

Definition 1.5 [3, definition 1.3] A planar *complex analytic (algebraic) billiard* is a finite collection of complex irreducible[1] analytic (algebraic) curves-"mirrors" $a_1, \dots, a_k \subset \mathbb{CP}^2$. We assume that no mirror a_j is an isotropic line and set $a_0 = a_k$, $a_{k+1} = a_1$. A *k-periodic billiard orbit* is a collection of points $A_j \in a_j$, $A_{k+1} = A_1$, $A_k = A_0$, such that for every $j = 1, \dots, k$ one has $A_j \neq A_{j+1}$, the tangent line $T_{A_j} a_j$ is not isotropic and the complex lines $A_{j-1}A_j$ and A_jA_{j+1} are transverse to it and symmetric with respect to it. (Properly saying, we have to take points A_j together with prescribed branches of curves a_j at A_j: this specifies the line $T_{A_j} a_j$ in unique way, if A_j is a self-intersection point of the curve a_j.) The complex lines A_jA_{j+1} are called the *edges* of the orbit.

Remark 1.6 In a real billiard the reflection of a ray from the boundary is uniquely defined: the reflection is made at the first point where the ray meets the boundary. In the complex case, the reflection of lines with respect to a complex analytic curve is a multivalued mapping *(correspondence)* of the space of lines in $\mathbb{CP}^2$: we do not have a canonical choice of intersection point of a line with the curve. Moreover, the notion of interior domain does not exist in the complex case, since the mirrors have real codimension two.

Definition 1.7 [3, definition 1.4] A complex analytic billiard $a_1, \dots, a_k$ is *k-reflective,* if it has an open set of periodic orbits. In more detail this means that there exists an open set of pairs $(A_1, A_2) \in a_1 \times a_2$ extendable to k-periodic orbits $A_1 \dots A_k$. (Then the latter property automatically holds for every other pair of neighbor mirrors a_j, a_{j+1}.)

Problem (Complexified version of Ivrii's conjecture) [3, section 1]. *Classify all the k-reflective complex analytic (algebraic) billiards.*

It is known that there exist 4-reflective complex planar analytic and even algebraic billiards, see [14, p.59, corollary 4.6] and [3, section 1]. Their complete classification is given in [3, 4]. Their existence implies existence of k-reflective algebraic billiards for all $k \equiv 0(mod 4)$, see [3, remark 1.5].

Conjecture. There are no k-reflective complex analytic (algebraic) planar billiards for odd k.

The next two theorems partially confirm this conjecture.

Theorem 1.8 *Every planar complex analytic billiard with three mirrors is not 3-reflective.*

[1] By *irreducible* complex analytic curve in a complex manifold we mean an analytic curve holomorphically parametrized by a connected Riemann surface.

Theorem 1.9 *Let a planar complex algebraic billiard have odd number k of mirrors, and let each mirror contain no isotropic point at infinity. Then the billiard is not k-reflective.*

Theorem 1.8 is the complexification of the above-mentioned results by M.Rychlik et al on triangular orbits in real billiards, see [2, 11, 13, 16, 18]. Theorem 1.9 has immediate application to the real Ivrii's conjecture.

Corollary 1.10 *Consider a real planar billiard with piecewise-algebraic boundary. Let the complexifications of its algebraic pieces contain no isotropic point at infinity. Then the set of its odd-periodic orbits has measure zero.*

The corollary follows immediately from Theorem 1.9 and Remark 1.1.

Theorem 1.9 is proved in Section 3. Theorem 1.8 is proved in Section 4.

In Subsection 5.1 we present applications of Theorems 1.9 and 1.8 to the so-called real analytic pseudo-billiards: the billiards where the reflection preserves the angle, as in the usual billiard, but allows to cross the mirror. In Subsection 5.2 we provide applications to a particular case of Plakhov's invisibility conjecture.

The proofs of Theorems 1.9 and 1.8 are based on the following elementary fact.

Proposition 1.11 *The symmetry with respect to a non-isotropic line permutes the isotropic directions: the image of an isotropic line through the isotropic point I_1 at infinity passes through the other isotropic point I_2.*

Proposition 1.11 follows from [3, Proposition 2.4].

Corollary 1.12 *Let a periodic orbit in a complex planar analytic billiard have finite vertices, and at least one of its edges be isotropic. Then all the edges are isotropic, and their directions (corresponding isotropic points at infinity) are intermittent, see Fig.1. In particular, the period is even.*

Given an irreducible analytic curve $a \subset \mathbb{CP}^2$, by $\hat{a}$ we denote the Riemann surface parametrizing its maximal analytic extension bijectively, except for self-intersections; it is called its maximal normalization, see Subsection 2.1 for more details.

For the proof of Theorems 1.8 and 1.9 we lift the open set U_0 of periodic billiard orbits to the product of the maximal normalizations $\hat{a}_1 \times \cdots \times \hat{a}_k$ and consider its closure $U = \overline{U_0}$ in the latter product. This is an analytic subset with only two-dimensional irreducible components, see Subsection 2.2. It is non-empty, if and only if the billiard is k-reflective.

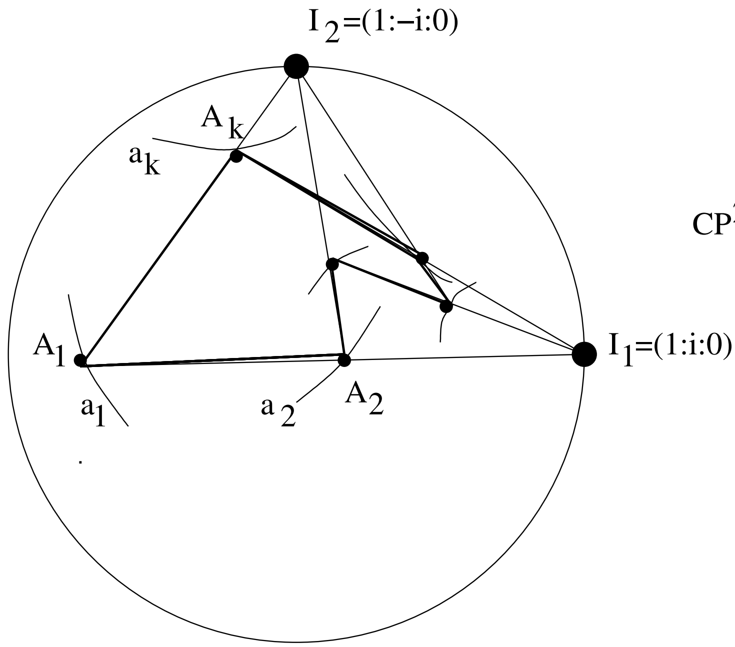

Figure 1: A periodic orbit with isotropic edges of intermittent directions

We prove Theorem 1.9 by contradiction. Supposing the contrary, i.e., the existence of an open set of k-periodic orbits, or equivalently, $U \neq \emptyset$, we take a one-parameter family Γ of k-gons $A_1 \dots A_k \in U$ with an isotropic edge A_1A_2. Its existence follows immediately from algebraicity. We show that a generic k-gon in Γ is a finite orbit with an isotropic edge, as in Corollary 1.12. To do this, it suffices to show that $A_j \not\equiv const$ and $A_j \not\equiv A_{j+1}$ on Γ for every j. This is the place we use the second technical assumption of Theorem 1.9 that $I_{1,2} \notin a_j$. This together with Corollary 1.12 implies that the period should be even, – a contradiction.

We prove Theorem 1.8 also by contradiction. Given an analytic billiard a, b, c, supposing its 3-reflectivity, we prove the existence of a one-dimensional analytic family Γ of orbits ABC with one isotropic edge AB; the vertices A and B vary along the curve Γ and $A \not\equiv B$ on Γ. This is the main technical part of the proof. To this end, we show that each mirror is either a rational curve, or a parabolic Riemann surface. This is done by considering the neighbor edge correspondence $(A, B) \mapsto (B, C)$ defined by U and proving its bimeromorphicity. Then we consider the two following cases:

Case 1): the vertex C varies along the curve Γ. We show that $A, B \not\equiv C$. This implies that a generic triangle $ABC \in \Gamma$ is a finite periodic orbit as in Corollary 1.12, and we get a contradiction, as above.

Case 2): $C \equiv const$ along the curve Γ. Then it follows immediately that C is an isotropic point at infinity. This together with [3, proposition 2.14] (recalled below as Proposition 2.8) implies that at least one of the edges AC or BC should coincide with the tangent line $T_C c$. This implies that all the vertices A, B, C are constant along the curve Γ, – a contradiction.

2 Maximal analytic extension and complex reflection law

2.1 Maximal analytic extension

Recall that a germ $(a, A) \subset \mathbb{CP}^n$ of analytic curve is *irreducible*, if it is the image of a germ of analytic mapping $(\mathbb{C}, 0) \to \mathbb{CP}^n$, $0 \mapsto A$.

Definition 2.1 Consider two holomorphic mappings of Riemann surfaces S_1, S_2 with base points $s_1 \in S_1$ and $s_2 \in S_2$ to $\mathbb{CP}^n$, $f_j : S_j \to \mathbb{CP}^n$, $j = 1, 2$, $f_1(s_1) = f_2(s_2)$. We say that $f_1 \leq f_2$, if there exists a holomorphic mapping $h : S_1 \to S_2$, $h(s_1) = s_2$, such that $f_1 = f_2 \circ h$. This defines a partial order on the set of classes of Riemann surface mappings to $\mathbb{CP}^n$ up to conformal reparametrization respecting base points.

Proposition 2.2 *Every irreducible germ of analytic curve in $\mathbb{CP}^n$ has maximal analytic extension. In more detail, let $(a, A) \subset \mathbb{CP}^n$ be an irreducible germ of analytic curve. There exists an abstract Riemann surface $\hat{a}$ with base point $\hat{A} \in \hat{a}$ (which we will call the* **maximal normalization** *of the germ a) and a holomorphic mapping $\pi_a : \hat{a} \to \mathbb{CP}^n$, $\pi_a(\hat{A}) = A$ with the following properties:*

- the image of germ at $\hat{A}$ of the mapping π_a is contained in a;

- π_a is a maximal mapping with the above property in the sense of Definition 2.1.

Moreover, the mapping π_a is unique up to composition with conformal isomorphism of Riemann surfaces respecting base points.

Proof An irreducible germ of an analytic curve a in an affine chart $\mathbb{C}^n \subset \mathbb{CP}^n$ is locally the graph of a germ of a (multivalued) analytic function $\alpha : \mathbb{C} \to \mathbb{C}^{n-1}$. The Riemann surface $\hat{a}$ of the maximal meromorphic extension of the germ α (taken together with branching points), see [12, Encadré XI, p.407], satisfies the statements of the proposition. $\Box$

Example 2.3 The maximal normalization of a projective algebraic curve is its usual normalization: a compact Riemann surface parametrizing the curve bijectively, except for self-intersections.

2.2 Complex reflection law

The material presented in this subsection is contained in [3, subsection 2.1].

We fix an Euclidean metric on $\mathbb{R}^2$ and consider its complexification: the complex-bilinear quadratic form $dz_1^2 + dz_2^2$ on the complex affine plane $\mathbb{C}^2 \subset \mathbb{CP}^2$. We denote the infinity line in $\mathbb{CP}^2$ by $\overline{\mathbb{C}}_\infty = \mathbb{CP}^2 \setminus \mathbb{C}^2$.

Definition 2.4 Let L be an isotropic line through a finite point x. A pair of lines through x is called *symmetric with respect to L,* if it is a limit of symmetric pairs of lines with respect to non-isotropic lines converging to L.

Lemma 2.5 *[3, lemma 2.3] Let L be an isotropic line through a finite point x. A pair of lines (L_1, L_2) through x is symmetric with respect to L, if and only if some of them coincides with L.*

Convention 2.6 Sometimes we identify a point (subset) in a with its preimage in the normalization $\hat{a}$ and denote both subsets by the same symbol. In particular, given a subset in $\mathbb{CP}^2$, say a line l, we set $\hat{a} \cap l = \pi_a^{-1}(a \cap l) \subset \hat{a}$. If $a, b \subset \mathbb{CP}^2$ are two curves, and $A \in \hat{a}$, $B \in \hat{b}$, $\pi_a(A) \neq \pi_b(B)$, then for simplicity we write $A \neq B$ and the line $\pi_a(A)\pi_b(B)$ will be referred to, as AB. For every $A \in \hat{a}$ the *local branch a_A of the curve a at A* is the irreducible germ of analytic curve given by the germ of normalization parametrization $\pi_a : (\hat{a}, A) \to (\mathbb{CP}^2, \pi_a(A))$. The tangent line $T_{\pi_a(A)} a_A$ will be referred to, as $T_A a$.

Definition 2.7 [3, definition 2.13] Let $a_1, \dots, a_k \subset \mathbb{CP}^2$ be an analytic (algebraic) billiard, and let $\hat{a}_1, \dots, \hat{a}_k$ be the maximal normalizations of its mirrors. The set P_k of k-periodic orbits lifted to the latter product is contained in the subset $Q_k \subset \hat{a}_1 \times \dots \times \hat{a}_k$ of (not necessarily periodic) k-orbits: the k-gons $A_1 \dots A_k$ such that for every $2 \leq j \leq k-1$ one has $A_j \neq A_{j\pm 1}$, the line $T_{A_j} a_j$ is not isotropic and the lines $A_j A_{j-1}$, $A_j A_{j+1}$ are symmetric with respect to it. Let $U_0 = \mathrm{Int}(P_k)$ denote the interior of the subset $P_k \subset Q_k$. The closure

$$U = \overline{U_0} \subset \hat{a}_1 \times \dots \times \hat{a}_k$$

in the product topology will be called the *k-reflective set.*

Proposition 2.8 *[3, proposition 2.14] The k-reflective set U is analytic (algebraic). The billiard is k-reflective, if and only if $U \neq \emptyset$. In this case each irreducible component of the set U is two-dimensional and each projection $U \to \hat{a}_j \times \hat{a}_{j+1}$ is a submersion on an open dense subset (epimorphic, if the billiard is algebraic). For every point $A_1 \dots A_k \in U$ and every j such that $A_{j\pm1} \neq A_j$ the* **complex reflection law** *holds:*

- if the tangent line $l_j = T_{A_j} a_j$ is not isotropic, then the lines $A_{j-1}A_j$ and A_jA_{j+1} are symmetric with respect to l_j;

- otherwise, if l_j is isotropic (finite or infinite), then at least one of the lines $A_{j-1}A_j$ or A_jA_{j+1} coincides with l_j.

3 Algebraic billiards: proof of Theorem 1.9

Definition 3.1 Let $a_1, \dots, a_k$ be a complex planar analytic (algebraic) billiard. A point $A \in a_j$ is *marked*, if it is either a cusp[2], or an isotropic tangency point of the mirror a_j. A point $A \in \mathbb{CP}^2$ is *double*, if it is either a self-intersection of a mirror, or an intersection point of two distinct mirrors. A point $A \in \hat{a}_j$ is marked, if it is a marked point of the local branch of the curve a_j at A, see Convention 2.6.

Suppose the contrary to Theorem 1.9: there exist an odd k and a k-reflective algebraic billiard $a_1, \dots, a_k$ with $I_{1,2} \notin a_j$ for every j. Let $U \subset \hat{a}_1 \times \dots \times \hat{a}_k$ denote its k-reflective set, which is an algebraic set with only two-dimensional irreducible components (Proposition 2.8).

Proposition 3.2 *There exists an irreducible algebraic curve Γ of k-gons $A_1 \dots A_k \in U$ such that $A_1 \not\equiv A_2$, $A_1, A_2 \not\equiv const$ along the curve Γ, and for every $A_1 \dots A_k \in \Gamma$ with $A_1 \neq A_2$ the line A_1A_2 is isotropic through the point I_1.*

Proof Recall that the projection $U \to \hat{a}_1 \times \hat{a}_2$ is epimorphic (Proposition 2.8): each pair $(A_1, A_2) \in \hat{a}_1 \times \hat{a}_2$ lifts to a k-gon $A_1 \dots A_k \in U$. The mirrors a_1 and a_2 do not coincide with one and the same line, since otherwise, there would exist no k-periodic orbit in the sense of Definition 1.5, as in [3, proof of Corollary 2.19]. Therefore, a generic line through the isotropic point I_1

[2]Everywhere in the paper by *cusp* we mean the singularity of an arbitrary irreducible singular germ of analytic curve, not necessarily the one given by equation $x^2 = y^3 + \dots$ in appropriate coordinates.

at infinity intersects a_1 and a_2 in at least two distinct points A_1 and A_2 respectively. The closure of liftings to $\hat{a}_1 \times \hat{a}_2$ of the latter pairs (A_1, A_2) is an algebraic curve H. Its projection preimage in U obviously contains an irreducible component Γ with non-constant projection to H. This Γ obviously satisfies the statements of the proposition. □

Proposition 3.3 *Let Γ be as in the above proposition. Then for every $j = 1, \dots, k$ one has $A_j \not\equiv A_{j+1}$, $A_j \not\equiv const$ along the curve Γ. For every $A_1 \dots A_k \in \Gamma$ with $A_j \neq A_{j+1}$ the line A_jA_{j+1} is an isotropic line through I_1 (I_2) if j is odd (respectively, even).*

Proof Induction on j.

Induction base: $A_1 \not\equiv A_2$, $A_1, A_2 \not\equiv$ *const* along the curve Γ, by Proposition 3.2, and the line A_1A_2 is isotropic through I_1, by definition.

Induction step. Let we have already shown that $A_{j-1} \not\equiv A_j$ and $A_{j-1} \not\equiv$ *const* along the curve Γ, and for every $A_1 \dots A_k \in \Gamma$ with $A_{j-1} \neq A_j$ the line $A_{j-1}A_j$ is isotropic, say, through I_1. Let us show that $A_j \not\equiv$ *const*, $A_j \not\equiv A_{j+1}$ and the line A_jA_{j+1} is isotropic through I_2, whenever $A_j \neq A_{j+1}$. Indeed, the isotropic line $A_{j-1}A_j$ is non-constant along Γ, as is A_{j-1}, and passes through the fixed isotropic point I_1. Therefore, its intersection points with the curve a_j vary, since $I_1 \notin a_j$ by assumption. This implies that $A_j \not\equiv$ *const*. Now suppose, by contradiction, that $A_j \equiv A_{j+1}$ on Γ; then $a_j = a_{j+1}$. Fix a k-gon $x = A_1 \dots A_k \in \Gamma$ with $A_{j-1} \neq A_j$ such that the tangent line $T_{A_j}a_j$ is not isotropic. The k-gon $x \in U$ is a limit of k-periodic billiard orbits $A_1^n \dots A_k^n$. By definition, $A_s^n \neq A_{s+1}^n$ for every s. The vertices A_j^n and A_{j+1}^n collide to the same limit A_j, hence the line $A_j^nA_{j+1}^n$ tends to the tangent line $T_{A_j}a_j$. On the other hand, $A_j^nA_{j+1}^n$ tends to the line through A_j symmetric to $A_{j-1}A_j$ with respect to $T_{A_j}a_j$. Hence, the latter limit line is isotropic through I_2 (Proposition 1.11), and at the same time, it coincides with a non-isotropic line $T_{A_j}a_j$. The contradiction thus obtained implies that $A_j \not\equiv A_{j+1}$ along Γ. The above argument also implies that if $A_j \neq A_{j+1}$, then the latter limit isotropic line through I_2 coincides with A_jA_{j+1}. The induction step is over. The proposition is proved. □

The curve Γ from Proposition 3.2 contains a finite k-periodic orbit with isotropic edges of intermittent directions, by Proposition 3.3. But then k should be even by Corollary 1.12. The contradiction thus obtained proves Theorem 1.9.

4 Triangular orbits: proof of Theorem 1.8

We prove Theorem 1.8 by contradiction. Suppose the contrary: there exists a 3-reflective analytic billiard a, b, c in $\mathbb{CP}^2$, let $U \subset \hat{a} \times \hat{b} \times \hat{c}$ be its 3-reflective set. The analytic subset $U \subset \hat{a} \times \hat{b} \times \hat{c}$ defines a correspondence $\psi_b : \hat{a} \times \hat{b} \to \hat{b} \times \hat{c}$: $(A, B) \mapsto (B, C)$ for every $ABC \in U$. First we show in the next proposition that the correspondence ψ_b extends to a bimeromorphic isomorphism $\hat{a} \times \hat{b} \to \hat{b} \times \hat{c}$. This implies (Corollary 4.3) that each mirror is either a rational curve, or a parabolic Riemann surface. Afterwards we deduce that the mirrors are distinct (Proposition 4.4) and there exists a one-dimensional family Γ of triangles $ABC \in U$ with isotropic edges AB (Corollary 4.5). This will bring us to a contradiction analogously to the above proof of Theorem 1.9.

Proposition 4.1 *Let a, b, c, U and ψ_b be as above. The correspondence ψ_b extends to a bimeromorphic*[3] *isomorphism $\hat{a} \times \hat{b} \to \hat{b} \times \hat{c}$.*

Proof It suffices to show that the mapping ψ_b is meromorphic: the proof of the meromorphicity of its inverse is analogous. Consider the auxiliary mapping $Q_{ab} : \hat{a} \times \hat{b} \to \mathbb{CP}^2$ defined as follows. Take an arbitrary pair $(A, B) \in \hat{a} \times \hat{b}$ with $A \neq B$, non-isotropic tangent lines $T_A a$, $T_B b$ and such that $AB \neq T_A a, T_B b$. Set $Q_{ab}(A, B)$ to be the point of intersection of two lines: the images of the line AB under the symmetries with respect to the tangent lines $T_A a$ and $T_B b$. The mapping Q_{ab} extends to a meromorphic mapping $\hat{a} \times \hat{b} \to \mathbb{CP}^2$, by the algebraicity of the reflection law. (Possible indeterminacies correspond to isolated points where either $A = B$ is a double point, or one of the tangent lines $T_A a$ or $T_B b$ is isotropic and coincides with AB.) Note that $Q_{ab}(A, B) \in c$ for every (A, B) from the domain of the mapping Q_{ab}, since this holds for an open set of pairs (A, B) that extend to triangular orbits ABC: the third vertex C is found as the intersection point of the above symmetric images of the line AB. This implies that the mapping ψ_b extends to a meromorphic mapping $\hat{a} \times \hat{b} \to \hat{b} \times \hat{c}$ by the formula $\psi_b(A, B) = (B, \pi_c^{-1} \circ Q_{ab}(A, B))$. The proposition is proved. □

Corollary 4.2 *The projection $U \to \hat{a} \times \hat{b}$ is bimeromorphic and in particular, the 3-reflective set U is irreducible. The complement to its image is at most discrete.*

[3] Recall that a *meromorphic mapping* $M \to N$ between complex manifolds is a mapping holomorphic on the complement of an analytic subset in M such that the closure of its graph is an analytic subset in $M \times N$.

Proof The first statement of the corollary follows immediately from the proposition. Let us prove the second statement. The inverse of the projection being induced by a meromorphic mapping $Q_{ab} : \hat{a} \times \hat{b} \to \mathbb{CP}^2$, it is holomorphic outside the indeterminacy locus of the mapping Q_{ab}. The latter locus is at most discrete, see the above proof. The corollary is proved. □

Corollary 4.3 *Let a, b, c be a 3-reflective analytic billiard in* $\mathbb{CP}^2$. *Then the maximal normalization of each its mirror is either parabolic (having universal cover* $\mathbb{C}$*), or conformally equivalent to the Riemann sphere.*

Proof A Riemann surface has one of the two above types, if and only if it admits a nontrivial holomorphic family of conformal automorphisms. Thus, it suffices to show that the maximal normalization of each mirror has a nontrivial holomorphic family of automorphisms, or equivalently, has a nontrivial holomorphic family of conformal isomorphisms onto a given Riemann surface. Fix a non-marked point $B \in \hat{b}$ that represents a finite point in $\mathbb{CP}^2$. For every $A \in \hat{a}$ set $\phi_B(A) = \pi_c^{-1} \circ Q_{ab}(A, B) \in \hat{c}$. This yields a family of conformal isomorphisms $\phi_B : \hat{a} \to \hat{c}$ depending holomorphically on B from an open and dense subset in $\hat{b}$, by bimeromorphicity (Proposition 4.1). In particular, the Riemann surfaces $\hat{a}$ and $\hat{c}$ are conformally equivalent. Similarly, $S = \hat{a} \simeq \hat{b} \simeq \hat{c}$. If the family ϕ_B is nontrivial (non-constant in B), then the Riemann surface S is either parabolic, or the Riemann sphere, by the statement from the beginning of the proof. We claim that in the contrary case, when ϕ_B is independent on B, one has $b \simeq \overline{\mathbb{C}}$. Indeed, let $\phi = \phi_B$ be independent on B. Fix an arbitrary finite point $A \in \hat{a}$, set $C = \phi(A)$. Then *for every* $B \in \hat{b}$ the lines AB and BC are symmetric with respect to the tangent line $T_B b$. Hence, b is either a line, or a conic, by [3, proposition 2.32]. Thus, $b \simeq \overline{\mathbb{C}}$. This proves the corollary. □

Proposition 4.4 *Let a, b, c be a 3-reflective analytic billiard in* $\mathbb{CP}^2$. *Then its mirrors are pairwise distinct: one is not analytic extension of another.*

Proof Suppose the contrary, say, $a = b$. Then the 3-reflective set U contains an irreducible one-dimensional analytic subset Γ such that for every $ABC \in \Gamma$ one has $A = B$, and $A, B \not\equiv const$ along the curve Γ. This follows from the second statement of Corollary 4.2: the image of the projection $U \to \hat{a} \times \hat{b}$ covers the diagonal with at most a discrete subset deleted.

Case 1): $A \equiv B \not\equiv C$ on Γ. Then $AC \equiv BC \equiv T_A a$ on Γ, as in [3, proof of corollary 2.19]. Indeed, every triangle $ABC \in \Gamma$ with the tangent line

$T_A a$ being non-isotropic is a limit of triangular orbits $A^nB^nC^n$ with distinct colliding vertices $A^n, B^n \to A$. Thus, $A^nB^n \to T_A a$, hence $A^nC^n, B^nC^n \to T_A a = AC = BC$ by reflection law. This implies that $C \not\equiv const$ along the curve Γ, being the intersection point of the curve c with the tangent line to a at a variable point A. Therefore, the curve Γ contains triangles ABC such that $C \neq A = B$ and A, B, C are not marked points. This contradicts [3, proposition 2.18].

Case 2): $A \equiv B \equiv C$ on Γ. Then we similarly get a contradiction to the same proposition (cf. [15]). This proves Proposition 4.4. □

Corollary 4.5 *There exists a one-dimensional irreducible analytic subset $\Gamma \subset U$ such that $A \not\equiv B$, $A, B \not\equiv const$ along the curve Γ and for every $ABC \in \Gamma$ with $A \neq B$ the line AB is isotropic through I_1.*

Proof There exists a line L through I_1 intersecting the curves a and b in at least two distinct finite points A and B respectively. Or equivalently, the projection $\mathbb{CP}^2 \setminus I_1 \to \mathbb{CP}^1$ from the point I_1 sends some two distinct points $A \in a$ and $B \in b$ to the same point. Indeed, for every $g = a, b$ its composition with the normalization parametrization $\pi_g : \hat{g} \to \mathbb{CP}^2$ extends to a non-constant holomorphic mapping $\hat{g} \to \overline{\mathbb{C}} = \mathbb{CP}^1$. The non-constance follows from the assumption that g is not an isotropic line. The Riemann surface $\hat{g}$ being either parabolic or Riemann sphere (Corollary 4.3), the latter mapping $\hat{g} \to \overline{\mathbb{C}}$ takes all the values except for at most two (Picard's Theorem). This together with the inequality $a \neq b$ implies the existence of the above L, A and B. Deforming the isotropic line L one can achieve that (A, B) be the projection of a triangle $ABC \in U$ (the second statement of Corollary 4.2). The condition that either $A = B$, or the line AB is isotropic through I_1 defines a one-dimensional analytic subset in U containing ABC. The curve Γ we are looking for is its one-dimensional irreducible component containing ABC. The corollary is proved. □

Proof of Theorem 1.8. Let Γ be the same, as in Corollary 4.5. We consider the two following cases:

Case 1): $C \not\equiv const$ along the curve Γ. One has $A, B \not\equiv C$, since $a, b \neq c$, by Proposition 4.4. Then there exists a triangular billiard orbit $ABC \in \Gamma$ with finite vertices that are not marked points. Its edges are isotropic lines with intermittent directions, since AB is isotropic and by Corollary 1.12. Hence, the period of the triangular orbit should be even, by the same corollary, – a contradiction.

Case 2): $C \equiv const$ along the curve Γ. For every $ABC \in \Gamma$ with A, B being not marked points of the curves a and b, $A \neq B$ and $A, B \neq C$ the line AC is isotropic through I_2, being the reflection image of the isotropic line AB through I_1 with respect to T_Bb (Proposition 1.11). This implies that $C = I_2$, since $A \not\equiv const$, $C \equiv const$ on the curve Γ and the curve a is not an isotropic line. Thus, the projective tangent line T_Cc contains I_2 and hence, is isotropic. This implies that for every ABC as above one of the lines AC or BC coincides identically with T_Cc (Proposition 2.8, see Fig.2), and hence, some of the vertices A or B is constant along the curve Γ, – a contradiction to Corollary 4.5. The proof of Theorem 1.8 is complete. $\square$

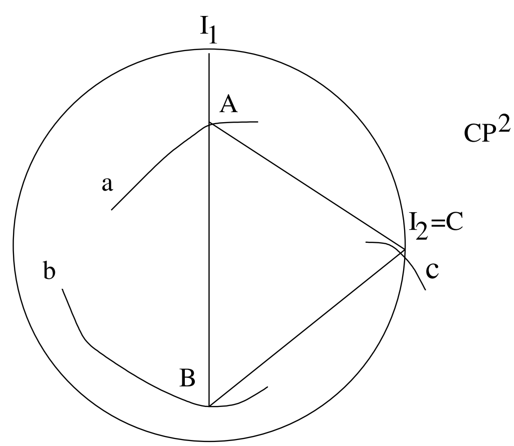

Figure 2: Triangular orbits with isotropic vertex C

5 Applications to pseudo-billiards and invisibility

5.1 On k-reflective analytic pseudo-billiards with odd k

Here by *real analytic curve* we mean a curve $a \subset \mathbb{RP}^2$ analytically parametrized by either $\mathbb{R}$, or S^1 that is not the infinity line. If a curve a has singularities (cusps or self-intersections), we consider its maximal real analytic extension $\pi_a : \hat{a} \to a$, where $\hat{a}$ is either $\mathbb{R}$, or S^1, see [6, lemma 37, p.302]. The parametrizing curve $\hat{a}$ will be called here the *real normalization.* The affine plane $\mathbb{R}^2 \subset \mathbb{RP}^2$ is equipped with Euclidean metric.

Definition 5.1 [3, remark 1.6] A triple of points $A, B, C \in \mathbb{R}^2$, $A \neq B$, $B \neq C$, and a line $L \subset \mathbb{R}^2$ through B satisfy the *usual real reflection law (skew real reflection law),* if the lines AB and BC are symmetric with respect to L, and also the points A *and* C *lie in the same half-plane (respectively, different half-planes)* with respect to the line L.

Remark 5.2 (loc.cit). A triple of real points $A, B, C \in \mathbb{R}^2$, $A \neq B$, $B \neq C$ and a line $L \subset \mathbb{R}^2$ through B satisfy the complex reflection law, i.e., the complex lines AB and BC are symmetric with respect to the line L, if and only if they satisfy either usual, or skew real reflection law.

Definition 5.3 [3, definition 6.1] A *real planar analytic (algebraic) pseudo-billiard* is a collection of k real irreducible analytic (algebraic) curves $a_1, \dots, a_k \subset \mathbb{RP}^2$. Its *$k$-periodic orbit* is a k-gon $A_1 \dots A_k$, $A_j \in a_j \cap \mathbb{R}^2$, such that for every $j = 1, \dots, k$ one has $A_j \neq A_{j\pm1}$, $A_jA_{j\pm1} \neq T_{A_j}a_j$ and the lines A_jA_{j-1}, A_jA_{j+1} are symmetric with respect to the tangent line $T_{A_j}a_j$. The latter means that for every j the triple A_{j-1}, A_j, A_{j+1} and the line $T_{A_j}a_j$ satisfy either usual, or skew real reflection law; we then say that usual (skew) reflection law is satisfied at A_j. See Fig.3 for $k = 3$. Here we set $a_{k+1} = a_1$, $A_{k+1} = A_1$, $a_0 = a_k$, $A_0 = A_k$. A real pseudo-billiard is called *k-reflective,* if it has an open set (i.e., a two-parameter family) of k-periodic orbits.

Theorem 5.4 *Let in a real algebraic planar pseudo-billiard $a_1, \dots, a_k$ the number k be odd and the complexification of each mirror a_j contain no isotropic point at infinity. Then the pseudo-billiard is not k-reflective.*

Theorem 5.5 *There are no 3-reflective real planar analytic pseudo-billiards.*

The latter theorems follow from Theorems 1.9 and 1.8 respectively and the fact that the complexification of a k-reflective planar analytic pseudo-billiard is a k-reflective complex billiard [3, remark 6.2].

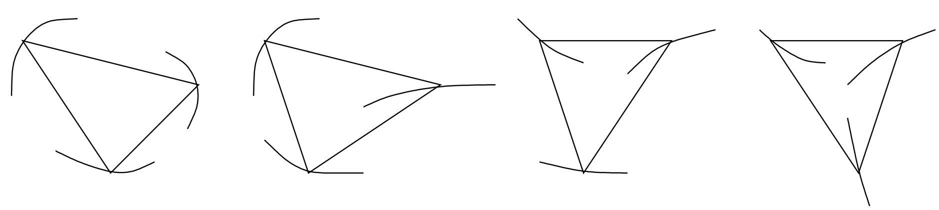

Figure 3: Triangular orbits of pseudo-billiards with three mirrors: their open sets are forbidden by Theorem 5.5.

Remark 5.6 The 4-reflective real planar analytic pseudo-billiards are classified in [3, 4].

5.2 Application to Plakhov's invisibility conjecture

This subsection is devoted to Plakhov's invisibility conjecture: the analogue of Ivrii's conjecture in the invisibility theory [8, conjecture 8.2]. We recall it below and show that it follows from a conjecture saying that no finite collection of germs of smooth curves can form a k-reflective pseudo-billiard with only two skew reflection laws at two neighbor mirrors. This shows that both invisibility and Ivrii's conjectures have the same complexification. For simplicity we present this relation in dimension two. We state and prove Corollaries 5.13 and 5.15 of Theorems 5.5 and 5.4 for planar Plakhov's invisibility conjecture.

Definition 5.7 Consider an arbitrary perfectly reflecting (may be disconnected) closed bounded body B in a Euclidean space. For every oriented line (ray) R take its first intersection point A_1 with the boundary ∂B and reflect R from the tangent hyperplane $T_{A_1}\partial B$. The reflected ray goes from the point A_1 and defines a new oriented line. Then we repeat this procedure. Let us assume that after a finite number of reflections the output oriented line coincides with the input line R and will not hit the body any more. Then we say that the body B *is invisible for the ray R,* see Fig.4. We call R a *ray of invisibility,* and the finite piecewise-linear curve bounded by the first and last reflection points will be called its *complete trajectory.*

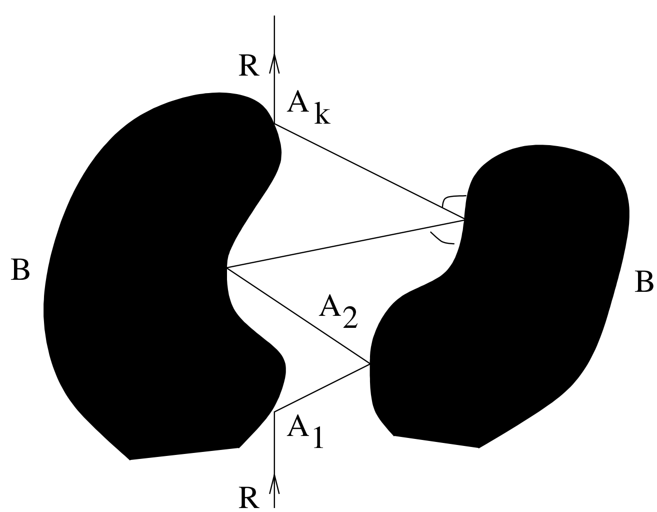

Figure 4: A body invisible in one direction.

Invisibility Conjecture (A.Plakhov, [8, conjecture 8.2, p.274].) *There is no body with piecewise C^∞ boundary for which the set of rays of invisibility has positive measure.*

Remark 5.8 As is shown by A.Plakhov in his book [8, section 8], there exist no body invisible for all rays. The same book contains a very nice survey on invisibility, including examples of bodies invisible for a finite number of (one-dimensional families of) rays. See also papers [1, 9, 10] for more results. The Invisibility Conjecture is open even in dimension 2. It is equivalent to the statement saying that there are no k-reflective bodies for every k, see the next definition.

Definition 5.9 A body B with piecewise-smooth boundary is called *k-reflective,* if the set of invisibility rays with k reflections has positive measure.

Definition 5.10 Let $a_1, \dots, a_k$ be a collection of (germs of) planar smooth curves. A k-gon $A_1 \dots A_k$ with $A_j \in a_j$, $A_{k+1} = A_1$, $A_0 = A_k$ is said to be a *k-invisible orbit*, if

- $A_j \neq A_{j+1}$ for every $j = 1, \dots, k$;
- the tangent line $T_{A_j} a_j$ is the exterior bisector of the angle $\angle A_{j-1} A_j A_{j+1}$ whenever $j \neq 1, k$, and it is its interior bisector for $j = 1, k$, see Fig.5.

We say that the collection $a_1, \dots, a_k$ is a *k-invisible billiard,* if the set of its k-invisible orbits has positive measure.

Proposition 5.11 *Let $k \in \mathbb{N}$ and $B \subset \mathbb{R}^2$ be a body such that no collection of k germs of its boundary forms a k-invisible billiard. Then the body B is not k-reflective.*

Proposition 5.11 is implicitly contained in [8, section 8].

Remark 5.12 A k-invisible billiard $a_1, \dots, a_k$ with analytic mirrors is a k-reflective planar analytic pseudo-billiard. It has an open set of k-periodic orbits with skew reflection law only at the mirrors a_1 and a_k.

Corollary 5.13 *There are no 3-reflective bodies in $\mathbb{R}^2$ with piecewise-analytic boundary.*

Remark 5.14 Corollary 5.13 is known to specialists. As it is stated in A.Plakhov's book [8] (after conjecture 8.2), Corollary 5.13 can be proved by adapting the proof of Ivrii's conjecture for triangular orbits. A.Plakhov's unpublished proof of Corollary 5.13 follows [18].

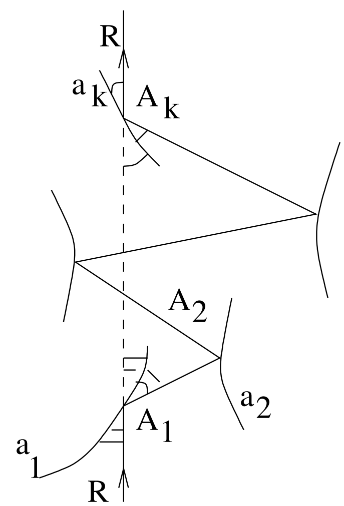

Figure 5: A k-invisible k-gon: skew reflection law at A_1 and A_k.

Corollary 5.15 *Let $B \subset \mathbb{R}^2$ be a body with piecewise-algebraic boundary, and let the complexifications of its algebraic pieces contain no isotropic point at infinity. Then B is not k-reflective for every odd k.*

Corollaries 5.13 and 5.15 follow from Proposition 5.11, Remark 5.12 and Theorems 5.5 and 5.4.

6 Acknowledgements

I am grateful to Yu.S.Ilyashenko, Yu.G.Kudryashov and A.Yu.Plakhov for attracting my attention to Ivrii's conjecture and invisibility and for helpful discussions.

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