On commuting billiards in higher-dimensional spaces of constant curvature

Alexey Glutsyuk

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Abstract

We consider two nested billiards in $\mathbb{R}^d$, $d \geq 3$, with C^2-smooth strictly convex boundaries. We prove that if the corresponding actions by reflections on the space of oriented lines commute, then the billiards are confocal ellipsoids. This together with the previous analogous result of the author in two dimensions solves completely the Commuting Billiard Conjecture due to Sergei Tabachnikov. The main result is deduced from the classical theorem due to Marcel Berger saying that in higher dimensions only quadrics may have caustics. We also prove versions of Berger's theorem and the main result for billiards in spaces of constant curvature: space forms.

Contents

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1 Introduction

1.1 Main result

Let $\Omega_a \Subset \Omega_b \subset \mathbb{R}^d$ be two nested bounded domains with smooth strictly convex boundaries $a = \partial\Omega_a$ and $b = \partial\Omega_b$. Consider the corresponding billiard transformations σ_a, σ_b acting on the space of oriented lines in space by reflection as follows. Each σ_f, $f = a, b$, acts as identity on the lines disjoint from f. For each oriented line l intersecting f we take its last intersection point x with f in the sense of orientation: the orienting arrow of the line l at x is directed outside Ω_f. The image $\sigma_f(l)$ is the line obtained by reflection of the line l from the hyperplane T_xf: the angle of incidence equals the angle of reflection. The line $\sigma_f(l)$ is oriented by a tangent vector at x directed inside Ω_f. This is a continuous mapping that is smooth on the space of lines intersecting f transversely.

Remark 1.1 The above action can be defined for a convex billiard in any Riemannian manifold; the billiard reflection acts on the space of oriented geodesics.

Recall, see, e.g., [2, 18], that a pencil of *confocal quadrics* in a Euclidean space $\mathbb{R}^d$ is a one-dimensional family of quadrics defined in some orthogonal coordinates $(x_1, \dots, x_d)$ by equations

$$\sum_{j=1}^{d} \frac{x_j^2}{a_j^2 + \lambda} = 1; \quad a_j \in \mathbb{R} \text{ are fixed; } \lambda \in \mathbb{R} \text{ is the parameter.}$$

It is known that *any two confocal elliptic or ellipsoidal billiards* commute [18, p.59, corollary 4.6], [20, p.58]. Sergei Tabachnikov stated the conjecture affirming the converse: any two commuting nested convex billiards are confocal ellipses (ellipsoids) [20, p.58]. In two dimensions his conjecture was proved by the author of the present paper in [10, theorem 5.21, p.231] for piecewise C^4-smooth boundaries. Here we prove it in higher dimensions in $\mathbb{R}^d$ and in spaces of constant curvature: space forms.

Theorem 1.2 *Let two nested strictly convex C^2-smooth closed hypersurfaces in $\mathbb{R}^d$, $d \geq 3$, be such that the corresponding billiard transformations commute. Then they are confocal ellipsoids.*

To extend Theorem 1.2 to spaces of constant curvature, let us recall the notions of space forms and (confocal) quadrics in them.

Definition 1.3 A *space form* is a complete connected Riemannian manifold of constant curvature.

Remark 1.4 We will deal only with simply connected space forms. It is well-known that they are the Euclidean space $\mathbb{R}^d$, the unit sphere $S^d \subset \mathbb{R}^{d+1}$ in the Euclidean space and the hyperbolic space $\mathbb{H}^d$ (up to normalization of the metric by constant scalar factor, which changes neither geodesics, nor reflections). It is known that the hyperbolic space $\mathbb{H}^d$ admits a standard model in the Minkovski space $\mathbb{R}^{d+1}$. Finally, each space form Σ we will be dealing with is realized as an appropriate hypersurface in the space $\mathbb{R}^{d+1}$ with coordinates $x = (x_0, \dots, x_d)$ equipped with a suitable quadratic form

$$< Gx, x >, \quad G \text{ is a symmetric } \ (d+1) \times (d+1) - \text{matrix}.$$

Here $< x, x >:= \sum_j x_j^2$.

Euclidean case: $G = \operatorname{diag}(0, 1, \dots, 1)$, $\Sigma = \mathbb{R}^d = \{x_0 = 1\}$.

Spherical case: $G = Id$, $\Sigma = S^d = \{< Gx, x >= 1\}$, $< Gx, x >= \sum_j x_j^2$.

Hyperbolic case: $G = \operatorname{diag}(-1, 1, \dots, 1)$, $\Sigma = \mathbb{H}^d = \{< Gx, x >= -1\} \cap \{x_0 > 0\}$.

The metric on each hypersurface Σ is the restriction to $T\Sigma$ of the quadratic form $< Gx, x >$ on the ambient space. It is well-known that the geodesics on Σ are its intersections with two-dimensional vector subspaces in $\mathbb{R}^{d+1}$. Completely geodesic k-dimensional submanifolds in Σ are its intersections with $(k+1)$-dimensional vector subspaces in $\mathbb{R}^{d+1}$.

Definition 1.5 [22, p. 84] A *quadric* in Σ is a hypersurface

$$S = \Sigma \cap \{< Qx, x >= 0\}, \ Q \text{ is a symmetric matrix}.$$

The *pencil of confocal quadrics* associated to a symmetric matrix Q is the family of quadrics

$$S_\lambda = \Sigma \cap \{< Q_\lambda x, x >= 0\}, \ Q_\lambda = (Q - \lambda G)^{-1}, \ \lambda \in \mathbb{R}.$$

Definition 1.6 A germ of C^2-smooth hypersurface S in a space form Σ at a point p is *strictly convex*, if it has quadratic tangency with its tangent completely geodesic hypersurface Γ_p: there exists a constant $C > 0$ such that for every $q \in S$ close to p one has

$$\mathrm{dist}(q, \Gamma_p) > C||q-p||^2; \text{ here } ||q-p|| = \mathrm{dist}(q,p).$$

Theorem 1.7 *Let $d \geq 3$, and let Σ be a simply connected d-dimensional space form: either $\mathbb{R}^d$, or the unit sphere, or the hyperbolic space. Let two nested strictly convex C^2-smooth closed hypersurfaces in Σ be such that the corresponding billiard transformations commute. Then they are confocal quadrics.*

Theorem 1.2 follows from Theorem 1.7.

Theorem 1.2 can be deduced from a classical theorem due to Marcel Berger [2] concerning billiards in $\mathbb{R}^d$, $d \geq 3$, which states that only billiards bounded by quadrics may have caustics (see Definition 1.8 for the notion of caustic), and the caustics are their confocal quadrics. To prove Theorem 1.7 in full generality, we extend Berger's theorem to the case of billiards in space forms (Theorem 1.10 stated in Subsection 1.3 and proved in Section 2) and then deduce Theorem 1.7 in Section 3. A local version of Theorem 1.7 will be proved in Section 4. In Section 5 we present some open problems.

1.2 Historical remarks

Commuting billiards are closely related to problems of classification of integrable billiards, see [20]. It is known that elliptic and ellipsoidal billiards are integrable, see [21, proposition 4], [18, chapter 4], and this also holds for non-Euclidean ellipsoids in sphere and in the Lobachensky (hyperbolic) space of any dimension, see [22, the corollary on p. 95]. The famous Birkhoff Conjecture states that in two dimensions the converse is true. Namely, it deals with the so-called *Birkhoff caustic-integrable* convex planar billiards with smooth boundary, that is, billiards for which there exists a foliation by closed caustics (a one-parameter family of nested closed caustics Γ_p, $p > 0$) in an interior neighborhood of the boundary, and the boundary itself is the leaf Γ_0 of this foliation. Birkhoff Conjecture states that the only Birkhoff caustic-integrable billiards are ellipses. Birkhoff Conjecture was first stated in print in Poritsky's paper [16], who proved it in loc. cit. under the additional assumption that for any two nested caustics in the above family Γ_p the smaller one is a caustic for the billiard in the bigger one. Poritsky's assumption implies that *the initial billiard map* in Γ_0 (being restricted to the set of those lines that are disjoint from some given caustic Γ_p with $p > 0$) *commutes with the billiard in every caustic* Γ_q. This follows by the arguments presented in [18, section 4, pp.58–59].

The set of lines intersecting the given convex billiard is a topological cylinder called the *phase cylinder.* One of the most famous results on Birkhoff Conjecture is a theorem of M.Bialy [3], who proved that if the phase cylinder of the billiard map is foliated (almost everywhere) by non-contractible closed curves which are invariant under the billiard map, then the boundary is a circle. In [4] he proved the same result for billiards on surfaces of non-zero constant curvature. A local version of Birkhoff Conjecture, for integrable deformations of ellipses was recently solved in [1, 13]. Recent solution of its polynomial version (stated and partially studied in [8]) is a result of papers [5, 6, 11, 12]. For a historical survey of Birkhoff Conjecture see [18, section 5, p.95], the recent surveys [7, 14], the papers [13, 11] and references therein. Dynamics in billiards in two and higher dimensions with piecewise smooth boundaries consisting of confocal quadrics was studied in [9].

1.3 Berger's theorem and its extension to billiards in space forms

Definition 1.8 Let a, b be two nested strictly convex closed hypersurfaces in a Riemannian manifold E: the hypersurface b bounds a relatively compact domain in E whose interior contains a. We say that a is a *caustic* for the hypersurface b, if the image of each oriented geodesic tangent to a by the reflection σ_b from b is again a geodesic tangent to a.

Remark 1.9 It is well-known that if a, b are two confocal ellipses (ellipsoids) in Euclidean space, then the smaller one is a caustic for the bigger one. In the plane this is the classical Proclus–Poncelet theorem. In higher dimensions this theorem is due to Jacobi, see [17, p.80]. Similar statement holds in any space form, see, e.g., [22, theorem 3].

We will deduce Theorem 1.7 from the following theorem, which implies that *in every space form only quadrics have caustics, and the caustics of each quadric S are exactly the quadrics confocal to S.*

Theorem 1.10 *Let $d \geq 3$, and let Σ be a d-dimensional simply connected space form. Let $S, U \subset \Sigma$ be germs of C^2-smooth hypersurfaces at points B and $A \neq B$ respectively with non-degenerate second fundamental forms. Let the geodesic AB be tangent to U at A and transversal to S at B. Let*

$C \in \Sigma \setminus \{B\}$, and let a vector tangent to the geodesic AB at B be reflected from the hyperplane $T_B S$ to a tangent vector to the geodesic BC. Let there exist a germ of C^2-smooth hypersurface V at C tangent to BC at C such that each geodesic close to AB and tangent to U be reflected from the hypersurface S to a geodesic tangent to V. Then S is a piece of a quadric b, and U, V are pieces of one and the same quadric confocal to b.

Remark 1.11 In the case, when $\Sigma = \mathbb{R}^d$, Theorem 1.10 was proved by Marcel Berger [2].

2 Caustics of hypersurfaces in space forms. Proof of Theorem 1.10

The proof of Theorem 1.10 for space forms essentially follows Berger's proof for the Euclidean case given in [2]. In Subsection 2.1 we first prove that the hypersurfaces U and V are pieces of the same quadric denoted by U. Then in Subsection 2.2 we show that S is a quadric confocal to U, using the fact that it is an integral hypersurface of a finite-valued hyperplane distribution: the field of symmetry hyperplanes in $T_x\Sigma$, $x \in \Sigma$, for the geodesic cones K_x circumscribed about the quadric U with vertex at x.

2.1 The hypersurfaces U and V and circumscribed cones

Theorem 2.1 *In the conditions of Theorem 1.10 the hypersurfaces U and V are pieces of one and the same quadric.*

Theorem 2.1 is proved below following [2]. As in loc. cit., we first prove that for every $y \in S$ the geodesic cone with vertex y tangent to U is a quadratic cone tangent to both U and V (Lemma 2.4). Afterwards we apply a result from [2] (stated below as Lemma 2.12 and proved in loc. cit. via arguments using projective duality), showing that if the latter statement holds, then U and V lie in the same quadric.

Let $\pi : \Sigma \to \mathbb{RP}^d$ denote the restriction to Σ of the tautological projection $\mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{RP}^d$. It is a diffeomorphism onto the image $\pi(\Sigma)$ in non-spherical cases and a degree two covering over $\mathbb{RP}^d$ in the spherical case. Let g denote the metric on $\pi(\Sigma)$ that is the (well-defined) pushforward of the space form metric. Note that the geodesics (completely geodesic subspaces) for the metric g are the intersections of projective lines (respectively, projective subspaces) with $\pi(\Sigma)$. In order to reduce the proof to the Euclidean case treated in [2], we use the following property of the metric g.

Proposition 2.2 *For every point $y \in \pi(\Sigma)$ there exist an affine chart $\mathbb{R}^d \subset \mathbb{RP}^d$ centered at y and a Euclidean metric on $\mathbb{R}^d$ (compatible with the affine structure) that has the same 1-jet at y, as the metric g.*

Proof Without loss of generality we consider that $y = (1 : 0 : \cdots : 0)$: the isometry group of the space form Σ acts transitively, and the projection π conjugates its action on Σ with its action on $\mathbb{RP}^d$ by projective transformations (since the isometry group is a subgroup in $GL_{d+1}(\mathbb{R})$). Thus, in the standard affine chart $\mathbb{R}^d = \{x_0 = 1\}$ the point y is the origin. The metric g is invariant under the orthogonal transformations of the chart $\mathbb{R}^d$, since the metric of the space form is invariant under the rotations around the x_0-axis. The metric g on $T_y\mathbb{R}^d$ coincides with the standard Euclidean metric of the chart $\mathbb{R}^d$, by definition. The two last statements together imply that the 1-jets of both metrics at $y = 0$ coincide. This proves the proposition. □

Corollary 2.3 *Let $U \subset \Sigma$ be a germ of hypersurface with non-degenerate second fundamental form. Then its projection $\pi(U)$ has non-degenerate second fundamental form in any affine chart $\mathbb{R}^d$ with respect to the standard Euclidean metric.*

Proof The corollary follows from Proposition 2.2 and invariance of the property of having non-degenerate second fundamental form under projective transformations. Indeed, each germ of projective hypersurface is tangent to some quadric with order 3 (which is not unique). The 2-jet of a quadric determines completely whether it is regular or not. Non-degeneracy of the second fundamental form is equivalent to regularity of the tangent quadric. The space of regular quadrics is invariant under projective transformations. This proves the corollary. □

In what follows in the present subsection we identify the hypersurfaces S, U, V and their points with their projection images: for simplicity the projection images $\pi(S)$, $\pi(U)$, $\pi(B)$ etc. will be denoted by the symbols S, U, B,...

Lemma 2.4 *Let $S, U, V \subset \mathbb{RP}^d$ be the tautological projection images of the same hypersurfaces in Σ, as in Theorem 1.10 (see the above paragraph). For every $y \in S$ there exists a quadratic cone $K_y \subset \mathbb{RP}^d$ (i.e., given by the zero locus of a homogeneous quadratic polynomial) with vertex at y that is tangent to both hypersurfaces U and V.*

The proof of Lemma 2.4 given below follows [2, section 2].

Let $\sigma_g : (T\mathbb{RP}^d)|_S \to (T\mathbb{RP}^d)|_S$ denote the involution acting as the symmetry of each space $T_y\mathbb{RP}^d$, $y \in S$, with respect to the hyperplane T_yS in the metric g. Its action on the projectivized tangent spaces $\mathbb{RP}^{d-1}_y = \mathbb{P}(T_y\mathbb{RP}^d)$ induces its action on the space of projective lines in $\mathbb{RP}^d \supset S$ intersecting S transversely and so that the intersection point is unique: if ℓ intersects S at a point y, then

$$\hat{\ell} := \sigma_g(\ell)$$

is the line through y that is symmetric to ℓ in the above sense.

For every $y \in \Sigma$ set

$M_y :=$ the space of projective lines through y that are tangent to U.

It suffices to prove the statement of Lemma 2.4 for an arbitrary point $y \in S$ satisfying the following statements.

Proposition 2.5 *(stated in [2, pp. 110-111]) There exists an open and dense subset of points $y \in S \subset \mathbb{RP}^d$ for which there exists an open and dense subset $M^0_y \subset M_y$ of lines ℓ satisfying the following statements:*

(i) the line ℓ is quadratically tangent to U, (i.e., ℓ is not an asymptotic direction of the hypersurface U at the tangency point);

(ii) the line $\hat{\ell} = \sigma_g(\ell)$ is quadratically tangent to V at a point, where the second fundamental form of the hypersurface V is non-degenerate;

(iii) the lines ℓ and $\hat{\ell}$ are transversal to T_yS and their above tangency points with U and V are distinct from the point y.

Proof Statement (i) holds for an open and dense subset of lines $\ell \in M_y$, since the second fundamental form of the hypersurface U is non-degenerate (by assumptions and Corollary 2.3). Statement (iii) also holds for a generic $\ell \in M_y$, whenever $y \notin U \cup V$. Let us show that statement (ii) also holds generically.

Let $y \in S$, $y \notin U \cup V$, and let ℓ be a line through y satisfying assumption (i). Then the cone K_y with vertex y containing ℓ and circumscribed about U is tangent to U along a $(n-2)$-dimensional submanifold $\mathcal{T}_U \subset U$.

The correspondence sending a point $p \in \mathcal{T}_U$ to the projective hyperplane tangent to U at p (i.e., to the projective hyperplane tangent to the cone along the line yp) is a local immersion to the space of hyperplanes through y. Or equivalently, the correspondence sending a line $L \subset K_y$ through y to the projective hyperplane tangent to K_y along L is a local immersion. This follows from non-degeneracy of the second fundamental form of the hypersurface U. This implies the similar statement for the symmetric cone

$\widehat{K_y} = \sigma_g(K_y)$ circumscribed about V: the correspondence sending each line $\widehat{L} \subset \widehat{K_y}$ through y to the hyperplane tangent to $\widehat{K_y}$ along the line $\widehat{L}$ is a local immersion to the space of hyperplanes through y.

Suppose now that a line $\widehat{L} \subset \widehat{K_y}$ through y is quadratically tangent to V at a point q. Then the above immersivity statement for the symmetric cone together with quadraticity of tangency imply non-degeneracy of the second fundamental form of the hypersurface V at the point q. It is clear that for a generic choice of the point $y \in S$ and a line $L \subset K_y$ through y the corresponding symmetric line $\widehat{L} = \sigma_g(L)$ is quadratically tangent to V. This proves the proposition. □

Convention 2.6 In the proof of Lemma 2.4 without loss of generality we consider that $y = B$, and there exists a line ℓ through B that is transversal to S and satisfies statements (i)–(iii) of Proposition 2.5. Without loss of generality we consider that A is the tangency point of the line ℓ with U, and C is the tangency point of the symmetric line $\hat{\ell} = \sigma_g(\ell)$ with V; $A, C \neq B$. Fix an affine chart $\mathbb{R}^d \subset \mathbb{RP}^d$ centered at B and equipped with an Euclidean metric whose 1-jet at B coincides with the 1-jet of the metric g (Proposition 2.2).

Consider a smooth deformation $x(t) \in S$ of the point B, $x(0) = B$, and a smooth deformation $p(t) \in U$ of the point A, $p(0) = A$, such that the line $\ell(t) = x(t)p(t)$ is tangent to U at $p(t)$; $t \in [0, 1)$. Then the line $\hat{\ell}(t) = \sigma_g(\ell(t))$ symmetric to $\ell(t)$ in the metric g is tangent to the hypersurface V at some point $q(t)$, $q(0) = C$, that depends smoothly on the parameter t (assumptions (i)–(iii)). We will show that the property that every deformation $x(t)$ extends to a pair of deformations $p(t)$ and $q(t)$ as above implies that the cone K_y tangent to both U and V is quadratic. To do this, consider the projective hyperplanes $\mathcal{U}$ and $\mathcal{V}$ through B containing the lines $\ell(0) = BA$ and $\hat{\ell}(0) = BC$ respectively: $\mathcal{U}$ is tangent to U at A, and $\mathcal{V}$ is tangent to V at C.

Remark 2.7 Let $\mathcal{U}$ and $\mathcal{V}$ be as above. The tangent subspaces $T_B\mathcal{U}, T_B\mathcal{V} \subset T_B\mathbb{RP}^d$ are σ_g-symmetric. Indeed, consider the germs of the cones circumscribed about the hypersurfaces U and V with vertex B and containing the lines $l(0)$ and $\hat{l}(0)$ respectively: we take the germs of the above cones at the latter lines. The σ_g-symmetry permutes the cones, by statement (ii) of Proposition 2.5, which holds for an open and dense set of lines throuh B. The hyperplanes $\mathcal{U}$ and $\mathcal{V}$ are tangent to the cones along the lines $l(0)$ and

$\hat{l}(0)$ respectively, by construction. Hence they are also σ_g-symmetric, as are the cones, and so are their tangent spaces $T_B\mathcal{U}$ and $T_B\mathcal{V}$.

The latter tangent spaces intersect on a codimension 2 subspace $H \subset T_B\mathbb{R}^d$ lying in T_BS, by symmetry:

$$H = T_B\mathcal{U} \cap T_BS = T_B\mathcal{V} \cap T_BS. \tag{2.1}$$

For every deformations $x(t)$, $p(t)$, $q(t)$ as above one has

$$u = x'(0) \in T_BS, \quad v = p'(0) \in T_A\mathcal{U}, \ w = q'(0) \in T_C\mathcal{V}. \tag{2.2}$$

This motivates the following definition

Definition 2.8 Let S be a germ of hypersurface at a point $B \in \mathbb{R}^d \subset \mathbb{RP}^d$. Let g be a positive definite scalar product on the bundle $T\mathbb{R}^d|_S$. Let ℓ be a projective line through B that is transversal to T_BS, and let $H \subset T_BS$ be a vector subspace of codimension one (codimension two in $T_B\mathbb{R}^d$). Let $A \in \ell$, $C \in \hat{\ell} = \sigma_g(\ell)$, $A, C \neq B$. Let $\mathcal{U}$ and $\mathcal{V}$ denote the projective hyperplanes through B that are tangent to H and such that $\ell \subset \mathcal{U}$, $\hat{\ell} \subset \mathcal{V}$. Let

$$u \in T_BS, \ u \neq 0, \ v \in T_A\mathcal{U}, \ w \in T_C\mathcal{V}.$$

We say that $(B, \ell, H, u, A, v, C, w)$ is a *Berger tuple* with base point B, if there exist germs of C^1-smooth curves of points $x(t) \in S$, $p(t), q(t) \in \mathbb{RP}^d$, $x(0) = B$, $p(0) = A$, $q(0) = C$, such that statements (2.2) hold and for every small t the lines $x(t)p(t)$ and $x(t)q(t)$ are σ_g-symmetric.

Proposition 2.9 *The property of being a Berger tuple depends only on the 1-jet of the metric g. Namely, let S be a germ of hypersurface at a point $B \in \mathbb{R}^d \subset \mathbb{RP}^d$. Let g_1 and g_2 be two positive definite scalar products on the bundle $(T\mathbb{R}^d)|_S$ that have the same 1-jet at B. Then any Berger tuple for the metric g_1 with base point B is a Berger tuple for the metric g_2 and vice versa.*

Proof The proposition follows from definition and smoothness of the dependence of the reflection σ_g on the parameters of the metric g: if two metrics have the same 1-jets at B, then the corresponding reflections acting in $T_y\mathbb{R}^d$ differ by a quantity $o(y - B)$. □

Theorem 2.10 *[2, section 2]. Let S be a germ of hypersurface at a point $B \in \mathbb{R}^d \subset \mathbb{RP}^d$. Consider the standard Euclidean metric on the affine chart $\mathbb{R}^d$, and let S have non-degenerate second fundamental form. Let ℓ be a line* *through B transversal to T_BS. Then there exist only a finite number $k \leq d-1$ of codimension one vector subspaces $H = H_1(\ell), \dots, H_k(\ell) \subset T_BS$ such that for every $u \in T_BS$, $u \neq 0$ the triple (ℓ, H, u) extends to a Berger tuple $(B, \ell, H, u, A, v, C, w)$ for the Euclidean metric. The number k depends only on the second fundamental form of the hypersurface S at B. The subspaces $H_j(\ell)$ are uniquely determined by the line ℓ and the second fundamental form.*

Proposition 2.11 *[2, p. 114]. In the conditions of Theorem 2.10 consider the tautological projection $\pi_B : \mathbb{R}^d \setminus \{B\} \to \mathbb{RP}^{d-1}$ to the space of lines through B. For every line ℓ through B the corresponding projection $\pi_B(\ell \setminus \{B\}) \in \mathbb{RP}^{d-1}$ will be denoted by $[\ell]$. For every ℓ transversal to S let $\Delta_j(\ell) \subset T_B\mathbb{RP}^d = \mathbb{R}^d$ denote the codimension 1 vector subspace spanned by $H_j(\ell)$ and ℓ. Let $\widetilde{\Delta}_j([\ell]) = \pi_B(\Delta_j(\ell) \setminus \{0\}) \subset \mathbb{RP}^{d-1}$ denote its tautological projection, which is a projective hyperplane through $[\ell]$. Set*

$$\mathcal{D}_j([\ell]) := T_{[\ell]}\widetilde{\Delta}_j([\ell]) \subset T_{[\ell]}\mathbb{RP}^{d-1}.$$

The subspaces $\mathcal{D}_1([\ell]), \dots, \mathcal{D}_k([\ell]) \subset T_{[\ell]}\mathbb{RP}^{d-1}$ form a k-valued hyperplane distribution $\mathcal{D}$ on $\mathbb{RP}^{d-1}$, whose all integral surfaces are quadrics. Moreover, let $\widetilde{S}$ be a quadric tangent to S at B with order 3: having the same second fundamental form at B. The π_B-preimages of the above quadrics in $\mathbb{RP}^{d-1}$ (i.e., the preimages of the integral hypersurfaces) are cones with vertex at B that are tangent to the quadrics confocal to $\widetilde{S}$.

Proof of Lemma 2.4. Let K be the cone with vertex at $y = B$ circumscribed about the hypersurface U. Let A be a point of tangency of the cone K with U. Set $\ell = BA$, $\hat{\ell} = \sigma_g(\ell)$. Let C denote the point of tangency of the line $\hat{\ell}$ with V. (We consider that the assumptions of Convention 2.6 hold.) Then for every germ of smooth curve $x(t) \subset S$, $x(0) = B$, there exist curves $p(t) \subset U$ and $q(t) \subset V$, $p(0) = A$, $q(0) = C$, such that the lines $x(t)p(t)$ and $x(t)q(t)$ are tangent to U and V at $p(t)$ and $q(t)$ respectively and σ_g-symmetric.

Let $\mathcal{U}, \mathcal{V} \subset \mathbb{RP}^d$ be the previously defined projective hyperplanes through B tangent to U and V at A and C respectively, and $H = T_B\mathcal{U} \cap T_B\mathcal{V} \subset T_BS$, see (2.1). The tuple $(B, \ell, H, x'(0), A, p'(0), C, q'(0))$ is a Berger tuple for the metric g, by definition. Therefore, it is also a Berger tuple for the Euclidean metric as well (Proposition 2.9). This together with Theorem 2.10 implies that $H = H_j(\ell)$ for some j.

The cone K is tangent along the line ℓ to the hyperplane generated by ℓ and $H = H_j$, by definition. Therefore, the tautological projection

$\widetilde{K} = \pi_B(K \setminus \{B\}) \subset \mathbb{RP}^{d-1}$ is tangent at $[\ell] = \pi_B(\ell \setminus \{B\})$ to the corresponding hyperplane $\mathcal{D}_j([\ell])$ from Proposition 2.11. Finally, $\widetilde{K}$ is an integral hypersurface of the multivalued hyperplane distribution $\mathcal{D}$ from Proposition 2.11, and hence, lies in a quadric $\Gamma(U)$. The preimage $\pi_B^{-1}(\Gamma(U))$ is a quadratic cone K_B with vertex B that contains K and is σ_g-symmetric, being a cone tangent to a quadric confocal to $\widetilde{S}$, see Proposition 2.11. (Recall that for any given quadric $\widetilde{S}$ and $B \in \widetilde{S}$ a cone with vertex B circumscribed about a quadric confocal to $\widetilde{S}$ is symmetric with respect to the hyperplane tangent to $\widetilde{S}$ at B.) Similarly, the punctured cone $\sigma_g(K) \setminus \{B\}$ tangent to V is projected to a quadric $\Gamma(V)$, and $\sigma_g(K)$ lies in a quadratic cone. The latter quadratic cone coincides with K_B, by symmetry. This proves Lemma 2.4. □

Lemma 2.12 *[2, section 3] Let U, V, S be C^2-smooth germs of hypersurfaces in $\mathbb{RP}^d$ with non-degenerate second fundamental forms. Let for every $x \in S$ there exist a quadratic cone K_x with vertex at x that is tangent to both U and V. Then U and V are pieces of one and the same quadric.*

Proof of Theorem 2.1. For every $y \in S$ close enough to B there exists a quadratic cone K_y with vertex at y circumscribed about both U and V (Lemma 2.4). Applying this statement to an open and dense subset of points $y \in S$ satisfying genericity assumptions from Convention 2.6 together with Lemma 2.12 yield that U and V are pieces of one and the same quadric. Theorem 2.1 is proved. □

2.2 Symmetry hyperplanes of circumscribed cones and confocal quadrics

Here we prove the following lemma and then deduce Theorem 1.10 from it.

Lemma 2.13 *(A generalization of an anologous statement in [2, p. 109].) Let Σ be a simply connected space form of dimension at least three. Let $U \subset \Sigma$ be a quadric with non-degenerate second fundamental form. For every $y \in \Sigma \setminus U$ let K_y denote the geodesic cone circumscribed about the quadric U with the vertex at y (i.e., the union of geodesics through y that are tangent to U). We identify the cone K_y with the cone $\widetilde{K}_y \subset T_y\Sigma$ of vectors tangent to the above geodesics via the exponential mapping* $\exp : T_y\Sigma \to \Sigma$. *Let $S \subset \Sigma$ be a germ of hypersurface at a point $B \notin U$ with non-degenerate second fundamental form such that for every $y \in S$ the cone $\widetilde{K}_y$ is symmetric with respect to the hyperplane T_yS. Then S is a quadric confocal to U.*

In the proof of Lemma 2.13 we use the following lemma. To state it, let us recall that the orthogonal polarity in $\mathbb{R}^{d+1}$ is the correspondence sending each vector subspace to its orthogonal complement with respect to the standard Euclidean scalar product. The orthogonal polarity in codimension one, which sends codimension one vector subspaces to their orthogonal lines, induces a projective duality $\mathbb{RP}^{d*} \to \mathbb{RP}^d$ sending hyperplanes to points. It sends each hypersurface $S \subset \mathbb{RP}^d$ to its dual S^*: the family of points dual to the hyperplanes tangent to S.

Definition 2.14 Consider a scalar product $< Gx, x >$ on $\mathbb{R}^{d+1}$ defining a space form. Orthogonality with respect to the latter scalar product will be called *G-orthogonality.* Let $V \subset \mathbb{R}^{d+1}$ be a subspace that is *not isotropic:* this means that the restriction to V of the scalar product $< Gx, x >$ is a non-degenerate quadratic form (or equivalently, that V is not tangent to the light cone $\{< Gx, x >= 0\}$). The *pseudo-symmetry* with respect to V is the linear involution $I_V : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ that preserves the above scalar product on $\mathbb{R}^{d+1}$ and whose fixed point set coincides with V: it acts trivially on V and as a central symmetry in the G-orthogonal subspace.

Lemma 2.15 *Let $V \subset \mathbb{R}^{d+1}$ be a non-isotropic vector subspace. Let $k < d+1$. Consider the action $I_{V,k} : G(k, d+1) \to G(k, d+1)$ of the pseudo-symmetry with respect to V on the Grassmannian of k-subspaces. The orthogonal polarity $L \mapsto L^\perp$ conjugates the actions $I_{V,k}$ and*[1] $I_{V^\perp, d+1-k}$.

Proof The lemma seems to be well-known to specialists. In three dimensions it follows from [8, formula (15), p.23], [15, formula (3.12), p.140]. Let us present its proof for completeness of presentation. As it is shown below, Lemma 2.15 is implied by the two following propositions.

Proposition 2.16 *Let G be a real symmetric $(d+1) \times (d+1)$-matrix such that $G^3 = G$. Let two non-isotropic subspaces $V, W \subset \mathbb{R}^{d+1}$ of complementary dimensions be G-orthogonal. Then their* **Euclidean** *orthogonal complements $V^\perp$ and $W^\perp$ are also non-isotropic and G-orthogonal.*

Proof The condition of the proposition implies that the restrictions of the linear operator G to V and W have zero kernels and

$$GV = W^\perp, \ GW = V^\perp. \tag{2.3}$$

[1]Everywhere below the orthogonality sign $\perp$ means orthogonality with respect to the standard Euclidean scalar product.

Thus, to prove G-orthogonality of the latter subspaces, it suffices to show that

$$< G^2 v, Gw >=< G^3 v, w >= 0 \text{ for every } v \in V \text{ and } w \in W.$$

The first equality follows from symmetry of the matrix G. The second one follows from G-orthogonality of the subspaces V and W and the equality $G^3 = G$. The subspaces (2.3) are non-isotropic, since the restrictions to them of the scalar product $< Gx, x >$ are isomorphic to its restrictions to V and W via the operator G: for every $v_1, v_2 \in V$ one has $< G(Gv_1), Gv_2 >=< Gv_1, v_2 >$, since $G^3 = G$. Proposition 2.16 is proved. □

Proposition 2.17 *Let $< Gx, x >$ be a scalar product on $\mathbb{R}^{d+1}$ defining a space form. Let $k \in \{1, \dots, d\}$, $V \subset \mathbb{R}^{d+1}$ be a non-isotropic subspace, and let $W \subset \mathbb{R}^{d+1}$ be its G-orthogonal completent*[2]*. Let $N_k(V) \subset G(k, d+1)$ denote the subset of those vector k-subspaces in $\mathbb{R}^{d+1}$ that are direct sums of some subspaces $\ell_1 \subset V$ and $\ell_2 \subset W$. The pseudo-symmetry I_V induces a non-trivial projective involution $\mathbb{RP}^d \to \mathbb{RP}^d$ and acts trivially on $N_k(V)$. Vice versa, every non-trivial projective involution acting trivially on $N_k(V)$ is the projectivization of the pseudo-symmetry I_V.*

Proof The first statement of the proposition is obvious. Let us prove the second one. Let $F : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be a linear transformation whose projectivization is a non-trivial involution acting trivially on $N_k(V)$. Without loss of generality we consider that $F^2 = \pm Id$. For every vector subspace $L \subset V$ of dimension between 1 and k the transformation F preserves the subset in $N_k(V)$ consisting of the k-subspaces containing L. Their intersection being equal to L, F preserves L. The same statement holds for $L \subset W$. Therefore, the restriction of the transformation F to any of the subspaces V and W is a homothety. The coefficients of the homotheties on V and W are equal to ± 1, since $F^2 = Id$ up to sign. The signs of the latter coefficients are opposite, since the projectivization of the transformation F is non-trivial. Hence, $F = \pm I_V$. This proves the proposition. □

Let us now return to the proof of Lemma 2.15. The action of a linear automorphism $F : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ on all the vector subspaces of all the dimensions is conjugated via the orthogonal polarity to the similar action

[2]The G-orthogonal complenent W to a non-isotropic subspace V is always a vector subspace complementary to V. In the non-Euclidean cases W is automatically non-isotropic. In the Euclidean case, when the matrix G is degenerate, W contains the kernel of the matrix G: the x_0-axis.

of the inverse $(F^*)^{-1}$ to the conjugate operator F^* (with respect to the Euclidean scalar product). In the case, when F is an involution, so is $F^* = (F^*)^{-1}$. Let W be the G-orthogonal complement of the subspace V.

Claim. *The conjugate operator $F = I_V^*$ acts trivially on $N_{d+1-k}(V^\perp)$.*
Proof The orthogonal polarity sends each k-subspace $\Pi = \ell_1 \oplus \ell_2 \in N_k(V)$, $\ell_1 \subset V$, $\ell_2 \subset W$, to the intersection of two subspaces $L_j = L_j(\Pi) = \ell_j^\perp$:

$$L_1 \supset V^\perp, \ L_2 \supset W^\perp, \ \Pi^\perp = L_1 \cap L_2, \tag{2.4}$$

$$\dim(\Pi^\perp) = \dim L_1 + \dim L_2 - (d+1) = d+1-k.$$

The transformation F fixes $\Pi^\perp$, by construction and since the pseudo-symmetry I_V fixes Π (Proposition 2.17). The intersection $\Pi^\perp$ is the direct sum of the subspaces $L_1 \cap W^\perp$ and $L_2 \cap V^\perp$, which follows from the inclusions (2.4). Hence, $\Pi^\perp$ lies in $N_{d+1-k}(V^\perp)$. Vice versa, each point in $N_{d+1-k}(V^\perp)$ can be represented as the intersection $\Pi^\perp$ of some subspaces L_1 and L_2 containing $V^\perp$ and $W^\perp$ respectively. Therefore, F acts trivially on all of $N_{d+1-k}(V^\perp)$. The claim is proved. □

The operator $F = I_V^*$ is a projectively non-trivial involution, as is I_V. It coincides with $I_{V^\perp}$ up to sign, by the claim and Proposition 2.17. This together with the discussion preceding the claim implies the statement of Lemma 2.15. □

Proof of Lemma 2.13. Consider the tautological projection $\pi : \mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{RP}^d$, the images $\pi(S), \pi(U) \subset \mathbb{RP}^d$ and the hypersurfaces in $\mathbb{RP}^d$ projective-dual to them with respect to the orthogonal polarity. For simplicity the latter projective-dual hypersurfaces will be denoted by S^* and U^* respectively. Let $\widetilde{S}, \widetilde{U}, \widetilde{S}^*, \widetilde{U}^* \subset \mathbb{R}^{d+1}$ denote the complete π-preimages in $\mathbb{R}^{d+1}$ of the hypersurfaces $\pi(S)$, $\pi(U)$, S^* and U^* respectively: the cones in $\mathbb{R}^{d+1} \setminus \{0\}$ defined by the latter hypersurfaces. Recall that $\pi(U)$ and U^* are dual quadrics; thus one can write

$$U^* = \{< Qx, x >= 0\}, \quad Q \text{ is a real symmetric } (d+1) \times (d+1) - \text{matrix}.$$

For every $y \in S$ let $\mathcal{T}_y S \subset \mathbb{RP}^d$ denote the projective hyperplane tangent to $\pi(S)$ at $\pi(y)$. Define the following vector subspaces in $\mathbb{R}^{d+1}$:

$$\Pi_y := \pi^{-1}(\mathcal{T}_y S) \cup \{0\} \subset \mathbb{R}^{d+1}, \ L_y := \Pi_y^\perp,$$

$$V_y := \text{ the one-dimensional subspace } \pi^{-1}(\pi(y)) \cup \{0\} \subset \Pi_y, \quad W_y := V_y^\perp.$$

The subspaces L_y and W_y are non-isotropic. Indeed, in the case, when Σ is non-Euclidean, this follows from obvious non-isotropicity of their orthogonal subspaces Π_y and V_y and Proposition 2.16. In the case, when Σ is Euclidean, if, to the contrary, either L_y, or W_y contained the x_0-axis, this would imply that either Π_y, or V_y lies in the coordinate $(x_1, \dots, x_d)$- subspace, and hence, is disjoint from Σ. This is obviously impossible.

Claim 1. *The quadric U^* is regular, i.e., the matrix Q is non-degenerate. The hyperplane section $\widetilde{U}^* \cap W_y$ is invariant under the pseudo-symmetry with respect to the one-dimensional vector subspace $L_y \subset W_y$.*

Proof The first statement (non-degeneracy) follows from non-degeneracy of the second fundamental form of the quadric U. The inclusion $L_y \subset W_y$ follows from definition. Recall that the cone $\widetilde{K}_y$ is symmetric with respect to the hyperplane T_yS, i.e., the preimage $\pi^{-1}(K_y)$ is pseudo-symmetric with respect to Π_y, by assumption. The latter statement is equivalent to the second statement of the claim, by duality and Lemma 2.15. $\square$

The restriction to the d-dimensional vector subspace W_y of the scalar product $< Gx, x >$ is non-degenerate (non-isotropicity), and there exist d values $\lambda = \lambda_1(y), \dots, \lambda_d(y)$ (taken with multiplicity, some of them may coincide) such that the restriction to W_y of the scalar product $< (Q-\lambda G)x, x >$ is degenerate. Thus, the d-dimensional vector subspace W_y is the G-orthogonal direct sum of kernels of the scalar products $< (Q - \lambda_j(y)G)x, x > |_{W_y}$.

Claim 2. *For every $y \in S$ the pseudo-symmetry line L_y lies in the kernel of some of the scalar products $< (Q - \lambda_j(y)G)x, x > |_{W_y}$.*

Proof The scalar product $< Qx, x > |_{W_y}$ is invariant under the pseudo-symmetry with respect to the line L_y. Indeed, the latter pseudo-symmetry is an involution preserving the zero locus (light cone) $\widetilde{U}^* \cap W_y = \{< Qx, x >= 0\} \cap W_y$ (Claim 1), and hence, it preserves the above scalar product up to sign. Let us show that the sign is also preserved. For an open and dense subset of points $y \in S$ one has $< Qx, x > \neq 0$ on $L_y \setminus \{0\}$: equivalently (via duality), the tangent hyperplane T_yS is not tangent to U. Indeed, the latter statement holds for an open and dense subset of points $y \in S$, since $S \cap U = \emptyset$ and a (germ of) hypersurface is uniquely defined by the family of its tangent hyperplanes (well-definedness of the dual hypersurface). Thus, for the above y the pseudo-symmetry fixes the non-zero quadratic form $< Qx, x > |_{L_y}$, since the points of the line L_y are fixed. This together with the above discussion implies that the above-mentioned sign, and hence the scalar product $< Qx, x > |_{W_y}$ are preserved for all $y \in S$.

For every $\lambda_j(y)$ the kernel of the form $< (Q-\lambda_j(y))x, x > |_{W_y}$ is invariant under the above pseudo-symmetry, by invariance of the scalar products $< Qx, x >$ and $< Gx, x >$. This is possible only in the case, when the pseudo-symmetry line L_y lies in some of the kernels, which form an orthogonal direct sum decomposition of the subspace W_y. This proves Claim 2. □

Remark 2.18 The subspace W_y and hence, the corresponding kernels from Claim 2 depend only on y and are well-defined for all $y \in \Sigma$.

Due to Claim 2, the following two cases are possible.

Case 1: for an open and dense subset S_0 of points $y \in S$ the line L_y coincides with a one-dimensional kernel corresponding to a simple eigenvalue $\lambda_j(y)$. Let us show that in this case S lies in a quadric confocal to U. Indeed then there exist a neighborhood $Y = Y(B) \subset \Sigma$ of the base point B of the hypersurface S and an open and dense subset $Y_0 \subset Y$ containing S_0 such that the correspondence $y \mapsto L_y$ extends to a family of lines depending analytically on $y \in Y_0$: these lines are some of the kernels mentioned in the above remark. This implies that the corresponding hyperplanes $\Pi_y := L_y^{\perp}$ also depend analytically on y and thus, induce a field of hyperplanes $T = T(y) = \Pi_y \cap T_y\Sigma$ on Y_0. The hypersurface S_0 is its integral hypersurface.

Subcase 1.1): U is a generic quadric. Then for a generic point $y \in \Sigma$ (here "generic" means "outside an algebraic subset")

- there are exactly d quadrics through y confocal to U, and any two of them are orthogonal at y;

- the corresponding eigenvalues $\lambda_j(y)$ are simple and the corresponding d kernels in W_y are one-dimensional.

Recall that the tangent hyperplanes at y of the above confocal quadrics are symmetry hyperplanes for the cone K_y, since U is a caustic for its confocal quadrics. Therefore, the orthogonal polarity $\Pi_y \mapsto L_y$ induces a one-to-one correspondence between the above tangent hyperplanes and kernels. This implies that for a generic $y \in Y_0$ the integral hypersurface of the hyperplane field T through y is a confocal quadric to U. Passing to limit, as y tends to a point of the integral hypersurface S, we get that S is a confocal quadric as well.

Subcase 1.2): U is a general regular quadric. Then it is a limit of generic quadrics U_n in the above sense. For each U_n the integral hypersurfaces of the corresponding above hyperplane field T_n are quadrics confocal to U_n. Passing to limit, as $n \to \infty$, we get the same statement for the hyperplane field T associated to U. Hence, S is a quadric confocal to U.

Case 2: there exists an open subset of points $y \in S$ for which L_y lies in at least two-dimensional kernel of the form $< (Q - \lambda_j(y))x, x > |_{W_y}$ corresponding to a multiple eigenvalue $\lambda_j(y)$. In this case the latter kernel contains at least two linearly independent vectors $w_1, w_2 \in W_y$, and by definition, both of them are orthogonal to the hyperplane W_y with respect to the scalar product $<(Q-\lambda G)x,x>$, $\lambda=\lambda_j(y)$. Hence, their appropriate non-zero linear combination $w=a_1w_1+a_2w_2$ is orthogonal to the whole ambient space $\mathbb{R}^{d+1}$ with respect to the same scalar product. Therefore, w lies in the kernel of the same scalar product taken on all of $\mathbb{R}^{d+1}$, and thus, λ is such that the matrix $Q-\lambda G$ is degenerate: then we'll call such a λ a *global eigenvalue*. The number of global eigenvalues λ is at most $d+1$, and all of them are independent on y.

Finally, there exist a global eigenvalue λ and an open subset $S_0\subset S$ such that for every $y\in S_0$ one has $<(Q-\lambda)x,x>\equiv 0$ on L_y, since L_y lies in the kernel of the restriction to W_y of the scalar product $<(Q-\lambda)x,x>$.

Thus, for $y\in S_0$ the projections $p(y)=\pi(L_y\setminus\{0\})\in\mathbb{RP}^d$ lie in a degenerate quadric $\Gamma\subset\mathbb{RP}^d$ defined by the equation $<(Q-\lambda)x,x>=0$. The points $p(y)$ form the dual hypersurface S_0^*, by definition. Hence, S_0^* lies in a degenerate quadric Γ. This contradicts non-degeneracy of the second fundamental form of the hypersurface S. Hence, the case under consideration is impossible. Lemma 2.13 is proved. □

Proof of Theorem 1.10. The hypersurfaces U and V lie in the same quadric in Σ, which will be now denoted by U (Theorem 2.1). The quadric U is a caustic for the hypersurface S: for every $y\in S$ the cone of geodesics through y that are tangent to U is symmetric with respect to the hyperplane tangent to T_yS. Therefore, S is a quadric confocal to U, by Lemma 2.13. This proves Theorem 1.10. □

3 Commuting billiards and caustics: proof of Theorem 1.7

Proposition 3.1 *Let Σ be a space form of constant curvature of dimension $d\geq 2$. Let two nested strictly convex C^2-smooth closed hypersurfaces $a,b\subset\Sigma$, $a\Subset\Omega_b$ (see the notations at the beginning of the paper) be such that the corresponding billiard transformations σ_a and σ_b commute. Then a is a caustic for the hypersurface b.*

Proof Let Π_a denote the open subset of geodesics in Σ that are disjoint from the hypersurface a. Its boundary $\partial\Pi_a$ consists of those geodesics that are tangent to a. A geodesic L is fixed by σ_a, if and only if $L\in\overline{\Pi}_a$, i.e., L is either disjoint from a, or tangent to a. In this case $\sigma_b\sigma_a(L)=\sigma_b(L)=\sigma_a\sigma_b(L)$, and thus, $\sigma_b(L)$ is a fixed point of the transformation σ_a. This implies that $\sigma_b(\overline{\Pi}_a)\subset\overline{\Pi}_a$. The subset $\overline{\Pi}_a$ is invariant under two transformations acting on oriented geodesics: the reflection σ_b and the transformation

J of the orientation change. The transformations J and $J\circ\sigma_b$ are involutions. Hence, they are homeomorphisms of the whole space of oriented geodesics in Σ. Their restrictions to the common invariant subset $\overline{\Pi}_a$ should be also a homeomorphism: an involution acting on a set is obviously always bijective. Therefore, each of them sends the boundary $\partial\Pi_a$ onto itself homeomorphically, and the same is true for their composition $\sigma_b = J\circ(J\circ\sigma_b)$: $\sigma_b(\partial\Pi_a)=\partial\Pi_a$. The latter equality means exactly that a is a caustic for the hypersurface b. The proposition is proved. □

Proof of Theorems 1.7 and 1.2. Let $a, b\subset\Sigma$ be two nested strictly convex C^2-smooth closed hypersurfaces in a space form Σ with commuting billiard transformations, $a\Subset\Omega_b$, $\dim\Sigma\geq 3$. Then a is a caustic for the hypersurface b, by Proposition 3.1. This means that for every points $B\in b$ and $A\in a$ such that the line AB is tangent to a at A the image $\sigma_b(AB)$ of the line AB (oriented from A to B) is a line through B tangent to a. Recall that a and b are strictly convex, which implies that their second fundamental forms are sign-definite and thus, non-degenerate. Therefore, for every A and B as above the germs at A and B of the hypersurfaces $U=a$ and $S=b$ respectively satisfy the conditions of Theorem 1.10, with V being the germ of the hypersurface a at its point D of tangency with the line $\sigma_b(AB)$. Hence, for every A and B as above the germ (S,B) lies in a quadric, and the germs (U,A), (V,D) lie in one and the same quadric confocal to S. This implies that b is a quadric, and a is a quadric confocal to b. Theorems 1.7, and 1.2 are proved. □

4 A tangential local version of Theorem 1.7

Theorem 4.1 *Let $d\geq 3$. Let (U,A), (S,B), (V,D) be germs of C^2-smooth hypersurfaces in a d-dimensional space form Σ at points A, B and D. Let $B\neq A, D$, and let U and S have non-degenerate second fundamental forms. For every $Z=U,S,V$ consider the action of the reflection σ_Z on the oriented geodesics that intersect Z, defined as at the beginning of the paper: we reflect the geodesic at its last intersection point with the hypersurface Z. Let L_0 be a geodesic through B transversal to S and quadratically tangent to U at A (we orient it from A to B), and let its image $\sigma_S(L_0)$ be quadratically tangent to V at D. Let W be a small neighborhood of the geodesic L_0 in the space of oriented geodesics; in particular, each point in W represents a geodesic intersecting S transversally. Let $\Pi_W\subset W$ denote the subset of those geodesics that intersect U. Let for every $L\in\Pi_W$ the image $\sigma_S(L)$ intersect V; more precisely, we suppose that the compositions $\sigma_S\circ\sigma_U$ and*

$\sigma_V \circ \sigma_S$ are well-defined on Π_W. Let the latter compositions be identically equal on Π_W. Then S lies in a quadric b, and U, V lie in one and the same quadric confocal to b.

Proof Every geodesic L tangent to U and close enough to L_0 lies in Π_W. Its image $\sigma_U(L)$ coincides with L (by definition), and hence, $\sigma_S \circ \sigma_U(L) = \sigma_S(L) = \sigma_V \circ \sigma_S(L)$. Thus, the geodesic $\sigma_S(L)$, which should intersect V by assumption, is fixed by σ_V. Hence, it is tangent to V (at the last point of its intersection with V). Finally, the germs of hypersurfaces U, S and V satisfy the conditions of Theorem 1.10. Therefore, S lies in a quadric b, and U, V lie in one and the same quadric confocal to b, by Theorem 1.10. This proves Theorem 4.1. $\square$

5 Open problems

The billiards in space forms are particular cases of the projective billiards introduced in [19]. The main results of the present paper (Theorem 1.10 extending Berger's result on caustics [2], Theorem 1.7 on commuting billiards) are proved for billiards in space forms. It would interesting to extend them to projective billiards.

Problem 1 (appeared as a result of our discussion with Sergei Tabachnikov). Let $S \subset \mathbb{R}^d$, $d \geq 3$ be a germ of hypersurface at a point B equipped with a field Λ of one-dimensional subspaces $\Lambda_y \subset T_y\mathbb{R}^d$, $y \in S$, transversal to S. Consider the family of linear involutions $\sigma_y : T_y\mathbb{R}^d \to T_y\mathbb{R}^d$, $y \in S$, that fix each point of the hyperplane T_yS and have Λ_y as an eigenline with eigenvalue -1. Let there exist two germs of hypersurfaces U and V at points $A, C \neq B$ respectively such that the lines AC, BC are tangent to U and V at points A and C respectively and for every $y \in S$ each line through y that is tangent to U is reflected by σ_y to a line tangent to V. (Thus defined action of the reflections σ_y on oriented lines transversal to S is called the **projective billiard transformation**, and the pair (S, Λ) is called a **projective billiard,** see [19].) Is it true that then U and V lie in one and the same quadric?

Problem 2 (S.Tabachnikov). Classify commuting nested pairs of projective billiards in $\mathbb{R}^d$, $d \geq 2$.

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References

[1] Avila, A.; De Simoi, J.; Kaloshin, V. *An integrable deformation of an ellipse of small eccentricity is an ellipse.* Ann. of Math. (2) **184** (2016), no. 2, 527–558.

[2] Berger, M. *Seules les quadriques admettent des caustiques.* Bull. Soc. Math. France **123** (1995), 107–116.

[3] Bialy, M. *Convex billiards and a theorem by E. Hopf.* Math. Z., **214(1)** (1993), 147–154.

[4] Bialy, M. *Hopf rigidity for convex billiards on the hemisphere and hyperbolic plane.* Discrete Contin. Dyn. Syst. **33** (2013), No. 9, 3903–3913.

[5] Bialy, M.; Mironov, A. *Angular billiard and algebraic Birkhoff conjecture.* Adv. in Math. **313** (2017), 102–126.

[6] Bialy, M.; Mironov, A. *Algebraic Birkhoff conjecture for billiards on Sphere and Hyperbolic plane.* J. Geom. Phys., **115** (2017), 150–156.

[7] M. Bialy, A. Mironov. *A survey on polynomial in momenta integrals for billiard problems.* Philos. Trans. Roy. Soc. A **376** (2018), no. 2131, 20170418, 19 pp.

[8] Bolotin, S.V. *Integrable billiards on surfaces of constant curvature.* Math. Notes **51** (1992), No. 1–2, 117–123.

[9] Dragović, V.; Radnović, M. *Integrable billiards and quadrics.* Russian Math. Surveys **65** (2010), no. 2, 319–379.

[10] Glutsyuk, A. *On 4-reflective complex analytic planar billiards.* J. Geom. Analysis, **27** (2017), 183–238.

[11] Glutsyuk, A.A. *On polynomially integrable billiards on surfaces of constant curvature.* To appear in J. Eur. Math. Soc.
Preprint: https://arxiv.org/abs/1706.04030

[12] Glutsyuk, A.A. *On two-dimensional polynomially integrable billiards on surfaces of constant curvature.* Doklady Mathematics, **98** (2018), No.1, 382–385.

[13] Kaloshin, V.; Sorrentino, A. *On local Birkhoff Conjecture for convex billiards.* Ann. of Math., **188** (2018), No. 1, 315–380.

[14] V. Kaloshin, A. Sorrentino. *On the integrability of Birkhoff billiards.* Philos. Trans. Roy. Soc. A **376** (2018), no. 2131, 20170419, 16 pp.

[15] Kozlov, V.V.; Treshchev, D.V. *Billiards. A genetic introduction to the dynamics of systems with impacts.* Translated from Russian by J.R.Schulenberger. Translations of Mathematical Monographs, **89**, American Mathematical Society, Providence, RI, 1991.

[16] Poritsky, H. *The billiard ball problem on a table with a convex boundary – an illustrative dynamical problem.* Ann. of Math. (2) **51** (1950), 446–470.

[17] Staude, O. *III, 22, Quadriques.* Encyclopédie des Sciences Mathématiques, Teubner, Gauthier-Villars, J.Gabay, 1904–1992, 1–162.

[18] Tabachnikov, S., *Geometry and Billiards.* Amer. Math. Soc. 2005.

[19] Tabachnikov, S., *Introducing projective billiards.* Erg. Th. and Dynam. Sys. **17** (1997), 957–976.

[20] Tabachnikov, S., *Commuting dual billiard maps.* Geometriae Dedicata, **53** (1994), 57–68.

[21] Veselov, A. P. *Integrable systems with discrete time, and difference operators.* Funct. Anal. Appl. **22** (1988), No. 2, 83–93.

[22] Veselov, A.P. *Confocal surfaces and integrable billiards on the sphere and in the Lobachevsky space.* J. Geom. Phys., **7** (1990), Issue 1, 81–107.