On commuting billiards in higher dimensions

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Abstract

We consider two nested billiards in $\mathbb{R}^n$, $n \geq 3$, with smooth strictly convex boundaries. We prove that if the corresponding actions by reflections on the space of oriented lines commute, then the billiards are confocal ellipsoids. This together with the previous analogous result of the author in two dimensions solves completely the Commuting Billiard Conjecture due to Sergei Tabachnikov. The main result is deduced from the classical theorem due to Marcel Berger saying that in higher dimensions only quadrics may have caustics.

1 Introduction

Let $\Omega_a \Subset \Omega_b \subset \mathbb{R}^n$ be two nested bounded domains with smooth strictly convex boundaries $a = \partial\Omega_a$ and $b = \partial\Omega_b$. Consider the corresponding billiard transformations σ_a, σ_b acting on the space of oriented lines in the plane by reflection as follows. Each σ_g, $g = a, b$, acts as identity on the lines disjoint from g. For each oriented line l intersecting g we take its last intersection point x with g in the sense of orientation: the orienting arrow of the line l at x is directed outside Ω_g. The image $\sigma_g(l)$ is the line obtained by reflection of the line l from the hyperplane $T_x g$: the angle of incidence equals the angle of reflection. The line $\sigma_g(l)$ is oriented by a tangent vector at x directed inside Ω_g. This is a continuous mapping that is smooth on the space of lines intersecting g transversely.

Recall, see, e.g., [2, 12], that a pencil of *confocal quadrics* in a Euclidean space $\mathbb{R}^n$ is a one-dimensional family of quadrics defined in some orthogonal

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coordinates $(x_1, \dots, x_n)$ by equations

$$\sum_{j=1}^{n} \frac{x_j^2}{a_j^2 + \lambda} = 1; \quad a_j \in \mathbb{R} \text{ are fixed; } \lambda \in \mathbb{R} \text{ is the parameter.}$$

It is known that *any two confocal elliptic or ellipsoidal billiards* commute [12, p.59, corollary 4.6], [13, p.58]. Sergei Tabachnikov stated the conjecture affirming the converse: any two commuting nested billiards are confocal ellipses (ellipsoids) [13, p.58]. In two dimensions his conjecture was proved by the author of the present paper in [6, theorem 5.21, p.231] for piecewise C^4-smooth boundaries. Here we prove it in higher dimensions.

Theorem 1.1 *Let two nested strictly convex C^2-smooth closed hypersurfaces in $\mathbb{R}^n$, $n \geq 3$ be such that the corresponding billiard transformations commute. Then they are confocal ellipsoids.*

In the next section Theorem 1.1 will be deduced from a classical theorem due to Marcel Berger concerning billiards in $\mathbb{R}^n$, $n \geq 3$, which states that only billiards bounded by quadrics may have caustics, see the next definition. In Subsection 3 we state and prove a tangential local version of Theorem 1.1.

Definition 1.2 Let a, b be two nested strictly convex closed hypersurfaces in $\mathbb{R}^n$, $a \Subset \Omega_b$, see the notations at the beginning of the paper. We say that a is a *caustic* for the hypersurface b, if the image of each oriented line tangent to a by the reflection σ_b is again a line tangent to a.

Example 1.3 It is well-known that if a, b are two confocal ellipses (ellipsoids), then the smaller one is a caustic for the bigger one. In the plane this is the classical Proclus–Poncelet theorem. In higher dimensions this theorem is due to Jacobi, see [11, p.80].

Commuting billiards are closely related to problems of classification of integrable billiards, see [13]. It is known that elliptic billiards and billiards in ellipsoids are integrable, see [14, proposition 4], [12, chapter 4], and this also holds for non-Euclidean ellipsoids in sphere and in the Lobachensky (hyperbolic) space of any dimension, see [15, the corollary on p. 95]. The famous Birkhoff Conjecture states that in two dimensions the converse is true. Namely, it deals with the so-called *Birkhoff caustic-integrable* convex planar billiards with smooth boundary, that is, billiards for which there exists a foliation by closed caustics in an interior neighborhood of the boundary. It states that the only Birkhoff caustic-integrable billiards are ellipses. Birkhoff Conjecture was first stated in print in Poritsky's paper [10], who proved it in loc. cit. under the additional assumption that the billiard in each closed caustic near the boundary has the same closed caustics, as the initial billiard. Poritsky's assumption implies that *the initial billiard map commutes with the billiard in any closed caustic;* this follows by the arguments presented in [12, pp.58–59]. One of the most famous results on Birkhoff Conjecture is a theorem of M.Bialy, who proved that if the phase cylinder of the billiard map is foliated (almost everywhere) by non-contractible closed curves which are invariant under the billiard map, then the boundary is a circle. A local version of Birkhoff Conjecture, for integrable deformations of ellipses was recently solved in [1, 9]. Its recently solved algebraic version is a result of papers [3, 4, 7], see also a short version [8] of the preprint [7]. For a historical survey of Birkhoff Conjecture see [12, p.95] and papers [9, 7] and references therein. Dynamics in billiards in two and higher dimensions with piecewise smooth boundary consisting of confocal quadrics was studied in [5].

2 Commuting billiards and caustics: proof of Theorem 1.1

Proposition 2.1 *Let $n \geq 3$. Let two nested strictly convex C^2-smooth closed hypersurfaces $a, b \subset \mathbb{R}^n$, $a \Subset \Omega_b$ (see the notations at the beginning of the paper) be such that the corresponding billiard transformations σ_a and σ_b commute. Then a is a caustic for the hypersurface b.*

Proof Let Π_a denote the open subset of lines in $\mathbb{R}^n$ that are disjoint from the hypersurface a. Its boundary $\partial\Pi_a$ consists of those lines that are tangent to a. A line L is fixed by σ_a, if and only if $L \in \overline{\Pi}_a$, i.e., L is either disjoint from a, or tangent to a. In this case $\sigma_b\sigma_a(L) = \sigma_b(L) = \sigma_a\sigma_b(L)$, and thus, $\sigma_b(L)$ is a fixed point of the transformation σ_a. This implies that $\sigma_b(\overline{\Pi}_a) \subset \overline{\Pi}_a$. The subset Π_a is invariant under two transformations acting on oriented lines: the reflection σ_b and the transformation J of the orientation change. The transformations J and $J \circ \sigma_b$ are involutions, thus sending Π_a to itself. Therefore, $J(\Pi_a) = J \circ \sigma_b(\Pi_a) = \Pi_a$. Hence, $\sigma_b(\Pi_a) = \Pi_a$, and thus, $\sigma_b(\partial\Pi_a) = \partial\Pi_a$. The latter equality means exactly that a is a caustic for the hypersurface b. The proposition is proved. □

As it is shown below, Theorem 1.1 is implied by Proposition 2.1 and the following theorem due to M.Berger.

Theorem 2.2 *[2] Let $n \geq 3$. Let $S, U \subset \mathbb{R}^n$ be germs of C^2-smooth hypersurfaces at points B and A respectively with non-degenerate second fundamental forms. Let the affine tangent line $T_A U$ go through B transversely to S. Let there exist a germ of C^2-smooth hypersurface V such that for every point $x \in U$ close to A the image of the affine tangent line $T_x U$ under the reflection from the hypersurface S be tangent to V. Then S is a piece of a quadric a, and U, V are pieces of one and the same quadric confocal to a.*

Proof of Theorem 1.1. Let $a, b \subset \mathbb{R}^n$ be the nested hypersurfaces under question with commuting billiard transformations, $a \Subset \Omega_b$. Then a is a caustic for the hypersurface b, by Proposition 2.1. This means that for every points $B \in b$ and $A \in a$ such that the line AB is tangent to a at A the image $\sigma_b(AB)$ of the line AB (oriented from A to B) is a line through B tangent to a. Recall that a and b are strictly convex, which implies that their second fundamental forms are sign-definite and thus, non-degenerate. Therefore, for every A and B as above the germs at A and B of the hypersurfaces $U = a$ and $S = b$ respectively satisfy the conditions of Theorem 2.2, with V being the germ of the hypersurface a at its point D of tangency with the line $\sigma_b(AB)$. Hence, for every A and B as above the germ (S, B) lies in a quadric, and the germs (U, A), (V, D) lie in one and the same quadric confocal to S. This implies that b is a quadric, and a is a quadric confocal to b. Theorem 1.1 is proved. □

3 A tangential local version of Theorem 1.1

Theorem 3.1 *Let $n \geq 3$. Let (U, A), (S, B), (V, D) be germs of hypersurfaces in $\mathbb{R}^n$ at points A, B and D, and let U and S have non-degenerate second fundamental forms. For every $G = U, S, V$ consider the action of the reflection σ_G on the oriented lines that intersect G, defined as at the beginning of the paper. Let the affine tangent line $L_0 = T_A U$ go through B transversely to S (we orient it from A to B), and let its image $\sigma_S(L_0)$ be tangent to V at D. Let W be a small neighborhood of the line L_0 in the space of oriented lines; in particular, each line in W intersects S. Let $\Pi_W \subset W$ denote the subset of those lines that intersect U. Let for every $L \in \Pi_W$ the image $\sigma_S(L)$ intersect V: thus, the compositions $\sigma_S \circ \sigma_U$ and $\sigma_V \circ \sigma_S$ are well-defined on Π_W. Let the latter compositions be identically equal on Π_W. Then S lies in a quadric b, and U, V lie in one and the same quadric confocal to b.*

Proof Every line L tangent to U and close enough to L_0 lies in Π_W. Its image $\sigma_U(L)$ is tangent to V. Indeed, $\sigma_S \circ \sigma_U(L) = \sigma_S(L) = \sigma_V \circ \sigma_S(L)$. Thus, the line $\sigma_S(L)$ intersects V and is invariant under the reflection from V. Hence, it is tangent to V (at the last point of its intersection with V). Finally, the germs of hypersurfaces U, S and V satisfy the conditions of Theorem 2.2. Therefore, S lies in a quadric b, and U, V lie in one and the same quadric confocal to b, by Theorem 2.2. This proves Theorem 3.1. □

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