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FOURIER EXPANSIONS AT CUSPS

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Abstract. In this article we study the number fields generated by the Fourier coefficients of modular forms at arbitrary cusps. We prove that these fields are contained in certain cyclotomic extensions of the field generated by the Fourier coefficients at ∞, and show that this bound is tight in the case of newforms with trivial Nebentypus. The main tool is an extension of a result of Shimura on the interplay between the actions of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{Aut}(\mathbb{C})$ on spaces of modular forms. We give two new proofs of this result: one based on products of Eisenstein series, and the other using the theory of algebraic modular forms.

1. Introduction

In this article we study the number fields generated by the Fourier coefficients of modular forms at the cusps of $X_1(N)$. To do this we study the connections between two actions on spaces of modular forms: the action of $\mathrm{GL}_2^+(\mathbb{Q})$ via the slash-operator and the action of $\mathrm{Aut}(\mathbb{C})$ on the Fourier coefficients of a modular form. A detailed study of these actions was conducted by Shimura in [16], where he proved a formula for the action of $\mathrm{Aut}(\mathbb{C})$ on $f|g$ for modular forms of even weight. In Theorem 3.3 we give an extension of his result to modular forms of any integral weight and provide two new proofs of it: one using a theorem of Khuri-Makdisi [11] on products of Eisenstein series, and the other using Katz's theory of algebraic modular forms [10].

We use this theorem to bound the fields generated by the Fourier coefficients of modular forms at the cusps. Let us assume for simplicity that f is a modular form in $M_k(\Gamma_0(N))$, and let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We show in Theorem 4.1 that the coefficients of $f|g$ lie in the cyclotomic extension $K_f(\zeta_{N'})$, where K_f is the number field generated by the coefficients of f, and $N' = N/\gcd(CD, N)$. In the case f has non-trivial Nebentypus, we show in Theorem 4.4 that the coefficients of $f|g$ belong to a 1-dimensional $K_f(\zeta_{N'})$-vector space, which is itself contained in an explicit cyclotomic extension $K_f(\zeta_M)$.

We apply these results in Section 5 to find number fields that contain the Atkin–Lehner pseudo-eigenvalues of a newform, recovering a result of Cohen in [4].

In Section 6 we discuss how to choose g among the matrices in $\mathrm{SL}_2(\mathbb{Z})$ that map ∞ to a given cusp $\alpha \in \mathbb{P}^1(\mathbb{Q})$ so that N' (or M) becomes minimal. Assuming that $f \in M_k(\Gamma_0(N))$ is an eigenfunction of the Atkin–Lehner operators, we describe how to further reduce N' by potentially replacing α with its image under a suitable Atkin–Lehner operator. The Fourier expansion of $f|g$ can then easily be obtained from another Fourier expansion $f|g'$ which has coefficients in the field $K_f(\zeta_{\gcd(\delta, N/\delta)})$, where $\delta = \gcd(C, N)$ is the denominator of the cusp $\alpha = A/C$. Note that $\mathbb{Q}(\zeta_{\gcd(\delta, N/\delta)})$ is the field of definition of the cusp α in the canonical model of $X_0(N)$ over $\mathbb{Q}$.

The second author was partially funded by the DFG-Forschergruppe 1920 and the LOEWE research unit "Uniformized Structures in Arithmetic and Geometry".

In the last section, we prove in Theorem 7.6 that if f is a newform for $\Gamma_0(N)$ then the number field provided by Theorem 4.1 is the best possible, in the sense that it is the number field generated by the coefficients of $f|g$.

Recently three algorithms for the computation of the Fourier expansion of $f|g$ have appeared: two algorithms in SageMath, one by Dan Collins [5] and another by Martin Dickson and the second author [8]. The third algorithm was implemented in PARI/GP by Karim Belabas and Henri Cohen [4]. While the first algorithm only uses numerical approximations of the Fourier coefficients in order to compute Petersson inner products, the latter two calculate the Fourier coefficients as algebraic numbers. The knowledge of the number field (or vector space) generated by the Fourier coefficients of $f|g$ could provide a significant speed-up for these calculations.

Acknowledgements: We thank Henri Cohen for encouraging us to write this article.
We are grateful to Abhishek Saha for pointing out an alternative approach to the Theorems in Section 4, sketched in Remark 4.8.

Notations. For any integer $k \in \mathbb{Z}$, we define the weight k action of $\mathrm{GL}_2^+(\mathbb{R})$ on functions $f : \mathcal{H} \to \mathbb{C}$ by

$$f|_k g(\tau) = \frac{\det(g)^{k/2}}{(c\tau+d)^k} f\Big(\frac{a\tau+b}{c\tau+d}\Big) \qquad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})\right).$$

We will usually omit k from the notation and just write $f|g$ for $f|_k g$.

The automorphism group $\mathrm{Aut}(\mathbb{C})$ acts on spaces of modular forms as follows: for any modular form $f(\tau) = \sum_n a_n e^{2\pi i n\tau/w}$, we let

$$f^\sigma(\tau) = \sum_n \sigma(a_n) e^{2\pi i n\tau/w} \qquad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

For any integer $N \geq 1$, we denote $\zeta_N = e^{2\pi i/N} \in \mathbb{C}$.

2. Eisenstein series

2.1. **Definitions.** We refer the reader to [9, §3] for more details on Eisenstein series.
For integers $k \geq 1$, $N \geq 1$ and $a, b \in \mathbb{Z}/N\mathbb{Z}$, define the series

$$E_{a,b}^{(k)}(\tau) = \frac{(k-1)!}{(-2\pi i)^k} \sum_{\substack{\omega \in \mathbb{Z}\tau+\mathbb{Z} \\ \omega \neq -(\tilde{a}\tau+\tilde{b})/N}} \frac{1}{(\omega + \frac{\tilde{a}\tau+\tilde{b}}{N})^k |\omega + \frac{\tilde{a}\tau+\tilde{b}}{N}|^{2s}}\Bigg|_{s=0}$$

where $\tilde{a}, \tilde{b}$ denote any representatives of a, b in $\mathbb{Z}$, and $\cdot|_{s=0}$ denotes analytic continuation to $s = 0$ (this is needed only when $k \in \{1, 2\}$). It follows from the definition that the weight k action of $\mathrm{SL}_2(\mathbb{Z})$ on these series is given by $E_{a,b}^{(k)}|g = E_{(a,b)g}^{(k)}$ for every matrix $g \in \mathrm{SL}_2(\mathbb{Z})$. In particular, the function $E_{a,b}^{(k)}$ is modular of weight k with respect to the principal congruence subgroup $\Gamma(N)$. If $k \neq 2$, then $E_{a,b}^{(k)}$ is a holomorphic Eisenstein series of weight k with respect to $\Gamma(N)$. If $k = 2$, then $\tilde{E}_{a,b}^{(2)} := E_{a,b}^{(2)} - E_{0,0}^{(2)}$ is a holomorphic Eisenstein series of weight 2 with respect to $\Gamma(N)$.

2.2. **Fourier expansions.** We refer the reader to [10, §3] and [15, Chap. VII] for proofs of the following facts.
If $k \neq 2$, then the Fourier expansion of $E^{(k)}_{a,b}$ is given by

$$E^{(k)}_{a,b}(\tau) = a_0(E^{(k)}_{a,b}) + \sum_{\substack{m,n \geq 1 \\ m \equiv a(N)}} \zeta_N^{bn} n^{k-1} q^{mn/N} + (-1)^k \sum_{\substack{m,n \geq 1 \\ m \equiv -a(N)}} \zeta_N^{-bn} n^{k-1} q^{mn/N} \qquad (q = e^{2\pi i \tau}).$$

If $k = 2$, then the Fourier expansion of $\tilde{E}^{(2)}_{a,b}$ is given by

$$\tilde{E}^{(2)}_{a,b}(\tau) = a_0(\tilde{E}^{(2)}_{a,b}) + \sum_{\substack{m,n \geq 1 \\ m \equiv a(N)}} \zeta_N^{bn} n q^{mn/N} + \sum_{\substack{m,n \geq 1 \\ m \equiv -a(N)}} \zeta_N^{-bn} n q^{mn/N} - 2 \sum_{m,n \geq 1} n q^{mn}.$$

The constant terms $a_0(E^{(k)}_{a,b})$ and $a_0(\tilde{E}^{(2)}_{a,b})$ are elements of $\mathbb{Q}(\zeta_N)$ and are given in [9, 3.10] and [3, §3]. We will not need the precise expressions since modularity determines them uniquely.

Proposition 2.1. Let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that $\sigma(\zeta_N) = \zeta_N^{\lambda}$ with $\lambda \in (\mathbb{Z}/N\mathbb{Z})^\times$. If $k \neq 2$, then

$$(E^{(k)}_{a,b}|g)^\sigma = (E^{(k)}_{a,b})^\sigma | g_\lambda,$$

where g_λ is any lift in $\mathrm{SL}_2(\mathbb{Z})$ of the matrix $\left(\begin{smallmatrix} A & \lambda B \\ \lambda^{-1} C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. If $k = 2$, then the same statement holds with $E^{(2)}_{a,b}$ replaced by $\tilde{E}^{(2)}_{a,b}$.

Proof. Note that $(E^{(k)}_{a,b})^\sigma = E^{(k)}_{a,\lambda b}$, so that

$$(E^{(k)}_{a,b}|g)^\sigma = E^{(k)}_{aA+bC,\lambda(aB+bD)} = E^{(k)}_{a,\lambda b} \Big| \begin{pmatrix} A & \lambda B \\ \lambda^{-1} C & D \end{pmatrix} = (E^{(k)}_{a,b})^\sigma | g_\lambda.$$

The argument for $\tilde{E}^{(2)}_{a,b}$ is similar. □

3. The actions of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{Aut}(\mathbb{C})$ on modular forms

In this section we investigate the connection between the natural actions of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{Aut}(\mathbb{C})$ on modular forms.
Let us first recall Khuri-Makdisi's result [11] giving generators of the graded algebra of modular forms. Let $\mathcal{R}_N$ be the subalgebra of $M_*(\Gamma(N)) = \bigoplus_{k \geq 0} M_k(\Gamma(N))$ generated by the Eisenstein series $E^{(1)}_{a,b}$ with $a, b \in \mathbb{Z}/N\mathbb{Z}$.

Theorem 3.1. [11] If $N \geq 3$, then $\mathcal{R}_N$ contains all modular forms on $\Gamma(N)$ of weight 2 and above. In other words, $\mathcal{R}_N$ misses only the cusp forms of weight 1 on $\Gamma(N)$.

Proof. This follows from combining [11, Theorem 3.5, Remark 3.14, Theorem 5.1]. The link between our notations and Khuri-Makdisi's notations is

$$E^{(1)}_{a,b}(\tau) = -\frac{1}{2\pi i} G_1\Big(\tau, \frac{a\tau + b}{N}\Big) = \frac{1}{2\pi i} \lambda_{(a\tau+b)/N},$$

see [11, Definition 2.1 and Corollary 3.13]. □

Remark 3.2. If $N = 2$, then the algebra $M_*(\Gamma(2))$ is generated by the weight 2 Eisenstein series $\tilde{E}^{(2)}_{1,0}$, $\tilde{E}^{(2)}_{0,1}$ and $\tilde{E}^{(2)}_{1,1}$, the only relation being $\tilde{E}^{(2)}_{1,0} + \tilde{E}^{(2)}_{0,1} + \tilde{E}^{(2)}_{1,1} = 0$, see [11, Remark 3.6]. Of course, if $N = 1$ then $M_*(\mathrm{SL}_2(\mathbb{Z}))$ is freely generated by the usual Eisenstein series of weight 4 and 6.

The following theorem is an extension of Shimura's result [16, Theorem 8] which deals with the case of modular forms of even weight. We note however, that Shimura works with a much wider class of functions, including Hilbert modular forms and also certain derivatives of them. His methods can be extended to provide a full proof of Theorem 3.3, but we will give two new proofs of it, the first using Khuri-Makdisi's Theorem 3.1. The second proof uses Katz's theory of algebraic modular forms and is given in the appendix along with a brief introduction to the theory of algebraic modular forms.

Theorem 3.3. Let $f \in M_k(\Gamma(N))$ be a modular form of weight $k \geq 1$ and level $\Gamma(N)$. Let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that $\sigma(\zeta_N) = \zeta_N^\lambda$ with $\lambda \in (\mathbb{Z}/N\mathbb{Z})^\times$. Then

$$(f|g)^\sigma = f^\sigma|g_\lambda,$$

where g_λ is any lift in $\mathrm{SL}_2(\mathbb{Z})$ of the matrix $\left(\begin{smallmatrix} A & \lambda B \\ \lambda^{-1}C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

First proof. Let us assume $N \geq 3$, and let $f \in \mathcal{R}_N$. The maps $h \mapsto (h|g)^\sigma$ and $h \mapsto h^\sigma|g_\lambda$ are both σ-linear ring homomorphisms. We are thus reduced to the case $f = E^{(1)}_{a,b}$, which follows from Proposition 2.1. If $f \in S_1(\Gamma(N))$, then f^2 and f^3 are in $\mathcal{R}_N$, so Theorem 3.3 holds for them. Using $f = f^3/f^2$, we get $(f|g)^\sigma = f^\sigma|g_\lambda$. In the case $N = 2$, we proceed similarly by applying Proposition 2.1 to $\tilde{E}^{(2)}_{a,b}$. Finally, the case $N = 1$ is trivial. □

Remark 3.4. If we restrict to modular forms on $\Gamma_1(N)$, then Theorem 3.3 also follows from the result of Borisov and Gunnells [2, Thm 5.15] that all modular forms of sufficiently large weight are toric.

By Theorem 3.3, the space $M_k(\Gamma(N); \mathbb{Q}(\zeta_N))$ of modular forms with coefficients in $\mathbb{Q}(\zeta_N)$ is stable under the weight k action of $\mathrm{SL}_2(\mathbb{Z})$. It is thus endowed with a right action of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. The Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ also acts on $M_k(\Gamma(N); \mathbb{Q}(\zeta_N))$, by means of the usual action on the Fourier expansion. But more is true: for any $f \in M_k(\Gamma(N); \mathbb{Q}(\zeta_N))$, let us define

$$f\Big|\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} = f^{\sigma_\lambda} \qquad (\lambda \in (\mathbb{Z}/N\mathbb{Z})^\times),$$

where $\sigma_\lambda \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ is the automorphism defined by $\sigma_\lambda(\zeta_N) = \zeta_N^\lambda$. Then the above actions of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and $(\mathbb{Z}/N\mathbb{Z})^\times$ combine to give a right action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on $M_k(\Gamma(N); \mathbb{Q}(\zeta_N))$. Indeed

$$g\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} A & \lambda B \\ C & \lambda D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} A & \lambda B \\ \lambda^{-1}C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} g_\lambda \tag{1}$$

and Theorem 3.3 says precisely that both sides of this equality act in the same way on $M_k(\Gamma(N); \mathbb{Q}(\zeta_N))$. Note also that with this definition, the identities $E^{(k)}_{a,b}|g = E^{(k)}_{(a,b)g}$ for $k \neq 2$ and $\tilde{E}^{(2)}_{a,b}|g = \tilde{E}^{(2)}_{(a,b)g}$ are true for any $g \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

Remark 3.5. Let $\tilde{Y}(N)$ be the model of $\Gamma(N)\backslash\mathcal{H}$ over $\mathbb{Q}(\zeta_N)$ constructed in [17, Chapter 6]. The automorphism group of the $\mathbb{Q}$-scheme $\tilde{Y}(N)$ contains $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. This gives a right action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on the function field $\mathbb{Q}(\zeta_N)(\tilde{Y}(N))$, which is a subfield of $\mathbb{Q}(\zeta_N)((q^{1/N}))$. The action of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is the slash action of weight 0, and the action of the diagonal matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ with $\lambda \in (\mathbb{Z}/N\mathbb{Z})^\times$ coincides with the natural action of σ_λ. Shimura's original proof of Theorem 3.3 relies on this fact.

Corollary 3.6. Let $f \in M_k(\Gamma_0(N), \chi)$ be a nonzero modular form of weight $k \geq 1$, level $N \geq 1$ and Nebentypus character χ. Then the field K_f generated by the Fourier coefficients of f contains the field $\mathbb{Q}(\chi)$ generated by the values of χ.

Proof. We have to prove that every $\sigma \in \mathrm{Aut}(\mathbb{C}/K_f)$ fixes $\mathbb{Q}(\chi)$. Let $g \in \Gamma_0(N)$. Then $f|g = \chi(g)f$. Applying σ, we get $(f|g)^\sigma = \chi(g)^\sigma f$. But Theorem 3.3 implies

$$(f|g)^\sigma = f^\sigma|g_\lambda = f|g_\lambda = \chi(g_\lambda)f = \chi(g)f,$$

so that $\chi = \chi^\sigma$. □

Remark 3.7. Corollary 3.6 can also be proved using Katz's theory of algebraic modular forms (see the Appendix), noting that the diamond operators $\langle\delta\rangle$, $\delta \in (\mathbb{Z}/N\mathbb{Z})^\times$ are defined over $\mathbb{Q}$, hence leave stable the space $M_k(\Gamma_1(N); K)$ of modular forms with coefficients in a fixed subfield K of $\mathbb{C}$.

4. Bounding the coefficient field of $f|g$

Theorem 4.1. Let $f \in M_k(\Gamma_1(N))$ be a modular form of integral weight $k \geq 1$ on $\Gamma_1(N)$. Let K_f be the subfield of $\mathbb{C}$ generated by the Fourier coefficients $a_n(f)$, $n \geq 1$. Let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

(1) The modular form $f|_k g$ has coefficients in $K_f(\zeta_M)$ with $M = N/\gcd(C, N)$.
(2) If $f \in M_k(\Gamma_0(N))$ then $f|_k g$ has coefficients in $K_f(\zeta_{N'})$ with $N' = N/\gcd(CD, N)$.

Proof. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$. By Theorem 3.3, a sufficient condition for $f|g$ being fixed by σ is given by $f^\sigma = f = f|g_\lambda g^{-1}$, where $\sigma(\zeta_N) = \zeta_N^\lambda$. We have

$$g_\lambda g^{-1} \equiv \begin{pmatrix} AD - \lambda BC & AB(\lambda - 1) \\ CD(\lambda^{-1} - 1) & AD - \lambda^{-1}BC \end{pmatrix} \bmod N. \tag{2}$$

We see that $g_\lambda g^{-1} \in \Gamma_0(N)$ if and only if $\lambda \equiv 1 \bmod N'$. If $f \in M_k(\Gamma_0(N))$, then $f|g$ is fixed by every $\sigma \in \mathrm{Aut}(\mathbb{C}/K_f(\zeta_{N'}))$, hence has coefficients in $K_f(\zeta_{N'})$, which proves (2).
Furthermore $AD - \lambda BC = 1 + BC(1 - \lambda)$ so that $g_\lambda g^{-1} \in \Gamma_1(N)$ if and only if $\lambda \equiv 1 \bmod N'$ and $\lambda \equiv 1 \bmod N/\gcd(BC, N)$. Since B and D are coprime, the conjunction of these conditions is equivalent to $\lambda \equiv 1 \bmod N/\gcd(C, N)$. This proves (1). □

We now turn to modular forms with characters. We will actually bound not only the field of coefficients of $f|g$, but also the *vector space* generated by the coefficients of $f|g$.
In order to state our results, we need some more notation. Let $f \in M_k(\Gamma_0(N), \chi)$, where χ is a Dirichlet character of conductor m dividing N, and let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Put $N' = N/\gcd(CD, N)$, $m' = m/\gcd(BC, m)$ and $M = \mathrm{lcm}(N', m')$. Let $K = K_f(\zeta_{N'})$ and $L = K_f(\zeta_M)$. Since $L = K(\zeta_{m'})$, the extension L/K is abelian and its Galois group $G = \mathrm{Gal}(L/K)$ identifies with a subgroup G' of $(\mathbb{Z}/m'\mathbb{Z})^\times$ by means of the cyclotomic character $\lambda : G \to (\mathbb{Z}/m'\mathbb{Z})^\times$. Since $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(\mathbb{Q}(\zeta_{m'})/K')$ with $K' = K \cap \mathbb{Q}(\zeta_{m'})$, the subgroup $G' \subset (\mathbb{Z}/m'\mathbb{Z})^\times$ corresponds to the subfield $K' \subset \mathbb{Q}(\zeta_{m'})$.

Lemma 4.2. The map $\chi_g : G \to \mathbb{C}^\times$ defined by

$$\chi_g(\sigma) = \chi(AD - \lambda(\sigma)^{-1}BC) \qquad (\sigma \in G)$$

is a group homomorphism.

Proof. Let $\sigma \in G$. Note that $m'BC$ is divisible by m, so that $AD - \lambda(\sigma)^{-1}BC$ is well-defined in $\mathbb{Z}/m\mathbb{Z}$. Let $\lambda_N(\sigma) \in (\mathbb{Z}/N\mathbb{Z})^\times$ denote the cyclotomic character modulo N. Since $\lambda_N(\sigma) \equiv 1 \bmod N'$, the identity (2) shows that $g_{\lambda_N(\sigma)}g^{-1}$ is upper-triangular modulo N. It follows that $AD - \lambda_N(\sigma)^{-1}BC \in (\mathbb{Z}/N\mathbb{Z})^\times$ and thus $AD - \lambda(\sigma)^{-1}BC \in (\mathbb{Z}/m\mathbb{Z})^\times$. Therefore the map χ_g is well-defined.
Let us show that χ_g is a group homomorphism. We may write χ_g as the composition

$$G \xrightarrow{\lambda} G' \xrightarrow{\psi} (\mathbb{Z}/m\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$$

where ψ is defined by $\psi(\mu) = AD - \mu^{-1}BC$. Using the relation (1), we get the following identity in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$

$$g_{\mu\mu'}g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \mu^{-1} \end{pmatrix} g_{\mu'}g^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} g_\mu g^{-1} \qquad (\mu, \mu' \in (\mathbb{Z}/N\mathbb{Z})^\times).$$

Specialising to the case $\mu, \mu' \equiv 1 \bmod N'$ and comparing the bottom-right entries, we deduce that ψ is a group homomorphism. $\square$

Note that the character χ_g takes values in $\mathbb{Q}(\chi)^\times$, which is contained in $K_f^\times$ by Corollary 3.6. By the normal basis theorem, L is a free $K[G]$-module of rank 1. Since K_f is contained in K, the character χ_g cuts out a K-line L^{χ_g} in L, namely

(3) $$L^{\chi_g} = \{x \in L : \forall \sigma \in G, \sigma(x) = \chi_g(\sigma)x\}.$$

We are now ready to state our result.

Theorem 4.3. The modular form $f|g$ has coefficients in L^{χ_g}.

Proof. Let $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ with $\sigma(\zeta_N) = \zeta_N^\lambda$. Since $\lambda \equiv 1 \bmod N'$, we have $g_\lambda g^{-1} \in \Gamma_0(N)$. Then

(4) $$(f|g)^\sigma = f^\sigma|g_\lambda = f|g_\lambda g^{-1}g = \chi(AD - \lambda^{-1}BC)f|g = \chi_g(\sigma|_L)f|g.$$

In particular $f|g$ is fixed by $\mathrm{Aut}(\mathbb{C}/L)$, hence has coefficients in L. Moreover (4) shows that $f|g$ has coefficients in L^{χ_g}. $\square$

We summarise our result and make it slightly more precise as follows.

Theorem 4.4. Let $f \in M_k(\Gamma_0(N), \chi)$, where χ is a Dirichlet character of conductor m dividing N, and let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Put $N' = N/\gcd(CD, N)$, $m' = m/\gcd(BC, m)$ and $M = \mathrm{lcm}(N', m')$. Then $f|_k g$ has coefficients in $K_f(\zeta_M)$.
More precisely, let G' be the subgroup of $(\mathbb{Z}/m'\mathbb{Z})^\times$ corresponding to the abelian number field $K_f(\zeta_{N'}) \cap \mathbb{Q}(\zeta_{m'})$, and let ζ be any m'-th root of unity such that

$$c_{\chi,g} := \sum_{\mu \in G'} \chi(AD - \mu BC)\zeta^\mu$$

is nonzero (such a ζ always exists). Then $f|_k g$ has coefficient in $c_{\chi,g} \cdot K_f(\zeta_{N'})$.

Proof. The fact that $f|g$ has coefficients in $L = K_f(\zeta_M)$ was proved in Theorem 4.3. Let $K = K_f(\zeta_{N'})$ and let $\pi_{\chi_g} : L \to L^{\chi_g}$ be the K-linear projector associated to the linear character $\chi_g : G \to K^\times$. It is given explicitly by

$$\pi_{\chi_g}(x) = \frac{1}{|G|}\sum_{\tau\in G}\chi_g(\tau^{-1})\tau(x) \qquad (x \in L).$$

Since L is generated as a K-vector space by the m'-th roots of unity, there exists $\zeta \in \mu_{m'}$ such that $\pi_{\chi_g}(\zeta) \neq 0$. We set $c_{\chi,g} = |G| \cdot \pi_{\chi_g}(\zeta)$, so that

$$c_{\chi,g} = \sum_{\tau\in G}\chi_g(\tau^{-1})\tau(\zeta) = \sum_{\tau\in G}\chi_g(\tau^{-1})\zeta^{\lambda(\tau)} = \sum_{\mu\in G'}\chi(AD-\mu BC)\zeta^{\mu}.$$

By Theorem 4.3, the modular form $f|g$ has coefficients in $L^{\chi_g} = c_{\chi,g} \cdot K$. ☐

Remark 4.5. The choice $\zeta = \zeta_{m'}$ does not always work. For example, take the newform f of weight $k = 3$, level $N = 9$ and character χ of conductor $m = 9$, with $\chi(4) = \zeta_3$ and $\chi(-1) = -1$. We have $K_f = \mathbb{Q}(\chi) = \mathbb{Q}(\zeta_3)$. Taking $g = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 3 \end{smallmatrix}\right)$, we get $N' = 3$, $m' = 9$ and $G' = \{1, 4, 7\}$, so that $c_{\chi,g} = 0$ for $\zeta = \zeta_9$. On the other hand, for $\zeta = \zeta_9^2$ we get $c_{\chi,g} = 3\zeta_9^2$ and $f|g$ indeed has coefficients in $\zeta_9^2 \cdot \mathbb{Q}(\zeta_3) = \langle \zeta_9^2, \zeta_9^5\rangle_{\mathbb{Q}}$.

Remark 4.6. Theorem 4.4 says that the coefficients of $f|g$ belong to a vector space which has the same dimension as in the $\Gamma_0(N)$ case.

Remark 4.7. Theorem 4.3 also shows that the coefficients of $f|g$ lie in the fixed field $L^{\ker\chi_g}$. Let $\chi'_g : G' \to \mathbb{C}^\times$ be the character defined by $\chi'_g(\mu) = \chi(AD - \mu^{-1}BC)$ (using the notations of the proof of Lemma 4.2, we have $\chi'_g = \chi \circ \psi$ and $\chi_g = \chi'_g \circ \lambda$). Then the field $L^{\ker\chi_g}$ is equal to the composite $F \cdot K_f(\zeta_{N'})$, where F is the subfield of $\mathbb{Q}(\zeta_{m'})$ corresponding to the kernel of χ'_g.

Remark 4.8. An alternative approach towards proving Theorems 4.1 and 4.4 would be to use local Whittaker newforms as in [6]. In particular Proposition 3.3 in *loc. cit.* gives an explicit formula for the Fourier coefficients of $f|g$ in terms of Whittaker newforms and the Galois action on such newforms is described in the proof of Proposition 2.17.

5. Atkin-Lehner operators

For a divisor Q of N with $\gcd(Q, N/Q) = 1$ we define the Atkin-Lehner operator on $M_k(\Gamma_1(N))$ as follows. Choose $x, y, z, w \in \mathbb{Z}$ with $x \equiv 1 \bmod N/Q$ and $y \equiv 1 \bmod Q$ such that the matrix $W_Q = \left(\begin{smallmatrix} Qx & y \\ Nz & Qw \end{smallmatrix}\right)$ has determinant Q. Note that $W_Q = h_Q \left(\begin{smallmatrix} Q & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $h_Q = \begin{pmatrix} x & y \\ \frac{N}{Q}z & Qw \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. For a modular form $f \in M_k(\Gamma_1(N))$ we have

$$f|_k W_Q(\tau) = Q^{k/2}\left(f|_k h_Q\right)(Q\tau).$$

Therefore we can apply our previous results to find a module that contains the coefficients of $f|h_Q$, or equivalently the coefficients of $Q^{-k/2}f|W_Q$, and reprove a theorem of Cohen.

Corollary 5.1 (Theorem 2.6 in [4])**.** Let Q be a maximal divisor of N and let $f \in M_k(\Gamma_1(N))$ and K_f be the subfield of $\mathbb{C}$ generated by its Fourier coefficients. Then

(1) The modular form $f|_k W_Q$ has coefficients in $Q^{k/2} \cdot K_f(\zeta_Q)$.

(2) If $f \in M_k(\Gamma_0(N), \chi)$ for a character χ of conductor m, then $f|_k W_Q$ has coefficients in $Q^{k/2} G'(\chi_Q) \cdot K_f$, where $G'(\chi_Q)$ is the Gauss sum of the primitive character associated to the Q-part of χ.

Proof. The first statement follows directly from Theorem 4.1.
Now let $f \in M_k(\Gamma_0(N), \chi)$. We will prove that the coefficients of $f|h_Q$ lie in $G'(\chi_Q) \cdot K_f$, which is equivalent to the second statement. First we determine the character χ_{h_Q} from the previous section. Splitting χ as a product of its Q-part and its N/Q-part we observe

$$\chi_{h_Q}(\sigma) = \chi_Q(-\frac{N}{Q} zy\lambda(\sigma)^{-1})\chi_{N/Q}(Qwx) = \chi_Q(\lambda(\sigma)^{-1}) = \overline{\chi_Q(\sigma)}.$$

The conclusion now follows from Theorem 4.4. Since $N' = 1$, $m' = m_Q$, and for $\zeta = \zeta_{m'}$ we have $c_{\chi, h_Q} = G'(\chi_Q)$. ◻

Let $f \in S_k(\Gamma_0(N), \chi)$ be a newform. Then according to [1, §1] there exists a newform $\tilde{f} \in S_k(\Gamma_0(N), \overline{\chi_Q}\chi_{N/Q})$ and an algebraic number $\lambda_Q(f)$ of absolute value 1 such that

$$f|W_Q = \lambda_Q(f)\tilde{f}. \tag{5}$$

The number $\lambda_Q(f)$ is called the pseudo-eigenvalue of f at Q. By looking at the first Fourier coefficient in (5), we get the following result.

Corollary 5.2. If $f \in S_k(\Gamma_0(N), \chi)$ is a newform, then the pseudo-eigenvalue $\lambda_Q(f)$ is in $Q^{k/2} G'(\chi_Q) \cdot K_f$ and $\tilde{f}$ has coefficients in K_f.

This should be compared to the following theorem of Atkin-Li, where an explicit formula for $\lambda_Q(f)$ is derived in a special case.

Theorem 5.3. [1, Theorem 2.1] Let $f = \sum_n a_n e^{2\pi i n\tau} \in S_k(\Gamma_0(N), \chi)$ be a newform, q be a prime dividing N and $Q = N_q$. If $a_q \neq 0$, then

$$\lambda_Q(f) = Q^{k/2-1}\frac{G(\chi_Q)}{a_Q},$$

where $G(\chi_Q) = \sum_{u \in (\mathbb{Z}/Q\mathbb{Z})^\times} \chi_Q(u) e^{2\pi i u/Q}$ is the Gauss sum of χ_Q.

6. Optimising the coefficient field

We may reduce the number fields provided by Theorems 4.1 and 4.4 as follows. Let $\alpha \in \mathbb{P}^1(\mathbb{Q})$ be a cusp, and let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $g\infty = \alpha$. In order to compute the Fourier expansion of f at α, we may replace g by gT^u with $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $u \in \mathbb{Z}$. Then $f|gT^u$ depends only on the class of u modulo w, where w is the width of α on the appropriate modular curve. The following proposition gives the minimal value of the integer M from Theorem 4.4 when u varies in $\mathbb{Z}/w\mathbb{Z}$.

Proposition 6.1. In the notation of Theorem 4.4, let M' be the minimal value of M for gT^u as u varies. Then

$$M' = \frac{N_C}{\gcd(C, N)} \cdot m_{\overline{C}},$$

where $N_C = \prod_{p|C} p^{v_p(N)}$ is the C-part of N and $m_{\overline{C}} = m/m_C$ is the prime to C part of m. Moreover, M' is attained for any u such that N/N_C divides $uC + D$.

Proof. Replacing g with gT^u changes D to $uC+D$ and B to $uA+B$. Varying u, we need to determine the minimal value of

$$M_u = \operatorname{lcm}\left(\frac{N}{\gcd(C(uC+D),N)}, \frac{m}{\gcd(C(uA+B),m)}\right).$$

Since C and D are coprime we have $\gcd(C(uC+D),N) = \gcd(C,N)\gcd(uC+D,N)$, so $N/\gcd(C(uC+D),N)$ is divisible by $N_C/\gcd(C,N)$. Therefore $N_C/\gcd(C,N)$ divides M_u.
Let $p \nmid C$. Since $C(uA+B) = A(uC+D)-1$, we have

$$\begin{aligned}
v_p\left(\frac{N}{\gcd(C(uC+D),N)}\right) &= v_p(N) - \min(v_p(uC+D), v_p(N)),\\
v_p\left(\frac{m}{\gcd(C(uA+B),m)}\right) &= v_p(m) - \min(v_p(A(uC+D)-1), v_p(m)).
\end{aligned}$$

If $v_p(uC+D) \neq 0$, then $v_p(m/\gcd(C(uA+B),m)) = v_p(m)$. On the other hand, if $v_p(uC+D)=0$, then $v_p(N/\gcd(C(uC+D),N)) = v_p(N) \geq v_p(m)$. In all cases, we have $v_p(M_u) \geq v_p(m)$, which proves that $m_{\overline{C}}$ divides M_u. It follows that M_u is always divisible by $N_C/\gcd(C,N)\cdot m_{\overline{C}}$.
Now choose u such that $N_{\overline{C}} = N/N_C$ divides $uC+D$. This is possible because C and $N_{\overline{C}}$ are coprime. Then $\gcd(uC+D,N) = N_{\overline{C}}$ so that

$$\frac{N}{\gcd(C(uC+D),N)} = \frac{N}{\gcd(C,N)N_{\overline{C}}} = \frac{N_C}{\gcd(C,N)}.$$

Moreover $\gcd(C(uA+B),m) = \gcd(A(uC+D)-1,m)$ is coprime to $N_{\overline{C}}$ and thus divides m_C. It follows that

$$m_{\overline{C}} \mid \frac{m}{\gcd(C(uA+B),m)}$$

On the other hand, if $p \mid C$, then

$$v_p\left(\frac{m}{\gcd(C(uA+B),m)}\right) \leq v_p\left(\frac{m}{\gcd(C,m)}\right) \leq v_p\left(\frac{N_C}{\gcd(C,N)}\right).$$

Hence $M_u = (N_C/\gcd(C,N))\cdot m_{\overline{C}}$. ∎

In practice, we may further reduce the field of coefficients as follows. Let $f \in M_k(\Gamma_0(N))$ be an eigenvector of the Atkin–Lehner operators and $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. The denominator of the cusp $\alpha = g\infty = A/C$ of $X_0(N)$ is $\delta := \gcd(C,N)$.
Now let Q be a maximal divisor of N, and let W_Q be the associated Atkin-Lehner involution of $X_0(N)$. Using the notations of Section 5, if f is an eigenvector of W_Q with eigenvalue $\lambda_Q(f) \in \{\pm 1\}$, then we may write

$$\begin{aligned}
f|g = \lambda_Q(f) f|W_Q g &= \lambda_Q(f) f|h_Q \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\\
&= \lambda_Q(f) f|h_Q \begin{pmatrix} \frac{AQ}{\gcd(C,Q)} & s \\ \frac{C}{\gcd(C,Q)} & r \end{pmatrix}\begin{pmatrix} \gcd(C,Q) & rBQ - sD \\ 0 & \frac{Q}{\gcd(C,Q)} \end{pmatrix},
\end{aligned}$$

where r, s are chosen, so that $r\frac{AQ}{\gcd(C,Q)} - s\frac{C}{\gcd(C,Q)} = 1$.

The action of the upper triangular matrix $\begin{pmatrix} \gcd(C,Q) & rBQ-sD \\ 0 & \frac{Q}{\gcd(C,Q)} \end{pmatrix}$ on Fourier expansions is easily calculated. We now try to find Q such that $f|g'$ has coefficients in the minimal possible number field, where $g' = h_Q \begin{pmatrix} \frac{AQ}{\gcd(C,Q)} & s \\ \frac{C}{\gcd(C,Q)} & r \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.
Let $\delta = \delta_Q \delta_{\overline{Q}}$ where δ_Q is the Q-part of δ. Then the cusp $\alpha' = g'\infty = W_Q \alpha$ has denominator $\delta' = \frac{Q}{\delta_Q} \delta_{\overline{Q}}$ and we may choose Q such that $M' := N_{\delta'}/\delta'$ is minimal. Explicitly, the choice

$$Q = \prod_{\substack{p|N \\ 0 < v_p(\delta) \leq v_p(N)/2}} p^{v_p(N)}$$

gives the minimal value $M' = \gcd(\delta, N/\delta)$. By Proposition 6.1, there exists $v \in \mathbb{Z}$ such that the form $f|g'T^v$ has coefficients in $K_f(\zeta_{M'})$. Thus the problem of finding the Fourier expansion of $f|g$ reduces to finding the the eigenvalue $\lambda_Q(f) \in \{\pm 1\}$, calculating the Fourier expansion of $f|g'T^v$ which is over a potentially much smaller number field than that of $f|g$, and finally applying an upper triangular matrix to $f|g'T^v$. Some information, such as the vanishing order of $f|g$ or the absolute value of its Fourier coefficients, can be extracted directly from $f|g'T^v$ without further calculation.

7. Determining the exact number field

Our final goal is to determine the exact coefficient field of $f|g$ when f is a newform. In this section, we assume that f is a newform of (even) weight $k \geq 2$ on the group $\Gamma_0(N)$.
We will need the following theorems of Newman [12, 13] on the congruence subgroup $\Gamma_0(N)$, where N is a fixed integer ≥ 1.

Theorem 7.1. [12, Theorem 3] Every intermediate subgroup between $\Gamma_0(N)$ and $\mathrm{SL}_2(\mathbb{Z})$ is of the form $\Gamma_0(M)$ for some positive divisor M of N.

In the following, we denote by R the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

Corollary 7.2. Let M be a positive divisor of N. The group $\Gamma_0(M)$ is generated by $\Gamma_0(N)$ and $R^M = \left(\begin{smallmatrix} 1 & 0 \\ M & 1 \end{smallmatrix}\right)$.

Proof. Let Γ be the group generated by $\Gamma_0(N)$ and R^M. By Theorem 7.1, we have $\Gamma = \Gamma_0(M')$ for some M' dividing N. Since $R^M \in \Gamma_0(M')$, the integer M' divides M. Moreover Γ is contained in $\Gamma_0(M)$, so that M divides M'. It follows that $\Gamma = \Gamma_0(M)$. □

Theorem 7.3. [13] The normaliser of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ is equal to $\Gamma_0(N/s)$, where s is the largest divisor of 24 such that s^2 divides N. Moreover, the quotient group $\Gamma_0(N/s)/\Gamma_0(N)$ is cyclic of order s, generated by the class of $R^{N/s} = \left(\begin{smallmatrix} 1 & 0 \\ N/s & 1 \end{smallmatrix}\right)$.

Proof. The first assertion follows from [13, Theorem 1]. By Corollary 7.2, the group $\Gamma_0(N/s)$ is generated by $\Gamma_0(N)$ and $R^{N/s}$. It follows that the quotient group $\Gamma_0(N/s)/\Gamma_0(N)$ is generated by the class of $R^{N/s}$, and it is easy to see that this class has order s. □

Proposition 7.4. Let F be a nonzero element of the new subspace $S_k^{\mathrm{new}}(\Gamma_0(N))$. Then its stabiliser

$$\mathrm{Stab}(F) = \{g \in \mathrm{SL}_2(\mathbb{Z}) : F|g = F\}$$

is equal to $\Gamma_0(N)$.

Proof. Since $\mathrm{Stab}(F)$ contains $\Gamma_0(N)$, Theorem 7.1 implies that $\mathrm{Stab}(F) = \Gamma_0(M)$ for some positive divisor M of N. But F belongs to the new subspace, so we must have $M = N$. $\square$

Proposition 7.5. Let f be a newform of weight $k \geq 2$ on $\Gamma_0(N)$. Let $g \in \mathrm{SL}_2(\mathbb{Z})$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that $f|g = f^\sigma$. Then we have $f^\sigma = f$ and $g \in \Gamma_0(N)$.

Proof. By Proposition 7.4, the stabilisers of f and f^σ are both equal to $\Gamma_0(N)$. On the other hand $\mathrm{Stab}(f|g) = g^{-1}\mathrm{Stab}(f)g$, so that g normalises $\Gamma_0(N)$. By Theorem 7.3, we have $g \in \Gamma_0(N/s)$, and there exists an integer $m \in \mathbb{Z}$ such that $\Gamma_0(N)g = \Gamma_0(N)R^{mN/s}$. Hence $f|g = f|R^{mN/s}$.
We now make use of the Atkin-Lehner involution $W_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. Let $w \in \{\pm 1\}$ be the root number of f, defined by $f|W_N = wf$. Since W_N is defined over $\mathbb{Q}$, we also have $f^\sigma|W_N = wf^\sigma$. Applying W_N on both sides of the equality $f|g = f^\sigma$, we get

$$wf^\sigma = f^\sigma|W_N = f|gW_N = f|R^{mN/s}W_N = f|W_N\left(W_N^{-1}R^{mN/s}W_N\right) = wf|R'$$

with $R' = W_N^{-1}R^{mN/s}W_N = \left(\begin{smallmatrix} 1 & -m/s \\ 0 & 1 \end{smallmatrix}\right)$. The Fourier expansion of $f|R'$ is given by

$$f|R'(z) = f(z - m/s) = \sum_{n\geq 1} a_n(f)e^{-2\pi imn/s}e^{2\pi inz}.$$

Comparing the first term of the Fourier expansions, we get $e^{-2\pi im/s} = 1$. This implies that s divides m, $f^\sigma = f$ and $g \in \Gamma_0(N)$. $\square$

We are now in the position to determine the exact number field of $f|g$. This refines Theorem 4.1(2) for $\Gamma_0(N)$ newforms.

Theorem 7.6. Let f be a newform of weight $k \geq 2$ on $\Gamma_0(N)$. Let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then the field generated by the Fourier coefficients of $f|_k g$ is equal to $K_f(\zeta_{N'})$ with $N' = N/\gcd(CD, N)$.

Proof. We have to show that every automorphism of $\mathbb{C}$ fixing $f|g$ also fixes $K_f(\zeta_{N'})$. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that $(f|g)^\sigma = f|g$. Define $\lambda \in (\mathbb{Z}/N\mathbb{Z})^\times$ by $\sigma(\zeta_N) = \zeta_N^\lambda$. By Theorem 3.3, we have $f^\sigma|g_\lambda = f|g$, so that $f^\sigma = f|gg_\lambda^{-1}$. By Proposition 7.5, we have $f^\sigma = f$ and $gg_\lambda^{-1} \in \Gamma_0(N)$. It follows that σ fixes K_f, and we have already seen in the proof of Theorem 4.1 that the condition $gg_\lambda^{-1} \in \Gamma_0(N)$ is equivalent to $\lambda \equiv 1 \bmod N'$. Therefore σ fixes $K_f(\zeta_{N'})$. $\square$

Remark 7.7. An inspection of the proofs shows that Proposition 7.5 and Theorem 7.6 are valid for elements $f = \sum a_n q^n$ of the new subspace of $S_k(\Gamma_0(N))$ that are eigenfunctions of W_N and satisfy the following condition: if s denotes the largest divisor of 6 whose square divides N, then there exists $n \in \mathbb{N}$ which is coprime to s such that a_n is a non-zero rational number. One family of such forms is given by the traces $\sum_\sigma f^\sigma$ of newforms f, where the sum is over all embeddings of K_f into $\mathbb{C}$.

Question 7.8. What is the $\mathbb{Q}$-vector space (or K_f-vector space) generated by the Fourier coefficients of $f|g$? Furthermore, can we bound effectively the denominators of these coefficients? Note that the q-expansion principle implies that the Fourier expansion of $f|g$ lies in $\mathbb{Z}[[q^{1/N}]] \otimes K_f(\zeta_{N'})$, so that the denominators of $f|g$ are indeed bounded. In fact, by Remark 8.5, we know that the denominators of $f|g$ divide some fixed power of N.

Question 7.9. It would be interesting to generalise Theorem 7.6 to newforms with non-trivial character. Is the number field provided by Remark 4.7 best possible?

8. Appendix: Algebraic modular forms

Here we recall the theory of algebraic modular forms, in order to give a second proof of Theorem 3.3. For more details on this theory, see [10, Chap. II] and the references therein.

Definition 8.1. Let R be an arbitrary commutative ring, and let $N \geq 1$ be an integer. A *test object* of level N over R is a triple $T = (E, \omega, \beta)$ where E/R is an elliptic curve, $\omega \in \Omega^1(E/R)$ is a nowhere vanishing invariant differential, and β is a *level N structure* on E/R, that is an isomorphism of R-group schemes

$$\beta : (\mu_N)_R \times (\mathbb{Z}/N\mathbb{Z})_R \xrightarrow{\cong} E[N]$$

satisfying $e_N(\beta(\zeta, 0), \beta(1, 1)) = \zeta$ for every $\zeta \in (\mu_N)_R$. Here $\mu_N = \operatorname{Spec} \mathbb{Z}[t]/(t^N - 1)$ is the scheme of N-th roots of unity, and e_N is the Weil pairing on $E[N]$ [1].

If $\phi : R \to R'$ is a ring morphism, we denote by $T_{R'} = (E_{R'}, \omega_{R'}, \beta_{R'})$ the base change of T to R' along ϕ.
The isomorphism classes of test objects over $\mathbb{C}$ are in bijection with the set of lattices L in $\mathbb{C}$ endowed with a symplectic basis of $\frac{1}{N}L/L$ [10, 2.4]. Another example is given by the Tate curve $\operatorname{Tate}(q) = \mathbb{G}_m/q^{\mathbb{Z}}$ [7, §8]. It is an elliptic curve over $\mathbb{Z}((q))$ endowed with the canonical differential $\omega_{\text{can}} = dx/x$ and the level N structure $\beta_{\text{can}}(\zeta, n) = \zeta q^{n/N} \bmod q^{\mathbb{Z}}$. The test object $(\operatorname{Tate}(q), \omega_{\text{can}}, \beta_{\text{can}})$ is defined over $\mathbb{Z}((q^{1/N}))$.

Definition 8.2. An *algebraic modular form* of weight $k \in \mathbb{Z}$ and level N over R is the data, for each R-algebra R', of a function

$$F = F_{R'} : \{\text{isomorphism classes of test objects of level } N \text{ over } R'\} \to R'$$

satisfying the following properties:

1. $F(E, \lambda^{-1}\omega, \beta) = \lambda^k F(E, \omega, \beta)$ for every $\lambda \in (R')^\times$;
2. F is compatible with base change: for every morphism of R-algebras $\psi : R' \to R''$ and for every test object T of level N over R', we have $F_{R''}(T_{R''}) = \psi(F_{R'}(T))$.

We denote by $M_k^{\text{alg}}(\Gamma(N); R)$ the R-module of algebraic modular forms of weight k and level N over R.

Evaluating at the Tate curve provides an injective R-linear map

$$M_k^{\text{alg}}(\Gamma(N); R) \hookrightarrow \mathbb{Z}((q^{1/N})) \otimes_{\mathbb{Z}} R$$

called the q-expansion map. The *q-expansion principle* states that if R' is a subring of R, then an algebraic modular form $F \in M_k^{\text{alg}}(\Gamma(N); R)$ belongs to $M_k^{\text{alg}}(\Gamma(N); R')$ if and only if the q-expansion of F has coefficients in R'.
Algebraic modular forms are related to classical modular forms as follows. To any algebraic modular form $F \in M_k^{\text{alg}}(\Gamma(N); \mathbb{C})$, we associate the function $F^{\text{an}} : \mathcal{H} \to \mathbb{C}$ defined by

$$F^{\text{an}}(\tau) = F\Big(\frac{\mathbb{C}}{2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z}}, dz, \beta_\tau\Big)$$

with $\beta_\tau(\zeta_N^m, n) := [2\pi i(m + n\tau)/N]$.

[1] Our definition of the Weil pairing is the reciprocal of Silverman's definition [18, III.8]. With our definition, we have $e_N(1/N, \tau/N) = e^{2\pi i/N}$ on the elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with $\operatorname{Im}(\tau) > 0$.

Proposition 8.3. The map $F \mapsto F^{\mathrm{an}}$ induces an isomorphism between $M_k^{\mathrm{alg}}(\Gamma(N);\mathbb{C})$ and the space $M_k^!(\Gamma(N))$ of weakly holomorphic modular forms on $\Gamma(N)$ (that is, holomorphic on $\mathcal{H}$ and meromorphic at the cusps). Moreover, the q-expansion of F coincides with that of F^{an}.

We now interpret the action of $\mathrm{SL}_2(\mathbb{Z})$ on modular forms in algebraic terms. Let $F \in M_k^{\mathrm{alg}}(\Gamma(N);\mathbb{C})$ with $f = F^{\mathrm{an}}$, and let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. A simple computation shows that

$$(f|_k g)(\tau) = F\Big(\frac{\mathbb{C}}{2\pi i(\mathbb{Z}+\tau\mathbb{Z})}, dz, \beta'_\tau\Big) \tag{6}$$

where the level N structure β'_τ is given by

$$\beta'_\tau(\zeta_N^m, n) = \beta_\tau(\zeta_N^{md+nb}, mc+na). \tag{7}$$

Let $\psi : (\mathbb{Z}/N\mathbb{Z})^2 \to \mu_N(\mathbb{C}) \times \mathbb{Z}/N\mathbb{Z}$ be the isomorphism defined by $\psi(a,b) = (\zeta_N^b, a)$. Let us identify the level structure β_τ (resp. β'_τ) with the map $\alpha_\tau = \beta_\tau \circ \psi$ (resp. $\alpha'_\tau = \beta'_\tau \circ \psi$). Then (7) shows that

$$\alpha'_\tau(a,b) = \alpha_\tau((a,b)g). \tag{8}$$

What we have here is the right action of $\mathrm{SL}_2(\mathbb{Z})$ on the row space $(\mathbb{Z}/N\mathbb{Z})^2$, which induces a left action on the set of level N structures. As we will see, all this makes sense algebraically. For any $\mathbb{Z}[\zeta_N]$-algebra R, we denote by $\zeta_{N,R}$ the image of $\zeta_N = e^{2\pi i/N}$ under the structural morphism $\mathbb{Z}[\zeta_N] \to R$.

Lemma 8.4. If R is a $\mathbb{Z}[\zeta_N, 1/N]$-algebra, then there is an isomorphism of R-group schemes $(\mathbb{Z}/N\mathbb{Z})_R \xrightarrow{\cong} (\mu_N)_R$ sending 1 to $\zeta_{N,R}$.

Proof. Note that $(\mu_N)_R = \operatorname{Spec} R[t]/(t^N-1) = \operatorname{Spec} R[\mathbb{Z}/N\mathbb{Z}]$ and $(\mathbb{Z}/N\mathbb{Z})_R = \operatorname{Spec} R^{\mathbb{Z}/N\mathbb{Z}}$. If $R = \mathbb{C}$, then $\mathbb{C}[\mathbb{Z}/N\mathbb{Z}] \cong \mathbb{C}^{\mathbb{Z}/N\mathbb{Z}}$ because all irreducible representations of $\mathbb{Z}/N\mathbb{Z}$ have dimension 1. This isomorphism $\mathcal{F}_{\mathbb{C}}$ is given by the Fourier transform, and both $\mathcal{F}_{\mathbb{C}}$ and $\mathcal{F}_{\mathbb{C}}^{-1}$ have coefficients in $\mathbb{Z}[\zeta_N, 1/N]$ with respect to the natural bases. It follows that in general $R[\mathbb{Z}/N\mathbb{Z}] \cong R^{\mathbb{Z}/N\mathbb{Z}}$ and this isomorphism sends $[1]$ to $(\zeta_{N,R}^a)_{a\in\mathbb{Z}/N\mathbb{Z}}$. $\square$

Let R be a $\mathbb{Z}[\zeta_N, 1/N]$-algebra. We have an isomorphism of R-group schemes

$$\psi_R : (\mathbb{Z}/N\mathbb{Z})_R^2 \to (\mu_N)_R \times (\mathbb{Z}/N\mathbb{Z})_R$$

given by $\psi_R(a,b) = (\zeta_{N,R}^b, a)$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts from the right on the row space $(\mathbb{Z}/N\mathbb{Z})_R^2$ by R-automorphisms, and for $\alpha : (\mathbb{Z}/N\mathbb{Z})_R^2 \xrightarrow{\cong} E[N]$ we define

$$(g\cdot\alpha)(a,b) = \alpha((a,b)g) \qquad ((a,b) \in (\mathbb{Z}/N\mathbb{Z})^2). \tag{9}$$

Using ψ_R, we transport this to a left action of $\mathrm{SL}_2(\mathbb{Z})$ on the set of level N structures of an elliptic curve over R. Given a test object $T = (E,\omega,\beta)$ over R, we define $g\cdot T := (E,\omega,g\cdot\beta)$. For any $F \in M_k^{\mathrm{alg}}(\Gamma(N);R)$, we define $F|g \in M_k^{\mathrm{alg}}(\Gamma(N);R)$ by the rule $(F|g)(T) = F(g\cdot T)$ for any test object T over any R-algebra R'. The computation (6) then shows that the right action of $\mathrm{SL}_2(\mathbb{Z})$ on $M_k^{\mathrm{alg}}(\Gamma(N);\mathbb{C})$ corresponds to the usual slash action on $M_k^!(\Gamma(N))$.

Remark 8.5. The action of $\mathrm{SL}_2(\mathbb{Z})$ on algebraic modular forms over $\mathbb{Z}[\zeta_N, 1/N]$-algebras has the following consequence: if a classical modular form $f \in M_k(\Gamma(N))$ has Fourier coefficients in some subring A of $\mathbb{C}$, then for any $g \in \mathrm{SL}_2(\mathbb{Z})$, the Fourier expansion of $f|g$ lies in $\mathbb{Z}[[q^{1/N}]] \otimes A[\zeta_N, 1/N]$.

We now interpret the action of $\mathrm{Aut}(\mathbb{C})$ in algebraic terms (see [14, p. 88]). Let $\sigma \in \mathrm{Aut}(\mathbb{C})$. For any $\mathbb{C}$-algebra R, we define $R^\sigma := R \otimes_{\mathbb{C},\sigma^{-1}} \mathbb{C}$, which means that $(ax) \otimes 1 = x \otimes \sigma^{-1}(a)$ for all $a \in \mathbb{C}$, $x \in R$. We endow R^σ with the structure of a $\mathbb{C}$-algebra using the map $a \in \mathbb{C} \mapsto 1 \otimes a \in R^\sigma$. We denote by $\phi_\sigma : R \to R^\sigma$ the map defined by $\phi_\sigma(x) = x \otimes 1$. The map ϕ_σ is a ring isomorphism, but one should be careful that ϕ_σ is not a morphism of $\mathbb{C}$-algebras, as it is only σ^{-1}-linear. For any test object T over R, we denote by T^σ its base change to R^σ using the ring morphism ϕ_σ.
Let $F \in M_k^{\mathrm{alg}}(\Gamma(N);\mathbb{C})$ be an algebraic modular form. For any $\mathbb{C}$-algebra R, we define

$$\begin{aligned} F_R^\sigma : \{\text{isomorphism classes of test objects of level } N \text{ over } R\} &\to R \\ T &\mapsto \phi_\sigma^{-1}\big(F_{R^\sigma}(T^\sigma)\big). \end{aligned}$$

One may check that the collection of functions F_R^σ satisfies the conditions (1) and (2) above, hence defines an algebraic modular form $F^\sigma \in M_k^{\mathrm{alg}}(\Gamma(N);\mathbb{C})$. Moreover, since the Tate curve is defined over $\mathbb{Z}((q))$, one may check that the map $F \mapsto F^\sigma$ corresponds to the usual action of $\mathrm{Aut}(\mathbb{C})$ on the Fourier expansions of modular forms: for every $F \in M_k^{\mathrm{alg}}(\Gamma(N);\mathbb{C})$ and every $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have $(F^\sigma)^{\mathrm{an}} = (F^{\mathrm{an}})^\sigma$.
We finally come to the second proof of Theorem 3.3.

Proof. Let $f \in M_k(\Gamma(N))$ with corresponding algebraic modular form $F \in M_k^{\mathrm{alg}}(\Gamma(N);\mathbb{C})$. Let $g \in \mathrm{SL}_2(\mathbb{Z})$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$. We take as test object $T = (\mathrm{Tate}(q), \omega_{\mathrm{can}}, \beta_{\mathrm{can}})$ over $R = \mathbb{Z}((q^{1/N})) \otimes \mathbb{C}$. Since a modular form is determined by its Fourier expansion, and unravelling the definitions of $F|g$ and F^σ, it suffices to check that the test objects $g \cdot T^\sigma$ and $(g_\lambda \cdot T)^\sigma$ over R^σ are isomorphic. Since $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ acts only on the level structures of the test objects, we have to show that

$$g \cdot \beta_{\mathrm{can}}^\sigma \cong (g_\lambda \cdot \beta_{\mathrm{can}})^\sigma. \tag{10}$$

For any scheme X over R, let X^σ denote its base change to R^σ along ϕ_σ. Since ϕ_σ is a ring isomorphism, the canonical projection map $X^\sigma \to X$ is an isomorphism of schemes, and we also denote by $\phi_\sigma : X \to X^\sigma$ the inverse map.
Put $E = \mathrm{Tate}(q)$ and $\beta = \beta_{\mathrm{can}}$. Let $\alpha = \beta \circ \psi_R : (\mathbb{Z}/N\mathbb{Z})_R^2 \xrightarrow{\cong} E[N]$. By functoriality, the level structure β^σ is given by the following commutative diagram

$$\begin{tikzcd} (\mathbb{Z}/N\mathbb{Z})_R^2 \arrow[r, "\psi_R"] \arrow[d, dotted, "\gamma"'] \arrow[rr, bend left, "\alpha"] & (\mu_N)_R \times (\mathbb{Z}/N\mathbb{Z})_R \arrow[r, "\beta"] \arrow[d, "\cong"', "\phi_\sigma"] & E[N] \arrow[d, "\cong"', "\phi_\sigma"] \\ (\mathbb{Z}/N\mathbb{Z})_{R^\sigma}^2 \arrow[r, "\psi_{R^\sigma}"] \arrow[rr, bend right, "\alpha^\sigma"'] & (\mu_N)_{R^\sigma} \times (\mathbb{Z}/N\mathbb{Z})_{R^\sigma} \arrow[r, "\beta^\sigma"] & E^\sigma[N]. \end{tikzcd} \tag{11}$$

Let us compute the dotted arrow γ. Since ϕ_σ is σ^{-1}-linear, we have $\phi_\sigma(\zeta_{N,R}) = \zeta_{N,R^\sigma}^{\lambda^{-1}}$. It follows that

$$\phi_\sigma(\psi_R(a,b)) = \phi_\sigma(\zeta_{N,R}^b, a) = (\zeta_{N,R^\sigma}^{\lambda^{-1}b}, a) = \psi_{R^\sigma}(a, \lambda^{-1}b) \tag{12}$$

so that $\gamma(a,b) = (a, \lambda^{-1}b)$. We may thus express α^σ in terms of α by

$$\alpha^\sigma(a,b) = \phi_\sigma \circ \alpha \circ \gamma^{-1}(a,b) = \phi_\sigma \circ \alpha(a, \lambda b) = \phi_\sigma \circ \alpha\left((a,b)\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}\right). \tag{13}$$

Let us make explicit both sides of (10). By (9) and (13), the left hand side is given by

$$(g \cdot \alpha^\sigma)(a,b) = \alpha^\sigma((a,b)g) = \phi_\sigma \circ \alpha\left((a,b)g\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}\right). \tag{14}$$

Let us now turn to the right hand side of (10). By (9), we have $(g_\lambda \cdot \alpha)(a,b) = \alpha((a,b)g_\lambda)$. Applying the commutative diagram (11) with α replaced by $g_\lambda \cdot \alpha$, we get

$$(g_\lambda \cdot \alpha)^\sigma(a,b) = \phi_\sigma \circ (g_\lambda \cdot \alpha)\left((a,b)\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}\right) = \phi_\sigma \circ \alpha\left((a,b)\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix} g_\lambda\right). \tag{15}$$

Finally, we recall equation (1), which states $g\left(\begin{smallmatrix}1 & 0\\ 0 & \lambda\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & 0\\ 0 & \lambda\end{smallmatrix}\right) g_\lambda$.

□

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