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Abstract. We compute the p-adic étale and the pro-étale cohomologies of the Drinfeld half-space of any dimension. The main input is a new comparison theorem for the p-adic pro-étale cohomology of p-adic Stein spaces.

Contents

1. Introduction

Let p be a prime. Let $\mathscr{O}_K$ be a complete discrete valuation ring of mixed characteristic $(0,p)$ with perfect residue field k and fraction field K. Let F be the fraction field of the ring of Witt vectors

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$\mathscr{O}_F = W(k)$ of k. Let $\overline{K}$ be an algebraic closure of K and let $C = \widehat{\overline{K}}$ be its p-adic completion; let $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$.

1.1. **The p-adic étale cohomology of Drinfeld half-space.** This paper reports on some results of our research project that aims at understanding the p-adic (pro-)étale cohomology of p-adic symmetric spaces. The main question of interest being: does this cohomology realize the hoped for p-adic local Langlands correspondence in analogy with the ℓ-adic situation ? When we started this project we did not know what to expect and local computations were rather discouraging: geometric p-adic étale cohomology groups of affinoids and their interiors are huge and not invariant by base change to a bigger complete algebraically closed field. However there was one computation done long ago by Drinfeld that stood out. Let us recall it.

Assume that $[K : \mathbf{Q}_p] < \infty$ and let $\mathbb{H}_K = \mathbb{P}^1_K \setminus \mathbb{P}^1(K)$ be the Drinfeld half-plane, thought of as a rigid analytic space. It admits a natural action of $G := \mathrm{GL}_2(K)$.

Fact 1.1. (Drinfeld) *If ℓ is a prime number* (including $\ell = p$!)*, there exists a natural isomorphism of $G \times \mathscr{G}_K$-representations*

$$H^1_{\text{ét}}(\mathbb{H}_C, \mathbf{Q}_\ell(1)) \simeq (\mathrm{Sp}^{\mathrm{cont}}(\mathbf{Q}_\ell))^*,$$

where $\mathrm{Sp}^{\mathrm{cont}}(\mathbf{Q}_\ell) := \mathscr{C}(\mathbb{P}^1(K), \mathbf{Q}_\ell)/\mathbf{Q}_\ell$ is the continuous Steinberg representation of G with coefficients in $\mathbf{Q}_\ell$ equipped with a trivial action of $\mathscr{G}_K$ and $(-)^*$ denotes the weak topological dual.

The proof is very simple: it uses Kummer theory and vanishing of the Picard groups (of the standard Stein covering of $\mathbb{H}_K$) [11, 1.4]. This result was encouraging because it showed that the p-adic étale cohomology was maybe not as pathological as one could fear.

Drinfeld's result was generalized, for $\ell \neq p$, to higher dimensions by Schneider-Stuhler [59]. Let $d \geq 1$ and let $\mathbb{H}^d_K$ be the Drinfeld half-space of dimension d, i.e.,

$$\mathbb{H}^d_K := \mathbb{P}^d_K \setminus \bigcup_{H \in \mathscr{H}} H,$$

where $\mathscr{H}$ denotes the set of K-rational hyperplanes. We set $G := \mathrm{GL}_{d+1}(K)$. If $1 \leq r \leq d$, and if ℓ is a prime number, denote by $\mathrm{Sp}_r(\mathbf{Q}_\ell)$ and $\mathrm{Sp}^{\mathrm{cont}}_r(\mathbf{Q}_\ell)$ the generalized locally constant and continuous Steinberg $\mathbf{Q}_\ell$-representations of G (see Section 5.1.1), respectively, equipped with a trivial action of $\mathscr{G}_K$.

Theorem 1.2. (Schneider-Stuhler) *Let $r \geq 0$ and let $\ell \neq p$. There are natural $G \times \mathscr{G}_K$-equivariant isomorphisms*

$$H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Q}_\ell(r)) \simeq \mathrm{Sp}^{\mathrm{cont}}_r(\mathbf{Q}_\ell)^*, \quad H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbf{Q}_\ell(r)) \simeq \mathrm{Sp}_r(\mathbf{Q}_\ell)^*.$$

The computations of Schneider-Stuhler work for any cohomology theory that satisfies certain axioms, the most important being the homotopy property with respect to the open unit ball, which fails rather dramatically for the p-adic (pro-)étale cohomology since the p-adic étale cohomology of the unit ball is huge. Nevertheless, we prove the following result.

Theorem 1.3. *Let $r \geq 0$.*

(1) *There is a natural isomorphism of $G \times \mathscr{G}_K$-locally convex topological vector spaces (over $\mathbf{Q}_p$).*

$$H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Q}_p(r)) \simeq \mathrm{Sp}^{\mathrm{cont}}_r(\mathbf{Q}_p)^*.$$

These spaces are weak duals of Banach spaces.

(2) *There is a strictly exact sequence of $G \times \mathscr{G}_K$-Fréchet spaces*

$$0 \longrightarrow (\Omega^{r-1}(\mathbb{H}^d_K)/\ker d)\widehat{\otimes}_K C \longrightarrow H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbf{Q}_p(r)) \longrightarrow \mathrm{Sp}_r(\mathbf{Q}_p)^* \longrightarrow 0.$$

(3) *The natural map $H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Q}_p(r)) \to H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbf{Q}_p(r))$ identifies étale cohomology with the space of G-bounded vectors*[1] *in the pro-étale cohomology.*

[1]Recall that a subset X of a locally convex vector space over $\mathbf{Q}_p$ is called *bounded* if $p^n x_n \mapsto 0$ for all sequences $\{x_n\}, n \in \mathbf{N}$, of elements of X. In the above, x is called a *G-bounded vector* if its G-orbit is a bounded set.

Hence, the p-adic étale cohomology is given by the same dual of a Steinberg representation as its ℓ-adic counterpart and is invariant by scalar extension to bigger C's. However, the p-adic pro-étale cohomology is a nontrivial extension of the same dual of a Steinberg representation that describes its ℓ-adic counterpart by a huge space that depends very much on C.

Remark 1.4. In [11] we have generalized the above computation of Drinfeld in a different direction, namely, to the Drinfeld tower in dimension 1. We have shown that, if $K = \mathbf{Q}_p$, the p-adic local Langlands correspondence for de Rham Galois representations of dimension 2 (of Hodge-Tate weights 0 and 1 and not trianguline) can be realized inside the p-adic étale cohomology of the Drinfeld tower (see [11, Theorem 0.2] for a precise statement). The two important cohomological inputs were

(1) a p-adic comparison theorem that allows us to recover the p-adic pro-étale cohomology from the de Rham complex and the Hyodo-Kato cohomology; the latter being compared to the ℓ-adic étale cohomology computed, in turn, by non-abelian Lubin-Tate theory,
(2) the fact that the p-adic étale cohomology is equal to the space of G-bounded vectors in the p-adic pro-étale cohomology.

In contrast, here we obtain the third part of Theorem 1.3 only after proving the two previous parts. In fact, for a general rigid analytic variety, we do not know whether the natural map from p-adic étale cohomology to p-adic pro-étale cohomology is an injection (this is, of course, true for quasi-compact varieties).

Remark 1.5. The proof of Theorem 1.3 establishes a number of other isomorphisms (see Theorem 6.26) refining results of [59, 35, 14].

Remark 1.6. (i) For $r \geq d+1$, all spaces in Theorem 1.3 are 0.

(ii) For $1 \leq r \leq d$, the spaces on the left and on the right in the exact sequence in Theorem 1.3 describing the pro-étale cohomology of $\mathbb{H}^d_C$, despite being huge spaces, have some finiteness properties: they are both duals of admissible locally analytic representations of G (over C on the left and $\mathbf{Q}_p$ on the right), of finite length (on the left, this is due to Orlik and Strauch ([51] combined with [54])).

Remark 1.7. For small Tate twists ($r \leq p-1$), the Fontaine-Messing period map, which is an essential input in the proof of Theorem 1.3, is an isomorphism "on the nose". It is possible then that our proof of Theorem 1.3, with a better control of the constants, could give the integral p-adic étale cohomology of the Drinfeld half-space for small Tate twists, that is, a topological isomorphism

$$H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbf{F}_p(r)) \simeq \mathrm{Sp}_r(\mathbf{F}_p)^*.$$

It also seems likely that the same result holds for all twists, but proving so would probably require different techniques. We plan to address this problem in a future work.

1.2. **A comparison theorem for p-adic pro-étale cohomology.** The proof of Theorem 1.3 uses the result below, which is the main theorem of this paper and generalizes the above mentioned comparison theorem to rigid analytic Stein spaces[2] over K with a semistable reduction. Let the field K be as at the beginning of the introduction.

Theorem 1.8. *Let $r \geq 0$. Let X be a semistable Stein weak formal scheme[3] over $\mathcal{O}_K$. There exists a commutative $\mathcal{G}_K$-equivariant diagram of Fréchet spaces[4]*

$$\begin{array}{ccccccccc}
0 \longrightarrow & (\Omega^{r-1}(X_K)/\ker d)\widehat{\otimes}_K C & \longrightarrow & H^r_{\text{proét}}(X_C, \mathbf{Q}_p(r)) & \longrightarrow & (H^r_{\mathrm{HK}}(X_k)\widehat{\otimes}_F \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} & \longrightarrow 0 \\
 & \| & & \downarrow{\scriptstyle \widetilde{\beta}} & & \downarrow{\scriptstyle \iota_{\mathrm{HK}}\otimes\theta} & \\
0 \longrightarrow & (\Omega^{r-1}(X_K)/\ker d)\widehat{\otimes}_K C & \xrightarrow{d} & \Omega^r(X_K)_{d=0}\widehat{\otimes}_K C & \xrightarrow{\text{can}} & H^r_{\mathrm{dR}}(X_C) & \longrightarrow 0
\end{array}$$

[2]Recall that a rigid analytic space Y is Stein if it has an admissible affinoid covering $Y = \cup_{i\in\mathbf{N}} U_i$ such that $U_i \Subset U_{i+1}$. The key property we need is the acyclicity of cohomology of coherent sheaves.

[3]See Section 3.1.1 for the definition.

[4]The completed tensor product is taken with respect to a Stein covering of X_K; see Section 3.2.1 for details.

The rows are strictly exact, the maps $\widetilde{\beta}$ and $\iota_{\rm HK}\otimes\theta$ are strict (and have closed images). Moreover,

$$\ker(\widetilde{\beta})\simeq\ker(\iota_{\rm HK}\otimes\theta)\simeq(H^r_{\rm HK}(X_k)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^{r-1}}.$$

Here $H^r_{\rm HK}(X_k)$ is the overconvergent Hyodo-Kato cohomology of Grosse-Klönne [23], $\iota_{\rm HK}:H^r_{\rm HK}(X_k)\otimes_F K\xrightarrow{\sim}H^r_{\rm dR}(X_C)$ is the Hyodo-Kato isomorphism, $\mathbf{B}^+_{\rm st}$ is the semistable ring of periods defined by Fontaine, and $\theta:\mathbf{B}^+_{\rm st}\to C$ is Fontaine's projection.

Example 1.9. In the case the Hyodo-Kato cohomology vanish we obtain a particularly simple formula. Take, for example, the rigid affine space $\mathbb{A}^d_K$. For $r\geq 1$, we have $H^r_{\rm dR}(\mathbb{A}^d_K)=0$ and, by the Hyodo-Kato isomorphism, also $H^r_{\rm HK}(\mathbb{A}^d_K)=0$. Hence the above theorem yields an isomorphism

$$H^r_{\rm proét}(\mathbb{A}^d_C,\mathbf{Q}_p(r))\xleftarrow{\sim}(\Omega^{r-1}(\mathbb{A}^d_K)/\ker d)\widehat{\otimes}_K C.$$

This was our first proof of this fact but there is a more direct argument in [13]. Another approach, using relative fundamental exact sequences in pro-étale topology and their pushforwards to étale topology, can be found in [39].

Remark 1.10. (i) We think of the above theorem as a one-way comparison theorem, i.e., the pro-étale cohomology $H^r_{\rm proét}(X_C,\mathbf{Q}_p(r))$ is the pullback of the diagram

$$(H^r_{\rm HK}(X_k)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^r}\xrightarrow{\iota_{\rm HK}\otimes\theta}H^r_{\rm dR}(X_K)\widehat{\otimes}_K C\xleftarrow{\rm can}\Omega^r(X_K)_{d=0}\widehat{\otimes}_K C$$

built from the Hyodo-Kato cohomology and a piece of the de Rham complex.

(ii) When we started doing computations of pro-étale cohomology groups (for the affine line), we could not understand why the p-adic pro-étale cohomology seemed to be so big while the Hyodo-Kato cohomology was so small (actually 0 in that case): this was against what the proper case was teaching us. If X is proper, $\Omega^{r-1}(X_K)/\ker d=0$ and the upper line of the above diagram becomes

$$0\to H^r_{\rm proét}(X_C,\mathbf{Q}_p(r))\to(H^r_{\rm HK}(X_k)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^r}\to(H^r_{\rm dR}(X_K)\widehat{\otimes}\mathbf{B}^+_{\rm dR})/{\rm Fil}^r\to 0.$$

Hence the huge term on the left disappears, and an extra term on the right shows up. This seemed to indicate that there was no real hope of computing p-adic étale and pro-étale cohomologies of big spaces. It was learning about Drinfeld's result that convinced us to look further.

1.3. **Proof of Theorem 1.8.** The starting point of computations of pro-étale and étale cohomologies in these theorems is the same: the classical comparison theorem between p-adic nearby cycles and syntomic sheaves [66], [12]. When applied to the Stein spaces we consider here it yields:

Proposition 1.11. *Let X be a semistable Stein formal scheme over $\mathscr{O}_K$. Then the Fontaine-Messing period morphisms*

$$\alpha^{\rm FM}:{\rm R}\Gamma_{\rm syn}(X_{\mathscr{O}_C},\mathbf{Q}_p(r))\to{\rm R}\Gamma_{\rm proét}(X_C,\mathbf{Q}_p(r)),$$
$$\alpha^{\rm FM}:{\rm R}\Gamma_{\rm syn}(X_{\mathscr{O}_C},\mathbf{Z}_p(r))_{\mathbf{Q}}\to{\rm R}\Gamma_{\rm ét}(X_C,\mathbf{Q}_p(r))$$

are strict quasi-isomorphisms after truncation $\tau_{\leq r}$.

Here the crystalline geometric syntomic cohomology is that defined by Fontaine-Messing

$${\rm R}\Gamma_{\rm syn}(X_{\mathscr{O}_C},\mathbf{Z}_p(r)):=[{\rm R}\Gamma_{\rm cr}(X_{\mathscr{O}_C})^{\varphi=p^r}\to{\rm R}\Gamma_{\rm cr}(X_{\mathscr{O}_C})/F^r],\quad F^r{\rm R}\Gamma_{\rm cr}(X_{\mathscr{O}_C}):={\rm R}\Gamma_{\rm cr}(X_{\mathscr{O}_C},\mathscr{J}^{[r]}),$$

where the crystalline cohomology is absolute, i.e., over $W(k)$. The syntomic cohomology ${\rm R}\Gamma_{\rm syn}(X_{\mathscr{O}_C},\mathbf{Q}_p(r))$ is defined by taking ${\rm R}\Gamma_{\rm syn}(-,\mathbf{Z}_p(r))_{\mathbf{Q}}$ on quasi-compact pieces and then gluing.

The next step is to transform the syntomic cohomology (that works very well for defining period maps but is not terribly useful for computations) into Bloch-Kato type syntomic cohomology (whose definition is motivated by the Bloch-Kato's definition of Selmer groups; it involves much more concrete objects). This works well for the pro-étale topology[5] but only partially for the étale one. For the pro-étale topology,

[5] At least when X is associated to a weak formal scheme.

it is done by the top part of the commutative diagram

$$\begin{array}{ccccc}
\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_C},\mathbf{Q}_p(r)) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C},F)^{\varphi=p^r} & \xrightarrow{\mathrm{can}} & \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C},F)/F^r \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\mathrm{R}\Gamma_{\mathrm{syn}}(X_C,\mathbf{Q}_p(r)) & \longrightarrow & (\mathrm{R}\Gamma_{\mathrm{HK}}(X_k)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} & \xrightarrow{\iota_{\mathrm{HK}}\otimes\iota} & (\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r \\
\downarrow\theta & & \downarrow\theta\iota_{\mathrm{HK}} & & \downarrow\theta \\
\Omega^{\geq r}(X_K)\widehat{\otimes}C & \longrightarrow & \Omega^{\bullet}(X_K)\widehat{\otimes}_K C & \longrightarrow & \Omega^{\leq r-1}(X_K)\widehat{\otimes}_K C.
\end{array}$$

Here $\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C},F)$ and its filtrations are defined by the same procedure as $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_C},\mathbf{Q}_p(r))$ (starting from rational absolute crystalline cohomology). The horizontal triangles are distinguished (the top two by definition). The construction of the top vertical maps and the proof that they are isomorphisms is nontrivial and constitutes the technical heart of this paper. These maps are basically Künneth maps, that use the interpretation of period rings as crystalline cohomology of certain "base" rings (for example, $\mathbf{A}_{\mathrm{cr}}\simeq\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{O}_C)$), coupled with a rigidity of φ-eigenspaces of crystalline chomology, and followed by a change of topology (from crystalline to overconvergent) that can be done because X_K is Stein (hence X_k has proper and smooth irreducible components). To control the topology we work in the derived category of locally convex topological vector spaces over $\mathbf{Q}_p$ which, since $\mathbf{Q}_p$ is spherically complete, is reasonably well-behaved.

The bottom vertical maps in the diagram are induced by the projection $\theta:\mathbf{B}^+_{\mathrm{dR}}\to C$ and use the fact that, since X_K is Stein, we have $\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\simeq\Omega^{\bullet}(X_K)$. The diagram in Theorem 1.8 follows by applying H^r to the above diagram.

1.4. **Proof of Theorem 1.3.** To prove the pro-étale part of Theorem 1.3, by Theorem 1.8, it suffices to show that

$$(H^r_{\mathrm{HK}}(X_k)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\simeq\mathrm{Sp}_r(\mathbf{Q}_p)^*. \tag{1.12}$$

But we know from Schneider-Stuhler [59] that there is a natural isomorphism $H^r_{\mathrm{dR}}(X_K)\simeq\mathrm{Sp}_r(K)^*$ of G-representations. Moreover, we know that both sides are generated by standard symbols, i.e., cup products of symbols of K-rational hyperplanes thought of as invertible functions on X_K (this is because $\mathrm{Sp}_r(K)^*$, by definition, is generated by standard symbols and Iovita-Spiess [35] prove that so is $H^r_{\mathrm{dR}}(X_K)$) and that this isomorphism is compatible with symbols [35]. Coupled with the Hyodo-Kato isomorphism and the irreducibility of the representation $\mathrm{Sp}_r(K)^*$ this yields a natural isomorphism $H^r_{\mathrm{HK}}(X_k)\simeq\mathrm{Sp}_r(F)^*$. This isomorphism is unique once we impose that it should be compatible with the standard symbols. It follows that we have a natural isomorphism $H^r_{\mathrm{HK}}(X_k)^{\varphi=p^r}\simeq\mathrm{Sp}_r(\mathbf{Q}_p)^*$, which implies $H^r_{\mathrm{HK}}(X_k)\cong F\otimes_{\mathbf{Q}_p}H^r_{\mathrm{HK}}(X_k)^{\varphi=p^r}$ and (1.12).

The situation is more complicated for étale cohomology. Let X be a semistable Stein formal scheme over $\mathscr{O}_K$. An analogous computation to the one above yields the following quasi-isomorphism of distinguished triangles

$$\begin{array}{ccccc}
\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_C},\mathbf{Z}_p(r))_{\mathbf{Q}} & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C})^{\varphi=p^r}_{\mathbf{Q}} & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C})_{\mathbf{Q}}/F^r \\
\| & & \downarrow\wr & & \downarrow\wr \\
\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_C},\mathbf{Z}_p(r))_{\mathbf{Q}} & \longrightarrow & (\mathrm{R}\Gamma_{\mathrm{cr}}(X_k/\mathscr{O}^0_F)\widehat{\otimes}_{\mathscr{O}_F}\mathbf{A}_{\mathrm{st}})^{N=0,\varphi=p^r}_{\mathbf{Q}} & \xrightarrow{\gamma_{\mathrm{HK}}\otimes\iota} & (\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{\mathrm{cr},K})/F^r,
\end{array}$$

where $\mathscr{O}^0_F$ denotes $\mathscr{O}_F$ equipped with the log-structure induced by $1\mapsto 0$ and $\mathbf{A}_{\mathrm{st}}$, $\mathbf{A}_{\mathrm{cr},K}$ are certain period rings. But, in general, the map γ_{HK} is difficult to identify. In the case of Drinfeld half-space though its domain and target simplify significantly by the acyclicity of the sheaves of differentials proved by Grosse-Klönne [24, 26]. This makes it possible to describe it and, as a result, to compute the étale syntomic cohomology.

Let now X be the standard formal model of $\mathbb{H}^d_K$. Set $\mathrm{HK}_r := (\mathrm{R}\Gamma_{\mathrm{cr}}(X_k/\mathscr{O}_F^0)\widehat{\otimes}_{\mathscr{O}_F}\mathbf{A}_{\mathrm{st}})_{\mathbf{Q}}^{N=0,\varphi=p^r}$. We show that there are natural $G\times\mathscr{G}_K$-equivariant (quasi-)isomorphisms

$$
\begin{aligned}
(1.13)\qquad & H^r\mathrm{HK}_r \simeq H^0_{\mathrm{\acute{e}t}}(X_{\overline{k}}, W\Omega^r_{\log})_{\mathbf{Q}}, \quad H^{r-1}\mathrm{HK}_r \simeq (H^0_{\mathrm{\acute{e}t}}(X_{\overline{k}}, W\Omega^{r-1}_{\log})\widehat{\otimes}_{\mathscr{O}_F}\mathbf{A}_{\mathrm{cr}}^{\varphi=p})_{\mathbf{Q}},\\
& (\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{\mathrm{cr},K})/F^r \simeq \oplus_{r-1\geq i\geq 0}(H^0(X,\Omega^i)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{\mathrm{cr},K})/F^{r-i}[-i],
\end{aligned}
$$

where $W\Omega^r_{\log}$ is the sheaf of logarithmic de Rham-Witt differentials. They follow from the isomorphisms

$$
\begin{aligned}
(1.14)\qquad & H^i_{\mathrm{\acute{e}t}}(X_{\overline{k}}, W\Omega^r_{\log})\widehat{\otimes}_{\mathbf{Z}_p}W(\overline{k}) \xrightarrow{\sim} H^i_{\mathrm{cr}}(X_{\overline{k}}/W(\overline{k})^0),\\
& \iota_{\mathrm{HK}} : H^i_{\mathrm{cr}}(X_k/\mathscr{O}_F^0)\otimes_{\mathscr{O}_F}K \simeq H^i_{\mathrm{dR}}(X)\otimes_{\mathscr{O}_K}K.
\end{aligned}
$$

The second one is just the original Hyodo-Kato isomorphism from [30]. The first one is a restatement of the fact that X_k is pro-ordinary, which, in turn and morally speaking, follows from the fact that X_k is a normal crossing scheme whose all closed strata are classically ordinary (being products of blow-ups of projective spaces). Now, the acyclicity of the sheaves Ω^j_X and the fact that the differential is trivial on their global sections (both facts proved by Grosse-Klönne [24], [26]) imply (1.13).

Hence, we obtain the long exact sequence

$$
(H^0_{\mathrm{\acute{e}t}}(X_{\overline{k}}, W\Omega^{r-1}_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}^{\varphi=p})_{\mathbf{Q}}\xrightarrow{\gamma'_{\mathrm{HK}}}(H^0(X,\Omega^{r-1})\widehat{\otimes}_{\mathscr{O}_K}\mathscr{O}_C)_{\mathbf{Q}}\to H^r_{\mathrm{syn}}(X_{\mathscr{O}_C},\mathbf{Z}_p(r))_{\mathbf{Q}}\to H^0_{\mathrm{\acute{e}t}}(X_{\overline{k}}, W\Omega^r_{\log})_{\mathbf{Q}}\to 0
$$

We check that the map γ'_{HK} is surjective: (a bit surprisingly) the Hyodo-Kato isomorphism ι_{HK} above holds already integrally and $\gamma'_{\mathrm{HK}}=\iota_{\mathrm{HK}}\otimes\theta$, where $\theta:\mathbf{A}_{\mathrm{cr}}^{\varphi=p}\to\mathscr{O}_C$ is the canonical projection. This yields the isomorphism

$$
H^r_{\mathrm{syn}}(X_{\mathscr{O}_C},\mathbf{Z}_p(r))_{\mathbf{Q}}\xrightarrow{\sim} H^0_{\mathrm{\acute{e}t}}(X_{\overline{k}}, W\Omega^r_{\log})_{\mathbf{Q}}.
$$

Hence it remains to show that there exists a natural isomorphism

$$
(1.15)\qquad H^0_{\mathrm{\acute{e}t}}(X_{\overline{k}}, W\Omega^r_{\log})_{\mathbf{Q}}\simeq \mathrm{Sp}_r^{\mathrm{cont}}(\mathbf{Q}_p)^*.
$$

We do that showing that we can replace $\overline{k}$ by k and using the maps

$$
H^0_{\mathrm{\acute{e}t}}(X_k, W\Omega^r_{\log})\otimes_{\mathbf{Z}_p}K\xrightarrow{f} H^r_{\mathrm{dR}}(X_K)^{\ G\text{-bd}}\simeq(\mathrm{Sp}_r(K)^*)^{G\text{-bd}}\xleftarrow{\sim}\mathrm{Sp}_r^{\mathrm{cont}}(\mathbf{Q}_p)^*\otimes_{\mathbf{Q}_p}K.
$$

Here, the second isomorphism is that of Schneider-Stuhler. The map f (a composition of natural maps with the Hyodo-Kato isomorphism) is injective by pro-ordinarity of Y. It is surjective because $H^0_{\mathrm{\acute{e}t}}(X_k, W\Omega^r_{\log})$ is compact and nontrivial and $\mathrm{Sp}_r^{\mathrm{cont}}(\mathbf{Z}_p)/p\simeq\mathrm{Sp}_r(\mathbf{F}_p)$ is irreducible – a nontrivial fact proved by Grosse-Klönne [28, Cor. 4.3]. This yields an isomorphism (1.15).

Part (3) of Theorem 1.3 follows now easily from the two previous (compatible) parts and the fact that $(\mathrm{Sp}_r(K)^*)^{G\text{-bd}}\simeq\mathrm{Sp}_r^{\mathrm{cont}}(\mathbf{Q}_p)^*\otimes_{\mathbf{Q}_p}K$ and $H^r_{\mathrm{dR}}(X_K)^{\ G\text{-bd}}\simeq H^r_{\mathrm{dR}}(X)\otimes_{\mathscr{O}_K}K$ (the latter isomorphism uses the fact that X can be covered by G-translates of an open subscheme U such that U_K is an affinoid),

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1.4.1. *Notation.* Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of characteristic p. Let ϖ be a uniformizer of $\mathscr{O}_K$ that we fix in this paper. Let $\overline{K}$ be an algebraic closure of K and let $\mathscr{O}_{\overline{K}}$ denote the integral closure of $\mathscr{O}_K$ in $\overline{K}$. Let $W(k)$ be the ring of Witt vectors of k with fraction field F (i.e, $W(k)=\mathscr{O}_F$); let e be the ramification index of K over F. Set $\mathscr{G}_K=\mathrm{Gal}(\overline{K}/K)$, and let σ be the absolute Frobenius on $W(\overline{k})$.

We will denote by $\mathscr{O}_K$, $\mathscr{O}_K^\times$, and $\mathscr{O}_K^0$, depending on the context, the scheme $\mathrm{Spec}(\mathscr{O}_K)$ or the formal scheme $\mathrm{Spf}(\mathscr{O}_K)$ with the trivial, the canonical (i.e., associated to the closed point), and the induced by $\mathbf{N}\to\mathscr{O}_K, 1\mapsto 0$, log-structure, respectively.

We will denote by $\mathbf{A}_{\rm cr}, \mathbf{B}^+_{\rm cr}, \mathbf{B}^+_{\rm st}, \mathbf{B}^+_{\rm dR}$ the crystalline, semistable, and de Rham period rings of Fontaine. We have $\mathbf{B}^+_{\rm st} = \mathbf{B}^+_{\rm cr}[u]$ and $\varphi(u) = pu, N(u) = -1$. The embedding $\iota = \iota_\varpi : \mathbf{B}^+_{\rm st} \to \mathbf{B}^+_{\rm dR}$ is defined by $u \mapsto u_\varpi = \log([\varpi^\flat]/\varpi)$ and the Galois action on $\mathbf{B}^+_{\rm st}$ is induced from the one on $\mathbf{B}^+_{\rm dR}$ via this embedding.

Unless otherwise stated all formal schemes are p-adic, locally of finite type, and equidimensional. For a (p-adic formal) scheme X over $\mathscr{O}_K$, let X_0 denote the special fiber of X; let X_n denote its reduction modulo p^n.

If $f : C \to C'$ is a map in the dg derived category of a quasi-abelian category, we set

$$[\, C \xrightarrow{f} C' \,] := \operatorname{holim}(C \to C' \leftarrow 0).$$

2. Review of p-adic functional analysis

We gather here some basic facts from p-adic functional analysis that we use in the paper. Our main references are [60], [56], [16].

2.0.2. *Derived category of locally convex K-vector spaces.* A topological K-vector space[6] is called *locally convex* (*convex* for short) if there exists a neighbourhood basis of the origin consisting of $\mathscr{O}_K$-modules. Since K is spherically complete, the theory of such spaces resembles the theory of locally convex topological vector spaces over $\mathbf{R}$ or $\mathbf{C}$ (with some simplifications).

We denote by C_K the category of convex K-vector spaces. It is a quasi-abelian category[7] [56, 2.1.11]. Kernels, cokernels, images, and coimages are taken in the category of vector spaces and equipped with the induced topology [56, 2.1.8]. A morphism $f : E \to F$ is *strict* if and only if it is relatively open, i.e., for any neighbourhood V of 0 in E there is a neighbourhood V' of 0 in F such that $f(V) \supset V' \cap f(E)$ [56, 2.1.9].

Our convex K vector spaces are not assumed to be separated. We often use the following simple observation: *if F is separated and we have an injective morphism $f : E \to F$ then E is separated as well; if, moreover, F is finite dimensional and f is bijective then f is an isomorphism in C_K.*

The category C_K has a natural exact category structure: the admissible monomorphisms are embeddings, the admissible epimorphisms are open surjections. A complex $E \in C(C_K)$ is called *strict* if its differentials are strict. There are truncation functors on $C(C_K)$:

$$\tau_{\leq n}E := \cdots \to E^{n-2} \to E^{n-1} \to \ker(d^n) \to 0 \to \cdots$$
$$\tau_{\geq n}E := \cdots \to 0 \cdots \to \operatorname{coim}(d^{n-1}) \to E^n \to E^{n+1} \to \cdots$$

with cohomology objects

$$\widetilde{H}^n(E) := \tau_{\leq n}\tau_{\geq n}(E) = \cdots \to 0 \to \operatorname{coim}(d^{n-1}) \to \ker(d^n) \to 0 \to \cdots$$

We note that here $\operatorname{coim}(d^{n-1})$ and $\ker(d^n)$ are equipped naturally with the quotient and subspace topology, respectively. The cohomology $H^*(E)$ taken in the category of K-vector spaces we will call *algebraic* and, if necessary, we will always equip it with the quotient topology.

We will denote the bounded derived dg category of C_K by $\mathscr{D}^b(C_K)$. It is defined as the dg quotient [15] of the dg category $C^b(C_K)$ by the full dg subcategory of strictly exact complexes [47]. A morphism of complexes that is a quasi-isomorphism in $\mathscr{D}^b(C_K)$, i.e., its cone is strictly exact, will be called a *strict quasi-isomorphism*; this happens if and only if the induced morphism on cohomology groups is an isomorphism in C_K (for the sub-quotient topology). We will denote by $D^b(C_K)$ the homotopy category of $\mathscr{D}^b(C_K)$ [56, 1.1.5].

For $n \in \mathbf{Z}$, let $D^b_{\leq n}(C_K)$ (resp. $D^b_{\geq n}(C_K)$) denote the full subcategory of $D^b(C_K)$ of complexes that are strictly exact in degrees $k > n$ (resp. $k < n$)[8]. The above truncation maps extend to truncations

[6] For us, a *K-topological vector space* is a K-vector space with a linear topology.

[7] An additive category with kernels and cokernels is called *quasi-abelian* if every pullback of a strict epimorphism is a strict epimorphism and every pushout of a strict monomorphism is a strict monomorphism. Equivalently, an additive category with kernels and cokernels is called *quasi-abelian* if $\operatorname{Ext}(-,-)$ is bifunctorial.

[8] Recall [61, 1.1.4] that a sequence $A \xrightarrow{e} B \xrightarrow{f} C$ such that $fe = 0$ is called *strictly exact* if the morphism e is strict and the natural map $\operatorname{im} e \to \ker f$ is an isomorphism.

functors $\tau_{\leq n} : D^b(C_K) \to D^b_{\leq n}(C_K)$ and $\tau_{\geq n} : D^b(C_K) \to D^b_{\geq n}(C_K)$. The pair $(D^b_{\leq n}(C_K), D^b_{\geq n}(C_K))$ defines a t-structure on $D^b(C_K)$ by [61]. The heart $D^b(C_K)^{\heartsuit}$ is an abelian category $LH(C_K)$: every object of $LH(C_K)$ is represented (up to equivalence) by a monomorphism $f : E \to F$, where F is in degree 0, i.e., it is isomorphic to a complex $0 \to E \stackrel{f}{\to} F \to 0$; *if f is strict* this object is also represented by the cokernel of f (the whole point of this construction is to keep track of the two possibly different topologies on E: the given one and the one inherited by the inclusion into F).

We will denote by $\widetilde{H}^n : D^b(C_K) \to D^b(LH(C_K))$ the associated cohomological functors. We have an embedding $I : C_K \hookrightarrow LH(C_K)$, $E \mapsto (0 \to E)$, that induces an equivalence $D^b(C_K) \stackrel{\sim}{\to} D^b(LH(C_K))$ that is compatible with t-structures. The above t-structure pulls back to a t-structure on the derived dg category $\mathscr{D}^b(C_K)$. Note that if $\widetilde{H}^n(E)$ lies in the image of I then it is isomorphic to $H^n(E)$; we will say in that case that the cohomology $\widetilde{H}^n(E)$ *classical*.

We will often use the following simple facts:

(1) If, in the following short exact sequence in $C(LH(C_K))$, both A_1 and A_2 are in the essential image of I then so is A:
$$0 \to A_1 \to A \to A_2 \to 0.$$
(2) A complex $E \in C^b(C_K)$ is strictly exact in a specific degree if and only if $\mathscr{D}^b(I)(E)$ is exact in the same degree.

2.0.3. *Open Mapping Theorem.* Let $f : X \to Y$ be a continuous surjective map of locally convex K-vector spaces. We will need a well-known version of the Open Mapping Theorem that says that f is open if both X and Y are LF-spaces, i.e., countable inductive limits of Fréchet spaces[9].

If E, F are Fréchet, $f : E \to F$ is strict if and only if $f(E)$ is closed in F (the "if" part follows from the Open Mapping Theorem, the "only if" part from the fact that a Fréchet space is a metric space and a complete subspace of a metric space is closed).

2.0.4. *Tensor products.* Let V, W be two convex K-vector spaces. The abstract tensor product $V \otimes_K W$ can be equipped with several natural topologies among them the projective and injective tensor product topologies: $V \otimes_{K,\pi} W$ and $V \otimes_{K,\varepsilon} W$. Recall that the projective tensor product topology is universal for jointly continuous bilinear maps $V \times W \to U$; the injective tensor product topology, on the other hand, is defined by cross seminorms that satisfy a product formula and is the "weakest" topology with such property. There is a natural map $V \otimes_{K,\pi} W \to V \otimes_{K,\varepsilon} W$. We denote by $V \widehat{\otimes}_{K,\alpha} W$, $\alpha = \pi, \varepsilon$, the Hausdorff completion of $V \otimes_{K,\alpha} W$ with respect to the topology α.

Recall the following facts.

(1) The projective tensor product functor $(-) \otimes_{K,\pi} W$ preserves admissible epimorphisms; the injective tensor product functor $(-) \otimes_{K,\varepsilon} W$ preserves admissible monomorphisms.
(2) The natural map $V \otimes_{K,\pi} W \to V \otimes_{K,\varepsilon} W$ is an isomorphism[10] [55, Theorem 10.2.7]. In what follows we will often just write $V \otimes_K W$ for both products.
(3) From (1), (2), and the exactness properties of Hausdorff completion [67, Cor. 1.4], it follows that the tensor product functor $(-)\widehat{\otimes}_K W : C_K \to C_K$ is left exact, i.e., it carries strictly exact sequences
$$0 \to V_1 \to V_2 \to V_3 \to 0$$
to strictly exact sequences
$$0 \to V_1 \widehat{\otimes}_K W \to V_2 \widehat{\otimes}_K W \to V_3 \widehat{\otimes}_K W.$$
Moreover, the image of the last map above is dense [67, p.45]. It follows that this map is surjective if its image is complete as happens, for example, in the case when the spaces V_*, W are Fréchet [67, Cor. 1.7].

[9] We will say *strict* LF-spaces if the maps in the inductive system are strict inclusions. If the spaces involved are actually Banach, we will sometimes use the notation LB instead of LF.

[10] Here we used the fact that our field K is spherically complete.

(4) For $V = \varprojlim_n V_n$, there is a natural isomorphism

$$V\widehat{\otimes}_{K,\pi}W = (\varprojlim_n V_n)\widehat{\otimes}_{K,\pi}W \xrightarrow{\sim} \varprojlim_n (V_n\widehat{\otimes}_{K,\pi}W).$$

For products this is proved in [58, Prop. 9, p.192] and the general case follows from the fact that tensor product is left exact.

(5) Let $\{V_n\}$, $n \in \mathbf{N}$, be a regular[11] inductive system of Fréchet spaces with injective nuclear[12] transition maps. Then, for any Banach space W, we have an isomorphism [41, Theorem 1.3]

$$(\varinjlim_n V_n)\widehat{\otimes}_K W \xleftarrow{\sim} \varinjlim_n (V_n\widehat{\otimes}_K W).$$

2.0.5. *Acyclic inductive systems.* A locally convex K-vector space V is called *nuclear* [55, 8.5] if there exists a neighbourhood U of 0 "small" in the sense that, for every map $f : V \to B$, with B a Banach space, $f(U)$ is contained in the union of the translates of the unit ball of B by elements of a finite rank $\mathscr{O}_K$-module. Note that a Banach space is nuclear if and only if it is finite dimensional (take $f : B \to B$ mapping x to $p^{-N}x$, for N big depending on U).

In an Archimedean setting, nuclear spaces can be defined by requiring that the natural map from the projective tensor product to the injective tensor product with any locally convex space is an isomorphism. For a non-Archimedean spherically complete K this map is always an isomorphism, and such a definition would not ensure the good properties of nuclear spaces we want; hence the above definition.

Nuclear spaces are closed under the following operations [55, Theorem 8.5.7]: subspace and Hausdorff quotient, projective limit, tensor product, completion, countable inductive limit.

Nuclear Fréchet spaces are Montel [60, Cor. 19.3]. It follows that [71, Theorem 3.3], if $V = \varinjlim_n V_n$ is an inductive limit of nuclear Fréchet spaces with injective transition maps, then V is regular if and only if it is acyclic, i.e. if and only if $\mathrm{L}^1 \varinjlim_n V_n = 0$.

Lemma 2.1. *Let $A_n^\bullet$, $n \in \mathbf{N}$, be complexes of nuclear Fréchet spaces with strict differentials and cohomology groups that are finite dimensional Hausdorff. Assume that these complexes form a regular inductive system with injective nuclear transition maps $A_n^\bullet \to A_{n+1}^\bullet$. Let $A^\bullet = \varinjlim_n A_n^\bullet$. Then, for any Banach space W, we have*

$$\widetilde{H}^i(A^\bullet\widehat{\otimes}_K W) \simeq (H^iA^\bullet)\widehat{\otimes}_K W.$$

In particular, this cohomology is classical.

Proof. First, note that the inductive system $\{A_n^i\}, n \in \mathbf{N}, i \geq 0$, is acyclic. We claim that so is the inductive system $\{A_n^i\widehat{\otimes}_K W\}, n \in \mathbf{N}, i \geq 0$. We will use for that the following criterium of acyclicity [71, Theorem 1.1].

Proposition 2.2. *An inductive system $\{V_n\}_n, n \in \mathbf{N}$, of Fréchet spaces with injective transition maps is acyclic if and only if in every space V_n there is a convex neighbourhood U_n of 0 such that*

(1) *$U_n \subset U_{n+1}, n \in \mathbf{N}$, and*
(2) *For every $n \in \mathbf{N}$ there is $m > n$ such that all topologies of the spaces V_k, $k > m$, coincide on U_n.*

The inductive system $\{A_n^i\}, n \in \mathbf{N}, i \geq 0$, being acyclic satisfies the above criterium; to see that so does the system $\{A_n^i\widehat{\otimes}_K W\}, n \in \mathbf{N}, i \geq 0$, it is enough to use the U_n's chosen for the original system and to tensor them with a lattice in W.

Now, consider the following series of isomorphisms

$$\begin{aligned}\widetilde{H}^i(A^\bullet\widehat{\otimes}_K W) &\simeq \widetilde{H}^i(\varinjlim_n A_n^\bullet\widehat{\otimes}_K W) \simeq \varinjlim_n \widetilde{H}^i(A_n^\bullet\widehat{\otimes}_K W)\\ &\simeq \varinjlim_n (H^iA_n^\bullet)\widehat{\otimes}_K W \simeq (\varinjlim_n H^i(A_n^\bullet))\widehat{\otimes}_K W \simeq H^i(A^\bullet)\widehat{\otimes}_K W\end{aligned}$$

[11] Inductive system $\{V_n\}$, $n \geq 0$, with injective transition maps is called *regular* if for each bounded set B in $V = \varinjlim_n V_n$ there exists an n such that $B \subset V_n$ and B is bounded in V_n.

[12] A map $f : V \to W$ between two convex K-vector spaces is called *nuclear* if it can be factored $f : V \to V_1 \xrightarrow{f_1} W_1 \to W$, where the map f_1 is a compact map between Banach spaces.

The first and the fourth isomorphisms follow from the fact (5) recalled above; the second and the fifth one – from exactness of $\varinjlim$. It remains to show the third isomorphism, i.e., that

$$\widetilde{H}^i(A_n^\bullet\widehat{\otimes}_K W)\simeq (H^iA_n^\bullet)\widehat{\otimes}_K W,\quad n\in\mathbf{N}.$$

For that, write $A_n^\bullet := A_n^0\stackrel{d_0^n}{\to}A_n^1\stackrel{d_1^n}{\to}\cdots$. We have two strictly exact sequences

$$0\to\ker d_i^n\to A_n^i\to\operatorname{coim} d_i^n=\operatorname{im} d_i^n\to 0,$$
$$0\to\operatorname{im} d_{i-1}^n\to\ker d_i^n\to H^iA_n^\bullet\to 0.$$

Since $\ker d_i^n\simeq\operatorname{im} d_{i-1}^n\oplus H^iA_n^\bullet$ (use the Open Mapping Theorem and the fact that $H^iA_n^\bullet$ is finite dimensional),

$$\ker(d_i\otimes 1_W)\simeq(\ker d_i)\widehat{\otimes}_K W\simeq((\operatorname{im} d_{i-1})\widehat{\otimes}_K W)\oplus(H^iA^\bullet\widehat{\otimes}_K W).$$

It follows that it suffices to show that

$$(\operatorname{im} d_{i-1})\widehat{\otimes}_K W=\operatorname{im}(d_{i-1}\otimes 1_W).$$

But this is clear in view of the fact (3) recalled above since all the spaces involved are Fréchet.

□

3. Syntomic cohomologies

In the first part of this section we will define and study overconvergent syntomic cohomology. Its construction mimics the construction of Selmer groups by Bloch and Kato [6]. An analogous construction for schemes was given in [72]. In the second part of the section we modify the syntomic cohomology of Fontaine-Messing so that it also resembles Bloch-Kato Selmer groups.

3.1. **Overconvergent Hyodo-Kato cohomology.** We will review in this section the definition of the overconvergent Hyodo-Kato cohomology and the overconvergent Hyodo-Kato isomorphism due to Grosse-Klönne [23]. We will pay particular attention to topological issues.

3.1.1. *Dagger spaces and weak formal schemes.* We will review, very briefly, basic facts concerning dagger spaces and weak formal schemes. Our main references are [42, 20, 68], where the interested reader can find a detailed exposition.

We start with dagger spaces. For $\delta\in\mathbb{R}^+$, set

$$T_n(\delta)=K\{\delta^{-1}X_1,\ldots,\delta^{-1}X_n\}:=\{\sum_v a_vX^v\in K[[X_1,\ldots,X_n]]\mid\lim_{|v|\to\infty}|a_v|\delta^{|v|}=0\}.$$

Here $|v|=\sum_{i=1}^n v_i$, $v=(v_1,\ldots,v_n)\in\mathbf{N}^n$. We have $T_n:=K\{X_1,\ldots,X_n\}=T_n(1)$. If $\delta\in p^{\mathbf{Q}}$, this is an affinoid K-algebra; the associated Banach norm $|\bullet|_\delta: T_n(\delta)\to\mathbb{R}$, $|\sum a_vX^v|_\delta=\max_v|a_v|\delta^{|v|}$. We set

$$K[X_1,\ldots,X_n]^\dagger:=\bigcup_{\delta>1,\delta\in p^{\mathbf{Q}}}T_n(\delta)=\bigcup_{\delta>1}T_n(\delta)$$

It is a Hausdorff LF-algebra.

A *dagger algebra* A is a topological K-algebra isomorphic to a quotient of the overconvergent Tate algebra $K[X_1,\ldots,X_d]^\dagger$. It is canonically a Hausdorff LF-algebra [1, Cor. 3.2.4]. It defines a sheaf of topological K-algebras $\mathscr{O}^\dagger$ on $\operatorname{Sp}\widehat{A}$, $\widehat{A}$ being the p-adic completion of A, which is called a *dagger structure* on $\operatorname{Sp}\widehat{A}$. The pair $\operatorname{Sp}(A):=(|\operatorname{Sp}\widehat{A}|,\mathscr{O}^\dagger)$ is called a *dagger affinoid*.

A *dagger space*[13] X is a pair $(\widehat{X},\mathscr{O}^\dagger)$ where $\widehat{X}$ is a rigid analytic space over K and $\mathscr{O}^\dagger$ is a sheaf of topological K-algebras on $\widehat{X}$ such that, for some affinoid open covering $\{\widehat{U}_i\to\widehat{X}\}$, there are dagger structures U_i on $\widehat{U}_i$ such that $\mathscr{O}^\dagger|\widehat{U}_i\simeq\mathscr{O}^\dagger_{U_i}$. The set of global sections $\Gamma(X,\mathscr{O}^\dagger)$ has a structure of a convex K-vector space given by the projective limit $\varprojlim_Y\Gamma(Y,\mathscr{O}^\dagger|Y)$, where Y runs over all affinoid subsets of X. In the case of dagger affinoids this agrees with the previous definition.

Let $X=\operatorname{Sp}(A)\to Y=\operatorname{Sp}(B)$ be a morphism of affinoid dagger spaces and let $U\subset X$ be an affinoid subdomain. We write $U\Subset_Y X$ if there exists a surjection $\tau: B[X_1,\ldots,X_r]^\dagger\to A$ and $\delta\in p^{\mathbf{Q}},\delta>1$, such

[13]Sometimes called *rigid analytic space with overconvergent structure sheaf.*

that $U \subset \mathrm{Sp}(A[\delta^{-1}\tau(X_1),\ldots,\delta^{-1}\tau(X_r)]^{\dagger})$. A morphism $f: X \to Y$ of dagger (or rigid) spaces is called *partially proper* if f is separated and if there exist admissible coverings $Y = \bigcup Y_i$ and $f^{-1}(Y_i) = \bigcup X_{ij}$, all i, such that for every X_{ij} there exists an affinoid subset $\widetilde{X}_{ij} \subset f^{-1}(Y_i)$ with $X_{ij} \Subset_Y \widetilde{X}_{ij}$. A partially proper dagger space that is quasi-compact is called *proper*. This notion is compatible with the one for rigid spaces. In fact, the category of partially proper dagger spaces is equivalent to the category of partially proper rigid spaces [20, Theorem 2.27]. In particular, a rigid analytification of a finite type scheme over K is partially proper.

A dagger (or rigid) space X is called *Stein* if it admits an admissible affinoid covering $X = \bigcup_{i\in\mathbf{N}} U_i$ such that $U_i \subset^{\dagger} U_{i+1}$ for all i; we call the covering $U_i, i \in \mathbf{N}$, a Stein covering. Here the notation $U_i \subset^{\dagger} U_{i+1}$ means that the map $\widehat{U}_i = \mathrm{Sp}(C) \subset \widehat{U}_{i+1} = \mathrm{Sp}(D)$ is an open immersion of affinoid rigid spaces induced by a map $D \simeq T_n(\delta)/I \to C \simeq T_n/IT_n$ for some I and $\delta > 1$. Stein spaces are partially proper.

We pass now to weakly formal schemes; the relation between dagger spaces and weak formal schemes parallels [38] the one between rigid spaces and formal schemes due to Raynaud. A *weakly complete* $\mathscr{O}_K$*-algebra* $A^{\dagger}$ (with respect to (ϖ)) is an $\mathscr{O}_K$-algebra which is ϖ-adically separated and which satisfies the following condition: for any power series $f \in \mathscr{O}_K\{X_1,\ldots,X_n\}$, $f = \sum a_v X^v$, such that there exists a constant c for which $c(v_p(a_v)+1) \geq |v|$, all v, and for any n-tuple $x_1,\ldots,x_n \in A^{\dagger}$, the series $f(x_1,\ldots,x_n)$ converges to an element of $A^{\dagger}$. The *weak completion* of an $\mathscr{O}_K$-algebra A is the smallest weakly complete subalgebra $A^{\dagger}$ of $\widehat{A}$ containing the image of A.

A *weak formal scheme* is a locally ringed space $(X, \mathscr{O})$ that is locally isomorphic to an affine weak formal scheme. An *affine weak formal scheme* is a locally ring space $(X, \mathscr{O})$ such that $X = \mathrm{Spec}(A^{\dagger}/\varpi)$ for some weakly complete finitely generated $\mathscr{O}_K$-algebra $A^{\dagger}$ and the sheaf $\mathscr{O}$ is given on the standard basis of open sets by $\Gamma(X_{\overline{f}}, \mathscr{O}) = (A_f^{\dagger})^{\dagger}$, $f \in A^{\dagger}$. For a weak formal scheme X, flat over $\mathscr{O}_K$, the associated dagger space X_K is partially proper if and only if all irreducible closed subsets Z of X are proper over $\mathscr{O}_K$ [31, Remark 1.3.18].

A weak formal scheme over $\mathscr{O}_K$ is called *semistable* if, locally for the Zariski topology, it admits étale maps to the weak formal spectrum $\mathrm{Spwf}(\mathscr{O}_K[X_1,\ldots,X_n]^{\dagger}/(X_1\cdots X_r - \varpi))$, $1 \leq r \leq n$. We equip it with the log-structure coming from the special fiber. We have a similar definition for formal schemes. A (weak) formal scheme X is called *Stein* if its generic fiber X_K is Stein. It is called *Stein with a semistable reduction* if it has a semistable reduction over $\mathscr{O}_K$ (and then the irreducible components of $Y := X_0$ are proper and smooth) and there exist closed (resp. open) subschemes $Y_s, s \in \mathbf{N}$, (resp. $U_s, s \in \mathbf{N}$) of Y such that

1. each Y_s is a finite union of irreducible components,
2. $Y_s \subset U_s \subset Y_{s+1}$ and their union is Y,
3. the tubes $\{]U_s[_X\}, s \in \mathbf{N}$, form a Stein covering of X_K.

We will call the covering $\{U_s\}, s \in \mathbf{N}$, a Stein covering of Y. The schemes U_s, Y_s inherit their log-structure from Y (which is canonically a log-scheme log-smooth over k^0). The log-schemes Y_s are not log-smooth (over k^0) but they are ideally log-smooth, i.e., they have a canonical idealized log-scheme structure and are ideally log-smooth for this structure[14].

3.1.2. *Overconvergent Hyodo-Kato cohomology.* Let X be a semistable weak formal scheme over $\mathscr{O}_K$. We would like to define the overconvergent Hyodo-Kato cohomology as the rational overconvergent rigid cohomology of X_0 over $\mathscr{O}_F^0$:

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_0) := \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0).$$

The foundations of log-rigid cohomology missing [15] this has to be done by hand [23, 1].

[14]Recall [50] that an idealized log-scheme is a log-scheme together with an ideal in its log-structure that maps to zero in the structure sheaf. There is a notion of log-smooth morphism of idealized log-schemes. Log-smooth idealized log-schemes behave like classical log-smooth log-schemes. One can extend the definitions of log-crystalline, log-convergent, and log-rigid cohomology, as well as that of de Rham-Witt complexes to idealized log-schemes. In what follows we will often skip the word "idealized" if understood.

[15]See however [65].

Let Y be a fine k^0-log-scheme. Choose an open covering $Y=\cup_{i\in I}Y_i$ and, for every $i\in I$, an exact closed immersion $Y_i\hookrightarrow Z_i$ into a log-smooth weak formal $\mathscr{O}_F^0$-log-scheme Z_i. For each nonempty finite subset $J\subset I$ choose (perhaps after refining the covering) an exactification[16] [37, Prop. 4.10]

$$Y_J=\cap_{i\in J}Y_i\xrightarrow{\iota} Z_J\xrightarrow{f}\prod_{\mathscr{O}_F^0}(Z_i)_{i\in J}$$

of the diagonal embedding $Y_J\to\prod_{\mathscr{O}_F^0}(Z_i)_{i\in J}$. Let $\Omega^\bullet_{Z_J/\mathscr{O}_F^0}$ be the de Rham complex of the weak formal log-scheme Z_J over $\mathscr{O}_F^0$. This is a complex of sheaves on Z_J; tensoring it with F we obtain a complex of sheaves $\Omega^\bullet_{Z_{J,F}}$ on the F-dagger space $Z_{J,F}$. By [23, Lemma 1.2], the tube $]Y_J[_{Z_J}$ and the restriction $\Omega^\bullet_{]Y_J[_{Z_J}}:=\Omega^\bullet_{Z_{J,F}}|_{]Y_J[_{Z_J}}$ of $\Omega^\bullet_{Z_{J,F}}$ to $]Y_J[_{Z_J}$ depend only on the embedding system $\{Y_i\hookrightarrow Z_i\}_i$ not on the chosen exactification (ι,f). Equip the de Rham complex $\Gamma(]Y_J[_{Z_J},\Omega^\bullet)$ with the topology induced from the structure sheaf of the dagger space $]Y_J[_{Z_J}$.

For $J_1\subset J_2$, one has natural restriction maps $\delta_{J_1,J_2}:]Y_{J_2}[_{Z_{J_2}}\to]Y_{J_1}[_{Z_{J_1}}$ and $\delta^{-1}_{J_1,J_2}\Omega^\bullet_{]Y_{J_1}[_{Z_{J_1}}}\to\Omega^\bullet_{]Y_{J_2}[_{Z_{J_2}}}$. Well-ordering I, we get a simplicial dagger space $]Y_\bullet[_{Z_\bullet}$ and a sheaf $\Omega^\bullet_{]Y_\bullet[_{Z_\bullet}}$ on $]Y_\bullet[_{Z_\bullet}$. Consider the complex $\mathrm{R}\Gamma(]Y_\bullet[_{Z_\bullet},\Omega^\bullet)$. We equip it with the topology induced from the product topology on every cosimplicial level. In the derived category of F-vector spaces this complex is independent of choices made but we will make everything independent of choices already in the abelian category of complexes of F-vector spaces by simply taking limit over all the possible choices. We define a complex in $\mathscr{D}^b(C_F)$

$$\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^0):=\operatorname{hocolim}\Gamma(]Y_\bullet[_{Z_\bullet},\Omega^\bullet),\tag{3.1}$$

where the limit is over the data that we have described above. Note that the data corresponding to affine coverings form a cofinal system. We set

$$\widetilde{H}^i_{\mathrm{rig}}(Y/\mathscr{O}_F^0):=\widetilde{H}^i\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^0),\quad H^i_{\mathrm{rig}}(Y/\mathscr{O}_F^0):=H^i\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^0).$$

The complex $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^0)$ is equipped with a Frobenius endomorphism φ defined by lifting Frobenius to the schemes Z_i in the above construction. In the case Y is log-smooth over k^0 we also have a monodromy endomorphism[17] $N=\mathrm{Res}(\nabla(\mathrm{dlog}\,0))$ defined by the logarithmic connection satisfying $p\varphi N=N\varphi$.

Proposition 3.2. *Let Y be a semistable scheme over k with the induced log-structure* [23, 2.1].

1. *If Y is quasi-compact then $H^*_{\mathrm{rig}}(Y/\mathscr{O}_F^0)$ is a finite dimensional F-vector space with its unique locally convex Hausdorff topology.*
2. *The endomorphism φ on $H^*_{\mathrm{rig}}(Y/\mathscr{O}_F^0)$ is a homeomorphism.*
3. *If k is finite then $H^*_{\mathrm{rig}}(Y/\mathscr{O}_F^0)$ is a mixed F-isocrystal, i.e., the eigenvalues of φ are Weil numbers.*

Proof. All algebraic statements concerning the cohomology are proved in [23, Theorem 5.3]. They follow immediately from the following weight spectral sequence [23, 5.2, 5.3] that reduces the statements to the analogous ones for (classically) smooth schemes over k

$$E_1^{-k,i+k}=\bigoplus_{j\geq 0,j\geq -k}\prod_{N\in\Theta_{2j+k+1}}H^{i-2j-k}_{\mathrm{rig}}(N/\mathscr{O}_F)\Rightarrow H^i_{\mathrm{rig}}(X_0/\mathscr{O}_F^0).\tag{3.3}$$

Here Θ_j denotes the set of all intersections N of j different irreducible components of X that are equipped with trivial log-structure. By assumptions they are smooth over k.

Let us pass to topology. Recall the following fact (that we will repeatedly use in the paper)

Lemma 3.4. *(*[20, Lemma 4.7],[21, Cor. 3.2]*) Let Y be a smooth Stein space or a smooth affinoid dagger space. All de Rham differentials $d_i:\Omega^i(Y)\to\Omega^{i+1}(Y)$ are strict and have closed images.*

Remark 3.5. The above lemma holds also for log-smooth Stein spaces with the log-structure given by a normal crossing divisor. The proof in [21, Cor. 3.2] goes through using the fact that for such quasi-compact log-smooth spaces the rigid de Rham cohomology is isomorphic to the rigid de Rham cohomology

[16] Recall that *an exactification* is an operation that turns closed immersions of log-schemes into exact closed immersions.

[17] The formula that follows, while entirely informal, should give the reader an idea about the definition of the monodromy. The formal definition can be found in [48, formula (37)].

of the open locus where the log-structure is trivial (hence it is finite dimensional and equipped with the canonical Hausdorff topology).

We claim that, in the notation used above, if Y_J is affine, then the complex

$$\Gamma(]Y_J[_{Z_J},\Omega^\bullet)=\mathrm{R}\Gamma(]Y_J[_{Z_J},\Omega^\bullet)$$

has finite dimensional algebraic cohomology H^* whose topology is Hausdorff. Moreover, its cohomology $\widetilde{H}^*$ is classical. Indeed, note that, using the contracting homotopy of the Poincaré Lemma for an open ball, we may assume that the tube $]X_{0,J}[_{Z_J}$ is the generic fiber of a weak formal scheme lifting $X_{0,J}$ to $\mathscr{O}_F^0$. Now, write $H^i=\ker d_i/\operatorname{im} d_{i-1}$ with the induced quotient topology. By the above lemma, the natural map $\operatorname{coim} d_{i-1}\to\operatorname{im} d_{i-1}$ is an isomorphism and $\operatorname{im} d_{i-1}$ is closed in $\ker d_i$. Hence $\widetilde{H}^i$ is classical and $\widetilde{H}^i\stackrel{\sim}{\to}H^i$ is Hausdorff, as wanted.

Note that, by the above, a map between two de Rham complexes associated to two (different) embeddings of Y_I is a strict quasi-isomorphism. This implies that, if Y is affine, all the arrows in the system (3.1) are strict quasi-isomorphisms and the cohomology of $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^0)$ is isomorphic to the cohomology of $\Gamma(]Y_\bullet[_{Z_\bullet},\Omega^\bullet)$ for any embedding data.

This proves claim (1) of our proposition for affine schemes; the case of a general quasi-compact scheme can be treated in the same way (choose a covering by a finite number of affinoids). Claim (2) follows easily from claim (1). □

Remark 3.6. In an analogous way to $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^0)$ we define complexes $\mathrm{R}\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_K^\times)\in\mathscr{D}^b(C_K)$. For a quasi-compact Y, their cohomology groups are classical; they are finite K-vector spaces with their canonical Hausdorff topology.

3.1.3. *Overconvergent Hyodo-Kato isomorphism.* Set $r^+:=k[T], r^\dagger:=\mathscr{O}_F[T]^\dagger$ with the log-structure associated to T. Let X be a log-scheme over $r^+:=k[T]$ (in particular, we allow log-schemes over k^0). Assume that there exists an open covering $X=\cup_{i\in I}X_i$ and, for every i, an exact closed immersion $X_i\hookrightarrow\widetilde{X}_i$ into a log-scheme log-smooth over $\tilde{r}:=\mathscr{O}_F[T]$. For the schemes in this paper such embeddings will exist locally. For each nonempty finite subset $J\subset I$, choose an exactification (product is taken over $\tilde{r}$)

$$X_J:=\cap_{i\in J}X_i\stackrel{\iota}{\hookrightarrow}\widetilde{X}_J\stackrel{f}{\to}\prod_{i\in J}\widetilde{X}_i$$

of the diagonal embedding as in Section 3.1.2.

Let $\mathscr{X}_J$ be the weak completion of $\widetilde{X}_J$. Define the de Rham complex $\Omega^\bullet_{\mathscr{X}_J/r^\dagger}$ as the weak completion of the de Rham complex $\Omega^\bullet_{\widetilde{X}_J/\tilde{r}}$. The tube $]X_J[_{\mathscr{X}_J}$ with the complex $(\Omega^\bullet_{\mathscr{X}_J/r^\dagger}\otimes\mathbf{Q})|_{]X_J[_{\mathscr{X}_J}}$ is independent of the chosen factorization (ι,f). For varying J one has natural transition maps, hence a simplicial dagger space $]X_\bullet[_{\mathscr{X}_\bullet}$ and a complex

$$(\Omega^\bullet_{\mathscr{X}_\bullet/r^\dagger}\otimes\mathbf{Q})|_{]X_\bullet[_{\mathscr{X}_\bullet}}\tag{3.7}$$

One shows that in the derived category of vector spaces over $\mathbf{Q}_p$

$$\mathrm{R}\Gamma(]X_\bullet[_{\mathscr{X}_\bullet},\Omega^\bullet_{\mathscr{X}_\bullet/r^\dagger}\otimes\mathbf{Q}|_{]X_\bullet[_{\mathscr{V}_\bullet}})$$

is independent of choices. We make it though functorial as a complex by going to limit over all the choices and define a complex in $\mathscr{D}^b(C_F)$

$$\mathrm{R}\Gamma_{\mathrm{rig}}(X/r^\dagger):=\operatorname{hocolim}\Gamma(]X_\bullet[_{\mathscr{X}_\bullet},(\Omega^\bullet_{\mathscr{X}_\bullet/r^\dagger}\otimes\mathbf{Q})|_{]X_\bullet[_{\mathscr{X}_\bullet}}),$$

where the index set runs over the data described above.

Cohomology $\mathrm{R}\Gamma_{\mathrm{rig}}(X/r^\dagger)$ is equipped with a Frobenius endomorphism φ defined by lifting mod p Frobenius to the schemes $\widetilde{X}_i$ in the above construction in a manner compatible with the Frobenius on $r^\dagger$ induced by $T\mapsto T^p$. If X is log-smooth over k^0, we also have a monodromy endomorphism $N=\operatorname{Res}(\nabla(\operatorname{dlog}T))$ defined by the logarithmic connection satisfying $p\varphi N=N\varphi$. The map $p_0:\mathrm{R}\Gamma_{\mathrm{rig}}(X/r^\dagger)\to\mathrm{R}\Gamma_{\mathrm{rig}}(X/\mathscr{O}_F^0)$ induced by $T\mapsto 0$ is compatible with Frobenius and monodromy.

For a general (simplicial) log-scheme with boundary $(\overline{X}, X)$ over r^+ that satisfies certain mild condition[18] the definition of the rigid cohomology $\mathrm{R\Gamma_{rig}}((\overline{X}, X)/r^\dagger)$ is analogous. For details of the construction we refer the reader to [23, 1.10] and for the definition of log-schemes with boundary to [22].

Let X_0 be a semistable scheme over k with the induced log-structure [23, 2.1]. Let $\{X_i\}_{i\in I}$ be the irreducible components of X_0 with induced log-structure. Denote by $M_\bullet$ the nerve of the covering $\coprod_{i\in I} X_i \to X_0$. We define the complex $\mathrm{R\Gamma_{rig}}(M_\bullet/\mathscr{O}_F^0) \in \mathscr{D}^b(C_K)$ in an analogous way to $\mathrm{R\Gamma_{rig}}(X_0/\mathscr{O}_F^0)$ using the embedding data described in [23, 1.5].

Lemma 3.8. *Let $\mathscr{O}$ denote $\mathscr{O}_F^0$ or $\mathscr{O}_K^\times$. The natural map*

$$\mathrm{R\Gamma_{rig}}(X_0/\mathscr{O}) \to \mathrm{R\Gamma_{rig}}(M_\bullet/\mathscr{O})$$

is a strict quasi-isomorphism.

Proof. It suffices to argue locally, so we may assume that there exists an exact embedding of X_0 into a weak formal scheme X that is log-smooth over $\mathscr{O}$.

First we prove that above map is a quasi-isomorphism. The complex $\mathrm{R\Gamma_{rig}}(M_\bullet/\mathscr{O})$ can be computed by de Rham complexes on the tubes $]M_J[_X$, where, for a nonempty subset $J\subset I$, we set $M_J = \cap_{j\in J} X_j$ with the induced log-structure. To compute $\mathrm{R\Gamma_{rig}}(X_0/\mathscr{O})$, recall that, for a weak formal scheme X and a closed subscheme Z of its special fiber, if $Z=\cup_{i\in I} Z_i$ is a finite covering by closed subschemes of Z, then the dagger space covering $]Z[_X = \cup_{i\in I}]Z_i[_X$ is admissible open [23, 3.3]. Hence $\mathrm{R\Gamma_{rig}}(X_0/\mathscr{O})$ can be computed as the de Rham cohomology of the nerve of the covering $X_K=\cup_{i\in I}]M_i[_X$. Since the two above mentioned simplicial de Rham complexes are equal, we are done.

Now, strictness of the above quasi-isomorphism follows from the fact that the cohomology groups of the left complex are finite dimensional vector spaces (over F or K) with their canonical Hausdorff topology and so are the cohomology groups of the right complex (basically by the same argument using the quasi-isomorphism $\mathrm{R\Gamma_{rig}}(M_J/\mathscr{O}) \xrightarrow{\sim} \mathrm{R\Gamma_{rig}}(M_J^{\mathrm{tr}}/\mathscr{O})$ [23, Lemma 4.4], where M_J^{tr} denotes the open set of M_J, where the horizontal log-structure is trivial.) $\square$

Let $J\subset I$ and $M=M_J=\cap_{j\in J}X_j$. Grosse-Klönne [23, 2.2] attaches to M finitely many log-schemes with boundary $(P_M^{J'}, V_M^{J'})$, $\emptyset \subsetneq J' \subset J$. We think of $(P_M^{J'}, V_M^{J'})$ as the vector bundle $V_M^{J'}$ on M (built from the log-structure corresponding to J') that is compactified by the projective space bundle $P_M^{J'}$. It is a log-scheme with boundary over r^+ which, in particular, means that $V_M^{J'}$ is a genuine log-scheme over r^+ (however this is not the case for $P_M^{J'}$). We note, that in application to the Hyodo-Kato isomorphism for M all we need are index sets J' with just one element. This construction of Grosse-Klönne corresponds to defining the Hyodo-Kato isomorphism using not the deformation space r_ϖ^{PD} as in the classical constructions but its compactification (a projective space). The key advantage being that the cohomology of the structure sheaf of the new deformation space is now trivial.

The following proposition is the main result of [23].

Proposition 3.9. *Let $\emptyset\neq J'\subset J\subset I$ and let $\mathscr{O}_F(0)=\mathscr{O}_F^0, \mathscr{O}_F(\varpi)=\mathscr{O}_K^\times$. The map*

$$\mathrm{R\Gamma_{rig}}((P_M^{J'}, V_M^{J'})/r^\dagger)\otimes_F F(a) \to \mathrm{R\Gamma_{rig}}(M/\mathscr{O}_F(a)),\quad a=0,\varpi,$$

defined by restricting to the zero section $M=M_J\to P_M^{J'}$ and sending $T\mapsto a$, is a strict quasi-isomorphism.

Proof. The algebraic quasi-isomorphism was proved in [23, Theorem 3.1]. To show that this quasi-isomorphism is strict we can argue locally, for X_0 affine. Then the cohomology of the complex on the right is a finite rank vector space over $F(a)$ with its natural locally convex and Hausdorff topology. Algebraic quasi-isomorphism and continuity of the restriction map imply that the cohomology of the complex on the left is Hausdorff as well. Since it is locally convex space the map has to be an isomorphism in $C_{F(a)}$, as wanted. $\square$

[18] The interested reader can find a description of this condition in [23, 1.10]. It will be always satisfied by the log-schemes we work with in this paper.

Example 3.10. We have found that the best way to understand the above proposition is through an example supplied by Grosse-Klönne himself in [23]. Let X_0 be of dimension 1 and let M be the intersection of two irreducible components. Hence the underlying scheme of M is equal to $\operatorname{Spec} k$. Let U be the two dimensional open unit disk over K with coordinates x_1, x_2, viewed as a dagger space. Consider its two closed subspaces: U^0 defined by $x_1x_2 = 0$ and U^{ϖ} defined by $x_1x_2 = \varpi$.

Let $\widetilde{\Omega}^{\bullet}_U$ be the de Rham complex of U with log-poles along the divisor U^0; let $\Omega^{\bullet}_U$ be its quotient by its sub-$\mathscr{O}_U$-algebra generated by $\operatorname{dlog}(x_1x_2)$. Denote by $\Omega^{\bullet}_{U^0}$ and $\Omega^{\bullet}_{U^{\varpi}}$ its restriction to U^0 and U^{ϖ}, respectively. We note that U^{ϖ} is (classically) smooth and that $\Omega^{\bullet}_{U^{\varpi}}$ is its (classical) de Rham complex. We view the k^0-log-scheme M as an exact closed log-subscheme of the formal log-scheme $\operatorname{Spf}(\mathscr{O}_F[[x_1, x_2]]/(x_1x_2))$ that is log-smooth over $\mathscr{O}_F^0$ or of the formal log-scheme $\operatorname{Spf}(\mathscr{O}_K[[x_1, x_2]]/(x_1x_2 - \varpi))$ that is log-smooth over $\mathscr{O}_K^{\times}$. The corresponding tubes are U^0 and U^{ϖ}. We have

$$\mathrm{R}\Gamma_{\mathrm{rig}}(M/\mathscr{O}_F^0)\otimes_F K = \mathrm{R}\Gamma(U^0, \Omega^{\bullet}), \quad \mathrm{R}\Gamma_{\mathrm{rig}}(M/\mathscr{O}_K^{\times}) = \mathrm{R}\Gamma(U^{\varpi}, \Omega^{\bullet}). \tag{3.11}$$

We easily see that $H^*(U^0, \Omega^{\bullet}) \simeq H^*(U^{\varpi}, \Omega^{\bullet})$; in particular, $H^1(U^0, \Omega^{\bullet}_{U^0}) = H^1(U^{\varpi}, \Omega^{\bullet})$ is a one dimensional K-vector space generated by $\operatorname{dlog} x_1$.

The quasi-isomorphism between the cohomologies in (3.11) is constructed via the following deformation space (P, V):

$$P = (\mathbb{P}^1_K \times \mathbb{P}^1_K)^{\mathrm{an}} = ((\operatorname{Spec}(K[x_1]) \cup \{\infty\}) \times (\operatorname{Spec}(K[x_2]) \cup \{\infty\}))^{\mathrm{an}}, \quad V := (\operatorname{Spec}(K[x_1, x_2]))^{\mathrm{an}}$$

Let $\widetilde{\Omega}^{\bullet}_P$ be the de Rham complex of P with log-poles along the divisor

$$(\{0\} \times \mathbb{P}^1_K) \cup (\mathbb{P}^1_K \times \{0\}) \cup (\{\infty\} \times \mathbb{P}^1_K) \cup (\mathbb{P}^1_K \times \{\infty\}).$$

The section $\operatorname{dlog}(x_1x_2) \in \widetilde{\Omega}^1_U(U) = \widetilde{\Omega}^1_P(U)$ extends canonically to a section $\operatorname{dlog}(x_1x_2) \in \widetilde{\Omega}^1_P(P)$. Let $\Omega^{\bullet}_P$ be the quotient of $\widetilde{\Omega}^{\bullet}_P$ by its sub-$\mathscr{O}_P$-algebra generated by $\operatorname{dlog}(x_1x_2)$. The natural restriction maps

$$\mathrm{R}\Gamma(U^0, \Omega^{\bullet}) \leftarrow \mathrm{R}\Gamma(P, \Omega^{\bullet}) \to \mathrm{R}\Gamma(U^{\varpi}, \Omega^{\bullet}), \quad 0 \leftarrow\!\shortmid T, T \mapsto \varpi,$$

are quasi-isomorphisms. This is because we have killed one differential of $\widetilde{\Omega}_P$ and the logarithmic differentials of $\mathbb{P}^1_{\mathscr{O}_K}$ are isomorphic to the structure sheaf hence have cohomology which is 1-dimensional in degree 0 and trivial otherwise.

Varying the index set J' in a coherent way one glues the log-schemes $(P_M^{J'}, V_M^{J'})$ into a simplicial r^+-log-scheme $(P_{\bullet}, V_{\bullet})$ with boundary. Set $\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/r^{\dagger}) := \mathrm{R}\Gamma_{\mathrm{rig}}((P_{\bullet}, V_{\bullet})/r^{\dagger})$. We have the corresponding simplicial log-scheme $M'_{\bullet}$ over k^0. There is a natural map $M_{\bullet} \to M'_{\bullet}$ (that induces a strict quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{rig}}(M'_{\bullet}/\mathscr{O}_F^0) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(M_{\bullet}/\mathscr{O}_F^0)$) and a natural map $M'_{\bullet} \to (P_{\bullet}, V_{\bullet})$. The following proposition is an immediate corollary of Proposition 3.9 and Lemma 3.8.

Proposition 3.12. *(*[23, Theorem 3.4]*)Let $a = 0, \varpi$. The natural maps*

$$\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F(a)) \to \mathrm{R}\Gamma_{\mathrm{rig}}(M'_{\bullet}/\mathscr{O}_F(a)) \leftarrow \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/r^{\dagger})\otimes_F F(a)$$

are strict quasi-isomorphisms.

Let X be a semistable weak formal scheme over $\mathscr{O}_K$. We define the overconvergent Hyodo-Kato cohomology of X_0 as $\mathrm{R}\Gamma_{\mathrm{HK}}(X_0) := \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0)$. Recall that the Hyodo-Kato map

$$\iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X_0) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)$$

is defined as the zigzag (using the maps from the above proposition)

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_0) = \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0) \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/r^{\dagger}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_0/r^{\dagger})\otimes_F K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K^{\times}) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X_K).$$

It yields the (overconvergent) Hyodo-Kato strict quasi-isomorphism

$$\iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X_0)\otimes_F K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X_K).$$

Remark 3.13. The overconvergent Hyodo-Kato map, as its classical counterpart, depends on the choice of the uniformizer ϖ. This dependence takes the usual form [66, Prop. 4.4.17].

3.2. **Overconvergent syntomic cohomology.** In this section we will define syntomic cohomology (a la Bloch-Kato) using overconvergent Hyodo-Kato and de Rham cohomologies of Grosse-Klönne and discuss the fundamental diagram that it fits into. We call this definition a la Bloch-Kato because it is inspired by Bloch-Kato's definition of local Selmer groups [6].

3.2.1. *Overconvergent geometric syntomic cohomology.* Let X be a semistable weak formal scheme over $\mathscr{O}_K$. Take $r\geq 0$. We define the overconvergent geometric syntomic cohomology of X_K by the following mapping fiber (taken in $\mathscr{D}^b(C_{\mathbf{Q}_p})$)

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_C,\mathbf{Q}_p(r)) := [[\mathrm{R}\Gamma_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}(\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r].$$

This is an overconvergent analog of the algebraic geometric syntomic cohomology studied in [48]. Here, we wrote $[\mathrm{R}\Gamma_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}$ for the homotopy limit of the commutative diagram[19]

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}} & \xrightarrow{\varphi-p^r} & \mathrm{R}\Gamma_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}}\\
\downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N}\\
\mathrm{R}\Gamma_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}} & \xrightarrow{p\varphi-p^r} & \mathrm{R}\Gamma_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}},
\end{array}$$

where the completed tensor products $\widehat{\otimes}$ are defined, using Section 3.1.2, as follows

$$\begin{aligned}
(3.14)\quad \Gamma(]X_{0,J}[_{Z_J},\Omega^\bullet)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}} &:= \oplus_{i\geq 0}\Gamma(]X_{0,J}[_{Z_J},\Omega^\bullet)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{cr}}u^i,\quad \mathbf{B}^+_{\mathrm{st}}=\oplus_{i\geq 0}\mathbf{B}^+_{\mathrm{cr}}u^i,\\
\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}} &:= \mathrm{hocolim}(\Gamma(]X_{0,\bullet}[_{Z_\bullet},\Omega^\bullet)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}}),\\
(\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r = (\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r &:= \mathrm{hocolim}((\Gamma(]X_{0,\bullet}[_{Z_\bullet},\Omega^\bullet)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r),
\end{aligned}$$

and similarly for the terms involved in the definition of the Hyodo-Kato map. Note that the inductive limits can be taken over affine coverings. Set

$$\mathrm{HK}(X_C,r) := [\mathrm{R}\Gamma_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r},\quad \mathrm{DR}(X_C,r) := (\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r.$$

Hence

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_C,\mathbf{Q}_p(r)) = [\mathrm{HK}(X_C,r)\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}\mathrm{DR}(X_C,r)].$$

Example 3.15. Assume that X is affine. We claim that, in the notation used above, the complex

$$\Gamma(]X_{0,J}[_{Z_J},\Omega^\bullet)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{cr}}$$

has classical cohomology equal to $H^*_{\mathrm{HK}}(X_{0,J})\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{cr}}$, a finite rank free module over $\mathbf{B}^+_{\mathrm{cr}}$. Indeed, as in the proof of Proposition 3.2, using the contracting homotopy of the Poincaré Lemma for an open ball, we may assume that the tube $]X_{0,J}[_{Z_J}$ is the generic fiber of a weak formal scheme lifting $X_{0,J}$ to $\mathscr{O}_F^0$. In which case we can appeal to Lemma 2.1.

It follows that a map between two de Rham complexes associated to two (different) embeddings of $X_{0,J}$ is a strict quasi-isomorphism (as $\mathbf{B}^+_{\mathrm{cr}}$ is a Banach space we can use the Open Mapping Theorem). This implies that all the arrows in the system (3.14) are strict quasi-isomorphisms and the cohomology of $\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}}$ is isomorphic to the cohomology of $\Gamma(]X_{0,\bullet}[_{Z_\bullet},\Omega^\bullet)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}}$ for any embedding data. By the above, we have a natural isomorphism

$$\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})\simeq H^i_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}}.\tag{3.16}$$

It is easy to see that it also holds for X_0 quasi-compact (choose a finite affine covering and use the above computations).

Lemma 3.17. *Let X_0 be quasi-compact. The above isomorphism induces a natural isomorphism*

$$\widetilde{H}^i(\mathrm{HK}(X_C,r))\simeq (H^i_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}$$

of Banach space.

[19] In general, in what follows we will use the brackets [ ] to denote derived eigenspaces and the brackets ( ) or nothing to denote the non-derived ones.

Proof. The argument here is similar to the one given in [48, Cor. 3.26] for the Beilinson-Hyodo-Kato cohomology but requires a little bit more care. We note that $H^i_{\rm HK}(X_0)$ is a finite dimensional (φ,N)-module (by Proposition 3.2). Recall that, for a finite (φ,N)-module M, we have the following short exact sequences

$$\begin{aligned}0 \to & M\otimes_F \mathbf{B}^+_{\rm cr} \xrightarrow{\beta} M\otimes_F \mathbf{B}^+_{\rm st} \xrightarrow{N} M\otimes_F \mathbf{B}^+_{\rm st}\to 0,\\ 0\to & (M\otimes_F \mathbf{B}^+_{\rm cr})^{\varphi=p^r}\to M\otimes_F \mathbf{B}^+_{\rm cr}\xrightarrow{p^r-\varphi} M\otimes_F \mathbf{B}^+_{\rm cr}\to 0.\end{aligned} \tag{3.18}$$

The first one follows, by induction on m such that $N^m=0$ on M, from the fundamental exact sequence, i.e., the same sequence for $M=F$. The map β is the (Frobenius equivariant) trivialization map defined as follows

$$\beta: M\otimes \mathbf{B}^+_{\rm cr}\xrightarrow{\sim}(M\otimes \mathbf{B}^+_{\rm st})^{N=0},\quad m\otimes b\mapsto \exp(Nu)m\otimes b. \tag{3.19}$$

We note here that it is not Galois equivariant; however this fact will not be a problem for us in this proof. The second exact sequence follows from [12, Remark 2.30].

We will first show that

$$\widetilde{H}^i([\mathrm{R}\Gamma_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st}]^{N=0})\simeq (H^i_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{N=0}. \tag{3.20}$$

Set $\mathrm{HK}:=\mathrm{R}\Gamma_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st}$. We have the long exact sequence

$$\xrightarrow{N}\widetilde{H}^{i-1}(\mathrm{HK})\to\widetilde{H}^i([\mathrm{HK}]^{N=0})\to\widetilde{H}^i(\mathrm{HK})\xrightarrow{N}\widetilde{H}^i(\mathrm{HK})\to\widetilde{H}^{i+1}([\mathrm{HK}]^{N=0})\to$$

By the isomorphism (3.16) and the exact sequence (3.18), it splits into the short exact sequences

$$0\to\widetilde{H}^i([\mathrm{HK}]^{N=0})\to H^i_{\rm HK}(X_0)\otimes_F\mathbf{B}^+_{\rm st}\xrightarrow{N} H^i_{\rm HK}(X_0)\otimes_F\mathbf{B}^+_{\rm st}\to 0$$

The isomorphism (3.20) follows. By (3.18), we also have $\widetilde{H}^i([\mathrm{HK}]^{N=0})\simeq H^i_{\rm HK}(X_0)\otimes_F\mathbf{B}^+_{\rm cr}$.

Now, set $D:=[\mathrm{HK}]^{N=0}$. We have the long exact sequence

$$\xrightarrow{\varphi-p^r}\widetilde{H}^{i-1}(D)\to\widetilde{H}^i([D]^{\varphi=p^r})\to\widetilde{H}^i(D)\xrightarrow{\varphi-p^r}\widetilde{H}^i(D)\to\widetilde{H}^{i+1}([D]^{\varphi=p^r})\to$$

Since $\widetilde{H}^i(D)\simeq H^i_{\rm HK}(X_0)\otimes_F\mathbf{B}^+_{\rm cr}$, the sequence (3.18) implies that the above long exact sequence splits into the short exact sequences

$$0\to\widetilde{H}^i([D]^{\varphi=p^r})\to H^i_{\rm HK}(X_0)\otimes_F\mathbf{B}^+_{\rm cr}\xrightarrow{\varphi-p^r} H^i_{\rm HK}(X_0)\otimes_F\mathbf{B}^+_{\rm cr}\to 0$$

Our lemma follows. ☐

Assume now that X is Stein and let $\{U_n\}, n\in\mathbf{N}$, be a Stein covering. Arguing as above we get that

$$\mathrm{R}\Gamma_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st}\xrightarrow{\sim}\mathrm{holim}_n(\mathrm{R}\Gamma_{\rm HK}(U_n)\widehat{\otimes}_F\mathbf{B}^+_{\rm st}).$$

It follows that its cohomology is classical and, by Section 2.0.4,

$$\widetilde{H}^i(\mathrm{R}\Gamma_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})\xrightarrow{\sim}\varprojlim_n(H^i_{\rm HK}(U_n)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})\simeq(H^i_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st}) \tag{3.21}$$

since $H^s\,\mathrm{holim}_n(H^i(U_n)\widehat{\otimes}_F\mathbf{B}^+_{\rm cr})=0$, $s\geq 1$, as the system $\{H^i_{\rm HK}(U_n)\}_{n\in\mathbf{N}}$ is Mittag-Leffler.

Lemma 3.22. *The cohomology $\widetilde{H}^i([\mathrm{R}\Gamma_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st}]^{N=0,\varphi=p^r})$ is classical. We have natural isomorphisms*

$$H^i([\mathrm{R}\Gamma_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st}]^{N=0,\varphi=p^r})\simeq(H^i_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^r}.$$

In particular, the space $H^i([\mathrm{R}\Gamma_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st}]^{N=0,\varphi=p^r})$ is Fréchet. Moreover,

$$\widetilde{H}^i([\mathrm{R}\Gamma_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st}]^{N=0})\simeq(H^i_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{N=0}\simeq H^i_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm cr},$$

where the last isomorphism is not, in general, Galois equivariant.

Proof. The first claim follows from the fact that the cohomology $\widetilde{H}^*(\mathrm{R}\Gamma_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})$ is classical. For the second claim, we argue as in the proof of Lemma 3.17 using analogs of the exact sequences (3.18) (for $M = H^i_{\mathrm{HK}}(X_0)$):

$$0 \to H^i_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{cr}} \xrightarrow{\beta} H^i_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}} \xrightarrow{N} H^i_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}} \to 0,$$
$$0 \to (H^i_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{cr}})^{\varphi=p^r} \to H^i_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{cr}} \xrightarrow{p^r-\varphi} H^i_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{cr}} \to 0. \tag{3.23}$$

These sequences are limits of sequences (3.18) applied to $H^i_{\mathrm{HK}}(U_n)$, $n\in\mathbf{N}$. We wrote $\beta := \varprojlim_n \beta_n$.

Finally, the last claim of the lemma is proved while proving the second claim.

□

Example 3.24. Assume that X is affine. In that case $\Omega^i(X_K)$ is an LB-space. The de Rham cohomology $\widetilde{H}^i_{\mathrm{dR}}(X_K)$ is a finite dimensional K-vector space with its natural Hausdorff topology. Computing as in the above example[20], we get

$$\begin{aligned}
\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}} &\simeq \Omega^\bullet(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}} = (\mathscr{O}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}} \to \Omega^1(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}} \to \cdots)\\
F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}}) &\simeq F^r(\Omega^\bullet(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}}) = (\mathscr{O}(X_K)\widehat{\otimes}_K F^r\mathbf{B}^+_{\mathrm{dR}} \to \Omega^1(X_K)\widehat{\otimes}_K F^{r-1}\mathbf{B}^+_{\mathrm{dR}} \to \cdots)\\
\mathrm{DR}(X_C,r) &= (\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r \simeq (\Omega^\bullet(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r\\
&= (\mathscr{O}(X_K)\widehat{\otimes}_K(\mathbf{B}^+_{\mathrm{dR}}/F^r) \to \Omega^1(X_K)\widehat{\otimes}_K(\mathbf{B}^+_{\mathrm{dR}}/F^{r-1}) \to \cdots \to \Omega^{r-1}(X_K)\widehat{\otimes}_K(\mathbf{B}^+_{\mathrm{dR}}/F^1))
\end{aligned}$$

In low degrees we have

$$\begin{aligned}
&\mathrm{DR}(X_C,0) = 0, \quad \mathrm{DR}(X_C,1) \simeq \mathscr{O}(X_K)\widehat{\otimes}_K C,\\
&\mathrm{DR}(X_C,2) \simeq (\mathscr{O}(X_K)\widehat{\otimes}_K(\mathbf{B}^+_{\mathrm{dR}}/F^2) \to \Omega^1(X_K)\widehat{\otimes}_K C).
\end{aligned}$$

The cohomology $\widetilde{H}^i\mathrm{DR}(X_C,r)$ is classical and, since $\mathbf{B}^+_{\mathrm{dR}}/F^i$ is a Banach space, it is an LB-space; for $i\geq r$, $\widetilde{H}^i\mathrm{DR}(X_C,r)=0$.

Assume now that X is Stein. Recall that the de Rham complex is built from Fréchet spaces. Since the de Rham differentials are strict and have closed images, the de Rham cohomology $\widetilde{H}^i_{\mathrm{dR}}(X_K)$ is classical and Fréchet as well. Since $\mathbf{B}^+_{\mathrm{dR}}$ is a Fréchet space and $\mathbf{B}^+_{\mathrm{dR}}/F^i$ is a Banach space, we can use Section 2.0.4 to conclude that the above computation works for Stein weak formal schemes as well. In particular, the cohomology $\widetilde{H}^i\mathrm{DR}(X_C,r)$ is classical and Fréchet as well.

Let X be affine or Stein. We can conclude from the above that our syntomic cohomology fits into the long exact sequence

$$\to \widetilde{H}^{i-1}\mathrm{DR}(X_C,r) \xrightarrow{\partial} \widetilde{H}^i_{\mathrm{syn}}(X_C,\mathbf{Q}_p(r)) \to (\widetilde{H}^i_{\mathrm{HK}}(X_0)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \xrightarrow{\iota_{\mathrm{HK}}\otimes\iota} \widetilde{H}^i\mathrm{DR}(X_C,r) \to$$

where all the terms but the syntomic one were shown to be classical and LB or Fréchet, respectively. It follows that the syntomic cohomology is also classical and we will show later that it has the remaining properties as well.

Example 3.25. Assume that X is affine or Stein and geometrically irreducible. For $r=0$, from the above computations (we note that $\mathrm{DR}(X_C,0)$ is strictly exact), we obtain the continuous bijection

$$H^0_{\mathrm{syn}}(X_C,\mathbf{Q}_p) \xrightarrow{\sim} (H^0_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})^{\varphi=p,N=0} \simeq \mathbf{B}^{+,\varphi=1}_{\mathrm{cr}} = \mathbf{Q}_p.$$

Since $\mathbf{Q}_p$ is separated (and clearly finite dimensional over $\mathbf{Q}_p$) this is an isomorphism.

For $r=1$, we obtain the following strictly exact sequence

$$H^0\mathrm{HK}(X_C,1) \xrightarrow{\iota_{\mathrm{HK}}\otimes\iota} \mathscr{O}_{X_K}\widehat{\otimes}_K C \to H^1_{\mathrm{syn}}(X_C,\mathbf{Q}_p(1)) \to (H^1_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})^{\varphi=p,N=0} \to 0$$

Since $H^0_{\mathrm{HK}}(X_0)=F$, we have $H^0\mathrm{HK}(X_C,1)=\mathbf{B}^{+,\varphi=p}_{\mathrm{cr}}$ and the map $H^0\mathrm{HK}(X_C,1)\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}\mathscr{O}_{X_K}\widehat{\otimes}_K C$ is induced by the map $\iota:\mathbf{B}^+_{\mathrm{cr}}\to\mathbf{B}^+_{\mathrm{dR}}/F^1$. Since we have the fundamental sequence

$$0 \to \mathbf{Q}_p(1) \to \mathbf{B}^{+,\varphi=p}_{\mathrm{cr}} \to \mathbf{B}^+_{\mathrm{dR}}/F^1 \to 0$$

[20]The computations are actually simpler because, unlike $\mathbf{B}^+_{\mathrm{st}}$, $\mathbf{B}^+_{\mathrm{dR}}$ is a Fréchet space.

we get the strictly exact sequence

$$0\to C\to \mathscr{O}(X_K)\widehat{\otimes}_K C\xrightarrow{\partial} H^1_{\rm syn}(X_C,\mathbf{Q}_p(1))\to (H^1_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{\varphi=p,N=0}\to 0$$

3.2.2. *Fundamental diagram.* We will construct the fundamental diagram that syntomic cohomology fits into. We start with an example.

Example 3.26. *Fundamental diagram; the case of* $r=1$. Assume that X is affine or Stein and geometrically irreducible. We claim that we have the following commutative diagram with strictly exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathscr{O}(X_K)\widehat{\otimes}_K C)/C & \xrightarrow{\partial} & H^1_{\rm syn}(X_C,\mathbf{Q}_p(1)) & \longrightarrow & (H^1_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{\varphi=p,N=0} & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\iota_{\rm HK}\otimes\theta} & & \\
0 & \longrightarrow & (\mathscr{O}(X_K)\widehat{\otimes}_K C)/C & \xrightarrow{d} & \Omega^1(X_K)_{d=0}\widehat{\otimes}_K C & \longrightarrow & H^1_{\rm dR}(X_K)\widehat{\otimes}_K C & \longrightarrow & 0
\end{array}$$

The top row is the strictly exact sequence from the Example 3.2.1. The bottom row is induced by the natural sequence defining $H^1_{\rm dR}(X_K)$. The map $\iota_{\rm HK}\otimes\iota$ is induced by the composition

$$\mathrm{R}\Gamma_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st}\xrightarrow{\iota_{\rm HK}\otimes\iota}\mathrm{R}\Gamma_{\rm dR}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\rm dR}\xrightarrow{\theta}\mathrm{R}\Gamma_{\rm dR}(X_K)\widehat{\otimes}_K C.$$

The map β is induced by the composition (the fact that the first map lands in F^1 is immediate from the definition of $\mathrm{R}\Gamma_{\rm syn}(X_C,\mathbf{Q}_p(1))$)

$$\mathrm{R}\Gamma_{\rm syn}(X_C,\mathbf{Q}_p(1))\to F^1(\mathrm{R}\Gamma_{\rm dR}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\rm dR})\xrightarrow{\theta}(0\to\Omega^1(X_K)\widehat{\otimes}_K C\to\Omega^2(X_K)\widehat{\otimes}_K C\to\cdots).$$

Clearly they make the right square in the above diagram commute. To see that the left square commutes as well it is best to consider the following diagram of maps of distinguished triangles

$$\begin{array}{ccccc}
\mathrm{R}\Gamma_{\rm syn}(X_C,\mathbf{Q}_p(1)) & \longrightarrow & [\mathrm{R}\Gamma_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st}]^{\varphi=p,N=0} & \xrightarrow{\iota_{\rm HK}\otimes\iota} & (\mathrm{R}\Gamma_{\rm dR}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\rm dR})/F^1\\
\downarrow{\scriptstyle\tilde{\beta}} & & \downarrow{\scriptstyle\iota_{\rm HK}\otimes\iota} & & \|\\
F^1(\mathrm{R}\Gamma_{\rm dR}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\rm dR}) & \longrightarrow & \mathrm{R}\Gamma_{\rm dR}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\rm dR} & \longrightarrow & (\mathrm{R}\Gamma_{\rm dR}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\rm dR})/F^1\\
\downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta}\\
\Omega^{\geq 1}(X_K)\widehat{\otimes}_K C[-1] & \longrightarrow & \Omega^{\bullet}(X_K)\widehat{\otimes}_K C & \longrightarrow & \mathscr{O}(X_K)\widehat{\otimes}_K C
\end{array}$$

The map $\tilde{\beta}$ is the map on mapping fibers induced by the commutative right square. We have $\beta=\theta\tilde{\beta}$. It remains to check that the map $\mathscr{O}(X_K)\widehat{\otimes}_K C\to\Omega^1(X_K)\widehat{\otimes}_K C$ induced from the bottom row of the above diagram is equal to d but this is easy.

And here is the general case.

Proposition 3.27. *Let X be an affine or a Stein weak formal scheme. Let $r\geq 0$. There is a natural map of strictly exact sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\Omega^{r-1}(X_K)/\ker d)\widehat{\otimes}_K C & \xrightarrow{\partial} & H^r_{\rm syn}(X_C,\mathbf{Q}_p(r)) & \longrightarrow & (H^r_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^r} & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\iota_{\rm HK}\otimes\theta} & & \\
0 & \longrightarrow & (\Omega^{r-1}(X_K)/\ker d)\widehat{\otimes}_K C & \xrightarrow{d} & \Omega^r(X_K)_{d=0}\widehat{\otimes}_K C & \longrightarrow & H^r_{\rm dR}(X_K)\widehat{\otimes}_K C & \longrightarrow & 0
\end{array}$$

Moreover, $\ker(\iota_{\rm HK}\otimes\theta)\simeq(H^r_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^{r-1}}$, $H^r_{\rm syn}(X_C,\mathbf{Q}_p(r))$ *is LB or Fréchet, respectively, and the maps* β, $\iota_{\rm HK}\otimes\theta$ *are strict and have closed images.*

Proof. The following map of strictly exact sequences (where Ω^i, $H^i_{\rm dR}$ and $H^i_{\rm HK}$ stand for $\Omega^i(X_K)$, $H^i_{\rm dR}(X_K)$ and $H^i_{\rm HK}(X_0)$ respectively) is constructed in an analogous way to the case of $r=1$ treated in the above example

$$\begin{array}{ccccccccc}
(H^{r-1}_{\rm HK}\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^r} & \longrightarrow & (\Omega^{r-1}/d\Omega^{r-2})\widehat{\otimes}_K C & \xrightarrow{\partial} & H^r_{\rm syn}(X_C,\mathbf{Q}_p(r)) & \longrightarrow & (H^r_{\rm HK}\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^r} & \to 0\\
\downarrow{\scriptstyle \iota_{\rm HK}\otimes\theta} & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle \iota_{\rm HK}\otimes\theta} & \\
0\to H^{r-1}_{\rm dR}\widehat{\otimes}_K C & \longrightarrow & (\Omega^{r-1}/d\Omega^{r-2})\widehat{\otimes}_K C & \xrightarrow{d} & \Omega^r_{d=0}\widehat{\otimes}_K C & \xrightarrow{\pi} & H^r_{\rm dR}\widehat{\otimes}_K C & \to 0
\end{array}$$

To prove the first claim of the proposition it suffices to show that the map $\iota_{\rm HK}\otimes\theta$ in degree $r-1$ is surjective. For that we will need the following lemma.

Lemma 3.28. *Let M be an effective*[21] *finite (φ,N)-module over F. The sequence*

$$0\to(M\otimes_F\mathbf{B}^+_{\rm st})^{\varphi=p^j,N=0}\xrightarrow{t}(M\otimes_F\mathbf{B}^+_{\rm st})^{\varphi=p^{j+1},N=0}\xrightarrow{1\otimes\theta}M\otimes_F C$$

is exact. Moreover, the right arrow is a surjection if the slopes of Frobenius are $\leq j$.

Proof. Using the trivialization (3.19) and the fact that $\theta(u)=0$, we get the following commutative diagram

$$\begin{array}{ccccccc}
0\longrightarrow & (M\otimes_F\mathbf{B}^+_{\rm st})^{\varphi=p^j,N=0} & \xrightarrow{t} & (M\otimes_F\mathbf{B}^+_{\rm st})^{\varphi=p^{j+1},N=0} & \xrightarrow{1\otimes\theta} & M\otimes_F C\\
 & \uparrow{\scriptstyle\beta}\wr & & \uparrow{\scriptstyle\beta}\wr & & \uparrow{\scriptstyle \rm Id}\wr\\
0\longrightarrow & (M\otimes_F\mathbf{B}^+_{\rm cr})^{\varphi=p^j} & \xrightarrow{t} & (M\otimes_F\mathbf{B}^+_{\rm cr})^{\varphi=p^{j+1}} & \xrightarrow{1\otimes\theta} & M\otimes_F C
\end{array}$$

Hence it suffices to prove the analog of our lemma for the bottom sequence.

First we will show that the following sequence

$$0\to(M\otimes_F\mathbf{B}^+_{\rm cr})^{\varphi=p^j}\xrightarrow{t}(M\otimes_F\mathbf{B}^+_{\rm cr})^{\varphi=p^{j+1}}\xrightarrow{1\otimes\theta}M\otimes_F C \tag{3.29}$$

is exact. Multiplication by t is clearly injective. To show exactness in the middle it suffices to show that

$$(M\otimes_F F^1\mathbf{B}^+_{\rm cr})^{\varphi=p^{j+1}}=(M\otimes_F t\mathbf{B}^+_{\rm cr})^{\varphi=p^{j+1}},$$

where $F^1\mathbf{B}^+_{\rm cr}:=\mathbf{B}^+_{\rm cr}\cap F^1\mathbf{B}^+_{\rm dR}$. Or that $(F^1\mathbf{B}^+_{\rm cr})^{\varphi=p^{j+1-\alpha}}=(t\mathbf{B}^+_{\rm cr})^{\varphi=p^{j+1-\alpha}}$. But this follows from the fact that

$$(F^1\mathbf{B}^+_{\rm cr})^{\varphi=p^{j+1-\alpha}}\subset\{x\in\mathbf{B}^+_{\rm cr}|\theta(\varphi^k(x))=0,\forall k\geq 0\}=t\mathbf{B}^+_{\rm cr}.$$

It remains to show that if the Frobenius slopes of M are $\leq j$ then the last arrow in the sequence (3.29) is a surjection. To see this, we note that all the terms in the sequence are C-points of finite dimensional BC spaces[22] and the maps can be lifted to maps of such spaces. It follows that the cokernel of multiplication by t is a finite dimensional BC space. We compute its Dimension [12, 5.2.2]:

$$\begin{aligned}
{\rm Dim}(M\otimes_F\mathbf{B}^+_{\rm cr})^{\varphi=p^{j+1}}-{\rm Dim}(M\otimes_F\mathbf{B}^+_{\rm cr})^{\varphi=p^j} &=\sum_{r_i\leq j+1}(j+1-r_i,1)-\sum_{r_i\leq j}(j-r_i,1)\\
&=((j+1)\dim_F M-t_N(M),\dim_F M)-(j\dim_F M-t_N(M),\dim_F M)=(\dim_F M,0).
\end{aligned}$$

Here r_i's are the slopes of Frobenius repeated with multiplicity, $t_N(M)=v_p(\det\varphi)$, and the second equality follows from the fact that the slopes of Frobenius are $\leq j$. Since this Dimension is the same as the Dimension of the BC space corresponding to $M\otimes_F C$, we get the surjection we wanted. □

Assume first that X is quasi-compact. By the above lemma, to prove that the map $\iota_{\rm HK}\otimes\theta$ is surjective in degree $r-1$ and that its kernel in degree r is isomorphic to $(H^r_{\rm HK}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st})^{\varphi=p^{r-1},N=0}$ it suffices to show that the slopes of Frobenius on $H^i_{\rm HK}(X_0)$ are $\leq i$. For that we use the weight spectral sequence (3.3) to reduce to showing that, for a smooth scheme Y over k, the slopes of Frobenius on the (classical) rigid cohomology $H^j_{\rm rig}(Y/F)$ are $\leq j$. But this is well-known [8, Théorème 3.1.2].

[21] We call a (φ,N)-module M *effective* if all the slopes of the Fobenius are ≥ 0.

[22] Which are called finite dimensional Banach Spaces in [9] and Banach-Colmez spaces in most of the literature.

Since the syntomic cohomology is the equalizer of the maps $\iota_{\rm HK}\otimes\theta$, whose target and domain are Banach, and π, whose domain is LB, it is an LB-space as well. Since the map $\iota_{\rm HK}\otimes\theta$ lifts to a map of finite Dimensional BC spaces it is strict and has a closed image. It follows that so does the pullback map β.

Assume now that X is Stein with a Stein covering $\{U_i\}, i\in{\bf N}$. Since the map $\iota_{\rm HK}\otimes\theta$ is the projective limit of the maps

$$(\iota_{\rm HK}\otimes\theta)_i:(H^*_{\rm HK}(U_i)\widehat{\otimes}_F{\bf B}^+_{\rm st})^{N=0,\varphi=p^r}\to H^*_{\rm dR}(]U_i[_X)\widehat{\otimes}_KC$$

the computation above yields the statement on the kernel in degree r. For the surjectivity in degree $r-1$, it remains to show the vanishing of $H^1\operatorname{holim}_i(H^r_{\rm HK}(U_i)\widehat{\otimes}_F{\bf B}^+_{\rm st})^{N=0,\varphi=p^{r-1}}$. But the system $\{(H^r_{\rm HK}(U_i)\widehat{\otimes}_F{\bf B}^+_{\rm st})^{N=0,\varphi=p^{r-1}}\}, i\in{\bf N}$, is Mittag-Leffler. Indeed, it can be lifted to a system of finite dimensional BC spaces with Dimensions (d_i,h_i), $d_i,h_i\geq 0$ [9, Prop. 10.6]. The images of the terms in the system in a fixed BC space form a chain with decreasing dimensions D (in lexicographical order). Since the height h of any BC subspace of these spaces is also ≥ 0 [10, Lemma 2.6], they stabilize, as wanted.

Since the maps $(\iota_{\rm HK}\otimes\theta)_i$ are strict and have closed images it follows that the projective limit map $\iota_{\rm HK}\otimes\theta$ inherits these properties and then so does the pullback map β. Finally, since the syntomic cohomology is the equalizer of the maps $\iota_{\rm HK}\otimes\theta$ and π of Fréchet spaces it is Fréchet. □

Remark 3.30. Assume that X is affine. The image of the map $\iota_{\rm HK}\otimes\theta$ in degree r in the fundamental diagram is the C-points of a finite dimensional BC space that is the cokernel of the map:

$$(H^r_{\rm HK}(X_0)\otimes_F\mathbb{B}^+_{\rm st})^{N=0,\varphi=p^{r-1}}\xrightarrow{t}(H^r_{\rm HK}(X_0)\otimes_F\mathbb{B}^+_{\rm st})^{N=0,\varphi=p^r}.$$

Its Dimension is equal to

$$\sum_{r_i\leq r}(r-r_i,1)-\sum_{r_i\leq r-1}(r-1-r_i,1)=\sum_{r_i\leq r-1}(1,0)+\sum_{r-1<r_i\leq r}(r-r_i,1).$$

3.3. **Crystalline syntomic cohomology.** The classical crystalline syntomic cohomology of Fontaine-Messing and the related period map to étale cohomology generalize easily to formal schemes. We define them and then modify this syntomic cohomology in the spirit of Bloch-Kato to make it more computable.

3.3.1. *Definition a la Fontaine-Messing.* Let X be a semistable p-adic formal scheme over $\mathscr{O}_K$. This means that, locally for the Zariski topology, $X=\operatorname{Spf}(R)$, where R is the p-adic completion of an algebra étale over $\mathscr{O}_K\{T_1,\ldots,T_n\}/(T_1\cdots T_m-\varpi)$. We equip X with the log-structure induced by the special fiber. Set $\overline{X}:=X_{\mathscr{O}_C}$.

For $r\geq 0$, we have the *geometric syntomic cohomology* of Fontaine-Messing [17]

$$\mathrm{R}\Gamma_{\rm syn}(\overline{X},{\bf Z}/p^n(r)):=[F^r\mathrm{R}\Gamma_{\rm cr}(\overline{X}_n)\xrightarrow{\varphi-p^r}\mathrm{R}\Gamma_{\rm cr}(\overline{X}_n)],\quad F^r\mathrm{R}\Gamma_{\rm cr}(\overline{X}_n):=\mathrm{R}\Gamma_{\rm cr}(\overline{X}_n,\mathscr{J}^{[r]})$$
$$\mathrm{R}\Gamma_{\rm syn}(\overline{X},{\bf Z}_p(r)):=\operatorname{holim}_n\mathrm{R}\Gamma_{\rm syn}(\overline{X},{\bf Z}/p^n(r)).$$

Crystalline cohomology used here is the absolute one, i.e., over $\mathscr{O}_{F,n}$ and $\mathscr{J}^{[r]}$ is the r'th level of the crystalline Hodge filtration sheaf. We have

$$\begin{aligned}\mathrm{R}\Gamma_{\rm syn}(\overline{X},{\bf Z}_p(r))_{\bf Q}&=[F^r\mathrm{R}\Gamma_{\rm cr}(\overline{X})_{\bf Q}\xrightarrow{\varphi-p^r}\mathrm{R}\Gamma_{\rm cr}(\overline{X})_{\bf Q}]\\&=[\mathrm{R}\Gamma_{\rm cr}(\overline{X})_{\bf Q}^{\varphi=p^r}\to\mathrm{R}\Gamma_{\rm cr}(\overline{X})_{\bf Q}/F^r]\end{aligned}\tag{3.31}$$

and similarly with ${\bf Z}_p$ and ${\bf Z}/p^n$ coefficients.

The above geometric syntomic cohomology is related, via period morphisms, to the étale cohomology of the rigid space X_C (see below) and hence allows to describe the latter using differential forms. To achieve the same for pro-étale cohomology, we need to modify the definition of syntomic cohomology a bit. Consider the complexes of sheaves on X associated to the presheaves (U is an affine Zariski open in X and $\overline{U}:=U_{\mathscr{O}_C}$)

$$\mathscr{A}_{\rm cr}:=(U\mapsto(\operatorname{holim}_n\mathrm{R}\Gamma_{\rm cr}(\overline{U}_n))_{\bf Q}),\quad F^r\mathscr{A}_{\rm cr}:=(U\mapsto(\operatorname{holim}_nF^rR\Gamma_{\rm cr}(\overline{U}_n))_{\bf Q}),$$
$$\mathscr{S}(r):=(U\mapsto\mathrm{R}\Gamma_{\rm syn}(\overline{U},{\bf Z}_p(r))_{\bf Q}).$$

We have

$$\mathscr{S}(r) = [F^r\mathscr{A}_{\mathrm{cr}} \xrightarrow{\varphi - p^r} \mathscr{A}_{\mathrm{cr}}] = [\mathscr{A}_{\mathrm{cr}}^{\varphi = p^r} \to \mathscr{A}_{\mathrm{cr}}/F^r].$$

We define

$$\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}) := \mathrm{R}\Gamma(X, \mathscr{A}_{\mathrm{cr}}),\quad F^r\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}) := \mathrm{R}\Gamma(X, F^r\mathscr{A}_{\mathrm{cr}}),\quad \mathrm{R}\Gamma_{\mathrm{syn}}(\overline{X}, \mathbf{Q}_p(r)) := \mathrm{R}\Gamma(X, \mathscr{S}(r)).$$

Hence

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\overline{X}, \mathbf{Q}_p(r)) = [F^r\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X}) \xrightarrow{\varphi - p^r} \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X})] = [\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X})^{\varphi = p^r} \to \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{X})/F^r]. \tag{3.32}$$

There is a natural map

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\overline{X}, \mathbf{Z}_p(r))_{\mathbf{Q}} \to \mathrm{R}\Gamma_{\mathrm{syn}}(\overline{X}, \mathbf{Q}_p(r)). \tag{3.33}$$

It is a quasi-isomorphism in the case X is of finite type but not in general (since in the case of $\mathbf{Z}_p(r)$ we do all computations on U's as above integrally and invert p at the very end and in the case of $\mathbf{Q}_p(r)$ we invert p already on each U).

By proceeding just as in the case of overconvergent syntomic cohomology (and using crystalline embedding systems instead of dagger ones) we can equip both complexes in (3.33) with a natural topology for which they become complexes of Banach spaces over $\mathbf{Q}_p$ in the case X is quasi-compact [23].

The defining mapping fibers (3.31) and (3.33) can be taken in $\mathscr{D}^b(C_{\mathbf{Q}_p})$. Moreover, the change of topology map in (3.33) is continuous (and a strict quasi-isomorphism if X is of finite type).

3.3.2. *Period map.* We are interested in syntomic cohomology a la Fontaine-Messing because of the following comparison [66, 12].

Proposition 3.34. *Let X be a semistable finite type formal scheme over $\mathscr{O}_K$. The Fontaine-Messing period map* [17]

$$\alpha_{\mathrm{FM}} : \mathrm{R}\Gamma_{\mathrm{syn}}(\overline{X}, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_C, \mathbf{Q}_p(r))$$

is a quasi-isomorphism after truncation $\tau_{\leq r}$.

We equip the pro-étale and étale cohomologies $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))$, and $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_C, \mathbf{Q}_p(r))$ with a natural topology by proceeding as in the case of overconvergent rigid cohomology by using as local data compatible $\mathbf{Z}/p^n$-free complexes[24]. If X is quasi-compact, we obtain in this way complexes of Banach spaces over $\mathbf{Q}_p$.

Corollary 3.35. *Let X be a semistable formal scheme over $\mathscr{O}_K$. There is a natural Fontaine-Messing period map*

$$\alpha_{\mathrm{FM}} : \mathrm{R}\Gamma_{\mathrm{syn}}(\overline{X}, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r)) \tag{3.36}$$

that is a strict quasi-isomorphism after truncation $\tau_{\leq r}$.

Proof. Cover X with quasi-compact formal schemes and invoke Proposition 3.34; we obtain a quasi-isomorphism

$$\alpha_{\mathrm{FM}} : \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(\overline{X}, \mathbf{Q}_p(r)) \to \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_C, \mathbf{Q}_p(r)).$$

To see that it is strict, recall that the period map in Proposition 3.34 is actually defined modulo p^n and it is a p^N-quasi-isomorphism for a universal constant N, i.e, the kernel and cokernel on cohomology are annihilated by p^N. This passes to holim_n and implies that, on cohomology, there is a p^N-inverse (possibly different universal constant N) to the period map. Hence the strict isomorphism after inverting p we wanted.

It remains to show that, for a quasi-compact X, the natural map

$$\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_C, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))$$

[23] We note that $\mathscr{O}_K$ being syntomic over $\mathscr{O}_F$, all the integral complexes in sight are in fact p-torsion free.

[24] Such complexes can be found, for example, by taking the system of étale hypercovers.

is a (strict) quasi-isomorphism. From [62, Cor. 3.17] we know that this is true with $\mathbf{Z}/p^n$-coefficients. This implies that we have a sequence of quasi-isomorphisms

$$\begin{aligned}\mathrm{R}\Gamma_{\text{ét}}(X_C,\mathbf{Z}_p(r)) &= \operatorname{holim}_n \mathrm{R}\Gamma_{\text{ét}}(X_C,\mathbf{Z}/p^n(r)) \simeq \operatorname{holim}_n \mathrm{R}\Gamma_{\text{proét}}(X_C,\mathbf{Z}/p^n(r))\\ &\simeq \mathrm{R}\Gamma_{\text{proét}}(X_C,\operatorname{holim}_n \mathbf{Z}/p^n(r)) \simeq \mathrm{R}\Gamma_{\text{proét}}(X_C,\varprojlim_n \mathbf{Z}/p^n(r)) = \mathrm{R}\Gamma_{\text{proét}}(X_C,\mathbf{Z}_p(r)),\end{aligned}$$

where the third quasi-isomorphism follows from the fact that $\mathrm{R}\Gamma$ and holim commute and the fourth one follows from [62, Prop. 8.2].

It remains to show that

$$\mathrm{R}\Gamma_{\text{proét}}(X_C,\mathbf{Z}_p(r))\otimes\mathbf{Q} \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{proét}}(X_C,\mathbf{Q}_p(r)).$$

But, since $|X_C|$ is quasi-compact, the site $X_{C,\text{proét}}$ is coherent [62, Prop. 3.12]. Hence its cohomology commutes with colimits of abelian sheaves, yielding the above quasi-isomorphism. □

3.3.3. *Period rings* $\widehat{\mathbf{B}}^+_{\rm st}$, $\mathbf{A}_{{\rm cr},K}$. We denote by r^+_ϖ the algebra $\mathscr{O}_F[[T]]$ with the log-structure associated to T. Sending T to ϖ induces a surjective morphism $r^+_\varpi\to\mathscr{O}_K^\times$. We denote by $r^{\rm PD}_\varpi$ the p-adic divided power envelope of r^+_ϖ with respect to the kernel of this morphism. Frobenius is defined by $T\mapsto T^p$, monodromy by $T\mapsto T$.

We will recall the definition of the ring of periods $\widehat{\mathbf{B}}^+_{\rm st}$ [66, p.253]. Let

$$\widehat{\mathbf{A}}_{{\rm st},n} := H^0_{\rm cr}(\mathscr{O}^\times_{\overline{K},n}/r^{\rm PD}_{\varpi,n}),\quad \widehat{\mathbf{A}}_{\rm st} := \varprojlim_n \widehat{\mathbf{A}}_{{\rm st},n},\quad \widehat{\mathbf{B}}^+_{\rm st} := \widehat{\mathbf{A}}_{\rm st}[1/p].$$

We note that $\widehat{\mathbf{B}}^+_{\rm st}$ is a Banach space over F (which makes it easier to handle topologically than $\mathbf{B}^+_{\rm st}$). The ring $\widehat{\mathbf{A}}_{{\rm st},n}$ has a natural action of $\mathscr{G}_K$, Frobenius φ, and a monodromy operator N. We have a morphism $\mathbf{A}_{{\rm cr},n}\to\widehat{\mathbf{A}}_{{\rm st},n}$ induced by the map $H^0_{\rm cr}(\mathscr{O}_{C,n}/\mathscr{O}_{F,n})\to H^0_{\rm cr}(\mathscr{O}^\times_{C,n}/r^{\rm PD}_{\varpi,n})$. Both it and the natural map $r^{\rm PD}_{\varpi,n}\to\widehat{\mathbf{A}}_{{\rm st},n}$ are compatible with all the structures (Frobenius, monodromy, and Galois action). Moreover, we have the exact sequence

$$0\to\mathbf{A}_{{\rm cr},n}\to\widehat{\mathbf{A}}_{{\rm st},n}\xrightarrow{N}\widehat{\mathbf{A}}_{{\rm st},n}\to 0. \tag{3.37}$$

We can view $\widehat{\mathbf{A}}_{{\rm st},n}$ as the ring of the PD-envelope of the closed immersion

$$\operatorname{Spec}\mathscr{O}^\times_{C,n}\hookrightarrow\operatorname{Spec}(\mathbf{A}^\times_{{\rm cr},n}\otimes_{\mathscr{O}_{F,n}} r^+_{\varpi,n})$$

defined by the maps $\theta:\mathbf{A}_{{\rm cr},n}\to\mathscr{O}_{C,n}$ and $r^+_{\varpi,n}\to\mathscr{O}_{K,n}$, $T\mapsto\varpi$. Here $\mathbf{A}^\times_{{\rm cr},n}$ is $\mathbf{A}_{{\rm cr},n}$ equipped with the unique log-structure extending the one on $\mathscr{O}^\times_{C,n}$. This makes $\operatorname{Spec}\mathscr{O}^\times_{\overline{K},1}\hookrightarrow\operatorname{Spec}\widehat{\mathbf{A}}_{{\rm st},n}$ into a PD-thickening in the crystalline site of $\mathscr{O}^\times_{\overline{K},1}$. It follows [36, Sec. 3.9] that

$$\widehat{\mathbf{A}}_{{\rm st},n}\simeq\mathrm{R}\Gamma_{\rm cr}(\mathscr{O}^\times_{C,n}/r^{\rm PD}_{\varpi,n}). \tag{3.38}$$

There is a canonical $\mathbf{B}^+_{\rm cr}$-linear isomorphism $\mathbf{B}^+_{\rm st}\xrightarrow{\sim}\widehat{\mathbf{B}}^{+,N-{\rm nilp}}_{\rm st}$ compatible [36, Theorem 3.7] with the action of $\mathscr{G}_K$, φ, and N.

We will now pass to the definition of the ring of periods $\mathbf{A}_{{\rm cr},K}$ [66, 4.6]. Let

$$\mathbf{A}_{{\rm cr},K,n} := H^0_{\rm cr}(\mathscr{O}^\times_{\overline{K},n}/\mathscr{O}^\times_{K,n}),\quad \mathbf{A}_{{\rm cr},K} := \varprojlim_n \mathbf{A}_{{\rm cr},K,,n}.$$

The ring $\mathbf{A}_{{\rm cr},K,n}$ is a flat $\mathbf{Z}/p^n$-module and $\mathbf{A}_{{\rm cr},K,n+1}\otimes\mathbf{Z}/p^n\simeq\mathbf{A}_{{\rm cr},K,n}$; moreover, it has a natural action of $\mathscr{G}_K$. These properties generalize to $H^0_{\rm cr}(\mathscr{O}^\times_{\overline{K},n}/\mathscr{O}^\times_{K,n},\mathscr{J}^{[r]})$, for $r\in\mathbf{Z}$, and we have $H^i_{\rm cr}(\mathscr{O}^\times_{\overline{K},n}/\mathscr{O}^\times_{K,n},\mathscr{J}^{[r]})=0$, $i\geq 1$, $r\in\mathbf{Z}$. Set

$$F^r\mathbf{A}_{{\rm cr},K,n} := H^0_{\rm cr}(\mathscr{O}^\times_{\overline{K},n}/\mathscr{O}^\times_{K,n},\mathscr{J}^{[r]}),\quad F^r\mathbf{A}_{{\rm cr},K} := \varprojlim_n F^r\mathbf{A}_{{\rm cr},K,n}.$$

We have

$$F^r\mathbf{A}_{{\rm cr},K,n}\simeq\mathrm{R}\Gamma_{\rm cr}(\mathscr{O}^\times_{K,n},\mathscr{J}^{[r]}),\quad F^r\mathbf{A}_{{\rm cr},K,n}/F^s\simeq\mathrm{R}\Gamma_{\rm cr}(\mathscr{O}^\times_{K,n},\mathscr{J}^{[r]}/\mathscr{J}^{[s]}),\quad r\leq s.$$

The natural map $\mathrm{gr}^r_F\,\mathbf{A}_{\mathrm{cr},n}\to\mathrm{gr}^r_F\,\mathbf{A}_{\mathrm{cr},K,n}$ is a p^a-quasi-isomorphism for a universal constant a [66, Lemma 4.6.2]. There is a natural $\mathscr{G}_K$-equivariant map $\iota:\widehat{\mathbf{B}}^+_{\mathrm{st}}\to\mathbf{B}^+_{\mathrm{dR}}$ induced by the maps

$$p_\varpi:\widehat{\mathbf{B}}^+_{\mathrm{st}}\to\mathbf{A}_{\mathrm{cr},K},\quad (\mathbf{A}_{\mathrm{cr}}/F^r)_{\mathbf{Q}}\xrightarrow{\sim}\mathbf{A}_{\mathrm{cr},K}/F^r,$$

where p_ϖ denotes the map induced by sending $T\mapsto\varpi$.

3.3.4. *Definition a la Bloch-Kato.* Crystalline geometric syntomic cohomology a la Fontaine-Messing can be often described in a very simple way using filtered de Rham complexes and Hyodo-Kato cohomology (if that one can be defined) and the period rings $\mathbf{B}^+_{\mathrm{st}},\mathbf{B}^+_{\mathrm{dR}}$.This was done for proper algebraic and analytic varieties in [48, 12]. In this section we adapt the arguments from loc. cit. to the case of some non-quasi-compact rigid varieties. The de Rham term is the same, the Hyodo-Kato term is more complicated, and the role of the period ring $\mathbf{B}^+_{\mathrm{st}}$ is played by $\widehat{\mathbf{B}}^+_{\mathrm{st}}$.

Let $r\geq 0$. For a semistable formal scheme X over $\mathscr{O}_K$, we define the *crystalline geometric syntomic cohomology a la Bloch-Kato* (as an object in $\mathscr{D}^b(C_{\mathbf{Q}_p})$)

$$\mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(X_{\mathscr{O}_C},\mathbf{Q}_p(r)):=[[\mathrm{R}\Gamma_{\mathrm{cr}}(X/r^{\mathrm{PD}}_\varpi)\widehat{\otimes}_{r^{\mathrm{PD}}_{\varpi,\mathbf{Q}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}\xrightarrow{p_\varpi\otimes\iota}(\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r].$$

Here $\mathrm{R}\Gamma_{\mathrm{cr}}(X/r^{\mathrm{PD}}_\varpi)$ is defined in analogous way to $\mathrm{R}\Gamma_{\mathrm{cr}}(X)$ (hence it is rational), the completed tensor products are defined by the procedure we have used in Section 3.2.1.

Proposition 3.39. *There exists a functorial quasi-isomorphism in* $\mathscr{D}^b(C_{\mathbf{Q}_p})$

$$\iota_{\mathrm{BK}}:\mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(X_{\mathscr{O}_C},\mathbf{Q}_p(r))\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_C},\mathbf{Q}_p(r)).$$

Proof. The comparison map ι_{BK} will be induced by a pair of maps $(\iota^1_{\mathrm{BK}},\iota^2_{\mathrm{BK}})$, basically Künneth cup product maps, that make the following diagram commute (in $\mathscr{D}^b(C_{\mathbf{Q}_p})$).

$$\begin{array}{ccc}[\mathrm{R}\Gamma_{\mathrm{cr}}(X/r^{\mathrm{PD}}_\varpi)\widehat{\otimes}_{r^{\mathrm{PD}}_{\varpi,\mathbf{Q}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0} & \xrightarrow{p_\varpi\otimes\iota} & (\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r\\ \wr\downarrow\iota^1_{\mathrm{BK}} & & \wr\downarrow\iota^2_{\mathrm{BK}}\\ \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C}) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C})/F^r\end{array}$$

(i) *Construction of the map* ι^1_{BK}. We may assume that X is quasi-compact. Consider the following sequence of maps in $\mathscr{D}^b(C_F)$

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X/r^{\mathrm{PD}}_\varpi)\widehat{\otimes}_{r^{\mathrm{PD}}_{\varpi,\mathbf{Q}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}\xrightarrow{\cup}\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C}/r^{\mathrm{PD}}_\varpi)\leftarrow\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C}).\tag{3.40}$$

We claim that the cup product map is a quasi-isomorphism. Indeed, the proof of an analogous result in the case of schemes [66, Prop. 4.5.4] goes through in our setting. Recall the key points. By (3.38) and the fact that $\mathbf{A}_{\mathrm{st},n}$ is flat over $r^{\mathrm{PD}}_{\varpi,n}$, it suffices to prove that the Künneth morphism

$$\cup:\quad \mathrm{R}\Gamma_{\mathrm{cr}}(X_n/r^{\mathrm{PD}}_{\varpi,n})\otimes^L_{r^{\mathrm{PD}}_{\varpi,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{O}^\times_{\overline{K},n}/r^{\mathrm{PD}}_{\varpi,n})\to\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}},n}/r^{\mathrm{PD}}_{\varpi,n})\tag{3.41}$$

is a quasi-isomorphism. By unwinding both sides one finds a Künneth morphism for certain de Rham complexes. It is a quasi-isomorphism because these complexes are "flat enough" which follows from the fact that the map $X_{\mathscr{O}_{\overline{K},n}}\to\mathscr{O}^\times_{\overline{K},n}$ is log-syntomic. This finishes the argument.

Both maps in (3.40) are compatible with the monodromy operator N. Moreover, we have the distinguished triangle in $\mathscr{D}^b(C_F)$ [36, Lemma 4.2]

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K},n}}/\mathscr{O}_{F,n})\to\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K},n}}/r^{\mathrm{PD}}_{\varpi,n})\xrightarrow{N}\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K},n}}/r^{\mathrm{PD}}_{\varpi,n}).\tag{3.42}$$

It follows that the last map in (3.40) is a quasi-isomorphism after taking the (derived) $N=0$ part. Hence applying $N=0$ to the terms in (3.40) we obtain a functorial quasi-isomorphism in $\mathscr{D}^b(C_F)$

$$\iota^1_{\mathrm{BK}}:[\mathrm{R}\Gamma_{\mathrm{cr}}(X/r^{\mathrm{PD}}_\varpi)\widehat{\otimes}_{r^{\mathrm{PD}}_{\varpi,\mathbf{Q}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C}).\tag{3.43}$$

(ii) *Construction of the map* ι^2_{BK}. We may assume that X is quasi-compact. Consider the maps

$$(3.44)\qquad (\mathrm{R}\Gamma_{\mathrm{cr}}(X_n/\mathscr{O}^{\times}_{K,n})\otimes^{L}_{\mathscr{O}_{K,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{O}^{\times}_{\overline{K},n}/\mathscr{O}^{\times}_{K,n}))/F^r \xrightarrow{\cup} \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K},n}}/\mathscr{O}^{\times}_{K,n})/F^r \leftarrow \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K},n}})/F^r.$$

The cup product map is a Künneth map and it is a quasi-isomorphism for the same reason as the map (3.41). The second map – the change of base map from $\mathscr{O}_F$ to $\mathscr{O}_K^{\times}$ – is a quasi-isomorphism (up to a canonical constant) by [48, Cor. 2.4]. Rationally, the above maps induce a map

$$\iota^2_{\mathrm{BK}}:(\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C})/F^r.$$

Clearly, ι^2_{BK} is a strict quasi-isomorphism.

Compatibility of the maps $\iota^1_{\mathrm{BK}},\iota^2_{\mathrm{BK}}$ can be easily inferred from the natural commutative diagram

$$\begin{array}{ccccc}
 & & \xrightarrow{\qquad\qquad p_{\varpi}\otimes\iota\qquad\qquad} & & \\
[\mathrm{R}\Gamma_{\mathrm{cr}}(X/r^{\mathrm{PD}}_{\varpi})\widehat{\otimes}_{r^{\mathrm{PD}}_{\varpi}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0} & \xrightarrow{p_{\varpi}\otimes p_{\varpi}} & (\mathrm{R}\Gamma_{\mathrm{cr}}(X/\mathscr{O}_K^{\times})\widehat{\otimes}_K\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{O}_C^{\times}/\mathscr{O}_K^{\times}))/F^r & \xleftarrow{\sim} & (\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r\\
\wr\big\downarrow\cup & & \wr\big\downarrow\cup & \swarrow\iota^2_{\mathrm{BK}} & \\
\iota^1_{\mathrm{BK}}\quad[\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C}/r^{\mathrm{PD}}_{\varpi})]^{N=0} & \xrightarrow{p_{\varpi}} & \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C}/\mathscr{O}_K^{\times})/F^r & & \\
\wr\big\uparrow & & \wr\big\uparrow & & \\
\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C}) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_C})/F^r. & &
\end{array}$$

∎

4. Comparison of syntomic cohomologies

We have defined two geometric syntomic cohomologies: the crystalline one and the overconvergent one. We will show now that they are naturally isomorphic and that, as a result, pro-étale cohomology fits into a fundamental diagram.

4.1. **Comparison morphism.** Let X be a semistable weak formal scheme over $\mathscr{O}_K$. Let $\widehat{X}$ be the associated formal scheme. The purpose of this section is to prove that the change of convergence map from overconvergent syntomic cohomology to crystalline syntomic cohomology is a strict quasi-isomorphism assuming that X is Stein.

Theorem 4.1. *There is a functorial map in* $\mathscr{D}^b(C_{\mathbf{Q}_p})$

$$\iota_{\mathrm{rig}}:\mathrm{R}\Gamma_{\mathrm{syn}}(X_C,\mathbf{Q}_p(r))\to\mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(\widehat{X}_{\mathscr{O}_C},\mathbf{Q}_p(r)).$$

It is a quasi-isomorphism if X is Stein.

Proof. (i) *Passage from* $\mathbf{B}^+_{\mathrm{st}}$ *to* $\widehat{\mathbf{B}}^+_{\mathrm{st}}$. In the definition of syntomic cohomology we will change $\mathbf{B}^+_{\mathrm{st}}$ to $\widehat{\mathbf{B}}^+_{\mathrm{st}}$. That is, we will show that the natural map $\iota_{\mathrm{st}}:\mathbf{B}^+_{\mathrm{st}}\to\widehat{\mathbf{B}}^+_{\mathrm{st}}$ induces a commutative diagram (in $\mathscr{D}^b(C_{\mathbf{Q}_p})$)

$$(4.2)\qquad \begin{array}{ccc}
[\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r} & \xrightarrow{\iota_{\mathrm{HK}}\otimes\iota} & (\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r\\
\wr\big\downarrow 1\otimes\iota_{\mathrm{st}} & & \big\|\\
[\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0,\varphi=p^r} & \xrightarrow{\iota_{\mathrm{HK}}\otimes\iota} & (\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r.
\end{array}$$

The fact that this diagram is commutative is clear since the composition $\mathbf{B}^+_{\mathrm{st}}\xrightarrow{\iota_{\mathrm{st}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}\xrightarrow{\iota}\mathbf{B}^+_{\mathrm{dR}}$ is just the original map $\iota:\mathbf{B}^+_{\mathrm{st}}\to\mathbf{B}^+_{\mathrm{dR}}$. To see that the map $1\otimes\iota_{\mathrm{st}}$ is a strict quasi-isomorphism it suffices to show that so is the map

$$1\otimes\iota_{\mathrm{st}}:[\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}}]^{N=0}\xrightarrow{\sim}[\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0}.$$

For that, we may assume that X_0 is affine. Then, we can argue as in Example 3.15, using the fact that $\widehat{\mathbf{B}}^+_{\rm st}$ is a Banach space, and we have the short exact sequence

$$0\to \mathbf{B}^+_{\rm cr}\to \widehat{\mathbf{B}}^+_{\rm st}\stackrel{N}{\to}\widehat{\mathbf{B}}^+_{\rm st}\to 0$$

to get compatible isomorphisms

$$\widetilde{H}^i([\mathrm{R}\Gamma_{\rm rig}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\mathbf{B}^+_{\rm st}]^{N=0})\simeq (H^i_{\rm HK}(X_0)\otimes_F\mathbf{B}^+_{\rm st})^{N=0},$$
$$\widetilde{H}^i([\mathrm{R}\Gamma_{\rm rig}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st}]^{N=0})\simeq (H^i_{\rm HK}(X_0)\otimes_F\widehat{\mathbf{B}}^+_{\rm st})^{N=0}.$$

Since the monodromy on $H^i_{\rm HK}(X_0)$ is nilpotent and $\mathbf{B}^+_{\rm st}=\widehat{\mathbf{B}}^{+,N-{\rm nilp}}_{\rm st}$, the natural map

$$(H^i_{\rm HK}(X_0)\otimes_F\mathbf{B}^+_{\rm st})^{N=0}\to (H^i_{\rm HK}(X_0)\otimes_F\widehat{\mathbf{B}}^+_{\rm st})^{N=0}.$$

is an isomorphism. As both sides are Banach spaces, this is a strict isomorphism, as wanted.

(ii) *Definition of maps* $\iota^1_{\rm rig}$ *and* $\iota^2_{\rm rig}$.

Here, exceptionally, we assume $r\geq -1$. The quasi-isomorphism (4.2) allows us to induce the comparison map $\iota_{\rm rig}$ in our theorem by a pair of maps $(\iota^1_{\rm rig},\iota^2_{\rm rig})$ defined below that make the following diagram commute (in $\mathscr{D}^b(C_{\mathbf{Q}_p})$)

$$\begin{array}{ccc}
[\mathrm{R}\Gamma_{\rm rig}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st}]^{\varphi=p^r} & \xrightarrow{\iota_{\rm HK}\otimes\iota} & (\mathrm{R}\Gamma_{\rm dR}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\rm dR})/F^r\\
\downarrow{\scriptstyle \iota^1_{\rm rig}} & & \downarrow{\scriptstyle \iota^2_{\rm rig}}\\
[\mathrm{R}\Gamma_{\rm cr}(\widehat{X}/r^{\rm PD}_{\varpi})\widehat{\otimes}_{r^{\rm PD}_{\varpi,\mathbf{Q}}}\widehat{\mathbf{B}}^+_{\rm st}]^{\varphi=p^r} & \xrightarrow{p_{\varpi}\otimes\iota} & (\mathrm{R}\Gamma_{\rm dR}(\widehat{X}_K)\widehat{\otimes}_K\mathbf{B}^+_{\rm dR})/F^r.
\end{array}$$

The map $\iota^1_{\rm rig}$ is compatible with monodromy. Both maps $\iota^1_{\rm rig},\iota^2_{\rm rig}$ are quasi-isomorphisms if X is Stein.

($\star$) *Map* $\iota^2_{\rm rig}$. The map $\iota^2_{\rm rig}$, the easier of the two maps, is just the map from de Rham cohomology of a weak formal scheme to de Rham cohomology of its completion; in the case X is Stein, it is an isomorphism induced by the canonical identification of coherent cohomology of a partially proper dagger space and its rigid analytic avatar [20, Theorem 2.26].

($\star$) *Map* $\iota^1_{\rm rig}$. To define the map $\iota^1_{\rm rig}$, consider first the change of convergence map

$$\mathrm{R}\Gamma_{\rm rig}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st}\to \mathrm{R}\Gamma_{\rm cr}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st}. \tag{4.3}$$

It is compatible with Frobenius and monodromy. We claim that if X is Stein it is a strict quasi-isomorphism. We will reduce it to X_0 proper where it will be clear. To see this, take the subschemes $\{U_i\},\{Y_i\}, i\in\mathbf{N}$, of X_0 as in Section 3.1.1. We have strict quasi-isomorphisms

$$\mathrm{R}\Gamma_{\rm rig}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st}\stackrel{\sim}{\to}\operatorname{holim}_i\mathrm{R}\Gamma_{\rm rig}(U_i/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st}\stackrel{\sim}{\to}\operatorname{holim}_i\mathrm{R}\Gamma_{\rm rig}(Y_i/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st}.$$

The first one by Example 3.15, the second one trivially. We claim that we have similar strict quasi-isomorphisms for the crystalline cohomology

$$\mathrm{R}\Gamma_{\rm cr}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st}\stackrel{\sim}{\to}\operatorname{holim}_i\mathrm{R}\Gamma_{\rm cr}(U_i/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st}\stackrel{\sim}{\to}\operatorname{holim}_i\mathrm{R}\Gamma_{\rm cr}(Y_i/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st}.$$

The second one is again trivial. For the first one, we can argue almost exactly as in Example 3.15 by replacing the key fact that the overconvergent Hyodo-Kato cohomology is finite dimensional for quasi-compact schemes with the fact that for such schemes the crystalline Hyodo-Kato cohomology has a canonical integral structure which is independent of the embedding system used (i.e., different embeddings give rise to quasi-isomorphic integral complexes, hence to strictly quasi-isomorphic complexes rationally).

We also have

$$\widetilde{H}^j(\mathrm{R}\Gamma_{\rm rig}(Y_i/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st})\simeq H^j_{\rm rig}(Y_i/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st},\quad \widetilde{H}^j(\mathrm{R}\Gamma_{\rm cr}(Y_i/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st})\simeq H^j_{\rm cr}(Y_i/\mathscr{O}_F^0)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st}.$$

the first isomorphism by an argument analogous to the one used in Example 3.15, the second one, by a similar argument using the fact that the cohomology $H^j_{\rm cr}(Y_i/\mathscr{O}_F^0)$ is finite dimensional. In both cases the key input is the fact that the irreducible components of the idealized log-scheme Y_i (that is ideally log-smooth over k^0) are log-smooth and proper over k itself. It suffices now to point out that the change of topology map $H^j_{\mathrm{rig}}(Y_i/\mathscr{O}^0_F)\to H^j_{\mathrm{cr}}(Y_i/\mathscr{O}^0_F)$ is an isomorphism (again, reduce to the irreducible components).

Now we will define the following functorial quasi-isomorphism in $\mathscr{D}^b(C_{\mathbf{Q}_p})$

$$h_{\mathrm{cr}}:[\mathrm{R}\Gamma_{\mathrm{cr}}(X/r^{\mathrm{PD}}_{\varpi})\widehat{\otimes}_{r^{\mathrm{PD}}_{\varpi,\mathbf{Q}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{\varphi=p^r}\to[\mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}^0_F)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{\varphi=p^r}. \tag{4.4}$$

We may assume that X is quasi-compact. Let J be the kernel of the map $p_0:r^{\mathrm{PD}}_{\varpi}\to\mathscr{O}_F$, $T\mapsto 0$. This map is compatible with Frobenius and monodromy. Consider the exact sequence

$$0\to J_n\to r^{\mathrm{PD}}_{\varpi,n}\stackrel{p_0}{\to}\mathscr{O}_{F,n}\to 0.$$

Tensoring it with $\widehat{\mathbf{A}}_{\mathrm{st},n}$, we get the following exact sequence

$$0\to J_n\otimes_{r^{\mathrm{PD}}_{\varpi,n}}\widehat{\mathbf{A}}_{\mathrm{st},n}\to\widehat{\mathbf{A}}_{\mathrm{st},n}\stackrel{p_0}{\to}\mathscr{O}_{F,n}\otimes_{r^{\mathrm{PD}}_{\varpi,n}}\widehat{\mathbf{A}}_{\mathrm{st},n}\to 0$$

We used here the fact that $\widehat{\mathbf{A}}_{\mathrm{st},n}$ is flat over $r^{\mathrm{PD}}_{\varpi,n}$. Going to limit with n, we get the exact sequence

$$0\to J\widehat{\otimes}_{r^{\mathrm{PD}}_{\varpi}}\widehat{\mathbf{A}}_{\mathrm{st}}\to\widehat{\mathbf{A}}_{\mathrm{st}}\stackrel{p_0}{\to}\mathscr{O}_F\widehat{\otimes}_{r^{\mathrm{PD}}_{\varpi}}\widehat{\mathbf{A}}_{\mathrm{st}}\to 0 \tag{4.5}$$

Set $E:=J\widehat{\otimes}_{r^{\mathrm{PD}}_{\varpi}}\widehat{\mathbf{A}}_{\mathrm{st}}$.

Tensoring the last sequence with $\mathrm{R}\Gamma_{\mathrm{cr}}(X_n/r^{\mathrm{PD}}_{\varpi,n})$ and $\mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}^0_{F,n})$, respectively, we obtain the following distinguished triangles

$$E_n\otimes^L_{r^{\mathrm{PD}}_{\varpi,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(X_n/r^{\mathrm{PD}}_{\varpi,n})\to\widehat{\mathbf{A}}_{\mathrm{st},n}\otimes^L_{r^{\mathrm{PD}}_{\varpi,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(X_n/r^{\mathrm{PD}}_{\varpi,n})\stackrel{p_0}{\to}\mathscr{O}_{F,n}\otimes_{r^{\mathrm{PD}}_{\varpi,n}}\widehat{\mathbf{A}}_{\mathrm{st},n}\otimes^L_{r^{\mathrm{PD}}_{\varpi,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(X_n/r^{\mathrm{PD}}_{\varpi,n}),$$
$$E_n\otimes^L_{\mathscr{O}_{F,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}^0_{F,n})\to\widehat{\mathbf{A}}_{\mathrm{st},n}\otimes^L_{\mathscr{O}_{F,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}^0_{F,n})\stackrel{p_0}{\to}\mathscr{O}_{F,n}\otimes_{r^{\mathrm{PD}}_{\varpi,n}}\widehat{\mathbf{A}}_{\mathrm{st},n}\otimes^L_{\mathscr{O}_{F,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}^0_{F,n}).$$

The last terms in these triangles are quasi-isomorphic. Indeed, by direct local computations we see that the natural map

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X_n/r^{\mathrm{PD}}_{\varpi,n})\otimes^L_{r^{\mathrm{PD}}_{\varpi,n}}\mathscr{O}_{F,n}\to\mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}^0_{F,n})$$

is a quasi-isomorphism. Hence

$$\begin{aligned}\mathscr{O}_{F,n}\otimes_{r^{\mathrm{PD}}_{\varpi,n}}\widehat{\mathbf{A}}_{\mathrm{st},n}\otimes^L_{r^{\mathrm{PD}}_{\varpi,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(X_n/r^{\mathrm{PD}}_{\varpi,n}) &\simeq\widehat{\mathbf{A}}_{\mathrm{st},n}\otimes^L_{r^{\mathrm{PD}}_{\varpi,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}^0_{F,n})\\ &\simeq\widehat{\mathbf{A}}_{\mathrm{st},n}\otimes^L_{r^{\mathrm{PD}}_{\varpi,n}}\mathscr{O}_{F,n}\otimes^L_{\mathscr{O}_{F,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}^0_{F,n}).\end{aligned}$$

The complexes

$$[E\widehat{\otimes}_{r^{\mathrm{PD}}_{\varpi}}\mathrm{R}\Gamma_{\mathrm{cr}}(X/r^{\mathrm{PD}}_{\varpi,\mathbf{Q}})]^{\varphi=p^r},\quad [E\widehat{\otimes}_F\mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}^0_F)]^{\varphi=p^r} \tag{4.6}$$

are strictly acyclic[25]: this is an immediate consequence of the fact that Frobenius φ is highly topologically nilpotent on J (hence $p^r-\varphi$ is rationally invertible). This implies that the following maps

$$[\mathrm{R}\Gamma_{\mathrm{cr}}(X/r^{\mathrm{PD}}_{\varpi})\widehat{\otimes}_{r^{\mathrm{PD}}_{\varpi,\mathbf{Q}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{\varphi=p^r}\stackrel{p_0}{\to}[\mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}^0_F)\widehat{\otimes}_F(\widehat{\mathbf{B}}^+_{\mathrm{st}}/E_{\mathbf{Q}})]^{\varphi=p^r}\stackrel{1\otimes p_0}{\longleftarrow}[\mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}^0_F)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{\varphi=p^r} \tag{4.7}$$

are strict quasi-isomorphisms. We define the map h_{cr} to be equal to the above zigzag. It is compatible with the monodromy operator (for the first map in the zigzag use the fact that the monodromy operator is defined by compatible residue maps).

We define a map in $\mathscr{D}^b(C_{\mathbf{Q}_p})$

$$\iota^1_{\mathrm{rig}}:[\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}^0_F)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{\varphi=p^r}\to[\mathrm{R}\Gamma_{\mathrm{cr}}(X/r^{\mathrm{PD}}_{\varpi})\widehat{\otimes}_{r^{\mathrm{PD}}_{\varpi,\mathbf{Q}}}\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{\varphi=p^r} \tag{4.8}$$

as the composition of the maps in (4.3) and (4.4). Both maps being compatible with the monodromy operator so is ι^1_{rig}.

(iii) *Compatibility of the maps* $\iota^1_{\mathrm{rig}},\iota^2_{\mathrm{rig}}$.

Before we proceed a small digression about *convergent cohomology* that we will use. Contrary to the case of rigid cohomology, the theory of (relative) convergent cohomology is well developed [49, 63, 64]. Recall the key points. The set up is the following: the base $\mathscr{B}$ is a p-adic formal log-scheme over $\mathscr{O}_F$,

[25] In fact, they are both isomorphic to a trivial complex.

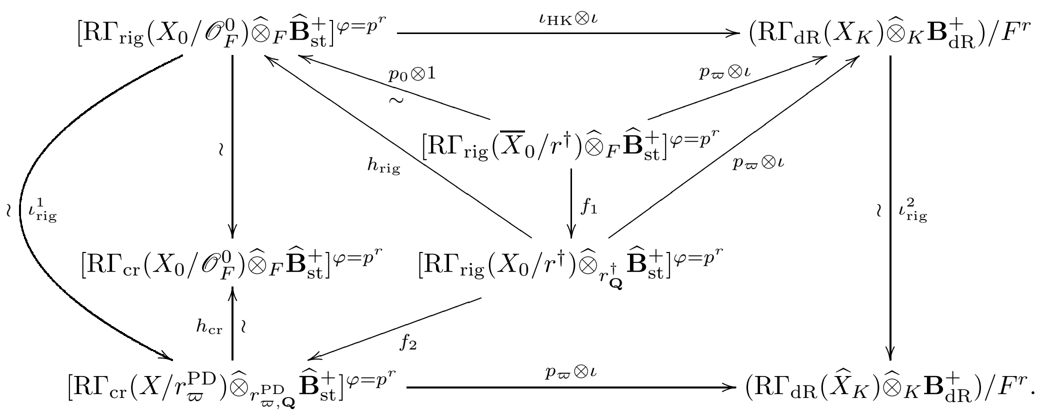
$B := \mathscr{B}_1$; we look at convergent cohomology of X over $\mathscr{Y}$, where X is a log-scheme over B and $\mathscr{Y}$ is a p-adic formal log-scheme over $\mathscr{B}$.

(1) There exist a convergent site defined in analogy with the crystalline site, where the role of PD-thickenings (analytically, objects akin to closed discs of a specific radius < 1) is played by enlargements (p-adic formal schemes; analytically, closed discs of any radius < 1). Convergent cohomology is defined as the cohomology of the rational structure sheaf on this site.

(2) *Invariance under infinitesimal thickenings.* If $i : X \to X'$ is a homeomorphic exact closed immersion then the pullback functor i^* induces a quasi-isomorphism on convergent cohomology [49, 0.6.1], [64, Prop. 3.1].

(3) *Poincaré Lemma.* It states that, locally, convergent cohomology can be computed by de Rham complexes of convergent tubes in p-adic formal log-schemes log-smooth over the base (playing the role of PD-envelopes) [64, Theorem 2.29]. Analytically, this means that the fixed closed discs used in the crystalline theory are replaced by open discs.

Let $r \geq 0$. Let us return now to the compatibility of the maps $\iota^1_{\rm rig}, \iota^2_{\rm rig}$. This can be shown by the commutative diagram



Here the map f_2 is the change of convergence map defined by the composition

$$f_2 : \mathrm{R}\Gamma_{\rm rig}(X_0/r^\dagger) \to \mathrm{R}\Gamma_{\rm conv}(X_0/\hat{r}) \xleftarrow{\sim} \mathrm{R}\Gamma_{\rm conv}(X_1/\hat{r}) \to \mathrm{R}\Gamma_{\rm cr}(X/r^{\rm PD}_\varpi),$$

where $\hat{r} := \mathscr{O}_F\{T\}$ and the subscript conv refers to convergent cohomology that we topologize in the same way we did rigid cohomology. The quasi-isomorphism is actually a natural isomorphism by the invariance under infinitisimal thickenings. The map f_2 is clearly compatible with the projection p_ϖ and the map $\iota^2_{\rm rig}$.

The map $h_{\rm rig}$ is defined in the same way as the map $h_{\rm cr}$: we just replace cr by rig and $r^{\rm PD}_\varpi$ by $r^\dagger$. To prove that the map $1 \otimes p_0$ in (4.7) is a strict quasi-isomorphism we use the fact that $\widehat{\mathbf{B}}^+_{\rm st}$ is Banach (see 2.0.4). We note however that in this case, in zigzag (4.7), the map p_0 is not a quasi-isomorphism. It is clear that the maps h_* are compatible.

The map f_1 is induced by the composition

$$\mathrm{R}\Gamma_{\rm rig}(\overline{X}_0/r^\dagger) = \mathrm{R}\Gamma_{\rm rig}((P_\bullet, V_\bullet)/r^\dagger) \to \mathrm{R}\Gamma_{\rm rig}(M'_\bullet/r^\dagger) \xrightarrow{\sim} \mathrm{R}\Gamma_{\rm rig}(M_\bullet/r^\dagger) \xleftarrow{\sim} \mathrm{R}\Gamma_{\rm rig}(X_0/r^\dagger).$$

The fact that the last quasi-isomorphism is strict needs a justification. We may assume that X_0 is affine and take a log-smooth lifting Y of X_0 to $r^\dagger$. Since the sheaf of differentials of Y_F is free we are reduced to showing strict acyclicity of the Čech complex of overconvergent functions for the covering corresponding to $\{M_i\}, i \in I$. Using a dagger presentation of Y_F, this complex can be written as an inductive limit of Čech complexes for analogous coverings of rigid analytic affinoids. Such complexes are strictly acyclic because we have the Open Mapping Theorem for Fréchet spaces. Inductive system being acyclic (see Section 2), we are done. For the above diagram we need though strictness of the last quasi-isomorphism but with terms tensored with $\mathbf{B}^+_{\rm st}$. For this we can use the same argument plus Lemma 2.1. Finally, it is easy to check (do it first without the period ring $\widehat{\mathbf{B}}^+_{\mathrm{st}}$) that the map f_1 makes the two small adjacent triangles in the above diagram commute. □

4.2. **Fundamental diagram.** Having the comparison theorem proved above, we can now deduce a fundamental diagram for pro-étale cohomology from the one for overconvergent syntomic cohomology.

Theorem 4.9. *Let X be a Stein semistable weak formal scheme over $\mathscr{O}_K$. Let $r \geq 0$. There is a natural map of strictly exact sequences of Fréchet spaces*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\Omega^{r-1}(X_K)/\ker d)\widehat{\otimes}_K C & \longrightarrow & H^r_{\mathrm{pro\acute{e}t}}(X_C,\mathbf{Q}_p(r)) & \longrightarrow & (H^r_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} & \longrightarrow & 0\\
 & & \Vert & & \big\downarrow{\scriptstyle\tilde{\beta}} & & \big\downarrow{\scriptstyle\iota_{\mathrm{HK}}\otimes\theta} & & \\
0 & \longrightarrow & (\Omega^{r-1}(X_K)/\ker d)\widehat{\otimes}_K C & \xrightarrow{d} & \Omega^r(X_K)_{d=0}\widehat{\otimes}_K C & \longrightarrow & H^r_{\mathrm{dR}}(X_K)\widehat{\otimes}_K C & \longrightarrow & 0
\end{array}$$

Moreover, the vertical maps have closed images, and $\ker\tilde{\beta} \simeq (H^r_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^{r-1}}$.

Proof. We define $\tilde{\beta} := p^{-r}\beta\iota_{\mathrm{rig}}^{-1}\iota_{\mathrm{BK}}^{-1}\alpha_{\mathrm{FM}}^{-1}$, using Corollary 3.35, Proposition 3.39, and Theorem 4.1; the twist by p^{-r} being added to make this map compatible with symbols. The theorem follows immediately from Proposition 3.27. □

Remark 4.10. The above diagram can be thought of as a one-way comparison theorem, i.e., the pro-étale cohomology $H^r_{\mathrm{pro\acute{e}t}}(X_C,\mathbf{Q}_p(r))$ is the pullback of the diagram

$$(H^r_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\theta} H^r_{\mathrm{dR}}(X_K)\widehat{\otimes}_K C \xleftarrow{\mathrm{can}} \Omega^r_{X_K,d=0}\widehat{\otimes}_K C$$

built from the Hyodo-Kato cohomology and a piece of the de Rham complex. For a striking comparison, recall that if X is a proper semistable formal scheme over $\mathscr{O}_K$ then the Semistable Comparison Theorem from [12] shows that we have the exact sequence

$$0\to H^r_{\mathrm{pro\acute{e}t}}(X_C,\mathbf{Q}_p(r))\to (H^r_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\iota}(H^r_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r, \tag{4.11}$$

i.e., the pro-étale cohomology $H^r_{\mathrm{pro\acute{e}t}}(X_C,\mathbf{Q}_p(r))$ is the pullback of the diagram

$$(H^r_{\mathrm{HK}}(X_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}\otimes\theta}(H^r_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r\longleftarrow 0.$$

Of course, in this case the étale and the pro-étale cohomologies agree. The sequence (4.11) is obtained in an analogous way to the top sequence in the fundamental diagram above. With the degeneration of the Hodge-de Rham spectral sequence and the theory of finite dimensional BC spaces forcing the injectivity on the left.

Remark 4.12. The following commutative diagram illustrates the relationship between syntomic cohomology of $\mathbf{Q}_p(r)$ and $\mathbf{Q}_p(r-1)$

$$\begin{array}{ccccccccccccccccc}
\cdots & \to & \mathrm{Syn}^{r-2}_{r-1} & \to & \mathrm{HK}^{r-2}_{r-1} & \to & \mathrm{DR}^{r-2}_{r-1} & \xrightarrow{\partial} & \mathrm{Syn}^{r-1}_{r-1} & \to & \mathrm{HK}^{r-1}_{r-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \to 0\\
\big\downarrow{\scriptstyle t} & & \wr\big\downarrow{\scriptstyle t} & & \big\downarrow{\scriptstyle t} & & \big\downarrow{\scriptstyle t} & & \wr\big\downarrow{\scriptstyle t} & & \big\downarrow{\scriptstyle t} & & \big\downarrow & & \big\downarrow & & \big\downarrow\\
\cdots & \to & \mathrm{Syn}^{r-2}_{r} & \to & \mathrm{HK}^{r-2}_{r} & \to & \mathrm{DR}^{r-2}_{r} & \xrightarrow{\partial} & \mathrm{Syn}^{r-1}_{r} & \to & \mathrm{HK}^{r-1}_{r} & \to & \mathrm{DR}^{r-1}_{r} & \xrightarrow{\partial} & \mathrm{Syn}^{r}_{r} & \to & \mathrm{HK}^{r}_{r} \to 0\\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\theta} & & \big\downarrow{\scriptstyle\theta} & & \big\downarrow & & \big\downarrow{\scriptstyle\theta} & & \big\downarrow{\scriptstyle\mathrm{Id}} & & \big\downarrow{\scriptstyle\mathrm{Id}} & & \big\downarrow{\scriptstyle\mathrm{Id}}\\
\cdots & \longrightarrow & 0 & \longrightarrow & H^{r-2}_{\mathrm{dR}} & \xrightarrow{\mathrm{Id}} & H^{r-2}_{\mathrm{dR}} & \longrightarrow & 0 & \longrightarrow & H^{r-1}_{\mathrm{dR}} & \to & \mathrm{DR}^{r-1}_{r} & \xrightarrow{\partial} & \mathrm{Syn}^{r}_{r} & \to & \mathrm{HK}^{r}_{r} \to 0
\end{array}$$

Here we wrote $\mathrm{HK}^i_r, \mathrm{DR}^i_r$, and Syn^i_r for the i'th cohomology of the complexes $\mathrm{HK}(X_C,r), \mathrm{DR}(X_C,r)$, and $\mathrm{R}\Gamma_{\mathrm{syn}}(X_C,\mathbf{Q}_p(r))$, respectively. We set $H^i_{\mathrm{dR}} := H^i_{\mathrm{dR}}(X_K)\widehat{\otimes}_K C$.

We claim that the rows of the above diagram are strictly exact. Indeed, the two top rows arise from the definition of syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X_C,\mathbf{Q}_p(r-1))$ and $\mathrm{R}\Gamma_{\mathrm{syn}}(X_C,\mathbf{Q}_p(r))$; the map between them is the multiplication by $t \in (\mathbf{B}^+_{\mathrm{cr}})^{\varphi=p} \cap F^1\mathbf{B}_{\mathrm{dR}}$. These rows are clearly strictly exact. It suffices now to show that the columns form short strictly exact sequences (with zeros at the ends). Indeed, for $i \leq r-1$, multiplication by t induces an isomorphism (using comparison with pro-étale cohomology)

$$\mathrm{Syn}^i_r \cong t\mathrm{Syn}^i_{r-1}$$

as well as the following strictly exact sequences

$$0 \to \mathrm{DR}^i_{r-1} \xrightarrow{t} \mathrm{DR}^i_r \to H^i_{\mathrm{dR}}(X_K)\widehat{\otimes}_K C \to 0, \quad r \geq i+2, \tag{4.13}$$
$$0 \to \mathrm{HK}^i_{r-1} \xrightarrow{t} \mathrm{HK}^i_r \to H^i_{\mathrm{dR}}(X_K)\widehat{\otimes}_K C \to 0.$$

The first strictly exact sequence follows from the strictly exact sequence

$$0 \to \mathrm{gr}_F^{r-i-1}\,\mathbf{B}^+_{\mathrm{dR}}\widehat{\otimes}_K(\Omega^i(X_K)/d\Omega^{i-1}(X_K)) \to \mathrm{DR}^i_r \to (\mathbf{B}^+_{\mathrm{dR}}/F^{r-i-1})\widehat{\otimes}_K H^i_{\mathrm{dR}}(X_K) \to 0;$$

the second one from Lemma 3.28.

4.3. **Examples.** We will now illustrate Theorem 4.9 with some simple examples.

4.3.1. *Affine space.* Let $d \geq 1$. Let $\mathbb{A}^d_K$ be the d-dimensional rigid analytic affine space over K. Recall that $H^r_{\text{ét}}(\mathbb{A}^d_C, \mathbf{Q}_p) = 0$ for $r \geq 1$ [2, Theorem 7.3.2]. On the other hand, as the following proposition shows, the pro-étale cohomology of $\mathbb{A}^d_C$ is highly nontrivial in nonzero degrees.

Proposition 4.14. *Let $r \geq 1$. There is a $\mathscr{G}_K$-equivariant isomorphism in $C_{\mathbf{Q}_p}$ (of Fréchet spaces)*

$$(\Omega^{r-1}(\mathbb{A}^d_K)/\ker d)\widehat{\otimes}_K C \xrightarrow{\sim} H^r_{\text{proét}}(\mathbb{A}^d_C, \mathbf{Q}_p(r)).$$

Remark 4.15. A simpler and more direct proof of this result (but still using syntomic cohomology) has been given in [13]. See [39] for another proof working directly with a fundamental exact sequence in the pro-étale topology.

Proof. Let $\mathscr{A}^d$ denote a semistable weak formal scheme over $\mathscr{O}_K$ such that $\mathscr{A}^d_K \simeq \mathbb{A}^d_K$. We will explain below how such a model $\mathscr{A}^d$ can be constructed. By Theorem 4.9, we have a $\mathscr{G}_K$-equivariant exact sequence (in $C_{\mathbf{Q}_p}$)

$$0 \longrightarrow (\Omega^{r-1}(\mathbb{A}^d_K)/\ker d)\widehat{\otimes}_K C \longrightarrow H^r_{\text{proét}}(\mathbb{A}^d_C, \mathbf{Q}_p(r)) \longrightarrow (H^r_{\mathrm{HK}}(\mathscr{A}^d_0)\widehat{\otimes}_F \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \longrightarrow 0$$

Recall that $H^r_{\mathrm{dR}}(\mathbb{A}^d_K) = 0$. Since, by the Hyodo-Kato isomorphism $H^r_{\mathrm{HK}}(\mathscr{A}^d_0)\otimes_F K \simeq H^r_{\mathrm{dR}}(\mathbb{A}^d_K)$, we have $H^r_{\mathrm{HK}}(\mathscr{A}^d_0) = 0$. Our proposition follows from the above exact sequence.

It remains to show that we can construct a semistable weak formal scheme $\mathscr{A}^d$ over $\mathscr{O}_K$ whose generic fiber is $\mathbb{A}^d_K$. For $d = 1$, we can define a model $\mathscr{A}^1$ using Theorem 4.9.1 of [18]. That theorem describes a construction of a formal semistable model for any analytic subspace $\mathbb{P}_K \setminus \mathscr{L}^*$, where $\mathscr{L}$ is an infinite compact subset of K-rational points of the projective line $\mathbb{P}_K$ and $\mathscr{L}^*$ is the set of its limit points. The proof of the theorem can be easily modified to yield a weak formal model. To define the model $\mathscr{A}^1$ we want we apply this theorem with $\mathscr{L} = \{\infty\} \cup \{\varpi^n | n \in \mathbf{Z}, n \leq 0\}$. We note that the special fiber of $\mathscr{A}^1$ is a half line of projective lines.

To construct a model $\mathscr{A}^d$ for $d > 1$, first we consider the d-fold product Y of the logarithmic weak formal scheme associated to $\mathscr{A}^1$. Product is taken over $\mathscr{O}^\times_K$. It is not, in general, a semistable scheme but it is log-smooth over $\mathscr{O}^\times_K$. Hence its singularities can be resolved using combinatorics of monoids describing the log-structure. In fact, using Lemma 1.9 of [57], one can define a canonical ideal sheaf of Y that needs to be blown-up to obtain a semistable model $\mathscr{A}^d$ we want. $\square$

4.3.2. *Torus.* Let $d \geq 1$. Let $\mathbb{G}^d_{m,K}$ be the d-dimensional rigid analytic torus over K. Let $\mathscr{Y}^d$ denote a semistable weak formal scheme over $\mathscr{O}_K$ such that $\mathscr{Y}^d_K \simeq \mathbb{G}^d_{m,K}$. Such a model $\mathscr{Y}^d$ exists. For $d = 1$, we can define a model $\mathscr{Y}^1$ using Theorem 4.9.1 of [18]; just as in the case of $\mathbb{A}^1_K$ above. More specifically, to define the model $\mathscr{Y}^1$ we want we apply this theorem with $\mathscr{L} = \{\infty, 0\} \cup \{\varpi^n | n \in \mathbf{Z}\}$. We note that the special fiber of $\mathscr{Y}^1$ is a line of projective lines. To construct a model $\mathscr{Y}^d$ for $d > 1$, we use products as above.

To make Theorem 4.9 explicit, we need to compute $(H^r_{\mathrm{HK}}(\mathscr{Y}^d_0)\widehat{\otimes}_F \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}$. For $d = 1$, we have

$$H^r_{\mathrm{dR}}(\mathbb{G}_{m,K}) = \begin{cases} K & \text{if } r = 0, \\ c_1^{\mathrm{dR}}(z)K & \text{if } r = 1, \\ 0 & \text{if } r > 1. \end{cases}$$

Here z is a coordinate of the torus and $c_1^{\mathrm{dR}}(z)$ is its de Rham Chern class, i.e. dz/z (see Appendix A). For $d>1$, we can use the Künneth formula to compute that $H^r_{\mathrm{dR}}(\mathbb{G}^d_{m,K})$ is a K-vector space of dimension $\binom{d}{r}$ generated by the tuples $c_1^{\mathrm{dR}}(z_{i_1})\cdots c_1^{\mathrm{dR}}(z_{i_r})$. Similarly, $H^r_{\mathrm{HK}}(\mathscr{Y}_0^d)$ is an F-vector space of dimension $\binom{d}{r}$ generated by the tuples $c_1^{\mathrm{HK}}(z_{i_1})\cdots c_1^{\mathrm{HK}}(z_{i_r})$. By Lemma A.7, the Hyodo-Kato and the de Rham symbols are compatible under the Hyodo-Kato map ι_{HK}.

Since $\varphi(c_1^{\mathrm{HK}}(z_{i_j}))=pc_1^{\mathrm{HK}}(z_{i_j})$ and $N(c_1^{\mathrm{HK}}(z_{i_j}))=0$, we get that

$$(H^r_{\mathrm{HK}}(\mathscr{Y}_0^d)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}=H^r_{\mathrm{HK}}(\mathscr{Y}_0^d)^{\varphi=p^r}=\wedge^r\mathbf{Q}_p^d$$

and that it is a $\mathbf{Q}_p$-vector space of dimension $\binom{d}{r}$ generated by the tuples $c_1^{\mathrm{HK}}(z_{i_1})\cdots c_1^{\mathrm{HK}}(z_{i_r})$. Hence, Theorem 4.9 gives us a map of $\mathscr{G}_K$-equivariant exact sequences (in $C_{\mathbf{Q}_p}$)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\Omega^{r-1}(\mathbb{G}^d_{m,K})/\ker d)\widehat{\otimes}_K C & \longrightarrow & H^r_{\mathrm{pro\acute{e}t}}(\mathbb{G}^d_{m,C},\mathbf{Q}_p(r)) & \longrightarrow & \wedge^r\mathbf{Q}_p^d & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle\tilde{\beta}} & & \downarrow{\scriptstyle\mathrm{can}} & & \\
0 & \longrightarrow & (\Omega^{r-1}(\mathbb{G}^d_{m,K})/\ker d)\widehat{\otimes}_K C & \xrightarrow{d} & \Omega^r(\mathbb{G}^d_{m,K})_{d=0}\widehat{\otimes}_K C & \longrightarrow & \wedge^r C^d & \longrightarrow & 0
\end{array}$$

4.3.3. *Curves.* Let X be a Stein curve over K with a semistable model $\mathscr{X}$ over $\mathscr{O}_K$. The diagram from Theorem 4.9 takes the following form[26]

$$\begin{array}{ccccccccccc}
0 & \to & C^{\pi_0(X)} & \to & \mathscr{O}(X)\widehat{\otimes}_K C & \xrightarrow{\exp} & H^1_{\mathrm{pro\acute{e}t}}(X_C,\mathbf{Q}_p(1)) & \to & (H^1_{\mathrm{HK}}(\mathscr{X}_0)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}})^{\varphi=p,N=0} & \to & 0\\
 & & \| & & \| & & \downarrow{\scriptstyle\mathrm{dlog}} & & \downarrow{\scriptstyle\iota_{\mathrm{HK}}\otimes\theta} & & \\
0 & \to & C^{\pi_0(X)} & \to & \mathscr{O}(X)\widehat{\otimes}_K C & \xrightarrow{d} & \Omega^1(X)\widehat{\otimes}_K C & \longrightarrow & H^1_{\mathrm{dR}}(X)\widehat{\otimes}_K C & \to & 0
\end{array}$$

5. Pro-étale cohomology of Drinfeld half-space

We will use the fundamental diagram of Theorem 4.9 to compute the p-adic pro-étale cohomology of the Drinfeld half-space.

Let K be a finite extension of $\mathbf{Q}_p$. Let $\mathbb{H}^d_K$, $d\geq 1$, be the d-dimensional Drinfeld half-space over K: the K-rigid space that is the complement in $\mathbb{P}^d_K$ of all K-rational hyperplanes. If $\mathscr{H}=\mathbb{P}((K^{d+1})^*)=\mathbb{P}^d(K)$ is the space of K-rational hyperplanes in K^{d+1} (this is a profinite set), we have

$$\mathbb{H}^d_K=\mathbb{P}^d_K\setminus\cup_{H\in\mathscr{H}}H.$$

The group $G:=\mathrm{GL}_{d+1}(K)$ acts on it. We will drop the subscript K if there is no danger of confusion. $\mathbb{H}^d_K$ is a rigid analytic Stein space hence also a dagger analytic Stein space. It has a (standard) G-equivariant semistable weak formal model $\widetilde{\mathbb{H}}^d_K$ [23, 6.1] (that is Stein).

5.1. **Generalized Steinberg representations.** We will briefly review the definitions and basic properties of the generalized Steinberg representations that we will need.

5.1.1. *Locally constant special representations.* Let B be the upper triangular Borel subgroup of G and $\Delta=\{1,2,...,d\}$. We identify the Weyl group W of G with the group of permutations of $\{1,2,...,d+1\}$ and with the subgroup of permutation matrices in G. Then W is generated by the elements $s_i=(i,i+1)$ for $i\in\Delta$.

For each subset J of Δ we let:
- W_J be the subgroup of W generated by the s_i with $i\in J$.
- $P_J=BW_JB$, the parabolic subgroup of G generated by B and W_J.
- $X_J=G/P_J$, a compact topological space.

If A is an abelian group and $J\subset\Delta$, let

$$\mathrm{Sp}_J(A)=\frac{\mathrm{LC}(X_J,A)}{\sum_{i\in\Delta\setminus J}\mathrm{LC}(X_{J\cup\{i\}},A)},$$

[26] We note here that the conditions of that theorem are always satisfied for curves.

where LC means locally constant (automatically with compact support since the X_J's are compact). This is a smooth G-module over A and we have a natural isomorphism $\mathrm{Sp}_J(A)=\mathrm{Sp}_J(\mathbf{Z})\otimes A$. For $J=\emptyset$ we obtain the usual Steinberg representation with coefficients in A, while for $J=\Delta$ we have $\mathrm{Sp}_J(A)=A$ (since X_J is a point). For $r\in\{0,1,...,d\}$ we use the simpler notation

$$\mathrm{Sp}_r=\mathrm{Sp}_{\{1,2,\dots,d-r\}}.$$

For $r>d$, we set $\mathrm{Sp}_r=0$.

Proposition 5.1. *If A is a field of characteristic 0 or p then the $\mathrm{Sp}_J(A)$'s (for varying J) are the irreducible constituents of $\mathrm{LC}(G/B,A)$, each occurring with multiplicity 1.*

Proof. This is due to Casselman in characteristic 0 (see [7, X, Theorem 4.11]) and to Grosse-Klönne [28, Cor. 4.3] in characteristic p. □

Remark 5.2. The proposition does not hold for A a field of characteristic $\ell\neq p$, see [70, III, Theorem 2.8].

The rigidity in characteristic p given by the previous theorem has consequences in characteristic 0 that will be very useful to us later on.

Corollary 5.3. *If J is a subset of Δ, then $\mathrm{Sp}_J(\mathscr{O}_K)$ is, up to a K^*-homothety, the unique G-stable $\mathscr{O}_K$-lattice in $\mathrm{Sp}_J(K)$.*

Proof. This follows easily from Proposition 5.1 and the fact that $\mathrm{Sp}_J(\mathscr{O}_K)$ is finitely generated over $\mathscr{O}_K[G]$, see [28, Cor. 4.5] for the details. □

5.1.2. *Topology.* If Λ is a topological ring, then $\mathrm{Sp}_J(\Lambda)$ has a natural topology: the space X_J being profinite, we can write $X_J=\varprojlim_n X_{n,J}$ for finite sets $X_{n,J}$ and then $\mathrm{LC}(X_J,\Lambda)=\varinjlim_n \mathrm{LC}(X_{n,J},\Lambda)$, each $\mathrm{LC}(X_{n,J},\Lambda)$ being a finite free Λ-module endowed with the natural topology. In particular, if Λ is a finite extension of $\mathbf{Q}_p$, this exhibits $\mathrm{Sp}_J(\Lambda)$ as an inductive limit of finite dimensional Λ-vector spaces, and the corresponding topology is the strongest locally convex topology on the Λ-vector space $\mathrm{Sp}_J(\Lambda)$, which is an LF-space.

Let $M^*:=\mathrm{Hom}_{\mathrm{cont}}(M,\Lambda)$ for any topological Λ-module M, and equip M^* with the weak topology. Then $\mathrm{Sp}_J(\Lambda)^*$ is naturally isomorphic to $\varprojlim_n \mathrm{LC}(X_{n,J},\Lambda)^*$, i.e. a countable inverse limit of finite free Λ-modules. In particular, if L is a finite extension of $\mathbf{Q}_p$ then $\mathrm{Sp}_J(L)^*$ is a nuclear Fréchet space (in fact a countable product of Banach spaces) and $\mathrm{Sp}_J(\mathscr{O}_L)^*$ is a compact $\mathscr{O}_L$-module, which is torsion free. Therefore $\mathrm{Sp}_J(\mathscr{O}_L)^*\otimes L$ has a natural structure of a weak dual of an L-Banach space.

5.1.3. *Continuous special representations.* Consider now the corresponding continuous special representation

$$\mathrm{Sp}_J^{\mathrm{cont}}(\Lambda)=\frac{\mathscr{C}(X_J,\Lambda)}{\sum_{\alpha\in\Delta\setminus J}\mathscr{C}(X_{J\cup\{\alpha\}},\Lambda)}.$$

Arguing as above, we see that, for any finite extension L of $\mathbf{Q}_p$, the space $\mathrm{Sp}_J^{\mathrm{cont}}(L)$ has a natural structure of an L-Banach space, with the unit ball given by $\mathrm{Sp}_J^{\mathrm{cont}}(\mathscr{O}_L)$. The action of G on all these spaces is continuous and we can recover $\mathrm{Sp}_J(L)$ from $\mathrm{Sp}_J^{\mathrm{cont}}(L)$ as the space of smooth vectors (for the action of G).

The rigidity in characteristic p given by Proposition 5.1 and Corollary 5.3 yields:

Corollary 5.4. *Let J be a subset of Δ and L a finite extension of $\mathbf{Q}_p$.*

a) The universal unitary completion of $\mathrm{Sp}_J(L)$ is $\mathrm{Sp}_J^{\mathrm{cont}}(L)$.

b) The space of G-bounded vectors in $\mathrm{Sp}_J(L)^*$ is $\mathrm{Sp}_J^{\mathrm{cont}}(L)^*$.

Proof. a) Follows from Corollary 5.3 and the fact that $\mathrm{St}_J^{\mathrm{cont}}(\mathscr{O}_L)$ is the p-adic completion of $\mathrm{Sp}_J(\mathscr{O}_L)$ (which in turn uses that $\mathrm{Sp}_J(A)=\mathrm{Sp}_J(\mathbf{Z})\otimes A$ for all A, and this is a free A-module).

b) Follows by duality from a).

□

Remark 5.5. One can also define a locally analytic generalized Steinberg representation $\mathrm{Sp}_J^{\mathrm{an}}(L)$ for any finite extension L or $\mathbf{Q}_p$ (or any closed subfield of complex numbers). It is naturally a space of compact type, whose dual is a nuclear Fréchet space. It contains $\mathrm{Sp}_J(L)$ as a closed subspace (it is closed because it is the space of vectors killed by the Lie algebra of G). The dual of $\mathrm{Sp}_J^{\mathrm{an}}(L)$ surjects onto the dual of $\mathrm{Sp}_J(L)$ and contains the dual of $\mathrm{Sp}_J^{\mathrm{cont}}(L)$ as a dense subspace. The big difference is that $\mathrm{Sp}_J^{\mathrm{an}}(L)$ is topologically reducible as a G-module. Its Jordan-Hölder constituents are described in [53].

5.2. **Results of Schneider-Stuhler.** We recall the cohomological interpretation of the representations $\mathrm{Sp}_r(\mathbf{Z})$, following [59]. Recall that $\mathscr{H}$ is the space of K-rational hyperplanes in K^{d+1}. For $r \in \{1, 2, ..., d\}$ we define simplicial profinite sets $Y_\bullet^{(r)}$, $\mathscr{T}_\bullet^{(r)}$ as follows:

- $Y_s^{(r)}$ is the closed subset of $\mathscr{H}^{s+1}$ consisting of tuples $(H_0, ..., H_s) \in \mathscr{H}^{s+1}$ with

$$\dim_K(\sum_{i=0}^{s} K\ell_{H_i}) \leq r,$$

where $\ell_{H_i} \in (K^{d+1})^*$ is any equation of H_i.

- $\mathscr{T}_s^{(r)}$ is the set of flags $W_0 \subset ... \subset W_s$ in $(K^{d+1})^*$ for which $\dim_K W_i \in \{1, ..., r\}$ for all i. This set has a natural profinite topology.

In both cases the face/degeneracy maps are the obvious ones, i.e. omit/double one hyperplane in a tuple, resp. a vector subspace in a flag. With the topology forgotten, $\mathscr{T}_\bullet^{(d)}$ is the Tits[27] (not Bruhat-Tits!) building of G.

The following result is due to Schneider and Stuhler:

Proposition 5.6. *For all $r \in \{1, 2, ..., d\}$ we have natural isomorphisms (where $\widetilde{H}$ denotes reduced cohomology)*

$$\widetilde{H}^{r-1}(|\mathscr{T}_\bullet^{(r)}|, \mathbf{Z}) \simeq \widetilde{H}^{r-1}(|Y_\bullet^{(r)}|, \mathbf{Z}) \simeq \mathrm{Sp}_r(\mathbf{Z}).$$

Proof. The isomorphism $\widetilde{H}^{r-1}(|\mathscr{T}_\bullet^{(r)}|, \mathbf{Z}) \simeq \widetilde{H}^{r-1}(|Y_\bullet^{(r)}|, \mathbf{Z})$ is proved in [59, Ch. 3, Prop 5]. To identify these objects with $\mathrm{Sp}_r(\mathbf{Z})$, assuming for simplicity $r > 1$ from now on, consider the clopen subset $\mathscr{N}\mathscr{T}_s^{(r)} \subset \mathscr{T}_s^{(r)}$ of $\mathscr{T}_s^{(r)}$ consisting of flags $W_0 \subset ... \subset W_s$ for which all inclusions are strict. Using the obvious isomorphism $\widetilde{H}^{r-1}(|\mathscr{N}\mathscr{T}_\bullet^{(r)}|, \mathbf{Z}) \simeq \widetilde{H}^{r-1}(|\mathscr{T}_\bullet^{(r)}|, \mathbf{Z})$ the result follows from the exact sequence[28]

$$\mathrm{LC}(\mathscr{N}\mathscr{T}_{r-2}^{(r)}, \mathbf{Z}) \to \mathrm{LC}(\mathscr{N}\mathscr{T}_{r-1}^{(r)}, \mathbf{Z}) \to H^{r-1}(|\mathscr{N}\mathscr{T}_\bullet^{(r)}|, \mathbf{Z}) \to 0$$

and the identifications

$$\mathscr{N}\mathscr{T}_{r-1}^{(r)} \simeq X_{\{1,2,...,d-r\}}, \quad \mathscr{N}\mathscr{T}_{r-2}^{(r)} \simeq \coprod_{i=d-r+1}^{d} X_{\{1,...,d-r,i\}}$$

□

Remark 5.7. For all $r \in \{1, 2, ..., d\}$ and all q there are natural isomorphisms

$$H^q(|\mathscr{N}\mathscr{T}_\bullet^{(r)}|, \mathbf{Z}) \simeq H^q(|\mathscr{T}_\bullet^{(r)}|, \mathbf{Z}) \simeq H^q(|Y_\bullet^{(r)}|, \mathbf{Z})$$

and these spaces are nonzero only for $q = 0, r-1$, with H^0 being given by $\mathbf{Z}$ for $r > 1$ and by $\mathrm{LC}(\mathbf{P}((K^{d+1})^*), \mathbf{Z})$ for $r = 1$. See [59, Ch. 3, Prop. 6] for the details.

The following theorem is one of the main results of [59]. See also [52] for a different argument (at least for a) and the compactly supported analogue of b)).

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[27] For instance, for $d = 1$ this is the set of ends of the tree.

[28] Recall that if $S_\bullet$ is any simplicial profinite set, then $H^*(|S_\bullet|, \mathbf{Z}) = H^*(\mathrm{LC}(S_\bullet, \mathbf{Z}))$, where $|S_\bullet|$ is the geometric realisation of $S_\bullet$ and $\mathrm{LC}(S_\bullet, \mathbf{Z})$ is the complex $(\mathrm{LC}(S_s, \mathbf{Z}))_s$, the differentials being given by the alternating sum of the maps induced by face maps in S.

Theorem 5.8. *(Schneider-Stuhler) Let $r \geq 0$.*

a) For a prime $\ell \neq p$, there are natural isomorphisms of $G \times G_K$-modules

$$H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Q}_\ell(r)) \simeq \mathrm{Sp}_r(\mathbf{Z}_\ell)^* \otimes \mathbf{Q}_\ell, \quad H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbf{Q}_\ell(r)) \simeq \mathrm{Sp}_r(\mathbf{Q}_\ell)^*.$$

b) There is a natural isomorphism of G-modules

$$H^r_{\mathrm{dR}}(\mathbb{H}^d_K) \simeq \mathrm{Sp}_r(K)^*.$$

Proof. Let H^* be any of the cohomologies occuring in the theorem. It has the properties required by Schneider-Stuhler [59, Ch. 2]. The crucial among them is the homotopy invariance property: if D is the 1-dimensional open unit disk then, for any smooth affinoid X, the projection $X \times D \to X$ induces a natural isomorphism $H^*(X) \xrightarrow{\sim} H^*(X \times D)$. For de Rham cohomology this is very simple (see the discussion preceding Prop. 3 in [59, Ch. 2]); for ℓ-adic étale and pro-étale cohomologies this follows from the "homotopy property" of ℓ-adic étale cohomology with respect to a closed disk [59, proof of Theorem 6.0.2], and the fact that ℓ-adic étale and pro-étale cohomologies are the same on affinoids.

We recall very briefly the key arguments, without going into the rather involved combinatorics. If $H \in \mathscr{H}$ and $n \geq 1$, let $U_n(H)$ be the open polydisk in the affine space $\mathbb{P}^d_K \setminus H$ given by[29] $|\ell_H(z)| > |\pi|^n$. The open subsets $U_n = \cap_{H \in \mathscr{H}} U_n(H)$ form a Stein covering of $\mathbb{H}^d_K$ and $U_n = \cap_{H \in \mathscr{H}_n} U_n(H)$, for a finite subset $\mathscr{H}_n$ of $\mathscr{H}$, in bijection with $\mathbb{P}^d((\mathscr{O}_K^{d+1}/\pi^n)^*)$. Writing $H^*(X, U)$ for the "cohomology with support in $X \setminus U$" (more precisely, the derived functors of the functor "sections vanishing on U"), a formal argument (see the discussion following [59, Ch.3, Cor. 5]) gives a spectral sequence

$$E_1^{-j,i}(n) = \bigoplus_{H_0,\dots,H_j \in \mathscr{H}_n} H^i(\mathbb{P}^d_K, U_n(H_0) \cup \dots \cup U_n(H_j)) \Rightarrow H^{i-j}(\mathbb{P}^d_K, U_n).$$

Now, $U_n(H_0) \cup \dots \cup U_n(H_j)$ is a locally trivial fibration over a projective space, whose fibers are open polydisks [59, Ch.1, prop.6]. Using the homotopy invariance of cohomology, one computes $H^i(\mathbb{P}^d_K, U_n(H_0) \cup \dots \cup U_n(H_j))$, in particular this is always equal to $A = H^0(\mathrm{Sp}(K))$ or 0 (with a simple combinatorial recipe allowing to distinguish the two cases). The spectral sequence simplifies greatly and[30] letting $n \to \infty$ gives (using also Proposition 5.6 and Remark 5.7) a spectral sequence

$$E_2^{-j,i} \Rightarrow H^{i-j}(\mathbb{P}^d_K, \mathbb{H}^d_K),$$

where

$$E_2^{-j,i} = \mathrm{Hom}_{\mathbf{Z}}(H^j(|\mathscr{T}_\bullet^{(\frac{i}{2})}|, \mathbf{Z}), A)$$

if $i \in [2, 2d]$ is even and $j \in \{0, \frac{i}{2} - 1\}$, and 0 otherwise. The analysis of this spectral sequence combined with Proposition 5.6 yields the cohomology groups $H^i(\mathbb{P}^d_K, \mathbb{H}^d_K)$. The result follows from the exact sequence

$$\dots \to H^i(\mathbb{P}^d_K) \to H^i(\mathbb{H}^d_K) \to H^{i+1}(\mathbb{P}^d_K, \mathbb{H}^d_K) \to H^{i+1}(\mathbb{P}^d_K) \to \dots.$$

□

Combining Theorem 5.8 and Corollary 5.4 yields:

Corollary 5.9. *The space of G-bounded vectors in $H^r_{\mathrm{dR}}(\mathbb{H}^d_K)$ is isomorphic to* $\mathrm{Sp}_r^{\mathrm{cont}}(K)^*$.

5.3. **Generalization of Schneider-Stuhler.** We will extend the results of Schneider-Stuhler to Hyodo-Kato cohomology. To do that we will use the description of the isomorphisms in Theorem 5.8 via symbols.

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[29]We use unimodular representatives for points of projective space and for linear forms giving equations of H.

[30]This is allowable as all modules involved are finite over the Artinian ring A.

5.3.1. *Results of Iovita-Spiess.* All the isomorphisms in Theorem 5.8 are rather abstract, but following Iovita-Spiess [35] one can make them quite explicit as follows. Let $\mathrm{LC}^c(\mathscr{H}^{r+1},\mathbf{Z})$ be the space of locally constant functions $f:\mathscr{H}^{r+1}\to\mathbf{Z}$ such that, for all $H_0,...,H_{r+1}\in\mathscr{H}$,

$$f(H_1,...,H_{r+1})-f(H_0,H_2,...,H_{r+1})+...+(-1)^{r+1}f(H_0,...,H_r)=0$$

and moreover, if ℓ_{H_i} are linearly dependent, then $f(H_0,...,H_r)=0$ (i.e., f vanishes on $Y_r^{(r)}$). Define analogously $\mathscr{C}^c(\mathscr{H}^{r+1},\mathbf{Z})$. It is not difficult to see that we have a natural isomorphism (see the proof of Proposition 5.6 for the notation used below)

$$\widetilde{H}^{r-1}(|\mathscr{N}\mathscr{T}_{\bullet}^{(r)}|,\mathbf{Z})\simeq\mathrm{LC}^c(\mathscr{H}^{r+1},\mathbf{Z})$$

and, in particular, (using Proposition 5.6) a natural isomorphism

$$\mathrm{Sp}_r(\mathbf{Z})\simeq\mathrm{LC}^c(\mathscr{H}^{r+1},\mathbf{Z}).$$

If S is a profinite set and A an abelian group, let $D(S,A)=\mathrm{Hom}(\mathrm{LC}(S,\mathbf{Z}),A)$ be the space of A-valued locally constant distributions on S. If L is a discrete valuation nonarchimedean field let $M(S,L)$ be the space of L-valued measures, i.e., bounded L-valued distributions. It has a natural topology that is finer than the subspace topology induced from $D(S,L)$ [35, Ch. 4].

The inclusion $\mathrm{LC}^c(\mathscr{H}^{r+1},\mathbf{Z})\subset\mathrm{LC}(\mathscr{H}^{r+1},\mathbf{Z})$ gives rise to a continuous strict surjection

$$D(\mathscr{H}^{r+1},A)\to\mathrm{Hom}(\mathrm{Sp}_r(\mathbf{Z}),A).$$

Define the space $D(\mathscr{H}^{r+1},A)_{\deg}$ of degenerate distributions as the kernel of this map. Combining this with the previous theorem we obtain surjections:

$$\begin{aligned}&D(\mathscr{H}^{r+1},K)\to H^s_{\mathrm{dR}}(\mathbb{H}^d_K),\quad M(\mathscr{H}^{r+1},\mathbf{Q}_\ell)\to H^r_{\text{ét}}(\mathbb{H}^d_C,\mathbf{Q}_\ell(r)),\\&D(\mathscr{H}^{r+1},\mathbf{Q}_\ell)\to H^r_{\text{proét}}(\mathbb{H}^d_C,\mathbf{Q}_\ell(s)).\end{aligned}$$

These surjections can be made explicit as follows. For each $(H_0,...,H_r)\in\mathscr{H}^{r+1}$, the invertible functions (on $\mathbb{H}^d_K$) $\frac{\ell_{H_1}}{\ell_{H_0}},...,\frac{\ell_{H_r}}{\ell_{H_0}}$ give rise (either by taking dlog and wedge-product or by taking the corresponding symbols in étale cohomology and then cup-product) to a symbol $[H_0,...,H_r]$ living in $H^r_{\mathrm{dR}}(\mathbb{H}^d_K)$, resp. in $H^r_{\text{ét}}(\mathbb{H}^d_C,\mathbf{Q}_\ell(r))$, resp. $H^r_{\text{proét}}(\mathbb{H}^d_C,\mathbf{Q}_\ell(r))$. For example, for de Rham cohomology $[H_0,...,H_r]$ is the class of the closed r-form

$$\mathrm{dlog}\frac{\ell_{H_1}}{\ell_{H_0}}\wedge...\wedge\mathrm{dlog}\frac{\ell_{H_r}}{\ell_{H_0}}$$

in $H^r_{\mathrm{dR}}(\mathbb{H}^d_K)$.

One shows that the following regulator map is well-defined

$$r_{\mathrm{dR}}:D(\mathscr{H}^{r+1},K)\to H^r_{\mathrm{dR}}(\mathbb{H}^d_K),\quad\mu\mapsto\int_{\mathscr{H}^{r+1}}[H_0,...,H_r]\mu(H_0,...,H_r).$$

The problem here is that the map $(H_0,...,H_r)\mapsto[H_0,...,H_r]$ is not locally constant on $\mathbb{H}^d_K$; however it is so on U_n (see the proof of Theorem 5.8 for the notation), for all n, which makes it possibe to give a meaning to the integral. The same integral works for ℓ-adic étale and pro-étale cohomologies yielding the regulator maps

$$r_{\text{ét}}:M(\mathscr{H}^{r+1},\mathbf{Q}_\ell)\to H^r_{\text{ét}}(\mathbb{H}^d_C,\mathbf{Q}_\ell(r)),\quad r_{\text{proét}}:D(\mathscr{H}^{r+1},\mathbf{Q}_\ell)\to H^r_{\text{proét}}(\mathbb{H}^d_C,\mathbf{Q}_\ell(r)).$$

This can be easily seen in the case of étale cohomology. For the pro-étale cohomology, the key point is that we can write

$$H^r_{\text{proét}}(\mathbb{H}^d_C,\mathbf{Q}_\ell(r))=\varprojlim_n H^r_{\text{proét}}(U_{n,C},\mathbf{Q}_\ell(r)),$$

where $H^r_{\text{proét}}(U_{n,C},\mathbf{Q}_\ell(r))$ is finite dimensional and the map $\mathscr{H}^{r+1}\to H^r_{\text{proét}}(\mathbb{H}^d_C,\mathbf{Q}_\ell(r))\to H^r_{\text{proét}}(U_{n,C},\mathbf{Q}_\ell(r))$, $(H_0,...,H_r)\to[H_0,...,H_r]$, is locally constant for all n, by arguing as in [35, Lemma 4.4]. All these regulators are continuous.

One can show that the above maps induce the isomorphisms in Theorem 5.8 by imitating the arguments in [35].

Theorem 5.10. *(Iovita-Spiess,* [35, Theorem 4.5]*) The following diagram of Fréchet spaces commutes*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D(\mathscr{H}^{r+1},K)_{\mathrm{deg}} & \longrightarrow & D(\mathscr{H}^{r+1},K) & \xrightarrow{r_{\mathrm{dR}}} & H^r_{\mathrm{dR}}(\mathbb{H}^d_K) & \longrightarrow & 0\\
 & & & & & \searrow^{\mathrm{can}} & \wr\big| & & \\
 & & & & & & \mathrm{Sp}_r(K)^* & &
\end{array}$$

and the sequence is strictly exact. Similarly for ℓ-adic étale and pro-étale cohomologies.

5.3.2. *Generalization of the results of Iovita-Spiess.* Set $X := \widetilde{\mathbb{H}}^d_K$ and $Y := \widetilde{\mathbb{H}}^d_{K,0}$. The above results of Iovita-Spiess can be generalized to Hyodo-Kato cohomology.

Lemma 5.11. *Let $r \geq 0$. There are natural isomorphisms of Fréchet spaces*

$$H^r_{\mathrm{HK}}(Y) \simeq \mathrm{Sp}_r(F)^*, \quad H^r_{\mathrm{HK}}(Y)^{\varphi=p^r} \simeq \mathrm{Sp}_r(\mathbf{Q}_p)^*$$

that are compatible with the isomorphism $H^r_{\mathrm{dR}}(X_K) \simeq \mathrm{Sp}_r(K)^*$ from Theorem 5.8 and the natural maps $\mathrm{Sp}_r(\mathbf{Q}_p)^* \to \mathrm{Sp}_r(F)^* \to \mathrm{Sp}_r(K)^*$.

Proof. We start with $H^r_{\mathrm{HK}}(Y)$. Consider the following diagram

$$\begin{array}{ccccccc}
 & \overset{\mathrm{can}}{\nearrow} & D(\mathscr{H}^{r+1},K) & \overset{r_{\mathrm{dR}}}{\searrow} & & \overset{\mathrm{can}}{\searrow} & \\
D(\mathscr{H}^{r+1},F) & \xrightarrow{r_{\mathrm{HK}}} & H^r_{\mathrm{HK}}(Y) & \xhookrightarrow{\iota_{\mathrm{HK}}} & H^r_{\mathrm{dR}}(X_K) & \xrightarrow{\sim} & \mathrm{Sp}_r(K)^*\\
 & \underset{\mathrm{can}}{\searrow} & \uparrow{\scriptstyle f} & & & \underset{\mathrm{can}}{\nearrow} & \\
 & & \mathrm{Sp}_r(F)^* & \hookrightarrow & & &
\end{array}$$

Here the regulator map r_{HK} is defined in an analogous way to the map r_{dR} but by using the overconvergent Hyodo-Kato Chern classes c_1^{HK} defined in the Appendix. It is continuous. The outer diagram clearly commutes. The small triangle commutes by Theorem 5.10. The square commutes by Lemma A.7. Chasing the diagram we construct the broken arrow, a continuous map $f : \mathrm{Sp}_r(F)^* \to H^r_{\mathrm{HK}}(Y)$ that makes the left bottom triangle commute; it is easy to check that it makes the right bottom triangle commute as well. This implies that the map f is injective. Since $H^r_{\mathrm{HK}}(Y)$ is topologically irreducible (use the Hyodo-Kato isomorphism), it is also surjective (use the fact that $\mathrm{Sp}_r(F)^*$ is closed in $\mathrm{Sp}_r(K)^*$).

The argument for $H^r_{\mathrm{HK}}(Y)^{\varphi=p^r}$ is similar. But first we need to show that the natural map

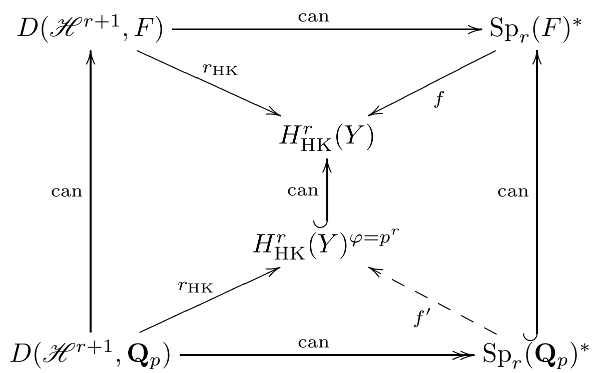
$$H^r_{\mathrm{HK}}(Y)^{\varphi=p^r} \otimes_{\mathbf{Q}_p} F \to H^r_{\mathrm{HK}}(Y) \tag{5.12}$$

is an injection. We compute

$$\begin{aligned}
H^r_{\mathrm{HK}}(Y)^{\varphi=p^r} \otimes_{\mathbf{Q}_p} F &\simeq (\varprojlim_s H^r_{\mathrm{HK}}(Y_s))^{\varphi=p^r} \otimes_{\mathbf{Q}_p} F \simeq (\varprojlim_s H^r_{\mathrm{HK}}(Y_s)^{\varphi=p^r}) \otimes_{\mathbf{Q}_p} F\\
&\simeq \varprojlim_s (H^r_{\mathrm{HK}}(Y_s)^{\varphi=p^r} \otimes_{\mathbf{Q}_p} F) \hookrightarrow \varprojlim_s H^r_{\mathrm{HK}}(Y_s) \simeq H^r_{\mathrm{HK}}(Y),
\end{aligned}$$

as wanted. For the injection above we have used the fact that all $H^r_{\mathrm{HK}}(Y_s)$ are finite dimensional over F.

We look now at the commutative diagram



The key point is that, as shown in Section A.2.1, the map r_{HK} restricted to $D(\mathscr{H}^{r+1}, \mathbf{Q}_p)$ factors through $H^r_{\mathrm{HK}}(Y)^{\varphi=p^r}$. Arguing as above we construct the continuous map f'. It is clearly injective. It is surjective by (5.12). ☐

5.4. **Pro-étale cohomology.** We are now ready to compute the p-adic pro-étale cohomology of $\mathbb{H}^d_C$. Let $r \geq 0$. Since the linearized Frobenius on $H^r_{\mathrm{HK}}(Y)$ is equal to the multiplication by q^r, where $q = |k|$. [23, Cor. 6.6] and $N\varphi = p\varphi N$ [23, Prop. 5.5], the monodromy operator is trivial on $H^r_{\mathrm{HK}}(Y)$. Hence the first isomorphism below is Galois equivariant.

$$\begin{aligned}(H^r_{\mathrm{HK}}(Y)\widehat{\otimes}_F \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} &\simeq (H^r_{\mathrm{HK}}(Y)\widehat{\otimes}_F \mathbf{B}^+_{\mathrm{cr}})^{\varphi=p^r} \simeq (H^r_{\mathrm{HK}}(Y)^{\varphi=p^r}\widehat{\otimes}_{\mathbf{Q}_p} \mathbf{B}^+_{\mathrm{cr}})^{\varphi=p^r}\\ &\simeq H^r_{\mathrm{HK}}(Y)^{\varphi=p^r}\widehat{\otimes}_{\mathbf{Q}_p} \mathbf{B}^{+,\varphi=1}_{\mathrm{cr}} \simeq \mathrm{Sp_r}(\mathbf{Q}_\mathrm{p})^*\widehat{\otimes}_{\mathbf{Q}_p} \mathbf{B}^{+,\varphi=1}_{\mathrm{cr}} = \mathrm{Sp_r}(\mathbf{Q}_\mathrm{p})^*.\end{aligned}$$

The second isomorphism follows from the proof of Lemma 5.11, the fourth one – from this lemma itself, and the third one is clear. Using the above isomorphisms and Lemma 5.11, the map $\iota_{\mathrm{HK}} \otimes \theta : (H^r_{\mathrm{HK}}(Y)\widehat{\otimes}_F \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \to H^r_{\mathrm{dR}}(X_K)\widehat{\otimes}_K C$ can be identified with the natural map $\mathrm{Sp}_r(\mathbf{Q}_p)^* \to \mathrm{Sp}_r(K)^*\widehat{\otimes}_K C$.

Similarly, we compute that

$$(H^r_{\mathrm{HK}}(Y)\widehat{\otimes}_F \mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^{r-1}} \simeq H^r_{\mathrm{HK}}(Y)^{\varphi=p^r}\widehat{\otimes}_{\mathbf{Q}_p} \mathbf{B}^{+,\varphi=p^{-1}}_{\mathrm{cr}} = 0.$$

Combined with Theorem 4.9, these yield the following theorem.

Theorem 5.13. *Let $r \geq 0$. There is a natural map of strictly exact sequences of $G \times \mathscr{G}_K$-Fréchet spaces (over $\mathbf{Q}_p$)*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\Omega^{r-1}(\mathbb{H}^d_K)/\ker d)\widehat{\otimes}_K C & \longrightarrow & H^r_{\mathrm{pro\acute{e}t}}(\mathbb{H}^d_C, \mathbf{Q}_p(r)) & \longrightarrow & \mathrm{Sp}_r(\mathbf{Q}_p)^* & \longrightarrow & 0\\ & & \| & & \downarrow{\scriptstyle\tilde{\beta}} & & \downarrow{\scriptstyle \mathrm{can}} & & \\ 0 & \longrightarrow & (\Omega^{r-1}(\mathbb{H}^d_K)/\ker d)\widehat{\otimes}_K C & \xrightarrow{d} & \Omega^r(\mathbb{H}^d_K)_{d=0}\widehat{\otimes}_K C & \longrightarrow & \mathrm{Sp}_r(K)^*\widehat{\otimes}_K C & \longrightarrow & 0\end{array}$$

where the vertical maps are closed immersions.

6. Étale cohomology of Drinfeld half-space

The purpose of this section is to compute the p-adic étale cohomology of the Drinfeld half-space. Using the following comparison theorem this reduces to the computation of syntomic cohomology.

Proposition 6.1. *Let X be a semistable formal scheme over $\mathscr{O}_K$. Let $r \geq 0$, $\overline{X} := X_{\mathscr{O}_C}$. There is a natural Fontaine-Messing period map*

$$\alpha_{\mathrm{FM}} : \mathrm{R}\Gamma_{\mathrm{syn}}(\overline{X}, \mathbf{Z}_p(r))_{\mathbf{Q}} \to \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_C, \mathbf{Q}_p(r))$$

that is a strict quasi-isomorphism after truncation $\tau_{\leq r}$.

Proof. This can be proved in a way similar to the proof of Corollary 3.35 of Proposition 3.34. ☐

6.1. **Cohomology of differentials.** We gather in this section computations of various bounded differential cohomologies of the Drinfeld half-space.

Let $\widetilde{X} := \widetilde{\mathbb{H}}^d_K$, $X := (\widetilde{\mathbb{H}}^d_K)^\wedge$ be the standard weak formal model, resp. formal model, of the Drinfeld half-space $\mathbb{H}^d_K$. It is equipped with an action of $G = \mathrm{GL}_{d+1}(K)$ compatible with the natural action on the generic fiber. Let $Y := X_0$, $\overline{Y} := Y_{\overline{k}}$. Let F^0 be the set of irreducible components of the special fiber Y. These are isomorphic smooth projective schemes over k that we see as log-schemes with the log-structure induced from Y. Let T be the central irreducible component of Y, i.e., the irreducible component with stabilizer $K^*\mathrm{GL}_{d+1}(\mathscr{O}_K)$. It is obtained from the projective space $\mathbb{P}^d_k$ by first blowing up all k-rational points, then – the strict transforms of k-rational lines, etc. For $0 \leq j \leq d-1$, let $\mathscr{V}^j_0$ be the set of all k-rational linear subvarieties Z of $\mathbb{P}^d_k$ with $\dim(Z) = j$ and let $\mathscr{V}_0 := \bigcup_{j=0}^{d-1} \mathscr{V}^j_0$. The set $\mathscr{V}$ of all strict transforms in T of elements of $\mathscr{V}_0$ is a set of divisors of T; together with the canonical log-structure of the log-point k^0, it induces the log-structure on T.

Let $\widetilde{\theta}_0, \ldots, \widetilde{\theta}_d$ be the standard projective coordinate functions on $\mathbb{P}^d_k$ and on T. For $i, j \in \{0, \ldots, d\}$ and $g \in G$ we call $g\,\mathrm{dlog}(\widetilde{\theta}_i/\widetilde{\theta}_j)$ a *standard logarithmic differential 1-form* on T; exterior products of such forms we call *standard logarithmic differential forms* on T.

6.1.1. *Cohomology of differentials on irreducible components.* As proved by Grosse-Klönne the sheaves of differentials on T are acyclic and the standard logarithmic differential forms generate the k-vector space of global differentials.

Proposition 6.2. *(*[24, Theorem 2.3, Theorem 2.8],[26, Prop. 1.1]*)*

(1) $H^i(T, \Omega^j) = 0$, $i > 0, j \geq 0$.
(2) *The k-vector space $H^0(T, \Omega^j)$, $j \geq 0$, is generated by standard logarithmic forms. In particular, it is killed by d.*
(3) $H^i_{\mathrm{cr}}(T/\mathscr{O}^0_F)$ *is torsion free and*

$$H^i_{\mathrm{cr}}(T/\mathscr{O}^0_F) \otimes_{\mathscr{O}_F} k = H^i_{\mathrm{dR}}(T) = H^0(T, \Omega^i_T).$$

We note here that, the underlying scheme of T being smooth, the crystalline cohomology $H^i_{\mathrm{cr}}(T/\mathscr{O}^0_F) = H^i_{\mathrm{cr}}(T'/\mathscr{O}_F)$, where T' is the underlying scheme of T equipped with the log-structure given by the elements of $\mathscr{V}$.

For $0 \leq j \leq d$, let $\mathbb{L}^j_T$ be the k-vector subspace of $\Omega^j_T(T^0)$, $T^0 := T \setminus \cup_{V \in \mathscr{V}} V$, generated by all j-forms η of the type

$$\eta = y_1^{m_1} \cdots y_j^{m_j}\, \mathrm{dlog}\, y_1 \wedge \cdots \wedge \mathrm{dlog}\, y_j$$

with $m_i \in \mathbf{Z}$ and $y_i \in \mathscr{O}^*_T(T^0)$ such that $y_j = \widetilde{\theta}_j/\widetilde{\theta}_0$ for an isomorphism of k-varieties $T \simeq \mathrm{Proj}(k[\widetilde{\theta}_0, \cdots, \widetilde{\theta}_d])$. By Theorem 6.2, $H^0(T, \Omega^j)$ is the k-vector subspace of $\mathbb{L}^j_T$ generated by all j-forms η as above with $m_i = 0$ for all $0 \leq i \leq j$.

Let $\mathbb{L}^j_T$, resp. $\mathbb{L}^{j,0}_T$, be the constant sheaf on T with values $\mathbb{L}^j_T$, resp. $H^0(T, \Omega^j_T)$. For a non-empty subset S of $\mathscr{V}$ such that $E = \cap_{V \in S} V$ is non-empty, define the subsheaf $\mathbb{L}^j_E$ of $\Omega^j_T \otimes \mathscr{O}_E$ as the image of the composite

$$\mathbb{L}^j_T \to \Omega^j_T \to \Omega^j_T \otimes \mathscr{O}_E.$$

Proposition 6.3. *(*[26, Theorem 1.2]*) The canonical maps*

$$\mathbb{L}^{j,0}_T \hookrightarrow \mathbb{L}^j_T \hookrightarrow \Omega^j_T, \quad \mathbb{L}^j_E \hookrightarrow \Omega^j_T \otimes \mathscr{O}_E$$

induce isomorphisms on Zariski cohomology groups.

6.1.2. *Cohomology of differentials on X and truncations of Y.* We quote an important result of Grosse-Klönne proving acyclicity of the sheaves of differentials on X and vanishing of the differential on their global sections.

Proposition 6.4. *(*[24, Theorem 4.5],[26, Prop. 4.5]*) Let $j \geq 0$.*

(1) *We have topological isomorphisms*[31]

$$H^i(X,\Omega^j_X)\simeq H^i(X,\Omega^j_X\otimes_{\mathscr{O}_K}k)=0,\quad i>0,$$
$$H^0(X,\Omega^j_X)\otimes_{\mathscr{O}_K}k\simeq H^0(X,\Omega^j_X\otimes_{\mathscr{O}_K}k).$$

(2) $d=0$ *on* $H^0(X,\Omega^j_X)$.

Corollary 6.5. *Let* $j\geq 0$. *We have* $H^j_{\rm dR}(X)\xleftarrow{\sim}H^0(X,\Omega^j_X)$. *In particular, it is torsion-free.*

Recall that the set of vertices of the Bruhat-Tits building BT of $\mathrm{PGL}_{d+1}(K)$ is the set of homothety classes of lattices in K^{d+1}. It corresponds to the set of irreducible components of Y. For $s\geq 0$, let BT_s denote the Bruhat-Tits building truncated at s, i.e., the simplicial subcomplex of BT supported on the vertices v such that the combinatorial distance $d(v,v_0)\leq s$, $v_0=[\mathscr{O}_K^{d+1}]$. Here, for a lattice L, $[L]$ denotes the homothety class of L. Let Y_s denote the union of the irreducible components corresponding to the vertices of BT_s. It is a closed subscheme of Y that we equip with the induced log-structure. We will sometimes write Y_∞ for the whole special fiber Y. We denote by $Y_s^\circ:=Y_s\setminus(Y_s\cap\overline{(Y\setminus Y_s)})$, where the bar denotes closure. We have immersions $Y_{s-1}\subset Y_s^\circ\subset Y_s$, where the first one is closed and the second one is open.

The above theorem can be generalized to the idealized log-schemes $Y_n, n\in\mathbf{N}$, in the following way.

Proposition 6.6. *Let* $j\geq 0$, $s\in\mathbf{N}$.

(1) $H^i(Y_s,\Omega^j)=0$ *for* $i>0$.
(2) $d=0$ *on* $H^0(Y_s,\Omega^j)$.

Proof. For the first claim, the argument is analogous to the one of Grosse-Klönne for $s=\infty$. We will sketch it briefly. Take $s\neq\infty$. Since $\Omega^j_{Y_s}$ is locally free over $\mathscr{O}_{Y_s}$, we have the Mayer-Vietoris exact sequence

$$0\to\Omega^j_{Y_s}\to\bigoplus_{Z\in F_s^0}\Omega^j_{Y_s}\otimes\mathscr{O}_Z\to\bigoplus_{Z\in F_s^1}\Omega^j_{Y_s}\otimes\mathscr{O}_Z\to\cdots$$

where F_s^r is the set of non-empty intersections of $(r+1)$ pairwise distinct irreducible components of Y_s and is a finite set (which is also the set of r-simplices of BT_s). By [24, Cor. 1.6], $H^i(Z,\Omega^j_{Y_s}\otimes\mathscr{O}_Z)=0$, $i>0$, for every $Z\in F_s^r$. Hence to show that $H^i(Y_s,\Omega^j_{Y_s})=0$, $i>0$, we need to prove that $H^i(\mathrm{BT}_s,\mathscr{F})=0$, for $i>0$, where $\mathscr{F}$ is the coefficient system on BT_s defined by $\mathscr{F}(Z)=H^0(Y,\Omega^j_{Y_s}\otimes\mathscr{O}_Z)$, for $Z\in F_s^r$. We will use for that an analog of Grosse-Klönne's acyclicity condition. For a lattice chain in BT_s

$$\varpi L_r\subsetneq L_1\subsetneq\cdots\subsetneq L_r$$

we call the ordered r-tuple $([L_1],\ldots,[L_r])$, a pointed $(r-1)$-simplex (with underlying $(r-1)$-simplex the unordered set $\{[L_1],\ldots,[L_r]\}$). Denote it by $\widehat{\eta}$ and consider the set

$$N_{\widehat{\eta}}=\{[L]\mid\varpi L_r\subsetneq L\subsetneq L_1\}.$$

We note that $N_{\widehat{\eta}}$ is a subset of vertices of BT_s. A subset M_0 of $N_{\widehat{\eta}}$ is called *stable* if, for all $L,L'\in M_0$, the intersection $L\cap L'$ also lies in M_0.

Lemma 6.7. *Let* $\mathscr{F}$ *be a cohomological coefficient system on* BT_s. *Let* $1\leq r\leq d$. *Suppose that for any pointed* $(r-1)$*-simplex* $\widehat{\eta}\in\mathrm{BT}_s$ *with underlying* $(r-1)$*-simplex* η *and for any stable subset* M_0 *of* $N_{\widehat{\eta}}$ *the following subquotient complex of the cochain complex* $C(\mathrm{BT}_s,\mathscr{F})$ *with values in* $\mathscr{F}$ *is exact*

$$\mathscr{F}(\eta)\to\prod_{z\in M_0}\mathscr{F}(\{z\}\cup\eta)\to\prod_{z,z'\in M_0,\{z,z'\}\in\mathrm{F}_s^1}\mathscr{F}(\{z,z'\}\cup\eta).$$

Then the r*-th cohomology group* $H^s(\mathrm{BT}_s,\mathscr{F})$ *of* $C(\mathrm{BT}_s,\mathscr{F})$ *vanishes.*

[31] Here and below, cohomology H^* without a subscript denotes Zariski cohomology. All the groups are profinite. This is because they are limits of cohomologies of the truncated log-schemes Y_s described below that are ideally log-smooth and proper.

Proof. For BT this is the main theorem of [25]. The argument used in its proof [25, Theorem 1.2] carries over to our case: when applied to a cocycle from BT_s, the recursive procedure of producing a coboundary in the proof of Theorem 1.2 in loc. cit. "does not leave" BT_s. □

Hence it suffices to check that the above condition is satisfied for our $\mathscr{F}$. But this was checked in [24, Cor. 1.6].

The second claim of the proposition follows from the diagram

$$H^0(Y_s,\Omega^j_{Y_s})\hookrightarrow \bigoplus_{Z\in F^0_s} H^0(Z,\Omega^j_{Y_s}\otimes\mathscr{O}_Z)$$

and Proposition 6.2. □

6.1.3. *Ordinary log-schemes.* A quick review of basic facts concerning ordinary log-schemes.

Let $W_n\Omega^\bullet_Y$ denotes the de Rham-Witt complex of Y/k^0 [29]. Recall first [32, Prop. II.2.1] that if T is a log-smooth and proper log-scheme over k^0, for a perfect field k of positive characterstic p, then $H^i_{\text{ét}}(T,W_n\Omega^j)$ is of finite length and we have $\mathrm{R}\Gamma_{\text{ét}}(T,W\Omega^j)\xrightarrow{\sim}\mathrm{holim}_n\,\mathrm{R}\Gamma_{\text{ét}}(T,W_n\Omega^j)$ for $W\Omega^j:=\varprojlim_n W_n\Omega^j$. It follows that $H^i_{\text{ét}}(T,W\Omega^j)\xrightarrow{\sim}\varprojlim_n H^i_{\text{ét}}(T,W_n\Omega^j)$. The module $M_{i,j}$ of p-torsion of this group is annihilated by a power of p and $H^i_{\text{ét}}(T,W\Omega^j)/M_{i,j}$ is a free $\mathscr{O}_F$-module of finite type [32, Theorem II.2.13]. However, $H^0_{\text{ét}}(T,W\Omega^j)$ is itself a free $\mathscr{O}_F$-module of finite type [32, Cor. II.2.17]. On the other hand, the complex $\mathrm{R}\Gamma_{\text{ét}}(T,W\Omega^\bullet)$ is perfect and $\mathrm{R}\Gamma_{\text{ét}}(T,W\Omega^\bullet)\otimes^L_{\mathscr{O}_F}\mathscr{O}_{F,n}\simeq\mathrm{R}\Gamma_{\text{ét}}(T,W_n\Omega^\bullet)$ [32, Theorem II.2.7].

Let V be a fine (idealized) log-scheme over k^0 that is of Cartier type. We have the subsheaves of boundaries and cocycles of Ω^j_V (thought of as sheaves on $V_{\text{ét}}$)

$$B^j_V:=\mathrm{im}(d:\Omega^{j-1}_V\to\Omega^j_V),\quad Z^j_V:=\ker(d:\Omega^j_V\to\Omega^{j+1}_V).$$

Assume now that V is proper and log-smooth. Recall that it is called *ordinary* if for all $i,j\geq 0$, $H^i_{\text{ét}}(V,B^j)=0$ (see [5], [33]).

We write $W_n\Omega^r_{-,\log}$ for the de Rham-Witt sheaf of logarithmic forms.

Proposition 6.8. *(*[40, Theorem 4.1]*) The following conditions are equivalent (we write $\overline{V}$ for $V_{\overline{k}}$).*

1. *V/k^0 is ordinary.*
2. *For $i,j\geq 0$, the inclusion $\Omega^j_{\overline{V},\log}\subset\Omega^j_{\overline{V}}$ induces a canonical isomorphism of $\overline{k}$-vector spaces*
$$H^i_{\text{ét}}(\overline{V},\Omega^j_{\log})\otimes_{\mathbf{F}_p}\overline{k}\xrightarrow{\sim}H^i_{\text{ét}}(\overline{V},\Omega^j).$$
3. *For $i,j,n\geq 0$, the canonical maps*
$$H^i_{\text{ét}}(\overline{V},W_n\Omega^j_{\log})\otimes_{\mathbf{Z}/p^n}W_n(\overline{k})\to H^i_{\text{ét}}(\overline{V},W_n\Omega^j),$$
$$H^i_{\text{ét}}(\overline{V},W\Omega^j_{\log})\otimes_{\mathbf{Z}_p}W(\overline{k})\to H^i_{\text{ét}}(\overline{V},W\Omega^j),$$
where $W\Omega^r_{\log}:=\varprojlim_n W_n\Omega^r_{\log}$, are isomorphisms.
4. *For $i,j\geq 0$, the de Rham-Witt Frobenius*
$$\mathrm{F}:H^i_{\text{ét}}(V,W\Omega^j)\to H^i_{\text{ét}}(V,W\Omega^j)$$
is an isomorphism.

Example 6.9. The above result implies that, by the Projective Space Theorem, projective spaces are ordinary, and, more generally, so are projectivizations of vector bundles [34, Prop. 1.4]. This implies, by the blow-up diagram, the following:

Proposition 6.10. *(*[34, Prop. 1.6]*) Let X be a proper smooth scheme over k. Let $Y\subset X$ be a smooth closed subscheme, $\widetilde{X}$ the blow-up of Y in X. Then X and Y are ordinary if and only if $\widetilde{X}$ is ordinary.*

And this, in turn, by the weight spectral sequence, implies the following:

Proposition 6.11. *(*[34, Prop. 1.10]*) Assume that $k=\overline{k}$. Let Y be a semistable scheme over k. Assume that it is a union of irreducible components Y_i, $1\leq i\leq r$ such that for all $I\subset\{1,\cdots,r\}$, the intersection Y_I is smooth and ordinary. Then Y, as a log-scheme over k^0, is ordinary.*

Proof. As suggested by Illusie in [34, Rem. 2.8], this can be proved using the weight spectral sequence

$$E_1^{-k,i+k} = \bigoplus_{j\geq 0, j\geq -k} H^{i-s-j}_{\text{ét}}(Y_{2j+k+1}, W\Omega^{s-j-k})(-j-k) \Rightarrow H^{i-s}_{\text{ét}}(Y, W\Omega^s).$$

Here Y_t denotes the intersection of t different irreducible components of Y that are equipped with the trivial log-structure. Such spectral sequences were constructed in [43, 3.23], [45, 4.1.1]. They are Frobenius equivariant (the Tate twist $(-j-k)$ refers to the twist of Frobenius by p^{j+k}) [45, Theorem 9.9]; hence, without the Twist twist, compatible with the de Rham-Witt Frobenius F.

Now, by assumptions, all the schemes Y_t are smooth and ordinary. It follows, by Proposition 6.8, that the Frobenius F induces an isomorphism on $E_1^{-k,i+k}$. Hence also on the abutment $H^{i-s}_{\text{ét}}(Y, W\Omega^s_Y)$, as wanted. □

We drop now the assumption that V is proper. Recall that we have the Cartier isomorphism

$$C : Z^j/B^j \xrightarrow{\sim} \Omega^j, \quad x^p \operatorname{dlog} y_1 \wedge \ldots \wedge \operatorname{dlog} y_j \mapsto x \operatorname{dlog} y_1 \wedge \ldots \wedge \operatorname{dlog} y_j.$$

Lemma 6.12. *Assume that $H^i_{\text{ét}}(V, \Omega^j) = 0$ and that $d = 0$ on $H^0_{\text{ét}}(V, \Omega^j)$ for all $i \geq 1$ and $j \geq 0$. Then V is ordinary [40, 4], i.e., for $i, j \geq 0$, we have $H^i_{\text{ét}}(V, B^j) = 0$.*

Proof. Consider the exact sequences

$$0 \to B^j \to Z^j \xrightarrow{f} \Omega^j \to 0, \quad 0 \to Z^j \to \Omega^j \to B^{j+1} \to 0, \tag{6.13}$$

where the map f is the composition $Z^j \to Z^j/B^j \xrightarrow{\sim} \Omega^j$ of the natural projection and the Cartier isomorphism. Since $H^i_{\text{ét}}(V, \Omega^j) = 0, i > 0$, the first exact sequence yields the isomorphisms

$$H^i_{\text{ét}}(V, B^j) \xrightarrow{\sim} H^i_{\text{ét}}(V, Z^j), \quad i \geq 2. \tag{6.14}$$

It also yields the long exact sequence

$$0 \to H^0_{\text{ét}}(V, B^j) \to H^0_{\text{ét}}(V, Z^j) \to H^0_{\text{ét}}(V, \Omega^j) \xrightarrow{\partial} H^1_{\text{ét}}(V, B^j) \to H^1_{\text{ét}}(V, Z^j) \to 0. \tag{6.15}$$

Since $d = 0$ on $H^0_{\text{ét}}(V, \Omega^j)$ and hence the natural map $H^0_{\text{ét}}(V, Z^j) \to H^0_{\text{ét}}(V, \Omega^j)$ is an isomorphism, the second exact sequence from (6.13) yields the isomorphisms (since we assumed $H^i_{\text{ét}}(V, \Omega^j) = 0$ for $i > 0$)

$$H^i_{\text{ét}}(V, B^{j+1}) \xrightarrow{\sim} H^{i+1}_{\text{ét}}(V, Z^j), \quad i \geq 0. \tag{6.16}$$

To prove the lemma, we will argue by increasing induction on j; the case of $j = 0$ being trivial since $B^0 = 0$. Assume thus that our lemma is true for j and all $i \geq 0$. We will show that this implies that it is true for $j+1$ and all $i \geq 0$. Since $H^1_{\text{ét}}(V, B^j) = 0$ by assumption, the exact sequence (6.15) implies that $H^1_{\text{ét}}(V, Z^j) = 0$. And this implies, by (6.14), that $H^i_{\text{ét}}(V, Z^r) = 0, i \geq 1$. This, in turn, yields, by (6.16), that $H^i_{\text{ét}}(V, B^{j+1}) = 0$, $i \geq 0$. This concludes the proof of the lemma. □

Corollary 6.17. *The idealized log-schemes $Y_s, s \in \mathbf{N} \cup \{\infty\}$, are ordinary.*

Proof. Use Lemma 6.12 and Proposition 6.6. □

Remark 6.18. Proposition 6.11 and Proposition 6.10 show that the underlying scheme of Y_s, for $s < \infty$, is (classically) ordinary by using the weight spectral sequence. One should be able to prove Corolalry 6.17 in an analogous way.

Before we continue, a small digression on topology. Let X be a Stein semistable formal scheme over $\mathscr{O}_K$, $Y := X_0$. Here and below the completed tensor products for complexes are defined (algebraically) as in the example:

$$\mathrm{R}\Gamma_{\mathrm{cr}}(Y/\mathscr{O}^0_{F,n}) \widehat{\otimes}_{\mathscr{O}_{F,n}} \widehat{\mathbf{A}}_{\mathrm{st},n} := \operatorname{holim}_i (\mathrm{R}\Gamma_{\mathrm{cr}}(U_i/\mathscr{O}^0_{F,n}) \otimes_{\mathscr{O}_{F,n}} \widehat{\mathbf{A}}_{\mathrm{st},n}),$$

where $\{U_i\}, i \in \mathbf{N}$, is a Stein covering of Y. For cohomology itself, we set[32]

$$H^j_{\mathrm{cr}}(Y/\mathscr{O}^0_{F,n}) \widehat{\otimes}_{\mathscr{O}_{F,n}} \widehat{\mathbf{A}}_{\mathrm{st},n} := \varprojlim_i (H^j_{\mathrm{cr}}(U_i/\mathscr{O}^0_{F,n}) \otimes_{\mathscr{O}_{F,n}} \widehat{\mathbf{A}}_{\mathrm{st},n}).$$

[32]This notation is abusive but should not lead to confusion.

This is clearly independent of the covering chosen (since two such coverings are nested in each other). We equip these complexes with topology in the following way: we put discrete product topology on the individual terms in the limits and the topology induced from the product topology on the limits themselves (that we represent as the standard mapping fiber or kernel of two products). We note here that this is consistent with the earlier definition of the topology on $\mathrm{R}\Gamma_{\mathrm{cr}}(U_i/\mathscr{O}^0_{F,n})$: U_i being quasi-compact we can represent this complex by an inductive limit of complexes equipped with discrete topology and quasi-isomorphic to each other.

Now, we come back to the Drinfeld half-space.

Lemma 6.19. *For $i\geq 1, j\geq 0$, we have*

1. $H^i_{\text{ét}}(Z,W_n\Omega^j)=0$, *for* $Z=Y,T$,
2. $d=0$ *on* $H^0_{\text{ét}}(T,W_n\Omega^j)$.
3. *For $V=Y,\overline{Y}$, the following sequence is strictly exact*[33]

$$0\to H^0(V,\Omega^j)\stackrel{V^n}{\to} H^0(V,W_{n+1}\Omega^j)\to H^0(V,W_n\Omega^j)\to 0,$$

Proof. For claim (1), we start with $Z=Y$. We have subsheaves

$$0=B^j_0\subset B^j_1\subset\ldots\subset Z^j_1\subset Z^j_0=\Omega^j_Y$$

such that $B^j_1=B^j_Y$, $Z^j_0=\Omega^j_Y$, $Z^j_1=Z^j_Y$ and for all n we have inverse Cartier isomorphisms

$$C^{-1}:B^j_n\stackrel{\sim}{\to}B^j_{n+1}/B^j_1,\quad C^{-1}:Z^j_n\stackrel{\sim}{\to}Z^j_{n+1}/B^j_1.$$

By Proposition 6.4 and Lemma 6.12, we have $H^i_{\text{ét}}(Y,B^j_1)=H^i_{\text{ét}}(Y,Z^j_1)=0$ for $i>0$, thus the same holds with B^j_1 and Z^j_1 replaced by B^j_n and Z^j_n. On the other hand, define R^j_n by the exact sequence

$$0\to R^j_n\to B^j_{n+1}\oplus Z^{j-1}_n\to B^j_1\to 0,\tag{6.20}$$

the last map being (C^n,dC^{n-1}). By the previous discussion, we have $H^i_{\text{ét}}(Y,R^j_n)=0$ for $i>0$. Hyodo and Kato prove [30, Theorem 4.4] that we have an exact sequence

$$0\to\frac{\Omega^j\oplus\Omega^{j-1}}{R^j_n}\to W_{n+1}\Omega^j\to W_n\Omega^j\to 0.\tag{6.21}$$

Note that $\frac{\Omega^j\oplus\Omega^{j-1}}{R^j_n}$ does not have higher cohomology since each of $\Omega^j,\Omega^{j-1},R^j_n$ has this property (use Proposition 6.4). Using the previous exact sequence, the result follows by induction on n (using that $W_1\Omega^j\simeq\Omega^j$).

In the case of $Z=T$ we argue in a similar way using Proposition 6.2 instead of Proposition 6.4.

For claim (2), since $\Gamma_{\text{ét}}(T,W_n\Omega^j)\hookrightarrow\Gamma_{\text{ét}}(T_{\overline{k}},W_n\Omega^j)$, we can pass to $T_{\overline{k}}$. But then, by ordinarity of $T_{\overline{k}}$, we have (see Proposition 6.8)

$$H^0_{\text{ét}}(T_{\overline{k}},W_n\Omega^j)\simeq H^0_{\text{ét}}(T_{\overline{k}},W_n\Omega^j_{\log})\otimes_{\mathbf{Z}/p^n}W_n(\overline{k})$$

and the latter group clearly has a trivial differential.

To prove claim (3), we note first that Lemma 6.12 applies to both Y and $\overline{Y}$. For Y this follows from Proposition 6.4. For $\overline{Y}$, we use Corollary 6.17 to write down a sequence of quasi-isomorphisms

$$\begin{aligned}\mathrm{R}\Gamma(\overline{Y},\Omega^j)&\simeq\mathrm{holim}_s\,\mathrm{R}\Gamma(\overline{Y}^{\circ}_s,\Omega^j)\simeq\mathrm{holim}_s\,\mathrm{R}\Gamma(\overline{Y}_s,\Omega^j)\\&\simeq\mathrm{holim}_s\,H^0(\overline{Y}_s,\Omega^j)\simeq\varprojlim_s H^0(\overline{Y}_s,\Omega^j).\end{aligned}$$

It follows that $H^i(\overline{Y},\Omega^j)=0$ for $i>0$. To see that $d=0$ on $H^0(\overline{Y},\Omega^j)$ we use the embedding $H^0(\overline{Y},\Omega^j)\hookrightarrow\prod_{C\in F^0}H^0(\overline{C},\Omega^j)$ and Proposition 6.2.

Now, set $V=Y,\overline{Y}$. By Lemma 6.12, we have $H^i_{\text{ét}}(V,B^j)=0$ $i,j\geq 0$. Note that, by (6.21), we have the exact sequence

$$0\to(H^0_{\text{ét}}(V,\Omega^j)\oplus H^0_{\text{ét}}(V,\Omega^{j-1}))/H^0_{\text{ét}}(V,R^j_n)\xrightarrow{(V^n,dV^n)}H^0_{\text{ét}}(V,W_{n+1}\Omega^j)\to H^0_{\text{ét}}(V,W_n\Omega^j)\to 0.$$

[33] Do not confuse V with the Verschiebung in V^n.

It remains to show that the natural map from $H^0_{\text{ét}}(V,\Omega^j)$ to the leftmost term is an isomorphism, or that, the natural map $H^0_{\text{ét}}(V,R^j_n)\to H^0_{\text{ét}}(V,\Omega^{j-1})$ is an isomorphism. The exact sequence (6.20) yields that the natural map $H^0_{\text{ét}}(V,R^j_n)\to H^0_{\text{ét}}(V,Z^{j-1}_n)$ is an isomorphism. It remains thus to show that so is the natural map $H^0_{\text{ét}}(V,Z^{j-1}_n)\to H^0_{\text{ét}}(V,\Omega^{j-1})$.

For that it suffices to show that the natural maps $H^0_{\text{ét}}(V,Z^{j-1}_{n+1})\to H^0_{\text{ét}}(V,Z^{j-1}_n)$, $n\geq 0$, are isomorphisms. We will argue by induction on $n\geq 0$. Since $d=0$ on $H^0_{\text{ét}}(V,\Omega^{j-1})$ this is clear for $n=0$. Assume now that this is true for $n-1$. We will show it for n itself. Consider the commutative diagram

$$\begin{array}{ccc} H^0_{\text{ét}}(V,Z^{j-1}_{n+1}) & \xrightarrow[\sim]{C} & H^0_{\text{ét}}(V,Z^{j-1}_n) \\ \Big\downarrow{\scriptstyle \text{can}} & & \Big\downarrow{\scriptstyle \text{can}\ \wr} \\ H^0_{\text{ét}}(V,Z^{j-1}_n) & \xrightarrow[\sim]{C} & H^0_{\text{ét}}(V,Z^{j-1}_{n-1}) \end{array}$$

The top and bottom isomorphisms follow from the isomorphism $C^{-1}:Z^j_i\xrightarrow{\sim} Z^j_{i+1}/B^j_1$, $i\geq 0$. The right vertical map is an isomorphism by the inductive assumption. We get that the left vertical map is an isomorphism, as wanted.

Finally, to see that the exact sequence in claim (3) is strictly exact note that for Y this follows from compactness of $H^0(Y,\Omega^j)$ and $H^0(Y,W_{n+1}\Omega^j)$ and for $\overline{Y}$ this follows from the case of Y by étale base change. □

We will need a generalization of the above results.

6.1.4. *Cohomology of differentials II.* We work with the Drinfeld half-space. We list the following cohomological identities.

Proposition 6.22. *Let $j\geq 0$. Let S be a topological $\mathscr{O}_K$-module and let R be a topological $W(k)$- or $W(\overline{k})$-module. Assume that S and R are orthonormalizable.*

(1) *The following natural maps are topological quasi-isomorphisms*[34]

$$\begin{aligned}
&\Gamma(X,\Omega^j_{X,n})\widehat{\otimes}_{\mathscr{O}_{K,n}}S_n\xrightarrow{\sim}\mathrm{R}\Gamma(X,\Omega^j_{X,n})\widehat{\otimes}_{\mathscr{O}_{K,n}}S_n,\\
&\Gamma_{\text{ét}}(Y,W_n\Omega^j)\widehat{\otimes}_{\mathscr{O}_{F,n}}R_n\xrightarrow{\sim}\mathrm{R}\Gamma_{\text{ét}}(Y,W_n\Omega^j)\widehat{\otimes}_{\mathscr{O}_{F,n}}R_n,\\
&\mathrm{R}\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n}R_n\xrightarrow{\sim}\mathrm{R}\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j)\widehat{\otimes}_{W_n(\overline{k})}R_n,\\
&\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n}R_n\xrightarrow{\sim}\mathrm{R}\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n}R_n.
\end{aligned}$$

(2) $d=0$ *on* $\Gamma_{\text{ét}}(X,\Omega^j_{X,n})\widehat{\otimes}_{\mathscr{O}_{K,n}}S_n$ *and on* $\Gamma_{\text{ét}}(Y,W_n\Omega^j_Y)\widehat{\otimes}_{\mathbf{Z}/p^n}R_n$.

Proof. In this proof, to lighten the notation, we will write simply $\mathrm{R}\Gamma(Z,\Omega^\bullet_n):=\mathrm{R}\Gamma(Z,\Omega^\bullet_{Z,n})$ for the de Rham cohomology of the log-scheme Z_n. We have the spectral sequence

$$E_2^{q,i}=\mathrm{R}^q\varprojlim_s(H^i(Y_s^\circ,\Omega^j_n)\otimes_{\mathscr{O}_{K,n}}S_n)\Rightarrow H^{q+i}(\mathrm{R}\Gamma(X,\Omega^j_n)\widehat{\otimes}_{\mathscr{O}_{K,n}}S_n).$$

Since the pro-systems

$$\{H^i(Y_s^\circ,\Omega^j_n)\otimes_{\mathscr{O}_{K,n}}S_n\},s\geq 0,\quad \{H^i(Y_s,\Omega^j_n)\otimes_{\mathscr{O}_{K,n}}S_n\},s\geq 0,$$

are equivalent (and $H^i(Y_s,\Omega^j_n)$ is of finite type since Y_s is ideally log-smooth and proper over k^0), they both have a trivial $\mathrm{R}^q\varprojlim_s$, $q>0$. Hence the spectral sequence degenerates and we have

$$H^i(\mathrm{R}\Gamma(X,\Omega^j_n)\widehat{\otimes}_{\mathscr{O}_{K,n}}S_n)\simeq\varprojlim_s(H^i(Y_s^\circ,\Omega^j_n)\otimes_{\mathscr{O}_{K,n}}S_n)\simeq\varprojlim_s(H^i(Y_s,\Omega^j_n)\otimes_{\mathscr{O}_{K,n}}S_n).$$

[34]Here by a *topological quasi-isomorphism* we mean a morphism of complexes that induces a homeomorphism on cohomology groups.

Moreover, using a basis $\{e_\lambda\}, \lambda\in I$, of S_n over $\mathscr{O}_{K,n}$, we get an embedding

$$\varprojlim_s (H^i(Y_s^\circ,\Omega^j_n)\otimes_{\mathscr{O}_{K,n}}S_n)\hookrightarrow \prod_{\lambda\in I}H^i(X,\Omega^j_n)e_\lambda$$

Since the latter groups are trivial for $i>0$, by Proposition 6.4, the vanishing of $H^i(\mathrm{R}\Gamma(X,\Omega^j_n)\widehat{\otimes}_{\mathscr{O}_{K,n}}S_n)$ follows. This embedding also shows that $d=0$ on H^0 in part (2) of the proposition.

The proof for the second map in part (1) of the proposition is analogous with Lemma 6.19 replacing Proposition 6.4.

For the proof for the third map in part (1) of the proposition, consider now the sequence of quasi-isomorphisms

$$\begin{aligned}\mathrm{R}\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n}R_n &= \operatorname{holim}_s(\mathrm{R}\Gamma_{\text{ét}}(\overline{Y}^\circ_s,W_n\Omega^j_{\log})\otimes_{\mathbf{Z}/p^n}R_n)\simeq \operatorname{holim}_s(\mathrm{R}\Gamma_{\text{ét}}(\overline{Y}_s,W_n\Omega^j_{\log})\otimes_{\mathbf{Z}/p^n}R_n)\\ &\xrightarrow{\sim} \operatorname{holim}_s(\mathrm{R}\Gamma_{\text{ét}}(\overline{Y}_s,W_n\Omega^j)\otimes_{W_n(\overline{k})}R_n)\simeq \operatorname{holim}_s(\mathrm{R}\Gamma_{\text{ét}}(\overline{Y}^\circ_s,W_n\Omega^j)\otimes_{W_n(\overline{k})}R_n)\\ &= \mathrm{R}\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j)\widehat{\otimes}_{W_n(\overline{k})}R_n.\end{aligned}$$

The second and the fourth quasi-isomorphisms are clear. The third quasi-isomorphism follows from the fact that, by Corollary 6.17, the log-scheme $\overline{Y}_s$ is ordinary and we have Proposition 6.8.

For the fourth quasi-isomorphism in part (1) of the proposition, use the second and the third one to reduce to showing that we have a natural quasi-isomorphism

$$\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n}R_n\simeq \Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j)\widehat{\otimes}_{W_n(\overline{k})}R_n.$$

But, by Proposition 6.8, we have

$$\begin{aligned}\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n}R_n &= \varprojlim_s(\Gamma_{\text{ét}}(\overline{Y}^\circ_s,W_n\Omega^j_{\log})\otimes_{\mathbf{Z}/p^n}R_n)\simeq \varprojlim_s(\Gamma_{\text{ét}}(\overline{Y}_s,W_n\Omega^j_{\log})\otimes_{\mathbf{Z}/p^n}R_n)\\ \simeq \varprojlim_s(\Gamma_{\text{ét}}(\overline{Y}_s,W_n\Omega^j)\otimes_{W_n(\overline{k})}R_n) &\simeq \varprojlim_s(\Gamma_{\text{ét}}(\overline{Y}^\circ_s,W_n\Omega^j)\otimes_{W_n(\overline{k})}R_n)=\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j)\widehat{\otimes}_{W_n(\overline{k})}R_n.\end{aligned}$$

Finally, for the topology, note that the only nontrivial statement is that the isomorphim

$$\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n}R_n\xrightarrow{\sim}\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j)\widehat{\otimes}_{W_n(\overline{k})}R_n$$

is topological. But, since this isomorphism holds already for for the individual schemes Y_s, this is clear.

It remains to show that $d=0$ on $\Gamma_{\text{ét}}(Y,W_n\Omega^j)\widehat{\otimes}_{\mathscr{O}_{F,n}}R_n$. Assume first that R is a $W(\overline{k})$-module. Arguing as above we obtain the embedding (notation as above)

$$\Gamma_{\text{ét}}(Y,W_n\Omega^j)\widehat{\otimes}_{\mathscr{O}_{F,n}}R_n\xrightarrow{\sim}\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n}\mathrm{R}_n\hookrightarrow\prod_{\lambda\in I}\Gamma_{\text{ét}}(\overline{Y},W_n\Omega^j_{\log})e_\lambda.$$

$d=0$ follows. If R is only a $W(k)$-module, we write

$$\Gamma_{\text{ét}}(Y,W_n\Omega^j)\widehat{\otimes}_{\mathscr{O}_{F,n}}R_n\hookrightarrow\Gamma_{\text{ét}}(Y,W_n\Omega^j)\widehat{\otimes}_{\mathscr{O}_{F,n}}(W_n(\overline{k})\otimes_{\mathscr{O}_{F,n}}R_n)$$

to obtain $d=0$ in this case as well. □

The following corollary lists analogs in the case of the Drinfeld half-space of properties of ordinary schemes from Proposition 6.8.

Corollary 6.23. (1) *For $i,j\geq 0$, we have a canonical topological isomorphism*[35]

$$H^i_{\text{ét}}(\overline{Y},\Omega^j_{\log})\widehat{\otimes}_{\mathbf{F}_p}\overline{k}\xrightarrow{\sim}H^i_{\text{ét}}(\overline{Y},\Omega^j).$$

(2) *For $i,j,n\geq 0$, the canonical maps*

$$H^i_{\text{ét}}(\overline{Y},W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n}W_n(\overline{k})\to H^i_{\text{ét}}(\overline{Y},W_n\Omega^j),$$
$$H^i_{\text{ét}}(\overline{Y},W\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}_p}W(\overline{k})\to H^i_{\text{ét}}(\overline{Y},W\Omega^j)$$

are topological isomorphisms[36].

[35] More specifically, topological isomorphism of projective limits of $\overline{k}$-vector spaces of finite rank.

[36] More specifically, topological isomorphisms of projective limits of $W_?(\overline{k})$-modules, free and of finite rank.

Proof. The first two quasi-isomorphisms are actually included in the above proposition. For the third quasi-isomorphism we need first to explain topology. We define it as usual by taking locally p-adic topology and p-adically completed tensor product. Now, both sides are nontrivial only in degree zero: by Lemma 6.19 and the second isomorphism of this corollary, the projective systems $\{H^0_{\text{ét}}(\overline{Y}, W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n} W_n(\overline{k})\}_n$ and $\{H^0_{\text{ét}}(\overline{Y}, W_n\Omega^j)\}_n$ are Mittag-Leffler. In degree zero we pass, as usual, to the limit over the truncated subschemes of the special fiber and there, since these subschemes are ordinary, we have a term-wise isomorphism, as wanted. □

Remark 6.24. There is an alternative argument which proves Proposition 6.22 and which does not use ordinarity of the truncated log-scheme Y_s. It starts with proving the above corollary. We present it in the Appendix.

Define the map

$$\iota_Y : H^i_{\text{ét}}(Y, W\Omega^\bullet_Y) \simeq H^i_{\text{cr}}(Y/\mathscr{O}^0_F) \to H^i_{\text{cr}}(Y/\mathscr{O}^0_F, F) \xleftarrow{\sim} H^i_{\text{rig}}(Y/\mathscr{O}^0_F) \xrightarrow{\iota_{\text{HK}}} H^i_{\text{rig}}(Y/\mathscr{O}^\times_K),$$

where $\mathrm{R}\Gamma_{\text{cr}}(X_0/\mathscr{O}^0_F)$ denotes the integral crystalline Hyodo-Kato cohomology and $H^i_{\text{cr}}(Y/\mathscr{O}^0_F, F)$ is defined in analogous way to $H^i\mathrm{R}\Gamma_{\text{cr}}(\overline{X})$ from Section 3.3.1, i.e., by taking $\mathrm{R}\Gamma_{\text{cr}}((\bullet)/\mathscr{O}^0_F)\otimes_{\mathscr{O}_F} F$ locally on Y and then globalizing it via hypercoverings. This map is continuous (to see that the first isomorphism in the definition of ι_Y is a homeomorphism, note that both sides are nontrivial only for $i = 0$, and there pass to W_n, where it is clear).

Proposition 6.25. (1) *The above map induces an injection*

$$\iota_Y : H^i_{\text{ét}}(Y, W\Omega^\bullet)\otimes_{\mathscr{O}_F} K \hookrightarrow H^i_{\text{rig}}(Y/\mathscr{O}^\times_K).$$

(2) *The canonical map*

$$H^i_{\text{dR}}(X)\otimes_{\mathscr{O}_K} K \to H^i_{\text{dR}}(X_K)$$

is injective.

Proof. For the first claim, it suffices to show that we have a commutative diagram

$$\begin{array}{ccc} H^i_{\text{ét}}(Y, W\Omega^\bullet) & \overset{\alpha}{\hookrightarrow} & \prod_{j\in\mathbf{N}} H^i_{\text{ét}}(C_j, W\Omega^\bullet) \\ \downarrow{\scriptstyle \iota_Y} & & \downarrow{\scriptstyle \prod_j \iota_{C_j}} \\ H^i_{\text{rig}}(Y/\mathscr{O}^\times_K) & \longrightarrow & \prod_{j\in\mathbf{N}} H^i_{\text{rig}}(C_j/\mathscr{O}^\times_K), \end{array}$$

where $C_j, j \in \mathbf{N}$, is the set of irreducible components of Y and the map ι_{C_j} is defined in an analogous way to the map ι_Y but by replacing the Hyodo-Kato map by the composition

$$H^i_{\text{rig}}(C_j/\mathscr{O}^0_F) \xrightarrow{\sim} H^i_{\text{rig}}(C^0_j/\mathscr{O}^0_F) \xrightarrow{\iota_{\text{HK}}} H^i_{\text{rig}}(C^0_j/\mathscr{O}^\times_K) \xleftarrow{\sim} H^i_{\text{rig}}(C_j/\mathscr{O}^\times_K).$$

Since the Hyodo-Kato map is compatible with Zariski localization the above diagram commutes.

We claim that we have natural isomorphisms

$$H^0_{\text{ét}}(Y, W\Omega^i) \xrightarrow{\sim} H^i_{\text{ét}}(Y, W\Omega^\bullet), \quad H^0_{\text{ét}}(C_j, W\Omega^i) \xrightarrow{\sim} H^i_{\text{ét}}(C_j, W\Omega^\bullet).$$

Indeed, set $Z = Y, C_j$. We have $H^0_{\text{ét}}(Z, W\Omega^i) = \varprojlim_n H^0_{\text{ét}}(Z, W_n\Omega^i)$. Since, by Proposition 6.22 and Lemma 6.19,

$$\mathrm{R}\Gamma_{\text{ét}}(Z, W_n\Omega^\bullet) \simeq \oplus_j H^0_{\text{ét}}(Z, W_n\Omega^j)[-j],$$

this implies that

$$H^i_{\text{ét}}(Z, W\Omega^\bullet) \simeq \varprojlim_n H^i_{\text{ét}}(Z, W_n\Omega^\bullet) \simeq \varprojlim_n H^0_{\text{ét}}(Z, W_n\Omega^i),$$

as wanted. In particular, there is no torsion.

It follows that the maps ι_{C_j} in the above diagram are injections: they are isomorphisms after tensoring the domains with K and the domains are torsion-free. The map α is an injection because so is, by definition, the map α' in the commutative diagram

$$\begin{array}{ccccc} H^0_{\text{ét}}(Y, W\Omega^i) & \xrightarrow{\alpha} & \prod_{j\in\mathbf{N}} H^0_{\text{ét}}(C_j, W\Omega^i) & \longrightarrow & \prod_{j\in\mathbf{N}} H^0_{\text{ét}}(C_j^0, W\Omega^i) \\ & \searrow^{\alpha'} & & & \uparrow \wr \\ & & & & H^0_{\text{ét}}(Y_{\text{tr}}, W\Omega^i_{Y_{\text{tr}}}), \end{array}$$

where Y_{tr} denotes the nonsingular locus of Y.

We note that the above computation shows also that the natural map $H^i_{\text{cr}}(Y/\mathscr{O}_F^0)\otimes_{\mathscr{O}_F} F \to H^i_{\text{cr}}(Y/\mathscr{O}_F^0, F)$ is an injection. This will be useful in proving the second claim of the proposition. Using the diagram (A.6) we can form a commutative diagram

$$\begin{array}{ccccc} H^i_{\text{dR}}(X)\otimes_{\mathscr{O}_K} K & & \xrightarrow{\text{can}} & & H^i_{\text{dR}}(X_K) \\ \iota_{\text{HK}} \uparrow \wr & & & & \iota_{\text{HK}} \uparrow \wr \\ H^i_{\text{cr}}(Y/\mathscr{O}_F^0)\otimes_{\mathscr{O}_F} K & \hookrightarrow & H^i_{\text{cr}}(Y/\mathscr{O}_F^0, F)\otimes_{\mathscr{O}_F} K & \xleftarrow{\sim} & H^i_{\text{rig}}(Y/\mathscr{O}_F^0)\otimes_{\mathscr{O}_F} K \end{array}$$

Here the first map ι_{HK} is the bounded Hyodo-Kato isomorphism described in the Appendix. Since the first bottom map is an injection so is the top map, as wanted. □

6.2. **Relation to Steinberg representations.** We proved in the previous section that, for all $i > 0$, the spaces $H^i_{\text{ét}}(Y, W\Omega^r)$ and $H^i_{\text{ét}}(\overline{Y}, W\Omega^r_{\log})$ vanish. The purpose of this section is to prove the following result describing the corresponding spaces for $i = 0$ in terms of generalized Steinberg representations.

Theorem 6.26. *Let $r \geq 0$.*

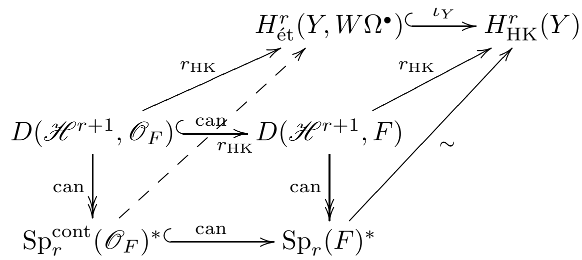
(1) *We have natural isomorphisms of locally convex topological vector spaces (more precisely, weak duals of Banach spaces)*
  (a) $H^0(Y, W\Omega^r)\otimes_{\mathscr{O}_F} F \simeq H^r(Y, W\Omega^\bullet)\otimes_{\mathscr{O}_F} F \simeq \mathrm{Sp}_r^{\text{cont}}(F)^*$,
  (b) $H^0_{\text{ét}}(Y, W\Omega^r_{\log})_{\mathbf{Q}} \simeq \mathrm{Sp}_r^{\text{cont}}(\mathbf{Q}_p)^*$,
  (c) $H^0(X, \Omega^r)\otimes_{\mathscr{O}_K} K \simeq H^r_{\text{dR}}(X)\otimes_{\mathscr{O}_K} K \simeq \mathrm{Sp}_r^{\text{cont}}(K)^*$,
  (d) $H^0_{\text{ét}}(\overline{Y}, W\Omega^r_{\log})_{\mathbf{Q}} \simeq \mathrm{Sp}_r^{\text{cont}}(\mathbf{Q}_p)^*$.
  They are compatible with the canonical maps between Steinberg representations and with the isomorphisms

$$H^r_{\text{dR}}(X_K) \simeq \mathrm{Sp}_r(K)^*, \quad H^r_{\text{HK}}(X) \simeq \mathrm{Sp}_r(F)^*$$

  from Theorem 5.10 and Lemma 5.11.

(2) *We have natural topological isomorphisms*
  (a) $H^0_{\text{ét}}(Y, W\Omega^r) \simeq H^r_{\text{ét}}(Y, W\Omega^\bullet) \simeq \mathrm{Sp}_r^{\text{cont}}(\mathscr{O}_F)^*$ *and* $H^0(Y, \Omega^r) \simeq \mathrm{Sp}_r(k)^*$,
  (b) $H^0_{\text{ét}}(Y, W\Omega^r_{\log}) \simeq \mathrm{Sp}_r^{\text{cont}}(\mathbf{Z}_p)^*$ *and* $H^0_{\text{ét}}(Y, \Omega^r_{\log}) \simeq \mathrm{Sp}_r(\mathbf{F}_p)^*$,
  (c) $H^0(X, \Omega^r) \simeq H^r_{\text{dR}}(X) \simeq \mathrm{Sp}_r^{\text{cont}}(\mathscr{O}_K)^*$,
  (d) $H^0_{\text{ét}}(\overline{Y}, W\Omega^r_{\log}) \simeq \mathrm{Sp}_r^{\text{cont}}(\mathbf{Z}_p)^*$ *and* $H^0_{\text{ét}}(\overline{Y}, \Omega^r_{\log}) \simeq \mathrm{Sp}_r(\mathbf{F}_p)^*$.
  They are compatible with the canonical maps between Steinberg representations and with the above isomorphisms.

Proof. Consider the following diagram



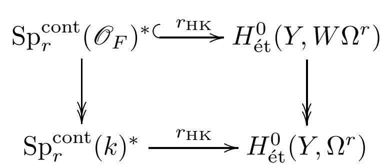
The bottom square clearly commutes. The first (continuous) regulator r_{HK} is defined by integrating the crystalline Hyodo-Kato Chern classes c_1^{HK} defined in the Appendix. By Section A.2.1 it makes the top square commute. The right triangle commutes by Lemma 5.11. It follows that there exists a broken arrow (we will call it r_{HK} as well) that makes the left triangle commute. This map is continuous and also clearly makes the adjacent square commute. Hence it is an injection. We will prove that it is an isomorphism after inverting p.

The above combined with Proposition 6.25 and Theorem 5.11 gives the embeddings

$$\mathrm{Sp}^{\text{cont}}_r(\mathscr{O}_F)^* \otimes_{\mathscr{O}_F} F \stackrel{r_{\text{HK}}}{\hookrightarrow} H^0_{\text{ét}}(Y, W\Omega^r) \otimes_{\mathscr{O}_F} F \stackrel{f}{\hookrightarrow} H^r_{\text{HK}}(Y) \simeq \mathrm{Sp}_r(F)^*.$$

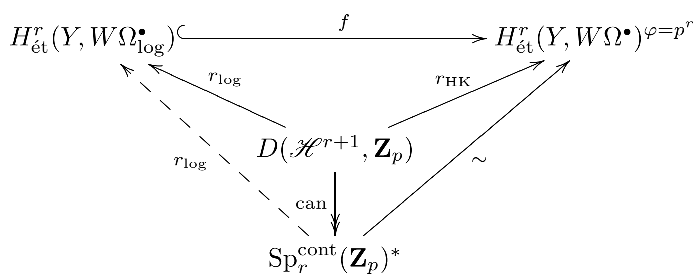
Their composite is the canonical embedding. The image of the map f must be in the subspace of G-bounded vectors of $\mathrm{Sp}_r(K)^*$, since $H^0_{\text{ét}}(Y, W\Omega^r)$ is compact (it is naturally an inverse limit of finite free $\mathscr{O}_F$-modules). That subspace is identified with $\mathrm{Sp}^{\text{cont}}_r(F)^* \simeq \mathrm{Sp}^{\text{cont}}_r(\mathscr{O}_F)^* \otimes_{\mathscr{O}_F} F$ by Corollary 5.9. It follows that the map r_{HK} is an isomorphism.

In fact, the above map r_{HK} is already an integral isomorphism (as stated in part (2a)). To see this, consider the commutative diagram



$H^0_{\text{ét}}(Y, W\Omega^r)$ is a G-equivariant lattice in $H^0_{\text{ét}}(Y, W\Omega^r)_{\mathbf{Q}} \simeq \mathrm{Sp}^{\text{cont}}_r(F)^*$ hence, by Corollary 5.3, it is homothetic to $\mathrm{Sp}^{\text{cont}}_r(\mathscr{O}_F)^*$. It follows that $H^0_{\text{ét}}(Y, \Omega^r) \simeq \mathrm{Sp}^{\text{cont}}_r(k)^*$ is irreducible. Moreover, the bottom map r_{HK} is nonzero: by construction of the top map r_{HK}, the symbol $\mathrm{dlog}\, z_1 \wedge \cdots \wedge \mathrm{dlog}\, z_r$ for coordinates $z_1, \ldots, z_r$ of $\mathbb{P}^d_K$ is in the image. It follows that it is an isomorphism hence so is the top map r_{HK} as well. Moreover, the latter is a topological isomorphism since the domain is compact. It follows that its rational version is a topological isomorphism as well, which proves part (1a) of the theorem.

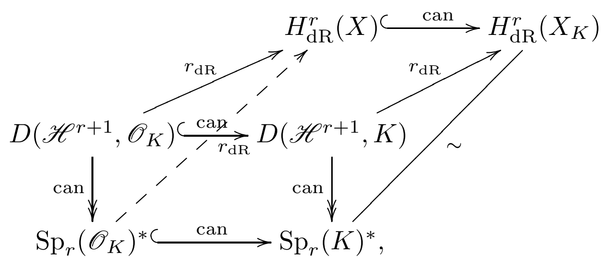
The proof of part (1b) is very similar to the proof of part (1a), so we will be rather brief. Consider the commutative diagram



Here the (continuous) regulator $r_{\log}$ is defined by integrating the crystalline logarithmic de Rham-Witt Chern classes $c_1^{\log}$ defined in the Appendix. Arguing as above we can construct the broken arrow, which is again a continuous map, making the whole diagram commute. It easily follows that both maps f and

$r_{\log}$ are isomorphisms. Now, that they are topological isomorphisms we argue first integrally, as for part (2b), and then rationally as for part (2a).

For part (1c) one repeats the argument starting with the following commutative diagram



where the continuous (bounded) regulator r_{dR} is defined by integrating the integral de Rham Chern classes c_1^{dR} defined in the Appendix, and using the fact that, by Proposition 6.25, we have

$$H^r_{\mathrm{dR}}(X)\otimes_{\mathscr{O}_K}K\hookrightarrow H^r_{\mathrm{dR}}(X)\simeq \mathrm{Sp}_r(K)^*$$

and, by Proposition 6.22, we have $H^r_{\mathrm{dR}}(X)\simeq H^0(X,\Omega^r)$. The integral part (2c) follows as above.

Parts (1d) and (2d) follow from parts (1b) and (2b) and the following lemma.

Lemma 6.27. *For $n\geq 1$, we have canonical topological isomorphisms*

$$H^0_{\text{ét}}(Y,W_n\Omega^r_{\log})\xrightarrow{\sim}H^0_{\text{ét}}(\overline{Y},W_n\Omega^r_{\log}),\quad H^0_{\text{ét}}(Y,W\Omega^r_{\log})\xrightarrow{\sim}H^0_{\text{ét}}(\overline{Y},W\Omega^r_{\log}).$$

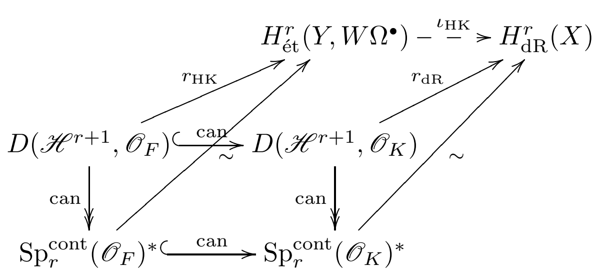
Proof. It suffices to prove the first isomorphism and, since both sides satisfy p-adic devissage, it suffices to do it for $n=1$. We have $H^0_{\text{ét}}(\overline{Y},\Omega^r_{\log})\xrightarrow{\sim}H^0_{\text{ét}}(\overline{Y},\Omega^r)^{C=1}$. On the other hand, by étale base change, we have a topological isomorphism $H^0_{\text{ét}}(\overline{Y},\Omega^r)\simeq H^0_{\text{ét}}(Y,\Omega^r)\widehat{\otimes}_k\overline{k}$. And parts (2a) and (2b) of the theorem show that the natural map $H^0_{\text{ét}}(Y,\Omega^r_{\log})\widehat{\otimes}_{\mathbf{F}_p}k\to H^0_{\text{ét}}(Y,\Omega^r)$ is a topological isomorphism. Hence, since $C=1$ on $H^0_{\text{ét}}(Y,\Omega^r_{\log})$, we obtain topological isomorphisms

$$H^0_{\text{ét}}(\overline{Y},\Omega^r_{\log})\xleftarrow{\sim}H^0_{\text{ét}}(\overline{Y},\Omega^r)^{C=1}\xleftarrow{\sim}(H^0_{\text{ét}}(Y,\Omega^r_{\log})\widehat{\otimes}_{\mathbf{F}_p}\overline{k})^{C=1}\xleftarrow{\sim}H^0_{\text{ét}}(Y,\Omega^r_{\log}),$$

as wanted. □

□

Remark 6.28. Consider the commutative diagram



The dotted arrow is defined to make the diagram commute. It is continuous. It can be thought of as an integral Hyodo-Kato map. Compatibilities used in the proof of Theorem 6.26 ensure that it is compatible with the bounded and the overconvergent Hyodo-Kato maps. Because the natural map $\mathrm{Sp}_r^{\mathrm{cont}}(\mathscr{O}_F)^*\otimes_{\mathscr{O}_F}\mathscr{O}_K\xrightarrow{\sim}\mathrm{Sp}_r^{\mathrm{cont}}(\mathscr{O}_K)^*$ is an isomorphism, we get the integral Hyodo-Kato (topological) isomorphism

$$\iota_{\mathrm{HK}}:H^r_{\text{ét}}(Y,W\Omega^\bullet)\otimes_{\mathscr{O}_F}\mathscr{O}_K\xrightarrow{\sim}H^r_{\mathrm{dR}}(X).$$

6.3. **Computation of syntomic cohomology.** We will prove in this section that the geometric syntomic cohomology of X can be computed using the logarithmic de Rham-Witt cohomology.

6.3.1. *Simplification of syntomic cohomology.* Let X now be a semistable Stein formal scheme over $\mathscr{O}_K$.

Lemma 6.29. *Let $r\geq -1$. There exist compatible natural strict quasi-isomorphisms*[37]

$$\begin{aligned}\iota_{\rm cr}:\quad & [\mathrm{R}\Gamma_{\rm cr}(\overline{X})_{\mathbf{Q}}]^{\varphi=p^r}\stackrel{\sim}{\to}[(\mathrm{R}\Gamma_{\rm cr}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_{\mathscr{O}_F}\widehat{\mathbf{A}}_{\rm st})_{\mathbf{Q}}]^{N=0,\varphi=p^r},\\ \iota_{\rm cr}:\quad & [\mathrm{R}\Gamma_{\rm cr}(X)_{\mathbf{Q}}]^{\varphi=p^r}\stackrel{\sim}{\to}[\mathrm{R}\Gamma_{\rm cr}(X_0/\mathscr{O}_F^0)_{\mathbf{Q}}]^{N=0,\varphi=p^r}.\end{aligned}$$

Proof. By (3.40), (3.41), we have a natural topological quasi-isomorphism

$$\iota^1_{\rm BK}:\quad [\mathrm{R}\Gamma_{\rm cr}(X_n/r^{\rm PD}_{\varpi,n})\widehat{\otimes}_{r^{\rm PD}_{\varpi,n}}\widehat{\mathbf{A}}_{{\rm st},n}]^{N=0}\stackrel{\sim}{\to}\mathrm{R}\Gamma_{\rm cr}(\overline{X}_n).$$

We can also adapt the proof of Theorem 4.1 to construct a natural strict quasi-isomorphism

$$h_{\rm cr}:\quad [(\mathrm{R}\Gamma_{\rm cr}(X/r^{\rm PD}_{\varpi})\widehat{\otimes}_{r^{\rm PD}_{\varpi}}\widehat{\mathbf{A}}_{\rm st})_{\mathbf{Q}}]^{\varphi=p^r}\stackrel{\sim}{\to}[(\mathrm{R}\Gamma_{\rm cr}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_{\mathscr{O}_F}\widehat{\mathbf{A}}_{\rm st})_{\mathbf{Q}}]^{\varphi=p^r}.$$

In fact, it suffices to note that the complexes (4.6) used in that proof have cohomology annihilated by p^N, for a constant $N=N(d,r)$, $d=\dim X_0$.

Define the first map in the lemma by $\iota_{\rm cr}:=h_{\rm cr}\iota_{\rm BK}^{-1}$. The definition of the second map $\iota_{\rm cr}$ is analogous (but easier: there is no need for the zigzag in the definition of $h_{\rm cr}$). □

Let $r\geq 0$. Set $\mathrm{DR}(\overline{X},r):=(\mathrm{R}\Gamma_{\rm dR}(X)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{{\rm cr},K})_{\mathbf{Q}}/F^r$. From the maps in (3.44) we induce a natural strict quasi-isomorphism

$$\iota^2_{\rm BK}:\mathrm{DR}(\overline{X},r)\stackrel{\sim}{\to}\mathrm{R}\Gamma_{\rm cr}(\overline{X})_{\mathbf{Q}}/F^r. \tag{6.30}$$

Computing as in Example 3.24, we get topological quasi-isomorphisms

$$\begin{aligned}\mathrm{R}\Gamma_{\rm dR}(X)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{{\rm cr},K} &\simeq \mathrm{R}\Gamma(X,\Omega^{\bullet}_X\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{{\rm cr},K})=\mathrm{R}\Gamma(X,(\mathscr{O}_X\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{{\rm cr},K}\to\Omega^1_X\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{{\rm cr},K}\to\cdots)),\\ F^r(\mathrm{R}\Gamma_{\rm dR}(X)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{{\rm cr},K}) &\simeq \mathrm{R}\Gamma(X,F^r(\Omega^{\bullet}_X\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{{\rm cr},K}))=\mathrm{R}\Gamma(X,(\mathscr{O}_X\widehat{\otimes}_{\mathscr{O}_K}F^r\mathbf{A}_{{\rm cr},K}\to\Omega^1_X\widehat{\otimes}_{\mathscr{O}_K}F^{r-1}\mathbf{A}_{{\rm cr},K}\to\cdots)),\\ (\mathrm{R}\Gamma_{\rm dR}(X)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{{\rm cr},K})/F^r &\simeq \mathrm{R}\Gamma(X,(\Omega^{\bullet}_X\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{{\rm cr},K})/F^r)\\ &= \mathrm{R}\Gamma(X,(\mathscr{O}_X\widehat{\otimes}_{\mathscr{O}_K}(\mathbf{A}_{{\rm cr},K}/F^r)\to\Omega^1_X\widehat{\otimes}_{\mathscr{O}_K}(\mathbf{A}_{{\rm cr},K}/F^{r-1})\to\cdots\to\Omega^{r-1}_X\widehat{\otimes}_{\mathscr{O}_K}(\mathbf{A}_{{\rm cr},K}/F^1)).\end{aligned}$$

In low degrees we have

$$\begin{aligned}&\mathrm{DR}(\overline{X},0)=0,\quad \mathrm{DR}(\overline{X},1)\simeq\mathrm{R}\Gamma(X,\mathscr{O}_X\widehat{\otimes}_{\mathscr{O}_K}\mathscr{O}_C)_{\mathbf{Q}},\\ &\mathrm{DR}(\overline{X},2)\simeq\mathrm{R}\Gamma(X,(\mathscr{O}_X\widehat{\otimes}_{\mathscr{O}_K}(\mathbf{A}_{{\rm cr},K}/F^2)\to\Omega^1_X\widehat{\otimes}_{\mathscr{O}_K}\mathscr{O}_C))_{\mathbf{Q}}.\end{aligned}$$

Set $\gamma_{\rm HK}:=(\iota^2_{\rm BK})^{-1}\iota_{\rm cr}^{-1}$. The above discussion yields the following strict quasi-isomorphism

$$\mathrm{R}\Gamma_{\rm syn}(\overline{X},\mathbf{Z}_p(r))_{\mathbf{Q}}\stackrel{\sim}{\to}\big[[(\mathrm{R}\Gamma_{\rm cr}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_{\mathscr{O}_F}\widehat{\mathbf{A}}_{\rm st})_{\mathbf{Q}}]^{N=0,\varphi=p^r}\xrightarrow{\gamma_{\rm HK}}(\mathrm{R}\Gamma_{\rm dR}(X)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{{\rm cr},K})_{\mathbf{Q}}/F^r\big]. \tag{6.31}$$

By construction, it is compatible with its pro-analog (3.32), i.e., we have a natural continuous map of distinguished triangles, where all the vertical maps are the canonical maps

$$\begin{array}{ccccc}\mathrm{R}\Gamma_{\rm syn}(\overline{X},\mathbf{Z}_p(r))_{\mathbf{Q}} & \longrightarrow & [(\mathrm{R}\Gamma_{\rm cr}(X_0/\mathscr{O}_F^0)\widehat{\otimes}_{\mathscr{O}_F}\widehat{\mathbf{A}}_{\rm st})_{\mathbf{Q}}]^{N=0,\varphi=p^r} & \xrightarrow{\gamma_{\rm HK}} & (\mathrm{R}\Gamma_{\rm dR}(X)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{{\rm cr},K})_{\mathbf{Q}}/F^r\\ \downarrow & & \downarrow & & \downarrow\\ \mathrm{R}\Gamma_{\rm syn}(\overline{X},\mathbf{Q}_p(r)) & \longrightarrow & [\mathrm{R}\Gamma_{\rm cr}(X_0/\mathscr{O}_F^0,F)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\rm st}]^{N=0,\varphi=p^r} & \xrightarrow{\gamma_{\rm HK}} & (\mathrm{R}\Gamma_{\rm dR}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\rm dR})/F^r.\end{array}$$

[37]For the definition of the completed tensor product we refer the reader to the digression before Lemma 6.19.

6.3.2. *Computation of the Hyodo-Kato part.* We come back now to the Drinfeld half-space.

Lemma 6.32. *We have the natural strict isomorphisms*

$$H^r([(\mathrm{R}\Gamma_{\mathrm{cr}}(Y/\mathscr{O}_F^0)\widehat{\otimes}_{\mathscr{O}_F}\widehat{\mathbf{A}}_{\mathrm{st}})_{\mathbf{Q}}]^{N=0,\varphi=p^r})\simeq H^0_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^r_{\log})_{\mathbf{Q}},\tag{6.33}$$
$$H^{r-1}([(\mathrm{R}\Gamma_{\mathrm{cr}}(Y/\mathscr{O}_F^0)\widehat{\otimes}_{\mathscr{O}_F}\widehat{\mathbf{A}}_{\mathrm{st}})_{\mathbf{Q}}]^{N=0,\varphi=p^r})\simeq (H^0_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^{r-1}_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}^{\varphi=p})_{\mathbf{Q}}.$$

Proof. By Proposition 6.22, we have topological quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{cr}}(Y/\mathscr{O}_F^0)\widehat{\otimes}_{\mathscr{O}_F}\widehat{\mathbf{A}}_{\mathrm{st}}\simeq \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(Y,W\Omega^\bullet)\widehat{\otimes}_{\mathscr{O}_F}\widehat{\mathbf{A}}_{\mathrm{st}}\xleftarrow{\sim}\Gamma_{\mathrm{\acute{e}t}}(Y,W\Omega^\bullet)\widehat{\otimes}_{\mathscr{O}_F}\widehat{\mathbf{A}}_{\mathrm{st}}$$
$$\xleftarrow{\sim}\bigoplus_{i\geq 0}\Gamma_{\mathrm{\acute{e}t}}(Y,W\Omega^i)\widehat{\otimes}_{\mathscr{O}_F}\widehat{\mathbf{A}}_{\mathrm{st}}[-i]\simeq\bigoplus_{i\geq 0}\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^i)\widehat{\otimes}_{W(\overline{k})}\widehat{\mathbf{A}}_{\mathrm{st}}[-i].$$

Let M be a finite type free (φ,N)-module over $W(\overline{k})$. Note that N is nilpotent. We claim that we have a natural short exact sequence

$$0\to M\otimes_{W(\overline{k})}\mathbf{A}_{\mathrm{cr}}\to M\otimes_{W(\overline{k})}\widehat{\mathbf{A}}_{\mathrm{st}}\xrightarrow{N}M\otimes_{W(\overline{k})}\widehat{\mathbf{A}}_{\mathrm{st}}\to 0.$$

Indeed, if $N=0$, this is clear from the short exact sequence (3.37). For a general M, we argue by induction on m such that $N^m=0$ using the short exact sequence

$$0\to M_0\to M\xrightarrow{N}M\to 0.$$

This is a sequence of finite type free (φ,N)-modules such that $N^{m-1}=0$. It follows that we have topological quasi-isomorphisms (reduce to the truncated log-schemes Y_s and pass to the limit)

$$[\mathrm{R}\Gamma_{\mathrm{cr}}(Y/\mathscr{O}_F^0)\widehat{\otimes}_{\mathscr{O}_F}\widehat{\mathbf{A}}_{\mathrm{st}}]^{N=0}\simeq\bigoplus_{i\geq 0}[\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^i)\widehat{\otimes}_{W(\overline{k})}\widehat{\mathbf{A}}_{\mathrm{st}}]^{N=0}[-i]\simeq\bigoplus_{i\geq 0}\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^i)\widehat{\otimes}_{W(\overline{k})}\mathbf{A}_{\mathrm{cr}}[-i]$$
$$\simeq\bigoplus_{i\geq 0}\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^i_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}[-i].$$

It remains to show that we have natural strict isomorphisms

$$[\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^r_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}]^{\varphi=p^r}_{\mathbf{Q}}\simeq\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^r_{\log})_{\mathbf{Q}}\tag{6.34}$$
$$[\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^{r-1}_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}]^{\varphi=p^r}_{\mathbf{Q}}\simeq(\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^{r-1}_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}^{\varphi=p})_{\mathbf{Q}}.$$

For $i\geq 0$, set $C_i:=\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^i_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}$. We claim that, for $j\geq i$, the classical eigenspace

$$C_i^{\varphi=p^j}=\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^i_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}^{\varphi=p^{j-i}}.\tag{6.35}$$

To see that, write, using the notation from the proof of Proposition 6.22, $H^0_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^i_{\log})\simeq\varprojlim_s H^0_{\mathrm{\acute{e}t}}(\overline{Y}_s,W\Omega^i_{\log})$ or, to simplify the notation, $A^i:=A=\varprojlim_s A_s$. Note that A_s is a finite type free $\mathbf{Z}_p$-module. Replace A_s with $B_s:=\cap_{s'>s}\mathrm{im}(A_{s'}\to A_s)$. Then the maps $B_{s+1}\to B_s$ are surjective and $A=\varprojlim_s B_s$. Choose basis of B_s, $s\geq 1$, compatible with the projections, i.e., the chosen basis of B_{s+1} includes a lift of the chosen basis of B_s. Using this basis we can write

$$A\simeq\mathbf{Z}_p^{I_1}\times\mathbf{Z}_p^{I_2}\times\mathbf{Z}_p^{I_3}\times\cdots,\quad A\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}\simeq\mathbf{A}_{\mathrm{cr}}^{I_1}\times\mathbf{A}_{\mathrm{cr}}^{I_2}\times\mathbf{A}_{\mathrm{cr}}^{I_3}\times\cdots.\tag{6.36}$$

Since the Frobenius on $A^i=\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^i_{\log})$ is equal to the multiplication by p^i we obtain the equality we wanted.

Consider now the following exact sequences

$$0\to C_r^{\varphi=p^r}\to C_r\xrightarrow{\varphi-p^r}C_r,\qquad 0\to C_{r-1}^{\varphi=p^r}\to C_{r-1}\xrightarrow{\varphi-p^r}C_{r-1}.$$

Since the map $\mathbf{A}_{\mathrm{cr}}\xrightarrow{p^i-\varphi}\mathbf{A}_{\mathrm{cr}}$, $i\geq 0$, is p^i-surjective, the maps $\varphi-p^r$ above are rationally surjective (use the basis (6.36) and the fact that the Frobenius on A^i is equal to the multiplication by p^i). Hence, rationally, the derived eigenspaces $[C_i]^{\varphi=p^r}$, $i=r,r-1$, are equal to the classical ones $C_i^{\varphi=p^r}$, $i=r,r-1$. Since, by (6.35), we have $C_r^{\varphi=p^r}=\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^r_{\log})$ and $C_{r-1}^{\varphi=p^r}=\Gamma_{\mathrm{\acute{e}t}}(\overline{Y},W\Omega^{r-1}_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}^{\varphi=p}$, the isomorphisms in (6.34) follow. □

6.3.3. *Computation of syntomic cohomology.*

Corollary 6.37. *Let $r \geq 0$. There exists a natural strict isomorphism*

$$H^r_{\mathrm{syn}}(\overline{X}, \mathbf{Z}_p(r))_{\mathbf{Q}} \simeq H^0_{\text{ét}}(\overline{Y}, W\Omega^r_{\log})_{\mathbf{Q}}.$$

Proof. We note that, by Proposition 6.22, there exist natural topological quasi-isomorphisms

$$\oplus_{i\geq 0}\Gamma(X,\Omega^i)\widehat{\otimes}_{\mathscr{O}_K} F^{r-i}\mathbf{A}_{\mathrm{cr},K}[-i] \xrightarrow{\sim} F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{\mathrm{cr},K}),$$
$$\oplus_{r-1\geq i\geq 0}\Gamma(X,\Omega^i)\widehat{\otimes}_{\mathscr{O}_K} \mathbf{A}_{\mathrm{cr},K}/F^{r-i}[-i] \xrightarrow{\sim} (\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{\mathrm{cr},K})/F^r.$$

This, combined with the strict quasi-isomorphisms (6.33), changes the map γ_{HK} from (6.31) into

$$\gamma'_{\mathrm{HK}}: \quad (\Gamma_{\text{ét}}(Y, W\Omega^{r-1}_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}^{\varphi=p})_{\mathbf{Q}} \to (\Gamma(X,\Omega^{r-1})\widehat{\otimes}_{\mathscr{O}_K}\mathscr{O}_C)_{\mathbf{Q}}.$$

Hence, we obtain the long strictly exact sequence

$$(\Gamma_{\text{ét}}(Y, W\Omega^{r-1}_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}^{\varphi=p})_{\mathbf{Q}} \xrightarrow{\gamma'_{\mathrm{HK}}} (\Gamma(X,\Omega^{r-1})\widehat{\otimes}_{\mathscr{O}_K}\mathscr{O}_C)_{\mathbf{Q}} \to H^r_{\mathrm{syn}}(\overline{X}, \mathbf{Z}_p(r))_{\mathbf{Q}} \to \Gamma_{\text{ét}}(\overline{Y}, W\Omega^r_{\log})_{\mathbf{Q}} \to 0$$

It suffices now to show that γ'_{HK} is surjective. For that we will need to trace carefully its definition. Consider thus the following commutative diagram

$$\begin{array}{ccc}
(\Gamma_{\text{ét}}(Y, W\Omega^{r-1}_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}^{\varphi=p})_{\mathbf{Q}}[-r+1] & \xrightarrow{\gamma'_{\mathrm{HK}}} & (\Gamma(X,\Omega^{r-1})\widehat{\otimes}_{\mathscr{O}_K}\mathscr{O}_C)_{\mathbf{Q}}[-r+1] \\
\downarrow & & \downarrow \\
[(\mathrm{R}\Gamma_{\mathrm{cr}}(Y/\mathscr{O}_F^0)\widehat{\otimes}_{\mathscr{O}_F}\widehat{\mathbf{A}}_{\mathrm{st}})_{\mathbf{Q}}]^{N=0,\varphi=p^r} & \xrightarrow{\gamma_{\mathrm{HK}}} & (\mathrm{R}\Gamma_{\mathrm{dR}}(X)\widehat{\otimes}_{\mathscr{O}_K}\mathbf{A}_{\mathrm{cr},K})_{\mathbf{Q}}/F^r \\
\downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\
[\mathrm{R}\Gamma_{\mathrm{cr}}(Y/\mathscr{O}_F^0, F)\widehat{\otimes}_F\widehat{\mathbf{B}}^+_{\mathrm{st}}]^{N=0,\varphi=p^r} & \xrightarrow{\gamma_{\mathrm{HK}}} & (\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r. \\
\uparrow{\scriptstyle \wr} & & {\scriptstyle \iota^2_{\mathrm{rig}}}\uparrow{\scriptstyle \wr} \\
[\mathrm{R}\Gamma_{\mathrm{HK}}(Y)\widehat{\otimes}_F\mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r} & \xrightarrow{\iota_{\mathrm{HK}}\otimes\iota} & (\mathrm{R}\Gamma_{\mathrm{dR}}(\widetilde{X}_K)\widehat{\otimes}_K\mathbf{B}^+_{\mathrm{dR}})/F^r
\end{array} \tag{6.38}$$

Here, the fact that the bottom square commutes follows from the proofs of Proposition 3.39 and Theorem 4.1. Taking H^{r-1} of the above diagram we obtain the outer diagram in the commutative diagram

$$\begin{array}{ccccc}
(\Gamma_{\text{ét}}(Y, W\Omega^{r-1}_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{A}_{\mathrm{cr}}^{\varphi=p})_{\mathbf{Q}} & \xrightarrow{\gamma'_{\mathrm{HK}}} & (\Gamma(X,\Omega^{r-1})\widehat{\otimes}_{\mathscr{O}_K}\mathscr{O}_C)_{\mathbf{Q}} & \searrow & \\
\downarrow & & \downarrow & & \\
H^{r-1}_{\mathrm{HK}}(Y)\widehat{\otimes}_F\mathbf{B}^{+,\varphi=p}_{\mathrm{cr}} & \xrightarrow{\iota_{\mathrm{HK}}\otimes\iota} & H^{r-1}_{\mathrm{dR}}(\widetilde{X}_K)\widehat{\otimes}_K C & \hookrightarrow & \Gamma(\widetilde{X}_K,\Omega^{r-1})\widehat{\otimes}_K C.
\end{array}$$

Since $d = 0$ on $(\Gamma(X,\Omega^{r-1})\widehat{\otimes}_{\mathscr{O}_K}\mathscr{O}_C)_{\mathbf{Q}}$, we get the shown factorization of the slanted map and the commutative square. This square is seen (by a careful chase of the diagram (6.38)) to be compatible with the projections $\theta: \mathbf{A}_{\mathrm{cr}}^{\varphi=p} \to \mathscr{O}_C$, $\theta: \mathbf{B}^{+,\varphi=p}_{\mathrm{cr}} \to C$. Using them, we obtain the commutative diagram

$$\begin{array}{ccc}
(\Gamma_{\text{ét}}(Y, W\Omega^{r-1}_{\log})\widehat{\otimes}_{\mathbf{Z}_p}\mathscr{O}_C)_{\mathbf{Q}} & \xrightarrow{\gamma_C} & (\Gamma(X,\Omega^{r-1})\widehat{\otimes}_{\mathscr{O}_K}\mathscr{O}_C)_{\mathbf{Q}} \\
\downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\
H^{r-1}_{\mathrm{HK}}(Y)\widehat{\otimes}_F C & \xrightarrow{\iota_{\mathrm{HK}}\otimes\iota} & H^{r-1}_{\mathrm{dR}}(\widetilde{X}_K)\widehat{\otimes}_K C.
\end{array}$$

and reduce our problem to showing that the induced map γ_C is surjective.

We will, in fact, show that γ_C is an isomorphism. Note that the right vertical map in the above diagram is injective: use that the natural map $H^{r-1}_{\mathrm{dR}}(X) \to H^{r-1}_{\mathrm{dR}}(X_K)$ is an injection (in particular the domain is torsion-fee). By Remark 6.28, the above diagram yields that $\gamma_C = (\iota_{\mathrm{HK}}\otimes 1)_{\mathbf{Q}}$. Since the integral Hyodo-Kato map is a topological isomorphism $\iota_{\mathrm{HK}}: \Gamma_{\text{ét}}(Y, W\Omega^{r-1}_{\log})\otimes_{\mathbf{Z}_p}\mathscr{O}_K \xrightarrow{\sim} \Gamma(X,\Omega^{r-1})$, so is γ_C, as wanted. $\square$

6.4. **Main theorem.** We are now ready to prove the following result. Recall that X, resp. $\widetilde{X}$, is the standard formal, resp. weak formal, model of the Drinfeld half-space $\mathbb{H}^d_K$, $Y=X_0$, $\overline{Y}=Y_{\overline{k}}$.

Theorem 6.39. *Let $r\geq 0$.*

(1) *There is a natural topological isomorphism of $G\times\mathscr{G}_K$-modules*

$$H^r_{\text{ét}}(X_C,\mathbf{Q}_p(r))\simeq \mathrm{Sp}_r^{\mathrm{cont}}(\mathbf{Q}_p)^*\simeq \mathrm{Sp}_r(\mathbf{Z}_p)^*\otimes\mathbf{Q}_p.$$

(2) *There are natural topological isomorphisms of G-modules*

$$H^r_{\mathrm{dR}}(X)\otimes_{\mathscr{O}_K}K\simeq \mathrm{Sp}_r^{\mathrm{cont}}(K)^*,\quad H^r_{\mathrm{cr}}(Y/\mathscr{O}_F^0)_{\mathbf{Q}}\simeq \mathrm{Sp}_r^{\mathrm{cont}}(F)^*,$$
$$H^i_{\text{ét}}(\overline{Y},W\Omega^r_{\overline{Y},\log})_{\mathbf{Q}}\simeq\begin{cases}\mathrm{Sp}_r^{\mathrm{cont}}(\mathbf{Q}_p)^* & \text{if } i=0,\\ 0 & \text{if } i>0.\end{cases}$$

(3) *The regulator maps*

$$r_{\text{ét}}:M(\mathscr{H}^{d+1},\mathbf{Q}_p)\to H^r_{\text{ét}}(X_C,\mathbf{Q}_p(r)),\quad r_{\mathrm{dR}}:M(\mathscr{H}^{d+1},K)\to H^r_{\mathrm{dR}}(X)\otimes_{\mathscr{O}_K}K,$$
$$r_{\mathrm{HK}}:M(\mathscr{H}^{d+1},F)\to H^r_{\mathrm{cr}}(Y/\mathscr{O}_F^0)_{\mathbf{Q}},\quad r_{\log}:M(\mathscr{H}^{d+1},\mathbf{Q}_p)\to H^0_{\text{ét}}(\overline{Y},W\Omega^r_{\log})_{\mathbf{Q}}$$

are continuous surjective maps (of weak duals of Banach spaces), compatible with the isomorphisms in (1) and (2), and their kernels are equal to the space of degenerate measures (defined as the intersection of the space of measures with the set of degenerate distributions).

(4) *The natural map*

$$H^r_{\text{ét}}(X_C,\mathbf{Q}_p(r))\to H^r_{\text{proét}}(X_C,\mathbf{Q}_p(r))$$

is an injection and identifies $H^r_{\text{ét}}(X_C,\mathbf{Q}_p(r))$ with the G-bounded vectors of $H^r_{\text{proét}}(X_C,\mathbf{Q}_p(r))$.

Proof. Point (3) follows from construction and Section A.2.2. Point (2) follows from Theorem 6.26 and Proposition 6.22. Point (1) follows from the comparison between étale and syntomic cohomologies in Proposition 6.1, the computation of syntomic cohomology in Corollary 6.37, Lemma 6.27, and the computation of the logarithmic de Rham-Witt cohomology in Theorem 6.26.

To prove point (4) consider the commutative diagram, where the bottom sequence is strictly exact:

$$\begin{array}{ccccccccc}
 & & & & H^r_{\text{ét}}(X_C,\mathbf{Q}_p(r)) & \xrightarrow{\sim} & \mathrm{Sp}_r^{\mathrm{cont}}(\mathbf{Q}_p)^* & & \\
 & & & & \big\downarrow{\scriptstyle\varepsilon} & & \big\downarrow{\scriptstyle\mathrm{can}} & & \\
0 & \longrightarrow & d\Omega^{r-1}(X_C) & \longrightarrow & H^r_{\text{proét}}(X_C,\mathbf{Q}_p(r)) & \longrightarrow & \mathrm{Sp}_r(\mathbf{Q}_p)^* & \longrightarrow & 0
\end{array}$$

Commutativity can be checked easily by looking at symbols. The change of topology map ε has image in $H^r_{\text{proét}}(X_C,\mathbf{Q}_p(r))^{G\text{-bd}}$ (since $\mathrm{Sp}_r^{\mathrm{cont}}(\mathbf{Z}_p)^*$ is compact). We need to show that this image is the whole of $H^r_{\text{proét}}(X_C,\mathbf{Q}_p(r))^{G\text{-bd}}$. For that, since $(\mathrm{Sp}_r(\mathbf{Q}_p)^*)^{G\text{-bd}}\simeq\mathrm{Sp}_r^{\mathrm{cont}}(\mathbf{Q}_p)^*$, it suffices to show that $(d\Omega^{r-1}(X_C))^{G\text{-bd}}=0$ or, equivalently, that the map $(\Omega^{r-1}(X_C)_{d=0})^{G\text{-bd}}\to H^{r-1}_{\mathrm{dR}}(X_C)$ is an injection. Since d is trivial on $\Omega^{r-1}(X_C)$, this amounts to showing that the map $\Omega^{r-1}(X_C)^{G\text{-bd}}\to H^{r-1}_{\mathrm{dR}}(X_C)$ is injective or that, by an analogous argument to the one we used in the proof of Proposition 6.22, so is the map $\Omega^{r-1}(X_K)^{G\text{-bd}}\to H^{r-1}_{\mathrm{dR}}(X_K)$.

Now, since, the map $\Omega^{r-1}(X)\otimes_{\mathscr{O}_K}K\to\Omega^{r-1}(X_K)^{G\text{-bd}}$ is an isomorphism (use the fact that X can be covered by G-translates of an open subscheme U such that U_K is an affinoid), it suffices to show that the map $\Omega^{r-1}(X)\otimes_{\mathscr{O}_K}K\to H^{r-1}_{\mathrm{dR}}(X_K)$ is an injection. But this we have done in Proposition 6.25. □

Appendix A. Symbols

We gather in this appendix a few basic facts concerning symbol maps and their compatibilities that we need in this paper. We use the notation from Section 6.

A.1. **Hyodo-Kato isomorphisms.** Let X be a semistable Stein weak formal scheme over $\mathscr{O}_K$. In the first part of this paper we have used the Hyodo-Kato isomorphism as defined by Grosse-Klönne in [23], $\iota_{\mathrm{HK}}: H^r_{\mathrm{HK}}(X_0)\otimes_F K \xrightarrow{\sim} H^r_{\mathrm{dR}}(X_K)$. But one can also try to use the original Hyodo-Kato isomorphism defined for quasi-compact formal schemes in [30]. Doing that we obtain two Hyodo-Kato isomorphisms. One that, modulo canonical identifications, turns out to be the same as the one of Grosse-Klönne, the other identifies bounded Hyodo-Kato and de Rham cohomologies.

Proposition A.1. *We have compatible Hyodo-Kato (topological) isomorphisms*

$$\iota_{\mathrm{HK}}: H^r_{\mathrm{cr}}(X_0/\mathscr{O}_F^0, F)\otimes_F K \xrightarrow{\sim} H^r_{\mathrm{dR}}(\widehat{X}_K),\quad \iota_{\mathrm{HK}}: (H^r_{\mathrm{cr}}(X_0/\mathscr{O}_F^0)\otimes_{\mathscr{O}_F}\mathscr{O}_K)_{\mathbf{Q}} \xrightarrow{\sim} H^r_{\mathrm{dR}}(\widehat{X})_{\mathbf{Q}}. \tag{A.2}$$

Proof. As mentioned above they are induced by the original Hyodo-Kato isomorphism [30]. We will describe them in more detail.

To start, assume that we have a quasi-compact semistable formal scheme Y over $\mathscr{O}_K$. We will work in the classical derived category. Recall that the Frobenius

$$r^{\mathrm{PD}}_{\varpi,n,\varphi}\otimes^L_{r^{\mathrm{PD}}_{\varpi,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(Y_1/r^{\mathrm{PD}}_{\varpi,n})\to \mathrm{R}\Gamma_{\mathrm{cr}}(Y_1/r^{\mathrm{PD}}_{\varpi,n}),\quad \mathscr{O}_{F,n,\varphi}\otimes^L_{\mathscr{O}_{F,n}}\mathrm{R}\Gamma_{\mathrm{cr}}(Y_0/\mathscr{O}_{F,n})\to \mathrm{R}\Gamma_{\mathrm{cr}}(Y_0/\mathscr{O}^0_{F,n})$$

has a p^N-inverse, for $N=N(d)$, $d=\dim Y_0$. This is proved in [30, 2.24]. Recall also that the projection $p_0:\mathrm{R}\Gamma_{\mathrm{cr}}(Y/r^{\mathrm{PD}}_{\varpi,n})\to \mathrm{R}\Gamma_{\mathrm{cr}}(Y_0/\mathscr{O}^0_{F,n})$ has a functorial (for maps between formal schemes and a change of n) and Frobenius-equivariant p^N-section

$$\iota_\varpi:\mathrm{R}\Gamma_{\mathrm{cr}}(Y_0/\mathscr{O}^0_{F,n})\to \mathrm{R}\Gamma_{\mathrm{cr}}(Y/r^{\mathrm{PD}}_{\varpi,n}),$$

i.e., $p_0\iota_\varpi=p^N$. This follows easily from the proof of Proposition 4.13 in [30]; the key point being that the Frobenius on $\mathrm{R}\Gamma_{\mathrm{cr}}(Y_0/\mathscr{O}^0_{F,n})$ is close to an isomorphism and the Frobenius on the PD-ideal of r^{PD}_ϖ is close to zero. Moreover, the resulting map

$$\iota_\varpi:\mathrm{R}\Gamma_{\mathrm{cr}}(Y_0/\mathscr{O}^0_{F,n})\otimes^L_{\mathscr{O}_{F,n}} r^{\mathrm{PD}}_{\varpi,n}\to \mathrm{R}\Gamma_{\mathrm{cr}}(Y/r^{\mathrm{PD}}_{\varpi,n})$$

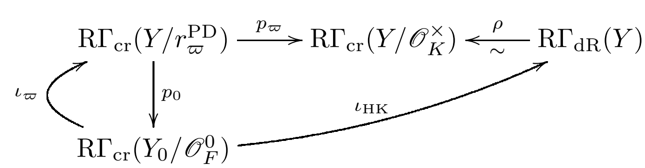
is a p^N-isomorphism [30, Lemma 5.2] and so is the composite

$$p_\varpi\iota_\varpi:\mathrm{R}\Gamma_{\mathrm{cr}}(Y_0/\mathscr{O}^0_{F,n})\otimes^L_{\mathscr{O}_{F,n}}\mathscr{O}_{K,n}\to \mathrm{R}\Gamma_{\mathrm{cr}}(Y/\mathscr{O}^\times_{K,n}).$$

Taking holim_n of the last map we obtain a map

$$p_\varpi\iota_\varpi:\mathrm{R}\Gamma_{\mathrm{cr}}(Y_0/\mathscr{O}^0_F)\otimes^L_{\mathscr{O}_F}\mathscr{O}_K\to \mathrm{R}\Gamma_{\mathrm{cr}}(Y/\mathscr{O}^\times_K)$$

that is a p^N-isomorphism. The Hyodo-Kato map is defined as $\iota_{\mathrm{HK}}=\rho^{-1}p_\varpi\iota_\varpi$. We have the commutative diagram



The induced map $\mathrm{R}\Gamma_{\mathrm{cr}}(Y_0/\mathscr{O}^0_F)\otimes_{\mathscr{O}_F}\mathscr{O}_K\to \mathrm{R}\Gamma_{\mathrm{dR}}(Y)$ is a p^N-isomorphism.

For a Stein semistable weak formal scheme X, we choose a Stein covering $\{U_s\}$, $s\in\mathbf{N}$, and define the compatible Hyodo-Kato maps

$$\begin{aligned}\iota_{\mathrm{HK}} &:= p^{-N}\varprojlim_s(\iota_{\mathrm{HK},U_s}\otimes\mathbf{Q}): H^i_{\mathrm{cr}}(X_0/\mathscr{O}^0_F,F)\to H^i_{\mathrm{dR}}(\widehat{X}_K),\\ \iota_{\mathrm{HK}} &:= p^{-N}(\varprojlim_s\iota_{\mathrm{HK},U_s})_{\mathbf{Q}}: H^i_{\mathrm{cr}}(X_0/\mathscr{O}^0_F)_{\mathbf{Q}}\to H^i_{\mathrm{dR}}(\widehat{X})_{\mathbf{Q}}.\end{aligned} \tag{A.3}$$

The twist by p^{-N} appears to make these maps compatible with symbols. We used here the fact that

$$\begin{aligned}&H^i_{\mathrm{cr}}(X_0/\mathscr{O}^0_F,F)\simeq\varprojlim_s H^i_{\mathrm{cr}}(U_s/\mathscr{O}^0_F,F),\quad H^i_{\mathrm{cr}}(X_0/\mathscr{O}^0_F)\simeq\varprojlim_s H^i_{\mathrm{cr}}(U_s/\mathscr{O}^0_F),\\ &H^i_{\mathrm{dR}}(\widehat{X}_K)\simeq\varprojlim_s H^i_{\mathrm{dR}}(]U_s[_{\widehat{X}_K}),\quad H^i_{\mathrm{dR}}(\widehat{X})_{\mathbf{Q}}\simeq(\varprojlim_s H^i_{\mathrm{dR}}(\widehat{X}|_{U_s}))_{\mathbf{Q}}.\end{aligned} \tag{A.4}$$

The third isomorphism, since X_K is Stein, is classical. The first two follow from the vanishing of the derived functors

$$H^1 \operatorname{holim}_s H^{i-1}_{\rm cr}(U_s/\mathcal{O}^0_F, F), \quad H^1 \operatorname{holim}_s H^{i-1}_{\rm cr}(U_s/\mathcal{O}^0_F),$$

which, in turn, follow from the fact that the pro-systems ($s \in \mathbf{N}$)

$$\{H^{i-1}_{\rm cr}(U_s/\mathcal{O}^0_F, F)\} = \{H^{i-1}_{\rm cr}(Y_s/\mathcal{O}^0_F, F)\}, \quad \{H^{i-1}_{\rm cr}(U_s/\mathcal{O}^0_F)\} = \{H^{i-1}_{\rm cr}(Y_s/\mathcal{O}^0_F)\}$$

are Mittag-Leffler. To show the last isomorphism in (A.4), it suffices to show the vanishining of $(H^1 \operatorname{holim}_s H^{i-1}_{\rm dR}(\widehat{X}|_{U_s}))_{\mathbf{Q}}$. For that, we will use the fact that the Hyodo-Kato map

$$H^{i-1}_{\rm cr}(U_s/\mathcal{O}^0_F) \otimes_{\mathcal{O}_F} \mathcal{O}_K \to H^{i-1}_{\rm dR}(\widehat{X}|_{U_s})$$

is a p^N-isomorphism. Which implies that so is the induced map

$$H^1 \operatorname{holim}_s (H^{i-1}_{\rm cr}(U_s/\mathcal{O}^0_F) \otimes_{\mathcal{O}_F} \mathcal{O}_K) \to H^1 \operatorname{holim}_s H^{i-1}_{\rm dR}(\widehat{X}|_{U_s}).$$

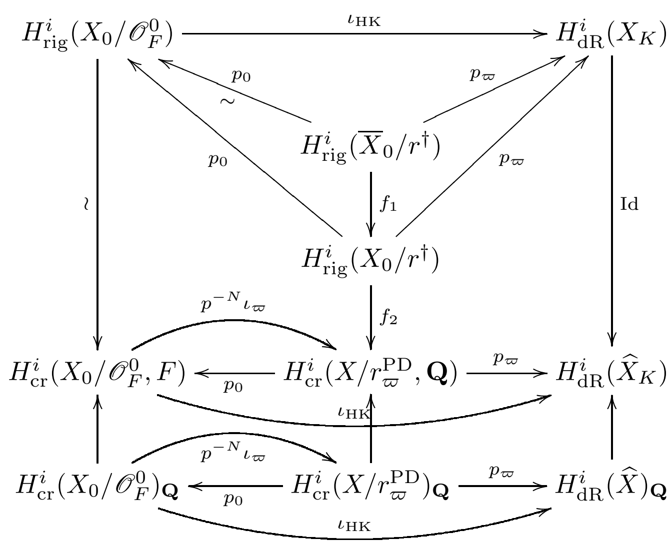
But, since the pro-system $\{H^{i-1}_{\rm cr}(U_s/\mathcal{O}^0_F)\} = \{H^{i-1}_{\rm cr}(Y_s/\mathcal{O}^0_F)\}$ is Mittag-Leffler, the first derived limit is trivial, as wanted.

Now, by definition, the Hyodo-Kato maps from (A.3) are topological isomorphisms. □

Corollary A.5. *The Hyodo-Kato isomorphisms from Proposition A.1 are compatible with the Grosse-Klönne Hyodo-Kato isomorphism.*

Proof. We may assume that X is quasi-compact. Then, this can be checked by the commutative diagram (we use the notation from Section 4.1)

(A.6)



□

A.2. **Definition of symbols.** We define now various symbol maps and show that they are compatible.

Let X be a semistable formal scheme over $\mathcal{O}_K$. Let M be the sheaf of monoids on X defining the log-structure, $M^{\rm gp}$ its group completion. This log-structure is canonical, in the terminology of Berkovich [4, 2.3], i.e., $M(U) = \{x \in \mathcal{O}_X(U) | x_K \in \mathcal{O}^*_{X_K}(U_K)\}$. This is shown in [4, Theorem 2.3.1], [3, Theorem 5.3] and applies also to semistable formal schemes with self-intersections. It follows that $M^{\rm gp}(U) = \mathcal{O}^*_{X_K}(U_K)$.

A.2.1. *Differential symbols.* We have the crystalline first Chern class maps of complexes of sheaves on $X_{\rm \acute{e}t}$ [66, 2.2.3]

$$c_1^{\rm st} : M^{\rm gp} \to M_n^{\rm gp} \to R\varepsilon_* \mathcal{J}^{[1]}_{X_n/r^{\rm PD}_{\varpi,n}}[1], \quad c_1^{\rm HK} : M^{\rm gp} \to M_0^{\rm gp} \to R\varepsilon_* \mathcal{J}^{[1]}_{X_0/\mathcal{O}^0_{F,n}}[1].$$

Here, the map ε is the projection from the corresponding crystalline-étale site to the étale site. These maps are clearly compatible. We get the induced functorial maps

$$c_1^{\mathrm{st}}:\Gamma(X_K,\mathscr{O}^*_{X_K})\to \mathrm{R}\Gamma_{\mathrm{cr}}(X/r_{\varpi}^{\mathrm{PD}},\mathscr{J}^{[1]})[1],\quad c_1^{\mathrm{HK}}:\Gamma(X_K,\mathscr{O}^*_{X_K})\to \mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}_F^0,\mathscr{J}^{[1]})[1].$$

The Hyodo-Kato classes above can be also defined using the de Rham-Witt complexes. That is, one can define (compatible) Hyodo-Kato Chern class maps [14, 2.1]

$$c_1^{\log}:\Gamma(X_K,\mathscr{O}^*_{X_K})\to \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_0,W\Omega^{\bullet}_{X_0,\log})[1],\quad c_1^{\mathrm{HK}}:\Gamma(X_K,\mathscr{O}^*_{X_K})\to \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X_0,W\Omega^{\bullet}_{X_0})[1].$$

They are compatible with the classical crystalline Hyodo-Kato Chern class maps above (use [19, I.5] and replace $\mathscr{O}^*$ by M^{gp}).

We also have the de Rham first Chern class map

$$c_1^{\mathrm{dR}}:M^{\mathrm{gp}}\to M_n^{\mathrm{gp}}\xrightarrow{\mathrm{dlog}}\Omega^{\bullet}_{X_n/\mathscr{O}^{\times}_{K,n}}[1].$$

It induces the functorial map

$$c_1^{\mathrm{dR}}:\Gamma(X_K,\mathscr{O}^*)\to \mathrm{R}\Gamma_{\mathrm{dR}}(X)[1].$$

It is evident that, by the canonical map $\mathrm{R}\Gamma_{\mathrm{dR}}(X)\to\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)$, this map is compatible with the rigid analytic class (defined using dlog) $c_1^{\mathrm{dR}}:\Gamma(X_K,\mathscr{O}^*)\to\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)[1]$.

Let now X be a semistable weak formal scheme over $\mathscr{O}_K$. The overconvergent classes

$$c_1^{\mathrm{HK}}:\Gamma(X_K,\mathscr{O}^*)\to \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/\mathscr{O}_F^0)[1],\quad c_1^{\mathrm{dR}}:\Gamma(X_K,\mathscr{O}^*)\to \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)[1]$$

are defined in an analogous way to the crystalline Hyodo-Kato classes and the rigid analytic de Rham classes (of $\widehat{X}_K$), respectively. Clearly they are compatible with those.

Lemma A.7. *Let X be a semistable Stein weak formal scheme over $\mathscr{O}_K$. The Hyodo-Kato maps*

$$\iota_{\mathrm{HK}}:H^1_{\mathrm{cr}}(X_0/\mathscr{O}_F^0)_{\mathbf{Q}}\to H^1_{\mathrm{dR}}(\widehat{X})_{\mathbf{Q}},\quad \iota_{\mathrm{HK}}:H^1_{\mathrm{cr}}(X_0/\mathscr{O}_F^0,F)\to H^1_{\mathrm{dR}}(\widehat{X}_K),$$
$$\iota_{\mathrm{HK}}:H^1_{\mathrm{rig}}(X_0/\mathscr{O}_F^0)\to H^1_{\mathrm{dR}}(X_K)$$

are continuous and compatible with the Chern class maps from $H^0(\widehat{X}_K,\mathscr{O}^*)$ and $H^0(X_K,\mathscr{O}^*)$.

Proof. Continuity follows from the construction.

For the first two maps, the proof of an analogous lemma in the algebraic setting goes through with only small changes [48, Lemma 5.1]. We will present it for the first map. We can assume that X is quasi-compact. Since $\iota_{\mathrm{HK}}=\rho^{-1}p_{\varpi}\iota'_{\varpi}$, $\iota'_{\varpi}=p^{-N}\iota_{\varpi}$, and the map p_{ϖ} is compatible with first Chern classes, it suffices to show the compatibility for the section ι'_{ϖ}. Let $x\in H^0(\widehat{X}_K,\mathscr{O}^*_{\widehat{X}_K})$. Since the map ι'_{ϖ} is a section of the map p_0 and the map p_0 is compatible with first Chern classes, we have that the element $\zeta\in H^1_{\mathrm{cr}}(X/r_{\varpi}^{\mathrm{PD}})_{\mathbf{Q}}$ defined as $\zeta=\iota'_{\varpi}(c_1^{\mathrm{HK}}(x))-c_1^{\mathrm{st}}(x)\in\ker p_0$. Since the map

$$\beta=\iota'_{\varpi}\otimes\mathrm{Id}:H^1_{\mathrm{cr}}(X_0/\mathscr{O}_F^0)_{\mathbf{Q}}\widehat{\otimes}_F r^{\mathrm{PD}}_{\varpi,\mathbf{Q}}\to H^1_{\mathrm{cr}}(X/r_{\varpi}^{\mathrm{PD}})_{\mathbf{Q}}$$

is surjective (see Section A.1), we can write $\zeta=\beta(\zeta')$ for $\zeta'\in H^1_{\mathrm{cr}}(X_0/\mathscr{O}_F^0)_{\mathbf{Q}}\widehat{\otimes}_F r^{\mathrm{PD}}_{\varpi,\mathbf{Q}}$. Since $p_0\beta(\zeta')=0$, we have $\zeta'\in\ker(\mathrm{Id}\otimes p_0)$. Hence $\zeta'=T\gamma$, $\gamma\in H^1_{\mathrm{cr}}(X_0/\mathscr{O}_F^0)_{\mathbf{Q}}\widehat{\otimes}_F r^{\mathrm{PD}}_{\varpi,\mathbf{Q}}$.

Since the map ι_{ϖ} is compatible with Frobenius and $\varphi(c_1^{\mathrm{HK}}(x))=pc_1^{\mathrm{HK}}(x)$, $\varphi(c_1^{\mathrm{st}}(x))=pc_1^{\mathrm{st}}(x)$, we have $\varphi(\zeta')=p\zeta'$. Since $\varphi(T\gamma)=T^p\varphi(\gamma)$ this implies that $\gamma\in\bigcap_{n=1}^{\infty}H^1_{\mathrm{cr}}(X_0/\mathscr{O}_F^0)_{\mathbf{Q}}\otimes_F T^n r^{\mathrm{PD}}_{\varpi,\mathbf{Q}}$, which is not possible unless γ (and hence ζ') are zero. This implies that $\zeta=0$ and this is what we wanted to show.

For the last map in the lemma, we use the diagram (A.6): an easy chase coupled with the listed above compatibilities gives us what we want. ☐

A.2.2. *Étale symbols and the period map.* We have the étale first Chern class maps (obtained from Kummer theory)

$$c_1^{\text{ét}}:\mathscr{O}^*_{X_K}\to \mathbf{Z}/p^n(1)[1],\quad c_1^{\text{ét}}:\Gamma(X_K,\mathscr{O}^*)\to \mathrm{R}\Gamma_{\text{ét}}(X_K,\mathbf{Z}/p^n(1))[1].$$

We also have the syntomic first Chern class maps [66, 2.2.3]

$$c_1^{\text{syn}}:\Gamma(X_K,\mathscr{O}^*)\to \mathrm{R}\Gamma_{\text{syn}}(\overline{X},\mathbf{Z}_p(1))[1]$$

that are compatible with the crystalline first Chern class maps $c_1^{\text{cr}}:\Gamma(X_K,\mathscr{O}^*)\to \mathrm{R}\Gamma_{\text{cr}}(\overline{X})[1]$. By [66, 3.2], they are also compatible with the étale Chern classes via the Fontaine-Messing period maps $\rho_{\text{FM}}:=p^{-r}\alpha_{\text{FM}}$:

$$\rho_{\text{FM}}:\mathrm{R}\Gamma_{\text{syn}}(\overline{X},\mathbf{Z}_p(r))_{\mathbf{Q}}\to \mathrm{R}\Gamma_{\text{ét}}(\overline{X},\mathbf{Q}_p(r)),\quad \rho_{\text{FM}}:\mathrm{R}\Gamma_{\text{syn}}(\overline{X},\mathbf{Q}_p(r))\to \mathrm{R}\Gamma_{\text{proét}}(\overline{X},\mathbf{Q}_p(r)).$$

Appendix B. Alternative proof of Corollary 6.23

We present in this appendix an alternative proof of Corollary 6.23 (hence also of Proposition 6.22 which easily follows from it) that does not use the ordinarity of the truncated log-schemes Y_s.

Let X/k^0 be a fine log-scheme of Cartier type. Recall that we have the subsheaves

$$Z^j_\infty=\cap_{n\geq 0}Z^j_n,\quad B^j_\infty=\cup_{n\geq 0}B^j_n$$

of $\Omega^j=\Omega^j_{X/k^0}$ (in what follows we will omit the subscripts in differentials if understood). Via the maps $C:Z^j_{n+1}\to Z^j_n$ (with kernels B^j_n), Z^j_∞ is the sheaf of forms ω such that $dC^n\omega=0$ for all n. This sheaf is acted upon by the Cartier operator C, and we recover

$$B^j_\infty=\cup_{n\geq 0}(Z^j_\infty)^{C^n=0},\quad \Omega^j_{\log}=(Z^j_\infty)^{C=1}.$$

The following result is proved in [32, 2.5.3] for classically smooth schemes. It holds most likely in much greater generality than the one stated below, but this will be sufficient for our purposes.

Lemma B.1. *Assume that X/k^0 is semi-stable (with the induced log structure) and that k is algebraically closed. Then the natural map of étale sheaves*

$$B^j_\infty\oplus(\Omega^j_{\log}\otimes_{\mathbf{F}_p}k)\to Z^j_\infty$$

is an isomorphism.

Proof. It suffices to show that, for X as above and affine, the map $B^j_\infty(X)\oplus(\Omega^j_{\log}(X)\otimes_{\mathbf{F}_p}k)\to Z^j_\infty(X)$ is an isomorphism. Take an open dense subset $j:U\hookrightarrow X$ which is smooth over k. Then Ω^i_X is a subsheaf of $j_*\Omega^i_U$ and so $Z^i_{\infty,X}$ is a subsheaf of $j_*Z^i_{\infty,U}$, giving an inclusion $Z^i_{\infty,X}(X)\subset Z^i_{\infty,U}(U)$. By a result of Raynaud [32, Prop. 2.5.2], $Z^i_{\infty,U}(U)$ is a union of finite dimensional k-vector spaces stable under C. We deduce that $Z^i_{\infty,X}(X)$ is also such a union.

The result follows now from the following basic result of semi-linear algebra (this is where the hypothesis that k is algebraically closed is crucial): if E is a finite dimensional k-vector space stable under C, then $E=E_{\text{nilp}}\oplus E_{\text{inv}}$, where $E_{\text{nilp}}=\cup_n E^{C^n=0}$, $E_{\text{inv}}=\cap_n C^n(E)$, and the natural map $E^{C=1}\otimes_{\mathbf{F}_p}k\to E_{\text{inv}}$ is an isomorphism. □

Proof. (of Corollary 6.23) (1) We prove this in several steps. We start with the case $i=0$ (the most delicate). By Lemma B.1,

$$H^0_{\text{ét}}(\overline{Y},Z^j_\infty)\simeq H^0_{\text{ét}}(\overline{Y},B^j_\infty)\oplus H^0_{\text{ét}}(\overline{Y},\Omega^j_{\log}\otimes_{\mathbf{F}_p}k).$$

We need the following intermediate result:

Lemma B.2. *We have $H^0_{\text{ét}}(\overline{Y},B^j_\infty)=0$.*

Proof. We note that $H^0_{\text{ét}}(\overline{Y}, B^j_n) = 0$ for all n: because we have $B^j_n \simeq B^j_{n+1}/B^j_1$ this follows from Lemma 6.12. This however does not allow us to deduce formally our lemma because $B^j_\infty = \cup_n B^j_n$ and $\overline{Y}$ is not quasi-compact. Instead, we argue as follows: the formation of the sheaves B^j_∞ being functorial, we have a natural map $\alpha : H^0_{\text{ét}}(\overline{Y}, B^j_\infty) \to \prod_{C\in F^0} H^0_{\text{ét}}(\overline{C}, B^j_\infty)$. It suffices to prove that α is injective and that $H^0_{\text{ét}}(\overline{T}, B^j_\infty) = 0$. To prove the injectivity of α, it suffices to embed both the domain and target of α in $H^0_{\text{ét}}(\overline{Y}, \Omega^j)$ and $\prod_C H^0_{\text{ét}}(\overline{C}, \Omega^j)$, and to use the injectivity of the natural map $H^0_{\text{ét}}(\overline{Y}, \Omega^j) \to \prod_C H^0_{\text{ét}}(\overline{C}, \Omega^j)$. Next, since $\overline{T}$ is quasi-compact,

$$H^0_{\text{ét}}(\overline{T}, B^j_\infty) = \varinjlim_n H^0_{\text{ét}}(\overline{T}, B^j_n) = 0,$$

the second equality being a consequence of Proposition 6.2 and Lemma 6.12 (as above, in the case of $\overline{Y}$). □

Consider now the sequence of natural maps

$$H^0_{\text{ét}}(\overline{Y}, \Omega^j_{\log})\widehat{\otimes}_{\mathbf{F}_p}\overline{k} \xrightarrow{\sim} H^0_{\text{ét}}(\overline{Y}, \Omega^j_{\log} \otimes_{\mathbf{F}_p} \overline{k}) \xrightarrow{\sim} H^0_{\text{ét}}(\overline{Y}, Z^j_\infty) \to H^0_{\text{ét}}(\overline{Y}, \Omega^j).$$

The first map is clearly a topological isomorphism, the second one is a topological isomorphism by Lemma B.2. Hence it remains to show that the last map is a topological isomorphism as well. Or that all the natural maps $H^0_{\text{ét}}(\overline{Y}, Z^j_n) \to H^0_{\text{ét}}(\overline{Y}, \Omega^j)$, $n \geq 1$, are topological isomorphisms. But this was done in the proof of Lemma 6.19. This gives the desired result for $i = 0$.

We prove next the result for $i > 0$, i.e., that $H^i_{\text{ét}}(\overline{Y}, \Omega^j_{\log})\widehat{\otimes}_{\mathbf{F}_p}\overline{k} = 0$ for $i \geq 1$. We start with showing that $H^i_{\text{ét}}(\overline{Y}, \Omega^j_{\log}) = 0$. The exact sequence

$$0 \to \Omega^j_{\log} \to \Omega^j/B^j_1 \xrightarrow{1-C^{-1}} \Omega^j/B^j_2 \to 0$$

yield the exact sequence

$$0 \to H^0_{\text{ét}}(\overline{Y}, \Omega^j_{\log}) \to H^0_{\text{ét}}(\overline{Y}, \Omega^j) \xrightarrow{1-C^{-1}} H^0_{\text{ét}}(\overline{Y}, \Omega^j) \to H^1_{\text{ét}}(\overline{Y}, \Omega^j_{\log}) \to 0$$

and $H^i_{\text{ét}}(\overline{Y}, \Omega^j_{\log}) = 0$ for $i > 1$. It suffices therefore to prove that $1 - C^{-1}$ is surjective on $H^0_{\text{ét}}(\overline{Y}, \Omega^j)$. For this, write $A_s = H^0_{\text{ét}}(\overline{Y}^\circ_s, \Omega^j_{\log})$. As we have already seen, we have an isomorphism

$$H^0_{\text{ét}}(\overline{Y}, \Omega^j) \simeq \varprojlim_s (A_s \otimes_{\mathbf{F}_p} \overline{k}) = H^0_{\text{ét}}(\overline{Y}, \Omega^j_{\log})\widehat{\otimes}_{\mathbf{F}_p}\overline{k}.$$

We have $C^{-1} = \varprojlim_s (1 \otimes \varphi)$ (φ being the absolute Frobenius on $\overline{k}$; note that $C - 1 = 0$ on A_s). To conclude that $1 - C^{-1}$ is surjective on $H^0_{\text{ét}}(\overline{Y}, \Omega^j)$, it suffices to pass to the limit in the exact sequences

$$0 \to A_s \to A_s \otimes_{\mathbf{F}_p} \overline{k} \xrightarrow{1-\varphi} A_s \otimes_{\mathbf{F}_p} \overline{k} \to 0,$$

whose exactness is ensured by the Artin-Schreier sequence for $\overline{k}$ and the fact that $(A_s)_s$ is Mittag-Leffler.

This shows that $H^i_{\text{ét}}(\overline{Y}, \Omega^j_{\log}) = 0$ for $i > 0$. Choosing a basis $(e_\lambda)_{\lambda\in I}$ of $\overline{k}$ over $\mathbf{F}_p$ we obtain an embedding

$$H^i_{\text{ét}}(\overline{Y}, \Omega^j_{\log})\widehat{\otimes}_{\mathbf{F}_p}\overline{k} \subset \prod_{\lambda\in I} H^i_{\text{ét}}(\overline{Y}, \Omega^j_{\log}) = 0,$$

which finishes the proof of (1).

(2) We prove the claim for W_n by induction on n, the case $n = 1$ being part (1). We pass from n to $n + 1$ using the strictly exact sequences

$$0 \to H^0(\overline{Y}, \Omega^j) \xrightarrow{V^n} H^0(\overline{Y}, W_{n+1}\Omega^j) \to H^0(\overline{Y}, W_n\Omega^j) \to 0,$$

$$0 \to H^0_{\text{ét}}(\overline{Y}, \Omega^j_{\log})\widehat{\otimes}_{\mathbf{F}_p}\overline{k} \xrightarrow{V^n} H^0_{\text{ét}}(\overline{Y}, W_{n+1}\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^{n+1}} W_{n+1}(\overline{k}) \to H^0_{\text{ét}}(\overline{Y}, W_n\Omega^j_{\log})\widehat{\otimes}_{\mathbf{Z}/p^n} W_n(\overline{k}) \to 0,$$

as well as the natural map between them. The first sequence is exact by Lemma 6.19. To show that the second sequence is exact, consider, as above, the exact sequences

$$0 \to H^0_{\text{ét}}(\overline{Y}^\circ_s, \Omega^j_{\log}) \xrightarrow{V^n} H^0_{\text{ét}}(\overline{Y}^\circ_s, W_{n+1}\Omega^j_{\log}) \to H^0_{\text{ét}}(\overline{Y}^\circ_s, W_n\Omega^j_{\log}) \to H^1_{\text{ét}}(\overline{Y}^\circ_s, \Omega^j_{\log}).$$

Tensoring over $\mathbf{Z}$ by $W_{n+1}(\overline{k})$, we can rewrite them as

$$0 \to H^0_{\text{ét}}(\overline{Y}^{\circ}_s, \Omega^j_{\log}) \otimes_{\mathbf{F}_p} \overline{k} \to H^0_{\text{ét}}(\overline{Y}^{\circ}_s, W_{n+1}\Omega^j_{\log}) \otimes_{\mathbf{Z}/p^{n+1}} W_{n+1}(\overline{k}) \to H^0_{\text{ét}}(\overline{Y}^{\circ}_s, W_n\Omega^j_{\log}) \otimes_{\mathbf{Z}/p^n} W_n(\overline{k})$$
$$\to H^1_{\text{ét}}(\overline{Y}^{\circ}_s, \Omega^j_{\log}) \otimes_{\mathbf{F}_p} \overline{k}.$$

To lighten the notation, write them simply as $0 \to A_s \to B_s \to C_s \to D_s$. Using that $(A_s)_s, (C_s)_s$ are finite $W_n(\overline{k})$-modules and that $\varprojlim_s D_s = H^1_{\text{ét}}(\overline{Y}, \Omega^j_{\log}) \widehat{\otimes}_{\mathbf{F}_p} \overline{k} = 0$ (as follows from (1)), we obtain the exact sequence

$$0 \to \varprojlim_s A_s \to \varprojlim_s B_s \to \varprojlim_s C_s \to 0,$$

which finishes the proof of (2) for W_n, $n \geq 1$. Passing to the limit over n gives us the proof for W. □

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