On polynomially integrable Birkhoff billiards on surfaces of constant curvature.

Alexey Glutsyuk

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Abstract

The algebraic version of Birkhoff Conjecture on integrable billiards on complete simply connected surfaces of constant curvature (plane, sphere, hyperbolic plane) was first stated, studied and solved in a particular case by Sergei Bolotin in 1990-1992. Here we present a complete solution of the algebraic version of Birkhoff Conjecture. Namely we show that every polynomially integrable real bounded planar billiard with C^2-smooth connected boundary is an ellipse. We extend this result to billiards with piecewise-smooth and not necessarily convex boundary on arbitrary two-dimensional surface of constant curvature: plane, sphere, Lobachevsky–Poincaré (hyperbolic) plane; each of them being modeled as a plane or a (pseudo-) sphere in $\mathbb{R}^3$ equipped with appropriate quadratic form. Namely, we show that a billiard is polynomially integrable, if and only if its boundary is a union of confocal conical arcs and appropriate geodesic segments. We also present a complexification of these results. These are joint results of Mikhail Bialy, Andrey Mironov and the author. The proof is split into two parts. The first part is given in two papers by Bialy and Mironov (in Euclidean and non-Euclidean cases respectively). Their geometric construction reduced the Algebraic Birkhoff Conjecture to a purely algebro-geometric problem to show that an irreducible algebraic curve in $\mathbb{CP}^2$ with certain properties is a conic. They have shown that its singular and inflection points lie in the complex light conic of the above-mentioned quadratic form. In the present paper we solve the above algebro-geometric problem completely.

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Contents

1 Introduction

1.1 Main results

The famous Birkhoff Conjecture deals with strictly convex bounded planar billiards with smooth boundary. Recall that a *caustic* of a planar billiard $\Omega \subset \mathbb{R}^2$ is a curve C such that each tangent line to C reflects from the boundary of the billiard to a line tangent to C. A billiard Ω is called *Birkhoff caustic-integrable,* if a neighborhood of its boundary in Ω is foliated by closed caustics, and the boundary $\partial\Omega$ is a leaf of this foliation. It is well-known that each elliptic billiard is integrable, see [40, section 4]. The **Birkhoff Conjecture** states the converse: *the only Birkhoff caustic-integrable convex bounded planar billiard with smooth boundary is an ellipse.*[1]

Let now Σ be a two-dimensional surface with a Riemannian metric, $\Omega \subset \Sigma$ be a connected domain[2] with piecewise smooth boundary. The *billiard flow B_t* acts on the tangent bundle $T\Sigma|_\Omega$ as follows. A point $(Q, P) \in T\Sigma|_\Omega$, $Q \in \Omega$, $P \in T_Q\Sigma$ moves along a trajectory of the geodesic flow of the surface Σ until Q hits the boundary $\partial\Omega$. While hitting the boundary, the point Q remains unchanged, and the velocity vector P is reflected from the boundary to the vector P^* according to the usual reflection law: the angle of incidence equals to the angle of reflection; $|P| = |P^*|$. Afterwards the new point (Q, P^*) again moves along a trajectory of the geodesic flow etc. The billiard flow thus defined, which can be viewed as a geodesic flow with impacts on $T\Sigma|_\Omega$, has an obvious first integral: the absolute value $|P|$ of the velocity. A strictly convex billiard Ω with smooth boundary is called *integrable in the Liouville sense,* if its flow has an additional first integral independent with

[1]This conjecture, classically attributed to G.Birkhoff, was published in print only in the paper [37] by H. Poritsky, who worked with Birkhoff as a post-doctoral fellow in late 1920-ths.

[2]Everywhere in the paper a billiard is a **connected domain** $\Omega \subset \Sigma$.

$|P|$ on the intersection with $T\Sigma|_{\overline{\Omega}}$ of a neighborhood of the unit tangent bundle to the boundary.

The notions of a caustic and Birkhoff caustic-integrability extend to the case of a strictly convex domain Ω on an arbitrary surface Σ equipped with a Riemannian metric, with lines replaced by geodesics. It is well-known that the *Liouville and Birkhoff caustic integrabilities are equivalent.*

There is an analogue of the Birkhoff Conjecture for billiards on a simply connected complete Riemannian surface of non-zero constant curvature: sphere or hyperbolic (Lobachevsky–Poincaré) plane. This is also an open problem.

The particular case of the Birkhoff Conjecture, when the additional first integral is supposed to be polynomial in the velocity components, motivated the next definition and conjecture.

Definition 1.1 Let Σ be a two-dimensional surface with Riemannian metric, and let $\Omega \subset \Sigma$ be a domain with piecewise smooth boundary. We say that the billiard in Ω is *polynomially integrable,* if its flow has a first integral on $T\Sigma|_{\Omega}$ that is a polynomial in the velocity P and whose restriction to the hypersurface $\{|P| = 1\}$ is non-constant.

Definition 1.2 Let Σ be as above, and let $\Omega \subset \Sigma$ be a domain with piecewise smooth boundary. We say that Ω is *analytically integrable,* if there exists a first integral analytic in P on a neighborhood in $T\Sigma|_{\Omega}$ of the zero section of the tangent bundle $T\Sigma|_{\Omega}$ that is not a function of just the modulus $|P|$. In addition, it is required that there exists a $r > 0$ such that the integral is defined for all (Q, P) with $Q \in \Omega$ and $|P| \leq r$ and its Taylor series in P converges uniformly in the above (Q, P).

Note that all the integrals under question, which are defined over an open domain Ω, should be invariant under the geodesic flow in Ω and under the reflections from its boundary.

Remark 1.3 The following facts are well-known:

- Analytic integrability implies polynomial integrability, since each homogeneous part in P of an analytic integral is a first integral itself, see [32, p. 107] (the converse is obvious);

- In the case, when Σ is a simply connected *complete surface of constant curvature* and the boundary $\partial\Omega$ is smooth and connected, polynomial integrability is equivalent to the existence of a polynomial integral as above in a neighborhood of the unit tangent bundle to $\partial\Omega$ in $T\Sigma|_{\Omega}$, by S.V.Bolotin's results [15, 16, 17], see Theorem 1.20 below. In this case each first integral that is just polynomial in P is *globally analytic* on $T\Sigma$, see [17, proof of proposition 2] and Theorem 1.20.

The **Algebraic Birkhoff Conjecture** states that *if a convex planar billiard with smooth boundary is polynomially integrable, then its boundary is a conic.* The Algebraic Birkhoff Conjecture together with its generalization to billiards with piecewise smooth (may be non-convex) boundaries on simply connected complete surfaces of constant curvature was first stated and studied by S.V.Bolotin [16, 17] and later studied in joint papers of M.Bialy and A.E.Mironov [10, 11, 12]. In the present paper we give a complete solution of the Algebraic Birkhoff Conjecture in full generality (Theorems 1.6 and 1.19).

Remark 1.4 The Algebraic Birkhoff Conjecture and its generalization are important and interesting themselves, *independently on a potential solution of the classical Birkhoff Conjecture.* They lie on the crossing of different domains of mathematics, first of all, dynamical systems, algebraic geometry and singularity theory. *They are not implied by the classical Birkhoff Conjecture.* For the general case of piecewise-smooth boundaries this is obvious. Even in the case of smooth convex boundary, while the algebraicity condition is a very strong restriction, the condition of just non-constance of a polynomial integral on the unit velocity level hypersurface is topologically weaker than the independence condition in the Liouville integrability, which requires independence of the additional integral and the energy on a whole neighborhood in $T\mathbb{R}^2|_{\overline{\Omega}}$ of the unit tangent bundle to the boundary.

Without loss of generality we consider simply connected complete surfaces Σ of constant curvature equal to 0 or ± 1: one can make non-zero constant curvature equal to ± 1 by multiplication of metric by constant factor; this changes neither geodesics, nor polynomial integrability. Thus, Σ is either the Euclidean plane, or the unit sphere, or the Lobachevsky–Poincaré hyperbolic plane. It is modeled as one of the three following surfaces in the space $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_3)$ equipped with the quadratic form

$$< Ax, x >, \ A \in \{\mathrm{diag}(1,1,0), \mathrm{diag}(1,1,\pm 1)\}, \ < x, x >= x_1^2 + x_2^2 + x_3^2.$$

- Euclidean plane: $\Sigma = \{x_3 = 1\}$, $A = \mathrm{diag}(1,1,0)$.

- The unit sphere: $\Sigma = \{x_1^2 + x_2^2 + x_3^2 = 1\}$, $A = Id$.

- The hyperbolic plane: $\Sigma = \{x_1^2 + x_2^2 - x_3^2 = -1\} \cap \{x_3 > 0\}$, $A = \mathrm{diag}(1,1,-1)$.

The metric of constant curvature on the surface Σ under question is induced by the quadratic form $< Ax, x >$. The *geodesics* on Σ are its intersections with two-dimensional vector subspaces in $\mathbb{R}^3$. The *conics* on Σ are its intersections with quadrics $\{< Cx, x >= 0\} \subset \mathbb{R}^3$, where C is a real symmetric 3×3-matrix.

Example 1.5 The billiard in a disk in $\mathbb{R}^2_{(x_1,x_2)}$ centered at 0 has first integral $x_1 P_2 - P_1 x_2$ linear in P. The billiard in any conic in any of the above surfaces Σ has an integral quadratic in P, see [17, proposition 1].

Theorem 1.6 *Let a billiard in Σ with C^2-smooth non-linear connected boundary be polynomially integrable. Then the billiard boundary is a conic.*

Corollary 1.7 *Every bounded polynomially integrable planar billiard with a C^2-smooth connected boundary is an ellipse.*

Below we extend the above theorem to billiards with countably piecewise smooth boundaries, see the following definition.

Definition 1.8 A domain $\Omega \subset \Sigma$ has *countably piecewise (C^2-) smooth boundary,* if $\partial\Omega$ consists of the two following parts:

- the *regular part:* an open and dense subset $\partial\Omega_{reg} \subset \partial\Omega$, where each point $X \in \partial\Omega_{reg}$ has a neighborhood $U = U(X) \subset \Sigma$ such that the intersection $U \cap \partial\Omega$ is a (C^2-) smooth one-dimensional submanifold in U;

- the *singular part:* the closed subset $\partial\Omega_{sing} = \partial\Omega \setminus \partial\Omega_{reg} \subset \partial\Omega$.

Remark 1.9 In the above definition the regular part of the boundary is always a dense and at most countable disjoint union of (C^2-) smooth arcs (taken without endpoints). The particular case of domains with piecewise smooth boundaries corresponds to the case, when the above union is finite, the arcs are smooth up to their endpoints and the singular part of the boundary is a finite set (which consists of endpoints and may be empty). For a general billiard with countably piecewise smooth boundary the billiard flow is well-defined on a residual set for all time values. In the case, when the singular part of the boundary has zero one-dimensional Hausdorff measure, the billiard flow is well-defined as a flow of measurable transformations.

Remark 1.10 The notions of polynomially (analytically) integrable billiards obviously extend to billiards with countably piecewise smooth boundaries, and these two notions are equivalent, as in the piecewise smooth case.

Definition 1.11 A billiard in $\mathbb{R}^2$ with countably piecewise smooth boundary is called *countably confocal,* if the regular part of its boundary consists of arcs of confocal conics and may be some straight-line segments such that

- at least one conical arc is present;

- in the case, when the common foci of the conics are distinct and finite (i.e., the conics are ellipses and (or) hyperbolas), the ambient line of each straight-line segment of the boundary is either the line through the foci, or the middle orthogonal line to the segment connecting the foci, see Fig. 1a);

- in the case, when the conics are concentric circles, the above ambient lines may be any lines through their common center, see Fig. 1b);

- in the case, when the conics are confocal parabolas, the ambient line of each straight-line segment of the boundary is either the common axis of the parabolas, or the line through the focus that is orthogonal to the axis, see Fig. 1 c), d).

Let us extend the above definition to the non-Euclidean case. To do this, let us recall the following definition.

Definition 1.12 [46, p.84]. Let $\Sigma \subset \mathbb{R}^3$ be one of the standard surfaces of constant curvature defined by a quadratic form $< Ax, x >$. Let B be a real symmetric 3×3-matrix that is not proportional to A. In the Euclidean case, when $A = \mathrm{diag}(1, 1, 0)$, we require in addition that the x_3-axis does not lie in $\mathrm{Ker}\, B$. The *pencil of confocal conics* in Σ defined by B consists of the conics

$$\Gamma_\lambda = \Sigma \cap \{< B_\lambda x, x >= 0\}, \ B_\lambda = (B - \lambda A)^{-1}. \tag{1.1}$$

For those λ, for which $\det(B - \lambda A) = 0$ and the kernel $K_\lambda = \mathrm{Ker}(B - \lambda A)$ is one-dimensional, we set Γ_λ to be the geodesic[3]

$$\Gamma_\lambda = \Sigma \cap K_\lambda^\perp. \tag{1.2}$$

In the case, when $dim K_\lambda = 2$, for every two-dimensional subspace $H \subset \mathbb{R}^3$ orthogonal to K_λ the intersection $\Sigma \cap H$ will be also called $\Gamma_\lambda = \Gamma_\lambda(H)$.

Remark 1.13 In the conditions of Definition 1.12 the confocal conic pencil is well-defined: $\det(B - \lambda A) \not\equiv 0$ as a function of λ. In the non-Euclidean cases this is obvious, since the matrix A is non-degenerate. In the Euclidean case one has $A = \mathrm{diag}(1, 1, 0)$ and the x_3-axis does not lie in $\mathrm{Ker}\, B$: that is, some of the matrix elements B_{13}, B_{23}, B_{33} is non-zero. One has

$$\det(B - \lambda A) = -\lambda^3 \det(A - \lambda^{-1} B)$$

[3]Everywhere below, whenever the contrary is not specified, the sign $\perp$ and the vector product are understood with respect to the standard Euclidean scalar product on $\mathbb{R}^3$.

$$= \lambda^2 B_{33} + \lambda(B_{13}^2 + B_{23}^2 - B_{33}(B_{11} + B_{22})) + \det B \not\equiv 0 : \qquad (1.3)$$

in the above right-hand side the identical vanishing of the coefficients at λ^2 and at λ would imply that $B_{33} = B_{13} = B_{23} = 0$, which is forbidden by our assumptions. Hence, $\det(B - \lambda A) \not\equiv 0$. Conversely, if in the Euclidean case the x_3-axis were contained in the kernel of the matrix B, then obviously $\det(B - \lambda A) \equiv 0$, and the confocal pencil would not be well-defined.

Remark 1.14 The matrix B is uniquely defined modulo transformation $B \mapsto \mu B - \lambda A$, $\mu \neq 0$ (i.e., modulo $\mathbb{R}A$ and up to constant factor) by the corresponding confocal pencil. In the Euclidean case, when $\Sigma = \{x_3 = 1\}$, $A = (1, 1, 0)$, the above notion of confocal conics coincides with the classical one. In the Euclidean case the kernel K_λ is two-dimensional for some λ, if and only if the confocal conics under question are concentric circles; then the corresponding geodesics $\Gamma_\lambda(H)$ are the lines through their common center.

Definition 1.15 A billiard $\Omega \subset \Sigma$ with a countably piecewise smooth boundary is *countably confocal,* if its boundary consists of arcs of confocal conics (at least one conical arc is present) and may be some geodesic segments so that the ambient geodesic of each geodesic segment of the boundary belongs to the following list of *admissible geodesics.* Here B is the matrix defining the confocal conic pencil given by the conical arcs in $\partial\Omega$.

1) Each geodesic Γ_λ in (1.2) and (or) $\Gamma_\lambda(H)$ (if any) is admissible.

2) In the case, when $B = Aa \otimes b + b \otimes Aa$ (modulo $\mathbb{R}A$, see Remark 1.14) where $a, b \in \mathbb{R}^3$, $< a, b >= 0$, the following geodesics are also admissible:

2a) in the subcase, when Σ is not the Euclidean plane: the geodesics

$$\{r \in \Sigma \mid \ < r, a >= 0\}, \ \{r \in \Sigma \mid \ < r, Ab >= 0\}; \qquad (1.4)$$

2b) in the subcase, when $\Sigma = \{x_3 = 1\}$ is Euclidean and b is not parallel to Σ: only Γ_λ in (1.2) and the first geodesic in (1.4) are admissible;

Note that the subcase in 2) when $\Sigma = \{x_3 = 1\}$ and b is parallel to Σ is impossible, since in this subcase the x_3-axis would lie in the kernel $\operatorname{Ker} B$, which is forbidded by our assumptions.

Remark 1.16 In the above subcase 2a) set $\widetilde{a} = Ab$, $\widetilde{b} = Aa$. Then $B = A\widetilde{a} \otimes \widetilde{b} + \widetilde{b} \otimes A\widetilde{a}$, and $< \widetilde{a}, \widetilde{b} >= 0$, since $A^2 = Id$. The geodesics in (1.4) are written in terms of the new vectors $\widetilde{a}$ and $\widetilde{b}$ in the opposite order. Thus, each geodesic of type (1.4) can be represented by the first equation in (1.4) for appropriate presentation $B = Aa \otimes b + b \otimes Aa$.

Confocal billiards with piecewise smooth boundaries were introduced by S.V.Bolotin [17], who proved their polynomial integrability with integrals of first, second or fourth degree. See the following proposition, whose proof presented in loc. cit. remains valid in the countably piecewise smooth case.

Proposition 1.17 *[17, proposition 1 in section 2; the theorem in section 4] Each countably confocal billiard is polynomially integrable: it has a non-trivial first integral that is either linear, or quadratic, or a degree 4 polynomial in the velocity components that is non-constant on the unit velocity hypersurface. The case of a degree 4 integral that cannot be reduced to a degree 2 integral is exactly the case, when the conics forming the billiard boundary are contained in a confocal pencil of types 2a) or 2b) from the above definition and the billiard boundary contains some of the admissible geodesics from (1.4) mentioned in 2a) and 2b) respectively.*

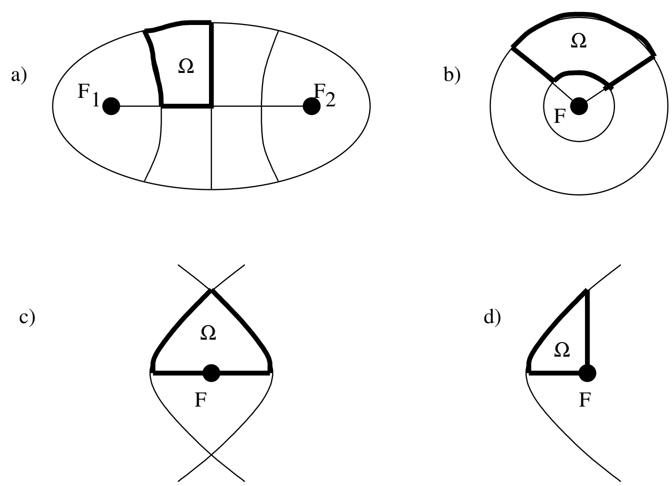

Figure 1: Examples of confocal planar billiards; F_1, F_2, F are the foci; the conics in c) and d) are parabolas. All of these billiards have quadratic integrals, except for the billiard at Fig. 1d), which has a degree 4 integral.

Example 1.18 A Euclidean billiard whose boundary contains an arc of parabola and a segment of the line through the focus that is orthogonal to the axis of the parabola, as at Fig. 1d), is exactly a billiard of type 2b), see the end of paper [17]; the above line is the first geodesic in (1.4). This example of a billiard having a degree 4 integral was first discovered in [38]. Analogous billiards on surfaces of non-zero constant curvature were constructed in [2].

The main result of the paper is the following theorem.

Theorem 1.19 [4] *Let a billiard in Σ with countably piecewise C^2-smooth boundary be polynomially (or equivalently, analytically) integrable, and let the regular part of its boundary contain at least one non-linear arc. Then the billiard is countably confocal.*

Theorem 1.19 is a joint result of M.Bialy, A.E.Mironov and the author. Its proof sketched below consists of the two following parts:

1) The papers [10, 11] of Bialy and Mironov, whose geometric construction reduced the proof of Theorem 1.19 to a purely algebro-geometric problem that was partially investigated by them.

2) The complete solution of the above-mentioned algebro-geometric problem obtained in the present paper (Theorem 1.23).

1.2 Sketch of proof of Theorem 1.19 and plan of the paper

In what follows a point $r \in \Sigma$ will be identified with its radius-vector in $\mathbb{R}^3$.

Theorem 1.20 *(S.V.Bolotin, see [16], [17, p.118; proof of proposition 2 on p.119], [33, chapter 5, section 3, proposition 5].) For every polynomially integrable billiard $\Omega \subset \Sigma$ with countably piecewise C^2-smooth boundary a polynomial integral non-constant on the unit velocity hypersurface $\{|P| = 1\}$ can be chosen to be a homogeneous polynomial $\Psi(M)$ of even degree in the components of the moment vector*

$$M = [r, P] = (x_2P_3 - x_3P_2, -x_1P_3 + x_3P_1, x_1P_2 - x_2P_1), \tag{1.5}$$

$$r = (x_1, x_2, x_3) \in \Sigma, \quad P = (P_1, P_2, P_3) \text{ is the velocity vector.}$$

(This statement is local and holds for reflection from an arbitrary smooth curve in Σ.) Each smooth arc of the boundary $\partial\Omega$ lies in an algebraic curve.

Theorem 1.21 *(see a more general theorem of S.V.Bolotin [17, section 4]). Let a billiard on Σ with a countably piecewise C^2-smooth boundary be polynomially integrable. Let its boundary contain a non-linear conical arc. Then the billiard is countably confocal.*

[4]Theorem 1.19 with a brief proof was announced in the author's note [26].

Remark 1.22 Theorems implying Theorems 1.20 and 1.21 were stated and proved in loc. cit. for piecewise smooth boundaries, but their proofs remain valid in the countably piecewise smooth case. To make the paper self-contained and to extend the main results to complex domain, we give a proof of Theorem 1.21 in Subsection 2.2. It follows the arguments from [17, section 4], but here it is done in the dual terms using results of Bialy and Mironov from [10, 11].

Let $\alpha \subset \partial\Omega$ be a non-linear smooth arc. By Bolotin's Theorem 1.21, for the proof of Theorem 1.19 it suffices to show that α contains a conical sub-arc. To do this, we use Bialy–Mironov construction of the dual billiard and their results presented in Subsection 2.1. Let us describe them briefly.

In what follows $\pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2$ denotes the tautological projection. Its complexification and restriction to Σ will be also denoted by π.

Recall that the standard Euclidean scalar product $< x, x >$ on $\mathbb{R}^3$ defines the *orthogonal polarity:* the correspondence sending each two-dimensional vector subspace in $\mathbb{R}^3$ to its Euclidean-orthogonal one-dimensional subspace. This together with the projection π induces a projective duality $\mathbb{RP}^{2*}_{(x_1:x_2:x_3)} \to \mathbb{RP}^2_{(M_1:M_2:M_3)}$ sending lines to points. (It is well-known that in the affine chart $(x_1 : x_2 : 1)$ the projective duality defined by the orthogonal polarity is the composition of the polar duality with respect to the unit circle and the central symmetry with respect to the origin.)

For simplicity, the curve dual to the projection $\pi(\alpha) \subset \mathbb{RP}^2$ with respect to the above projective duality will be denoted by α^* and called the *curve* Σ*-dual to* α.

Bialy and Mironov proved the following results in [10, 11]:

- Let $\Psi(M)$ be the homogeneous first integral of even degree $2n$ from Bolotin's Theorem 1.20. For every point $B \in \alpha^*$ the restriction to the projective tangent line $T_B\alpha^*$ of the rational function

$$G(M) = \frac{\Psi(M)}{< AM, M >^n} \tag{1.6}$$

is invariant under a special projective involution $T_B\alpha^* \to T_B\alpha^*$ fixing B: the so-called angular symmetry centered at B. More precisely, invariance of the function G is equivalent to the statement saying that for every $r \in \alpha$ the function $\Psi(M) = \Psi([r, v])$ in $v \in T_r\Sigma$ is invariant under reflection of the vector v from the line $T_r\alpha$.

- Consider the so-called *absolute:* the conic

$$\mathbb{I} = \{< AM, M >= 0\} \subset \mathbb{CP}^2_{(M_1:M_2:M_3)}. \tag{1.7}$$

The above angular symmetry coincides with the restriction to $T_B\alpha^*$ of the unique projective involution $\mathbb{CP}^2 \to \mathbb{CP}^2$ fixing B that fixes each line through B and permutes its intersection points with the conic $\mathbb{I}$: the so-called $\mathbb{I}$-*angular symmetry* centered at B.

- Concider the complex projective Zariski closure of the curve α^*, which is an algebraic curve, by Theorem 1.20. Each its non-linear irreducible component γ *generates a rationally integrable* $\mathbb{I}$*-angular billiard*, see Definition 2.10: for every point $B \in \gamma$ the restriction of a rational function G to the projective tangent line $T_B\gamma$ is invariant under the $\mathbb{I}$-angular symmetry centered at B; the function G is non-constant on $\mathbb{CP}^2$ and has poles in $\mathbb{I}$.

- For every curve γ generating a rationally integrable $\mathbb{I}$-angular billiard all its singular and inflection points (if any) lie in $\mathbb{I}$.

The main algebro-geometric result of the present paper, which implies the main results, is the following theorem.

Theorem 1.23 *Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic: either regular, or a union of two distinct lines. Every irreducible algebraic curve $\gamma \subset \mathbb{CP}^2$ generating a rationally integrable $\mathbb{I}$-angular billiard is a conic.*

For the proof of Theorem 1.23 we study *local branches* of the curve γ at points $C \in \gamma \cap \mathbb{I}$: the irreducible components of the germ (γ, C). Each local branch is holomorphically bijectively parametrized in so-called adapted affine coordinates by small complex parameter t as follows:

$$t \mapsto (t^q, ct^p(1 + o(1))), \quad \text{as } t \to 0; \quad q, p \in \mathbb{N}, \quad 1 \leq q < p, \quad c \neq 0.$$

In Section 4 we prove Theorem 4.1 giving a list of statements on p and q satisfied by local branches of appropriate type (see Cases 1) and 2) below). Afterwards in Section 5 we prove the following general algebro-geometric theorem. It states that Bialy–Mironov inclusions $Sing(\gamma), Infl(\gamma) \subset \mathbb{I}$ and the statements of Theorem 4.1 on local branches together imply that γ is a conic.

Theorem 1.24 *Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic: either regular, or a union of two distinct lines. Let $\gamma \subset \mathbb{CP}^2$ be an irreducible complex algebraic curve different from a line and from $\mathbb{I}$. Let all the singularities and inflection points (if any) of the curve γ lie in $\mathbb{I}$. Let for every $C \in \gamma \cap \mathbb{I}$ the local branches b of the curve γ at C satisfy the following statements:*

Case 1): C is a regular point of the conic $\mathbb{I}$. If b is tangent to $\mathbb{I}$, then it is quadratic: $p = 2q$. If b is transversal to $\mathbb{I}$, then it is regular and quadratic: $q = 1$, $p = 2$.

Case 2): $\mathbb{I}$ is a union of two distinct lines intersecting at C. If b is transversal to both lines, then b is subquadratic: $p \leq 2q$.

Then γ is a conic.

The proof of Theorem 1.24, which will be given in Section 5, is based on the ideas and arguments due to E.Shustin on plane curve invariants from the proof of its analogue for the case of outer billiards in [27].

The most technical part of the paper is the proof of statement (ii-b) of Theorem 4.1, which asserts that each local branch of the curve γ that is transversal to $\mathbb{I}$ and is based at a regular point of the conic $\mathbb{I}$ is regular and quadratic. Its proof is based on a remarkable formula of Bialy and Mironov for the Hessian of the function defining γ, see [10, theorem 6.1] and [11, formulas (16) and (32)]. This formula is recalled in Section 3 as formula (3.4). We use asymptotic formulas for both sides of Bialy–Mironov formula along the transversal local branches that are stated and proved in Subsection 3.4. In their proofs we use asymptotic formulas for the defining functions and their Hessians stated and proved in Subsections 3.2 and 3.3 respectively.

In Section 6 we prove the main results: Theorems 1.23, 1.19 and 1.6 and the complexification of Theorem 1.19 stated in the next subsection.

1.3 Complexification

Here we state a complexification of Theorem 1.19, which deals with the space $\mathbb{C}^3_{(x_1,x_2,x_3)}$ equipped with a quadratic form $< Ax, x >$, $x = (x_1, x_2, x_3)$, $A \in \{\text{diag}(1, 1, 0),\ \text{diag}(1, 1, \pm 1)\}$, and a complex surface $\Sigma \subset \mathbb{C}^3$.

- Euclidean case: $\Sigma = \{x_3 = 1\}$, $A = \text{diag}(1, 1, 0)$.

- Non-Euclidean case: $\Sigma = \Sigma_\pm = \{< Ax, x >= \pm 1\}$, $A = \text{diag}(1, 1, \pm 1)$.

We equip the surface Σ under question with the complex bilinear quadratic form induced by the form $< Ax, x >$ on its tangent planes.

Note that the surfaces $\Sigma_\pm$ are obtained one from the other by the transformation $(x_1, x_2, x_3) \mapsto (ix_1, ix_2, x_3)$, but the latter transformation changes the sign of the complex quadratic form $< Ax, x >$ on $T\Sigma_\pm$.

Recall that a one-dimensional subspace Λ in a complex linear space equipped with a $\mathbb{C}$-bilinear scalar product is *isotropic,* if each vector in Λ has zero scalar square. A holomorphic curve Λ in a complex manifold Σ equipped with a $\mathbb{C}$-bilinear quadratic form on $T\Sigma$ is *isotropic,* if for every $x \in \Lambda$ the tangent subspace $T_x\Lambda \subset T_x\Sigma$ is isotropic.

A *complex geodesic* is

- a non-isotropic line in $\Sigma = \mathbb{C}^2$ in the Euclidean case;

- the intersection of the surface Σ with a two-dimensional subspace in $\mathbb{C}^3$ that is not tangent to the light cone $\widehat{\mathbb{I}} = \{< Ax, x >= 0\}$ in the non-Euclidean case.

The reason to cross out the planes tangent to $\widehat{\mathbb{I}}$ is the following.

Proposition 1.25 *Consider the non-Euclidean case: $A = \mathrm{diag}(1, 1, \pm 1)$. For every two-dimensional vector subspace $H \subset \mathbb{C}^3$ tangent to the light cone $\widehat{\mathbb{I}}$ the intersection $H \cap \Sigma$ is a union of two parallel isotropic straight lines. Each isotropic holomorphic curve in Σ is a line contained in a two-dimensional vector subspace in $\mathbb{C}^3$ tangent to $\widehat{\mathbb{I}}$. For every $r \in \Sigma$ the one-dimensional isotropic vector subspaces in the plane $T_r\Sigma$ are exactly its intersections with two-dimensional vector subspaces in $\mathbb{C}^3$ containing r and tangent to $\widehat{\mathbb{I}}$: there are exactly two of them.*

Proof For every $r \in \Sigma$ the quadratic form on $T_r\Sigma$ induced by $< Ax, x >$ is non-degenerate, since $T_r\Sigma$ is $< Ax, x >$-orthogonal to the radius-vector of the point r and transversal to it: $< Ar, r >= \pm 1 \neq 0$. For every two-dimensional subspace H tangent to $\widehat{\mathbb{I}}$ the restriction of the form $< Ax, x >$ to H is non-zero and has a non-zero kernel K: the tangency line of the plane H with $\widehat{\mathbb{I}}$. Hence, in appropriate affine coordinates (z_1, z_2) on H centered at 0 one has $< Ax, x > |_H = z_1^2$, $K = \{z_1 = 0\}$, $H \cap \Sigma = \{z_1^2 = \pm 1\}$. Therefore, the intersection $H \cap \Sigma$ is a union of two lines parallel to K, which are thus isotropic. The first statement of the proposition is proved.

Let us now prove the third and the second statements of the proposition. For every point $r \in \Sigma$ the tangent plane $T_r\Sigma$ equipped with the quadratic form induced by $< Ax, x >$ contains two distinct one-dimensional isotropic vector subspaces, by non-degeneracy. Each ot them is the line of intersection of the plane $T_r\Sigma$ with a two-dimensional subspace H through r that is tangent to $\widehat{\mathbb{I}}$. This follows from the first statement of the proposition and the fact that there are two distinct 2-dimensional subspaces through r that are tangent to $\widehat{\mathbb{I}}$. This implies the third statement of the proposition. This also implies that each isotropic curve in Σ is locally a phase curve of a (double-valued) holomorphic line field defined by the above intersections. The only phase curves of the latter field are the isotropic lines in $H \cap \Sigma$, H being tangent to $\widehat{\mathbb{I}}$, by the first statement of the proposition and uniqueness theorem in ordinary differential equations. This proves the proposition. $\Box$

Definition 1.26 Consider the surface Σ in the non-Euclidean case. Let $\gamma \subset \Sigma$ be a complex geodesic. Let $\mathcal{G}_\gamma$ denote the stabilizer of the geodesic γ in the group of automorphisms $\mathbb{C}^3 \to \mathbb{C}^3$ preserving the form $< Ax, x >$. Its identity component, which will be denoted by $\mathcal{G}^0_\gamma$, will be called the *group of translations along the geodesic* γ. A translation of the complex Euclidean plane along a complex line L is the translation by a vector parallel to L.

Remark 1.27 In the above definition in the non-Euclidean case let $H \subset \mathbb{C}^3$ denote the corresponding two-dimensional vector subspace: $\gamma = H \cap \Sigma$. The geodesic γ is thus a regular conic in the plane H that is biholomorphically parametrized by $\mathbb{C}^*$. Its projective closure $\hat{\gamma}$ in $\mathbb{CP}^2 \supset H$ intersects the infinity line $\mathbb{CP}^2 \setminus H$ at two distinct points. The restrictions to γ of the translations along the geodesic γ are exactly those conformal automorphisms $\hat{\gamma} \to \hat{\gamma}$ that fix the latter intersection points. One has $\hat{\gamma} \simeq \overline{\mathbb{C}}$, $\gamma \simeq \mathbb{C}^*$. This yields to a natural isomorphism $\mathcal{G}^0_\gamma \simeq \mathbb{C}^*$.

Definition 1.28 A *complex billiard* on Σ is a collection (finite or infinite, countable or uncountable) of holomorphic curves $\Gamma_t \subset \Sigma$ distinct from isotropic lines (see [25, definition 1.3] for finite collections in the Euclidean case). A complex billiard is said to be *polynomially integrable,* if there exists a function $\Phi(r, P)$ on $T\Sigma$ that is polynomial in $P \in T_r\Sigma$ with the following properties:

- the restriction of the function Φ to the tangent bundle of every complex geodesic is invariant under the translations along the geodesic;

- for every point $r \in \Gamma_t$ such that the line $T_r\Gamma_t$ is non-isotropic for the quadratic form on $T_r\Sigma$ induced by $< Ax, x >$ the restriction $\Phi|_{T_r\Sigma}$ is invariant under the *symmetry* with respect to the complex line $T_r\Gamma_t$ (see [25, definition 2.1]): the unique $\mathbb{C}$-linear involution $T_r\Sigma \to T_r\Sigma$ that preserves the form induced by $< Ax, x >$ and fixes the points of the line $T_r\Gamma_t$.

Example 1.29 Consider a polynomially integrable billiard with countably piecewise smooth boundary in a real surface of constant curvature. Then the smooth part of the boundary is contained in a union of arcs of conics and segments of admissible geodesics (Theorem 1.19). Their complexifications form a complex billiard having a polynomial integral that is the complexification of the real polynomial integral of the real billiard: it can be chosen of degree no greater than four, see Proposition 1.17.

Remark 1.30 The confocality notion from Definition 1.12 for real conics extends to the case of complex conics in Σ without changes in both non-Euclidean and Euclidean cases with B being a complex symmetric matrix and $\lambda \in \mathbb{C}$. In the Euclidean case this complex confocality notion is equivalent to the one given in [25, definition 2.24], which follows from definition and Remark 1.14.

Remark 1.31 A pencil of confocal conics given by a matrix B is well-defined, if and only if inequality (1.3) holds: $\det(B - \lambda A) \not\equiv 0$ as a function of λ. In the real case inequality (1.3) is equivalent to the condition that the x_3-axis is not contained in the kernel of the matrix B, i.e., $(B_{13}, B_{23}, B_{33}) \neq (0, 0, 0)$, see Remark 1.13. In the complex case inequality (1.3) is equivalent to the following stronger condition: the equalities

$$B_{33} = 0, \; B_{23} = \pm i B_{13}, \; B_{13}^2 (B_{11} - B_{22} \pm 2i B_{12})$$

do not hold simultaneously.

Definition 1.32 A complex billiard Γ_t is said to be *confocal,* if the set of its curves different from complex geodesics is non-empty, all of them are confocal complex conics, and the complex geodesics from the family Γ_t belong to the lists of admissible complex geodesics given by Definition 1.15 (where now everything is complex: B, λ, a, b,...).

Remark 1.33 A priori, some complex curves Γ_λ in (1.2), $\Gamma_\lambda(H)$ and some intersections in (1.4) may be isotropic lines; then they are not complex geodesics, and we do not call them admissible. For example, in the non-Euclidean case let $\lambda \in \mathbb{C}$ be such that the kernel $K_\lambda = \mathrm{Ker}(B - \lambda A)$ is one-dimensional. The corresponding intersection $\Gamma_\lambda = K_\lambda^\perp \cap \Sigma$ is isotropic, if and only if $K_\lambda \subset \widehat{\mathbb{I}}$. This follows from Proposition 1.25 and since the Euclidean orthogonal $K_\lambda^\perp$ is tangent to the light cone $\widehat{\mathbb{I}}$ if and only if $K_\lambda \subset \widehat{\mathbb{I}}$: see the last statement of Corollary 2.15 in Subsection 2.2.

Theorem 1.34 *Every polynomially integrable complex billiard Γ_t on Σ containing at least one non-geodesic curve is confocal and has an integral $\Phi(r, P) = \Psi(M)$ (where $M = [r, P]$ is the complexified Euclidean vector product) that is a homogeneous polynomial in M of degree at most four. The integral can be chosen quadratic in M, except for the cases 2a), 2b) in Definition 1.15, when Γ_t contains a corresponding admissible complex geodesic of type (1.4): in this case there is an integral of degree four.*

Theorem 1.34 will be proved in Subsection 6.4.

1.4 Historical remarks

Existence of caustics in any strictly convex planar billiard with sufficiently smooth boundary was proved by V.F.Lazutkin [34]. Non-existence of caustics in higher-dimensional billiards with boundaries different from quadrics was proved by M.Berger [6].

The Birkhoff Conjecture was studied by many mathematicians. In 1950 H.Poritsky [37] proved it under the additional assumption that the billiard in each closed caustic near the boundary has the same closed caustics, as the initial billiard. Later in 1988 another proof of the same result was obtained by E.Amiran [5]. In 1993 M.Bialy [7] proved that if the phase cylinder of the billiard is foliated (almost everywhere) by continuous curves which are invariant under the billiard map, then the boundary curve is a circle. (Another proof of the same result was later obtained in [47].) In particular, Bialy's result implies Birkhoff Conjecture under the assumption that the foliation by caustics extends to the whole billiard domain punctured at one point: then the boundary is a circle. In 2012 he proved a similar result for billiards on the constant curvature surfaces [8] and also for magnetic billiards [9]. In 1995 A.Delshams and R.Ramirez-Ros suggested an approach to prove splitting of separatrices for generic perturbation of ellipse [19]. In 2013 D.V.Treschev [42] made a numerical experience indicating that there should exist analytic *locally integrable* billiards, with the billiard reflection map having a two-periodic point where the germ of its second iterate is analytically conjugated to a disk rotation. Recently Treschev studied the billiards from [42] in more detail in [43] and their multi-dimensional versions in [44]. A similar effect for a ball rolling on a vertical cylinder under the gravitation force was discovered in [3]: the authors have shown that the ratio between its vertical and horizontal oscillation periods is a universal irrational constant, the number $\sqrt{7/2}$. Recently V.Kaloshin and A.Sorrentino have proved a *local version* of the Birkhoff Conjecture [31]: *an integrable deformation of an ellipse is an ellipse.* (The case of ellipses with small extentricities was treated in the previous paper by A.Avila, J. De Simoi and V.Kaloshin [4].) A dynamical entropic version of Birkhoff Conjecture was stated and partially studied by J.-P.Marco in [35].

In 1988 A.P.Veselov proved that every billiard bounded by confocal quadrics in any dimension has a complete system of first integrals in involution that are quadratic in P [45, proposition 4]. In 1990 he studied a billiard in a non-Euclidean ellipsoid: in the sphere and in the Lobachevsky (i.e., hyperbolic) space of any dimension n. He proved its complete integrability and provided an explicit complete list of first integrals [46, the corollary on p. 95]. In the same paper he proved that all the sides of a billiard trajectory are tangent to the same $n-1$ quadrics confocal to the boundary of the ellipsoid and the billiard dynamics corresponds to a shift of the Jacobi variety corresponding to an appropriate hyperelliptic curve [46, theorems 3, 2 on p. 99]. The Algebraic Birkhoff Conjecture was studied by S.V.Bolotin, who proved in 1990 that in its conditions the billiard boundary lies in an algebraic curve [16]. In the same paper and in [17, section 4] he proved the conjecture under the assumption that at least one irreducible component of the ambient complex algebraic curve is non-linear and nonsingular. In 1992 he proved integrability of countably confocal billiards with piecewise smooth boundaries in two- and higher-dimensional spaces of constant curvature with integrals of degrees two or four in [17]. Recently M.Bialy and A.E.Mironov proved the planar Algebraic Birkhoff Conjecture in the case of integrals of degree four [12]. A version of the planar Algebraic Birkhoff Conjecture for *families* of billiards sharing the same polynomial integral (with boundaries depending continuously on one parameter) was solved in [1]: in loc. cit. it is sufficient to require that the union of the boundaries do not lie in an algebraic curve in $\mathbb{R}^2$, see [1, end of p.30]. Dynamics in countably confocal billiards with piecewise smooth boundaries in two and higher dimensions was studied in [20, 21, 22, 23, 24]. Dynamics in the so-called pseudo-integrable billiards (more precisely, confocal billiards with non-convex angles) was studied in [21, 22, 23, 24]. For further results on the Algebraic Birkhoff Conjecture and its version for magnetic billiards see the above-mentioned papers [10, 11, 12] by M.Bialy and A.E.Mironov, [13, 14] and references therein.

The analogue of the Birkhoff Conjecture for outer billiards was stated by S.L.Tabachnikov [41] in 2008. Its algebraic version was stated by Tabachnikov and proved by himself under genericity assumptions in the same paper, and recently solved completely in the joint work of the author of the present paper with E.I.Shustin [27].

2 Preliminaries: from polynomially integrable to $\mathbb{I}$-angular billiards

2.1 Reflection and $\mathbb{I}$-angular symmetry

Here we present results of M.Bialy and A.E.Mironov mentioned in Subsection 1.2 and give self-contained proofs of some of them.

Proposition 2.1 *(S.V.Bolotin, see [17, formula (15), p.23], [33, formula (3.12), p.140]). For every $r \in \Sigma$ the linear operator $\mathcal{M}_r : T_r\Sigma \to V_r = r^{\perp}$, $v \mapsto [r, v]$ is an isomorphism preserving the quadratic form $< Ax, x >$. Here the orthogonal complement and the vector product are taken with respect to the standard Euclidean scalar product, see Footnote 3.*

Definition 2.2 Let the space $\mathbb{R}^n$ be equipped with a quadratic form $<$

$Ax, x >$, A being a symmetric $n \times n$-matrix, and let $\ell \subset \mathbb{R}^n$ be a one-dimensional vector subspace such that $\ell \not\subset \{< Ax, x >= 0\}$. The *pseudo-symmetry* of the space $\mathbb{R}^n$ with respect to the line ℓ is the linear involution $\mathbb{R}^n \to \mathbb{R}^n$ preserving the quadratic form, fixing the points of the line ℓ and acting as central symmetry in its orthogonal complement with respect to the form. The definition of complex pseudo-symmetry of the space $\mathbb{C}^n$ is analogous.

Corollary 2.3 *For every $r \in \Sigma$ and one-dimensional subspace $\ell \subset T_r\Sigma$ the mapping $\mathcal{M}_r : T_r\Sigma \to V_r$, $v \mapsto M$ conjugates the pseudo-symmetry $T_r\Sigma \to T_r\Sigma$ with respect to the line ℓ and the pseudo-symmetry $V_r \to V_r$ with respect to the one-dimensional subspace orthogonal to both r and ℓ.*

Definition 2.4 Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic: either a smooth conic, or a union of two distinct lines. Let $B \in \mathbb{CP}^2 \setminus \mathbb{I}$. For every complex line L through B consider its complex projective involution fixing B and permuting its intersection points with $\mathbb{I}$. (If L is tangent to $\mathbb{I}$, the involution under question is the unique non-trivial projective involution $L \to L$ fixing B and the tangency point.) The transformation thus constructed for each L is a projective involution $\mathbb{CP}^2 \to \mathbb{CP}^2$ fixing B, which will be called the $\mathbb{I}$-*angular symmetry* with center B. See Fig. 2 in the Euclidean case.

Proposition 2.5 *Consider the space $\mathbb{C}^3_{(M_1,M_2,M_3)}$ equipped with a quadratic form $< AM, M >$, $dim(\mathrm{Ker}\, A) \leq 1$. The absolute $\mathbb{I} = \{< AM, M >= 0\} \subset \mathbb{CP}^2_{(M_1:M_2:M_3)}$, see (1.7), is either a regular conic, or a union of two distinct lines. The projectivization of a pseudo-symmetry $\mathbb{C}^3 \to \mathbb{C}^3$ with respect to a one-dimensional subspace ℓ is the $\mathbb{I}$-angular symmetry with center $\pi(\ell)$.*

The proposition follows from definition.

Theorem 2.6 *(see [11, theorem 1.3, p.151] in the non-Euclidean case). Let $\Omega \subset \Sigma$ be a polynomially integrable billiard with countably piecewise smooth boundary and a homogeneous polynomial integral $\Psi(M)$ of even degree. Let r be a point in a smooth arc in $\partial\Omega$. Set $V_r = r^\perp \subset \mathbb{R}^3$. Let $L \subset V_r$ be the one-dimensional subspace Euclidean-orthogonal to both r and the tangent line $T_r\partial\Omega$. The restriction $\Psi|_{V_r}$ is invariant under the pseudo-symmetry of the plane V_r equipped with the form $< Ax, x >$ with respect to the line L.*

Proof The polynomial integral $\Psi([r, v])$ is invariant under the action on v of the pseudo-symmetry $T_r\Sigma \to T_r\Sigma$ with respect to the line $\ell = T_r\partial\Omega$

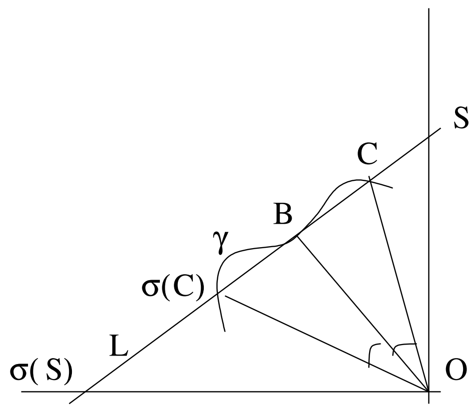

Figure 2: The $\mathbb{I}$-angular symmetry $\sigma : \mathbb{CP}^2 \to \mathbb{CP}^2$ with center B in the Euclidean case, when $\mathbb{I} = \{x_1^2 + x_2^2 = 0\}$: the action in the affine chart $\mathbb{C}^2_{(x_1,x_2)}$; $O = (0,0)$. The lines OC and $O\sigma(C)$ are symmetric with respect to the line OB. The projective lines OS and $O\sigma(S)$ are isotropic for the complex Euclidean metric $dx_1^2 + dx_2^2$ on $\mathbb{C}^2$, that is, $\mathbb{I} = OS \cup O\sigma(S)$.

(invariance under reflection). This together with Corollary 2.3 implies the statement of the theorem. □

Convention 2.7 Recall that for every curve $\alpha \subset \Sigma$ its Σ-*dual* is the curve $\alpha^* \subset \mathbb{RP}^2$ orthogonal-polar-dual to the projection $\pi(\alpha) \subset \mathbb{RP}^2$, see Subsection 1.2. For every $r \in \Sigma$ each one-dimensional vector subspace $\ell \subset T_r\Sigma$ is the intersection of the tangent plane $T_r\Sigma$ with a two-dimensional subspace $H \subset \mathbb{R}^3$ containing r. The intersection $\widehat{\ell} = H \cap \Sigma$ is the geodesic tangent to ℓ. The point $\pi(H^\perp) \in \mathbb{RP}^2$ will be called the *point* Σ-*dual* to the subspace ℓ and to the geodesic $\widehat{\ell}$. It will be denoted by $\widehat{\ell}^*$.

Theorem 2.8 *Let $\Omega \subset \Sigma$ be a polynomially integrable billiard with a countably piecewise smooth boundary. Let $\Psi(M)$ be its homogeneous polynomial integral of even degree $2n$. The function $G = \frac{\Psi(M)}{<AM,M>^n}$ from (1.6) treated as a rational function on $\mathbb{CP}^2_{(M_1:M_2:M_3)}$ satisfies the following statements.*

1) For every non-linear smooth arc $\alpha \subset \partial\Omega$, let $\alpha^* \subset \mathbb{RP}^2$ be its Σ-dual curve, for every point $C \in \alpha^*$ the restriction of the function G to the projective line $T_C\alpha^*$ is invariant under the $\mathbb{I}$-angular symmetry with center C. One has $G|_{\alpha^*} \equiv const.$

2) For every geodesic $\widehat{\ell} \subset \Sigma$ that contains a segment of the boundary $\partial\Omega$ the function G is invariant under the $\mathbb{I}$-angular symmetry of the whole projective plane $\mathbb{CP}^2$ with center $\widehat{\ell}^*$: the point Σ-dual to $\widehat{\ell}$.

Remark 2.9 A version of statement 1) of Theorem 2.8 in the Euclidean case was proved in [10, theorem 3] for convex domains with smooth boundary. But its proof remains valid in the general Euclidean case.

Proof of Theorem 2.8. Each point $C \in \alpha^*$ is dual to the projective line tangent to the curve $\pi(\alpha)$ at some point $\pi(r)$, $r \in \alpha$, by definition. Consider the projective line $T_C\alpha^*$ and set $V = \pi^{-1}(T_C\alpha^*) \cup \{0\} \subset \mathbb{R}^3$. It is the two-dimensional subspace orthogonal to the line Or, by definition. Set $L = \pi^{-1}(C) \cup \{0\} \subset V$: it is the one-dimensional subspace orthogonal to both lines $T_r\alpha$ and Or, by definition. The restrictions to V of both functions $\Psi(M)$ and $< AM, M >$ are invariant under the pseudo-symmetry of the plane V with respect to the line L, by Theorem 2.6 and isometry. Hence, the restriction to V of the ratio $G(M) = \frac{\Psi(M)}{<AM,M>^n}$ is also invariant. Therefore, the restriction to $\pi(V \setminus \{0\}) = T_C\alpha^*$ of the function G treated as a rational function on $\mathbb{CP}^2$ is invariant under the projectivized pseudo-symmetry, which coincides with the $\mathbb{I}$-angular symmetry centered at C, by Proposition 2.5. The equality $G|_{\alpha^*} \equiv const$ holds since the derivative of the function G at C along a vector tangent to $T_C\alpha^*$ vanishes. Indeed, the function $G|_{T_C\alpha^*}$, which is invariant under a projective involution fixing C, has zero derivative at C, similarly to vanishing of derivative of an even function at 0. Statement 1) is proved. The proof of statement 2) is analogous. In more detail, let $\Lambda \subset \Sigma$ be a geodesic whose segment $I \subset \Lambda$ is contained in $\partial\Omega$. For every point $Q \in I$ the projective line Q^* dual to $\pi(Q)$ passes through the point Λ^* Σ-dual to Λ. The restriction $G|_{Q^*}$ is invariant under the $\mathbb{I}$-angular symmetry with center Λ^*, as in the above argument. Therefore, this holds for the restriction of the function G to every complex line through Λ^*, and hence, on all of $\mathbb{CP}^2$, by uniqueness of analytic extension. Statement 2) is proved. □

Definition 2.10 Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic: either a regular conic, or a pair of distinct lines. Let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve different from a line and from $\mathbb{I}$. We say that γ *generates a rationally integrable* $\mathbb{I}$*-angular billiard,* if there exists a non-constant rational function G on $\mathbb{CP}^2$ with poles contained in $\mathbb{I}$ (called the *integral of the* $\mathbb{I}$*-angular billiard*) such that for every $C \in \gamma \setminus \mathbb{I}$ the restriction of the function G to the projective tangent line $T_C\gamma$ is invariant under the $\mathbb{I}$-angular symmetry with center C.

Corollary 2.11 *Let $\mathbb{I} \subset \mathbb{CP}^2_{(M_1:M_2:M_3)}$ be the absolute, see (1.7). Let $\Omega \subset \Sigma$ be a polynomially integrable billiard with homogeneous integral $\Psi(M)$ of even degree $2n$. Let $\alpha \subset \partial\Omega$ be a nonlinear C^2-smooth arc, let $\alpha^* \subset \mathbb{RP}^2 \subset \mathbb{CP}^2$ be its Σ-dual curve. The complex projective Zariski closure of the curve α^* is an algebraic curve. Each its non-linear irreducible component generates a rationally integrable $\mathbb{I}$-angular billiard with integral $G(M) = \frac{\Psi(M)}{<AM,M>^n}$, see (1.6).*

Proof The first statement of the corollary, which follows from Bolotin's Theorem 1.20, also follows from constance of the function G on α^*, see Statement 1) of Theorem 2.8. Its second statement follows from the invariance of the function G in Statement 1) of Theorem 2.8 by straightforward analytic extension argument. $\square$

Proposition 2.12 *Let an irreducible algebraic curve $\gamma \subset \mathbb{CP}^2$ generate a rationally integrable $\mathbb{I}$-angular billiard with the integral G. Then $G|_\gamma \equiv const$.*

The proof of the proposition repeats literally the above proof of the analogous statement from Theorem 2.8.

2.2 Duality and $\mathbb{I}$-angular billiards. Proof of Theorem 1.21

For the proof of Theorem 1.21 we use the well-known classical properties of the orthogonal polarity given by the following proposition and its corollary. We present the proof of the proposition for completeness of presentation.

Proposition 2.13 *Let B be a non-degenerate complex symmetric 3×3-matrix. Consider the complex space $\mathbb{C}^3$ with coordinates $x = (x_1, x_2, x_3)$ equipped with the complex-bilinear Euclidean quadratic form $dx_1^2 + dx_2^2 + dx_3^2$. The complex orthogonal-polar-dual to the conic in $\mathbb{CP}^2_{(x_1:x_2:x_3)}$ given by the equation $< Bx, x >= 0$ is the conic given by the equation $< B^{-1}x, x >= 0$.*

Proof Consider the cone $K = \{x \in \mathbb{C}^3 \setminus \{0\} \mid \; < Bx, x >= 0\}$ and its tautological projection $\Gamma = \pi(K) \subset \mathbb{CP}^2$, which is the conic under consideration. Let $x \in K$. The projective tangent line $L = T_{\pi(x)}\Gamma$ is defined by the tangent plane T_xK considered as a vector subspace in $\mathbb{C}^3$. It follows from definition that T_xK consists of those vectors v for which $< Bx, v >= 0$. Thus, $(T_xK)^\perp = \mathbb{C}(Bx)$, and the dual L^* is $\pi(Bx)$. Therefore, the dual Γ^* is the projection $\pi(B(K))$, which is obviously defined by the equation $< B(B^{-1}y), B^{-1}y >=< B^{-1}y, y >= 0$. This proves the proposition. $\square$

Definition 2.14 [46, p.84]. Let A, B be two real non-proportional symmetric 3×3-matrices. They define a *pseudo-Euclidean pencil* of conics in $\mathbb{RP}^2$: the conics given by the equation

$$\{< (B - \lambda A)M, M >= 0\} \subset \mathbb{RP}^2_{(M_1:M_2:M_3)}, \ \lambda \in \mathbb{R}.$$

The same pencil of complex conics in $\mathbb{CP}^2$ depending on $\lambda \in \mathbb{C}$ will be also called pseudo-Euclidean.

Corollary 2.15 *The Σ-duality transforms every confocal pencil of conics to the corresponding pseudo-Euclidean pencil. Namely, for every real symmetric 3×3-matrix B satisfying the conditions of Definition 1.12 for any two conics in Σ lying in the confocal pencil defined by B in Definition 1.12 their Σ-dual curves lie in conics belonging to the pseudo-Euclidean pencil defined by the same matrix B. In the non-Euclidean case, when the absolute $\mathbb{I}$ is a regular conic, $\mathbb{I}$ is self-dual with respect to complex orthogonal polarity.*

The first statement of the corollary is obvious. The self-duality follows from Proposition 2.13 and involutivity: $A^2 = Id$ in the non-Euclidean case.
Proof of Theorem 1.21. Let $\Omega \subset \Sigma$ be a polynomially integrable billiard. Let $\Psi(M_1, M_2, M_3)$ be a non-trivial homogeneous polynomial integral of the billiard Ω of even degree $2n$. Consider the affine chart $M_3 \neq 0$ on $\mathbb{CP}^2_{(M_1:M_2:M_3)}$ with coordinates (x, y): $x = \frac{M_1}{M_3}$, $y = \frac{M_2}{M_3}$. Set

$$\mathcal{Q}(x, y) =< AM, M >, \text{ where } M = (x, y, 1):$$

$$\mathcal{Q}(x, y) = x^2 + y^2 \text{ in the Euclidean case; otherwise } \mathcal{Q}(x, y) = x^2 + y^2 \pm 1.$$

In this affine chart the function G on $\mathbb{CP}^2$ from (1.6) takes the form

$$G(x, y) = \frac{F(x, y)}{(\mathcal{Q}(x, y))^n}, \ F(x, y) = \Psi(x, y, 1), \ \deg F \leq 2n.$$

In what follows for every conic $\alpha \subset \Sigma$ the corresponding complex conic containing its Σ-dual α^* will be denoted by $\widetilde{\alpha}^*$.

Let the boundary $\partial\Omega$ contain an arc of a conic α. Let $\mathcal{C}$ denote the confocal conic pencil containing α, and let $\mathcal{C}^*$ denote the corresponding (Σ-dual) pseudo-Euclidean pencil of conics containing $\widetilde{\alpha}^*$:

$$\kappa_\lambda = \{< B_\lambda X, X >= 0\} \subset \mathbb{R}^3_{(X_1, X_2, X_3)}, \ B_\lambda = (B - \lambda A)^{-1}, \ \mathcal{C}_\lambda = \kappa_\lambda \cap \Sigma;$$

$$\kappa^*_\lambda = \{< (B - \lambda A)M, M >= 0\} \subset \mathbb{C}^3_{(M_1, M_2, M_3)}, \ \mathcal{C}^*_\lambda = \pi(\kappa^*_\lambda \setminus \{0\}) \subset \mathbb{CP}^2,$$

$$\kappa_\infty^* = \widehat{\mathbb{I}} = \{< AM, M >= 0\} \subset \mathbb{C}^3, \ \mathcal{C}_\infty^* = \pi(\kappa_\infty^* \setminus \{0\}) = \mathbb{I}.$$

Claim 1. *Each C^2-smooth non-linear arc of the boundary $\partial\Omega$ lies in a conic confocal to α.*

Proof The conic $\widetilde{\alpha}^*$ generates a rationally integrable $\mathbb{I}$-angular billiard with integral G, by Corollary 2.11. On the other hand, it is known that the billiard on a conic α admits a non-trivial quadratic homogeneous first integral $\widetilde{\Phi} = \widetilde{\Phi}(M)$, see [17, proposition 1]. Set

$$\widetilde{F}(x,y) = \widetilde{\Phi}(x,y,1), \ \widetilde{G}(x,y) = \frac{\widetilde{F}(x,y)}{\mathcal{Q}(x,y)}.$$

Claim 2. *The level curves of the function $\widetilde{G}$ are conics from the pencil $\mathcal{C}^*$, and the function G is constant on each of them.*

Proof For every conic β confocal to α the quadratic integral $\widetilde{\Phi}$ is also an integral for the billiard on the conic β. This is well-known, see [17], and follows from the explicit formula [17, formula (12)] for the quadratic integral. Therefore, both corresponding complexified dual conics $\widetilde{\alpha}^*$ and $\widetilde{\beta}^*$ generate rationally integrable $\mathbb{I}$-angular billiards with a common quadratic rational integral $\widetilde{G}$ having first order pole at $\mathbb{I}$, by Corollary 2.11. Hence, $\widetilde{G}$ is constant on $\widetilde{\alpha}^*$ and $\widetilde{\beta}^*$, by Proposition 2.12. Thus, the integral $\widetilde{G}$ is constant on every conic from the complex pseudo-Euclidean pensil $\mathcal{C}^*$, since the above conics $\widetilde{\beta}^*$ with β being confocal to α form a real one-dimensional subfamily in $\mathcal{C}^*$. Let us normalize the integral $\widetilde{G}$ by additive constant (or equivalently, the integral $\widetilde{\Phi}$ by addition of $c < AM, M >$, $c = const$) so that $\widetilde{G}|_{\widetilde{\alpha}^*} \equiv 0$. After this normalization one has $\widetilde{F}|_{\widetilde{\alpha}^*} \equiv 0$: that is, $\widetilde{F}$ is the quadratic polynomial defining the conic $\widetilde{\alpha}^*$. On the other hand, $\widetilde{\alpha}^*$ generates a rationally integrable $\mathbb{I}$-angular billiard with integral G (Corollary 2.11). Hence, $G|_{\widetilde{\alpha}^*} \equiv c_1 = const$, by Proposition 2.12. Therefore,

$$G(x,y) = c_1 + G_1(x,y)\widetilde{G}(x,y),$$

$$G_1(x,y) = \frac{f_1(x,y)}{(\mathcal{Q}(x,y))^{n-1}}, \ \deg f_1 \leq 2n-2.$$

Hence, the fraction G_1 is also a rational integral of the $\mathbb{I}$-angular billiard generated by $\widetilde{\alpha}^*$, as are G and $\widetilde{G}$. Thus, $G_1|_{\widetilde{\alpha}^*} \equiv c_2 = const$, by Proposition 2.12. Similarly we get that

$$G_1(x,y) = c_2 + G_2(x,y)\widetilde{G}(x,y), \ G_2(x,y) = \frac{f_2(x,y)}{(\mathcal{Q}(x,y))^{n-2}},$$

$\deg f_2 \leq 2n-4$, and G_2 is an integral of the $\mathbb{I}$-angular billiard generated by $\widetilde{\alpha}^*$, as are G_1 and $\widetilde{G}$. Continuing this prodecure we get that G is a polynomial in $\widetilde{G}$. Hence, $G \equiv const$ on the level curves of the function $\widetilde{G}$, that is, on the conics from the pencil $\mathcal{C}^*$. Claim 2 is proved. $\Box$

Let ϕ be a non-linear C^2-smooth arc in $\partial\Omega$, and let $\phi^* \subset \mathbb{RP}^2 \subset \mathbb{CP}^2$ denote its Σ-dual curve. The curve ϕ^* lies in a level curve of the function G, by Theorem 2.8, statement 1). Hence, it lies in a finite union of conics from the pencil $\mathcal{C}^*$, since each level curve of the function G is a finite union of conics in $\mathcal{C}^*$ (follows from Claim 2). Therefore, ϕ lies in just one conic confocal to α, by smoothness, since any two intersecting confocal conics are orthogonal. This proves Claim 1. $\Box$

Now it remains to show that if $\partial\Omega$ contains geodesic segments, then their ambient geodesics are admissible with respect to the pencil $\mathcal{C}$, see Definition 1.15. As it is shown below, this is implied by the following proposition.

Proposition 2.16 *Let B be a symmetric real 3×3-matrix. Let $\mathcal{C}$ denote the corresponding pencil of confocal conics in Σ from Definition 1.12. The corresponding admissible geodesics in Σ from Definition 1.15 are exactly those geodesics $\widehat{l}$, for which the symmetry of the surface Σ with respect to $\widehat{l}$ leaves the pencil $\mathcal{C}$ invariant: the symmetry permutes confocal conics. Or equivalently, the geodesics $\widehat{l}$ for which the $\mathbb{I}$-angular symmetry with center $\widehat{l}^*$ Σ-dual to $\widehat{l}$ leaves the Σ-dual pseudo-Euclidean pencil $\mathcal{C}^*$ invariant.*

Remark 2.17 We will be using only the second statement of Proposition 2.16 characterizing admissible geodesics $\widehat{l}$ in terms of $\mathbb{I}$-angular symmetry with center $\widehat{l}^*$ of the pencil $\mathcal{C}^*$. Their characterization in terms of symmetry of the pencil $\mathcal{C}$ will be proved just for completeness of presentation.

Proof of Proposition 2.16. Let us first prove that for every given geodesic $\widehat{l} \subset \Sigma$ the two statements of the proposition are indeed equivalent. As it is shown below, this is implied by the following proposition.

Proposition 2.18 *Consider the action of the symmetry with respect to a given geodesic $\widehat{l} \subset \Sigma$ on the space of all the geodesics in Σ. The Σ-duality conjugates this action to the $\mathbb{I}$-angular symmetry with center $\widehat{l}^*$.*

Proof It suffices to prove the above conjugacy on the space of those geodesics that intersect $\widehat{l}$, by analyticity and since they form an open subset in the connected manifold of geodesics. Each geodesic through a point $r \in \widehat{l}$ is uniquely determined by its tangent line: a one-dimensional subspace

$\Lambda \subset T_r\Sigma$. Thus, it suffices to show that the Σ-duality conjugates the symmetry action on the projectized tangent plane $\mathbb{P}(T_r\Sigma)$ with the $\mathbb{I}$-angular symmetry centered at $\widehat{l}^*$. Indeed, the Σ-duality sends each one-dimensional subspace $\Lambda \subset T_r\Sigma$ to the point $\widehat{\Lambda}^* \in \mathbb{RP}^2$ represented by the one-dimensional vector subspace $\Lambda^r \subset \mathbb{R}^3$ orthogonal to both r and Λ (see Convention 2.7). The linear isomorphism $\mathcal{M}_r : T_r\Sigma \to V_r = r^\perp$, $v \mapsto [r, v]$ sends each subspace Λ to Λ^r and conjugates the pseudo-symmetries with respect to the lines $T_r\widehat{l} \subset T_r\Sigma$ and $(T_r\widehat{l})^r \subset V_r$, by definition and Corollary 2.3. Therefore, its projectivization realizes the Σ-duality $\mathbb{P}(T_r\Sigma) \to \mathbb{P}(V_r)$ and conjugates the action of the symmetry with respect to the line $T_r\widehat{l}$ on the source with the projectivized pseudo-symmetry of the image: the $\mathbb{I}$-angular symmetry with center $\widehat{l}^* = \pi((T_r\widehat{l})^r)$ (Proposition 2.5). Proposition 2.18 is proved. $\square$

Note that for every curve $\gamma \subset \Sigma$ the Σ-duality sends the family of geodesics tangent to γ to the Σ-dual curve γ^* (see Convention 2.7). This together with the above proposition implies equivalence of the two statements of Proposition 2.16. Thus, it suffices to prove its second statement: those geodesics $\widehat{l}$, for which the pseudo-Euclidean pencil $\mathcal{C}^*$ is invariant under the $\mathbb{I}$-angular symmetry with center $\widehat{l}^*$, are exactly the admissible geodesics from Definition 1.15.

Fix a geodesic $\widehat{l}$. Let $H \subset \mathbb{R}^3$ denote the two-dimensional vector subspace containing $\widehat{l}$. Fix a vector $a \in H^\perp \subset \mathbb{R}^3$, $a \neq 0$. It represents the Σ-dual $\widehat{l}^* = \pi(a)$. The vector a lies in a unique cone κ^*_λ with $\lambda \neq \infty$, since $< Aa, a > \neq 0$: otherwise, if $< Aa, a >= 0$, then the intersection $\widehat{l} = H \cap \Sigma$ would be empty. Indeed, in the Euclidean case the equality $< Aa, a >= 0$ on a real vector a holds exactly when a lies in the x_3-axis; then H is parallel to the plane Σ, $H \cap \Sigma = \emptyset$. In the non-Euclidean case the equality $< Aa, a >= 0$ implies that $A = \mathrm{diag}(1, 1, -1)$ and the projective line $a^* = \pi(H)$ is tangent to the real absolute $\{< Ax, x >= 0\} \subset \mathbb{RP}^2_{(x_1:x_2:x_3)}$, by self-duality (Corollary 2.15). Then H is tangent to the cone $\{< Ax, x >= 0\} \subset \mathbb{R}^3$ and hence, it is disjoint from the inner component containing Σ of its complement. Thus, $H \cap \Sigma = \emptyset$.

Without loss of generality we will consider that $a \in \kappa^*_0$, after replacing B by $B - \lambda A$ for appropriate λ, by the inequality $< Aa, a > \neq 0$. Let $S : \mathbb{C}^3 \to \mathbb{C}^3$ denote the pseudo-symmetry with respect to the line $\mathbb{C}a$.

Claim 3. *The pseudo-Euclidean pencil $\mathcal{C}^*$ is invariant under the $\mathbb{I}$-angular symmetry with center $\widehat{l}^*$, if and only if $S(\kappa^*_0) = \kappa^*_0$.*

Proof The above $\mathbb{I}$-angular symmetry is the projectivization of the pseudo-symmetry S. Therefore, invariance of the pencil $\mathcal{C}^*$ under the $\mathbb{I}$-angular symmetry is equivalent to the S-invariance of the family of cones κ^*_λ, that is, to the existence of an involution $h : \lambda \to h(\lambda)$ such that $S(\kappa^*_\lambda) = \kappa^*_{h(\lambda)}$. In the latter case one has $S(\kappa^*_0) = \kappa^*_0$, since $S(a) = a$, $a \in \kappa^*_0$ and $a \notin \kappa^*_\lambda$ for every $\lambda \neq 0$. Conversely, let $S(\kappa^*_0) = \kappa^*_0$. This means that the involution S sends the quadratic form $< Bx, x >$ to itself up to sign. Hence, $S(\kappa^*_\lambda) = \kappa^*_{\pm\lambda}$ for every λ, since S preserves the quadratic form $< AX, X >$. This together with the previous equivalence statement proves the claim. $\square$

Claim 4. *One has $S(\kappa^*_0) = \kappa^*_0$, if and only if κ^*_0 is a union of a pair of 2-planes through the origin in $\mathbb{C}^3$ that has one of the following types:*

α) both planes contain the line $\mathbb{C}a$ (they may coincide);

β) one plane in κ^*_0 contains the line $\mathbb{C}a$, and the other plane coincides with the two-dimensional subspace $H_A \subset \mathbb{C}^3$ that is orthogonal to the vector a with respect to the scalar product $< Ax, x >$.

Proof Every hyperplane $W \subset \mathbb{C}^3$ parallel to the plane H_A is S-invariant, and S acts there as the central symmetry with respect to the point C_W of intersection $W \cap (\mathbb{C}a)$. The S-invariance of the cone κ^*_0 is equivalent to the invariance of each intersection $I_W = W \cap \kappa^*_0$ under the latter symmetry for every W as above. The intersection I_W is either all of W, or a line through C_W, or a conic in W containing the center of its symmetry C_W, since $\mathbb{C}a \subset \kappa^*_0$. In the latter case I_W is a union of two lines through C_W, since a planar conic central-symmetric with respect to some its point C is a union of two lines through C (the lines under question may coincide). Note that all the intersections I_W with $W \neq H_A$ are naturally isomorphic between themselves via homotheties centered at the origin, since κ^*_0 is a cone. Therefore, the following two cases are possible.

α) I_W is a union of two (may be coinciding) lines through C_W for every W; then κ^*_0 is a union of two planes containing the line $\mathbb{C}a$.

β) I_W is a line for all $W \neq H_A$, and $I_W = W$ for $W = H_A$; then κ^*_0 is a union of the plane H_A and another plane containing $\mathbb{C}a$.

This proves the claim. $\square$

Now let us return to the proof of Proposition 2.16. The cone $\kappa^*_0 = \{< Bx, x >= 0\}$ is a union of two planes, by Claim 4.

Case α). The above planes both contain a, thus $a \in \operatorname{Ker} B$; $dim(\operatorname{Ker} B) = 1$, if the planes are distinct; $dim(\operatorname{Ker} B) = 2$, if they coincide. Hence, the hyperplane H orthogonal to a with respect to the standard Euclidean scalar product is orthogonal to $\operatorname{Ker} B$. Therefore, the geodesic $\widehat{l} = H \cap \Sigma$ is admissible of type 1) in Definition 1.15. Vice versa, each admissible geodesic of type 1) can be represented as above after replacing B by $B - \lambda A$.

Case β). Then the cone κ^*_0 is the union of the plane H_A and a plane Π containing the line $\mathbb{C}a$. The plane Π is the complexification of a real plane, which will be here also denoted by Π, since κ_0^* is defined by a quadratic equation over real numbers and H_A is the complexification of a real plane. Let $b \in \mathbb{R}^3 \setminus \{0\}$ denote a vector Euclidean-orthogonal to Π. Thus, $< a, b >= 0$. Note that the vector Aa is non-zero, since $< Aa, a >\neq 0$, as was shown above, and it is Euclidean-orthogonal to H_A, by definition. Therefore, $< BM, M >= c < Aa, M >< b, M >$, $c \in \mathbb{R} \setminus \{0\}$. Let us normalize the vectors a and b by constant factors so that $c = 2$. Then the quadratic form $< BM, M >$ can be represented in the tensor form as $Aa \otimes b + b \otimes Aa$. The plane H defining the geodesic $\widehat{l}$ is the plane orthogonal to the vector a, by definition. Hence, $\widehat{l}$ is an admissible geodesic of type 2) in Definition 1.15: the first geodesic in (1.4). Vice versa, each geodesic of type 2) can be represented as above, see Remarks 1.14, 1.16. Proposition 2.16 is proved. $\Box$

Let now $\widehat{l} \subset \Sigma$ be a geodesic whose some segment is contained in the boundary of the polynomially integrable billiard under question. The $\mathbb{I}$-angular symmetry with center $\widehat{l}^*$ leaves invariant the rational integral G, by Theorem 2.8. Hence, it permutes the level curves of the quadratic rational function $\widetilde{G}$, and the pencil $\mathcal{C}^*$ is invariant, by Claim 2. Thus, the geodesic $\widehat{l}$ is admissible, by Proposition 2.16. Theorem 1.21 is proved. $\Box$

3 Bialy–Mironov Hessian Formula and asymptotics of Hessians

The material of the present section will be used in Section 4 in the proof of Theorem 4.1, statement (ii-b). It includes:

- Bialy–Mironov Hessian Formula (3.4) recalled in Subsection 3.1;
- the asymptotics of its left- and right-hand sides along those local branches of the curve γ that are transversal to $\mathbb{I}$ (Subsection 3.4).

In the proof of the above asymptotics we use general asymptotic formulas

- for the defining function of an irreducible germ a of analytic curve along another irreducible germ b (Subsection 3.2);
- for the Hessian $H(f)$ of defining function of a given germ b along b (Subsection 3.3).

3.1 Bialy–Mironov formula

Let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve generating a rationally integrable $\mathbb{I}$-angular billiard with integral G. The function G has poles contained in $\mathbb{I}$ and is constant on γ, by Proposition 2.12. In what follows we normalize it so that $G|_\gamma \equiv 0$, and set

$$\Gamma = \{G = 0\} \supset \gamma.$$

Fix an affine chart $\mathbb{C}^2 \subset \mathbb{CP}^2$ with coordinates (x, y) such that the infinity line is not contained in $\mathbb{I}$. In this chart the function G takes the form

$$G(x,y) = \frac{F_1(x,y)}{(\mathcal{Q}(x,y))^n}, \text{ where } F_1 \text{ is a polynomial of degree at most } 2n,$$

$$\mathcal{Q}(x,y) \text{ is a fixed quadratic polynomial defining } \mathbb{I}: \ \mathbb{I} = \{\mathcal{Q} = 0\}.$$

Let $f(x, y)$ be the polynomial defining the curve γ, which is irreducible, as is γ: $\gamma = \{f = 0\}$, the differential df being non-zero on a Zariski open subset in γ. Recall that the polynomial F_1 vanishes on γ. Therefore,

$$F_1 = f^k g_1, \ k \in \mathbb{N}, \ g_1 \text{ is a polynomial coprime with } f. \tag{3.1}$$

Set

$$g = g_1^{\frac{1}{k}}, \ F = F_1^{\frac{1}{k}} = fg, \ m = \frac{n}{k}. \tag{3.2}$$

We consider the Hessian quadratic form of the function $f(x, y)$ evaluated on appropriately normalized tangent vector to $\gamma = \{f = 0\}$ at a point (x, y), namely, the skew gradient $(f_y, -f_x)$:

$$H(f) = f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2. \tag{3.3}$$

Theorem 3.1 *(see, [10, theorem 6.1], [11, formulas (16) and (32)]) The following formula holds for all $(x, y) \in \gamma$:*

$$g^3(x,y)H(f)(x,y) = H(gf) = c(\mathcal{Q}(x,y))^{3m-3}, \ c \equiv const \neq 0. \tag{3.4}$$

Remark 3.2 In 2008 S.Tabachnikov obtained a version of formula (3.4) with $k = 1$ and constant right-hand side for polynomially integrable outer billiards satisfying some non-degeneracy assumptions [41, p. 102]. Theorem 6.1 in [10] deals with a polynomially integrable planar billiard $\Omega \subset \mathbb{R}^2$ and a curve $\Gamma_1 \subset \mathbb{R}^2$ that is polar-dual to a non-linear smooth arc in $\partial\Omega$. The theorem deals with the absolute $\mathbb{I} = \{x^2 + y^2 = 0\}$ and states that formula (3.4) holds along the curve Γ_1. Then it holds automatically on every irreducible component γ of its complex Zariski closure. Its proof given in [10] remains valid for every irreducible algebraic curve γ generating a rationally integrable $\mathbb{I}$-angular billiard. The same remark concerns formulas (16) and (32) from the paper [11], which deal with the non-Euclidean case and the corresponding absolute $\mathbb{I} = \{x^2 + y^2 \pm 1 = 0\}$. These results from [10, 11] together cover Theorem 3.1 in the general case, since every conic different from a double line is projectively equivalent to some of the above absolutes.

Without loss of generality we will consider that G is an irreducible fraction, that is, its nominator $F_1(x,y)$ does not vanish identically on $\mathbb{I}$ in the case, when $\mathbb{I}$ is regular, and in case, when $\mathbb{I}$ is a union of two lines Λ_1 and Λ_2, one has $F_1 \not\equiv 0$ on each Λ_j. In the former case we can do this, by irreducibility of the conic $\mathbb{I}$: if F_1 vanishes on $\mathbb{I}$ with a certain multiplicity s, then we can divide both nominator and denominator in G by $(\mathcal{Q}(x,y))^s$ and achieve the desired property. In the latter case we can do this, by the fact that both lines Λ_1 and Λ_2 forming $\mathbb{I}$ enter the divisor of the function G (the zero-pole divisor) with the same multiplicity. Indeed, for every $u \in \gamma \setminus \mathbb{I}$ the tangent line $T_u\gamma$ intersects both lines Λ_1 and Λ_2, and their intersection points with the line $T_u\gamma$ are permuted by its $\mathbb{I}$-angular symmetry with center u, by definition. Both intersection points enter the divisor of the function $G|_{T_u\gamma}$ with the same multiplicity, by its invariance under the $\mathbb{I}$-angular symmetry. This implies the above statement on coincidence of multiplicities of the lines Λ_1 and Λ_2.

The above discussion implies that G has pole along each irreducible component of the conic $\mathbb{I}$. Therefore, no component in $\mathbb{I}$ is contained in Γ. We choose the above affine chart $\mathbb{C}^2_{(x,y)}$ so that the finite intersection $\Gamma \cap \mathbb{I}$ lies in $\mathbb{C}^2$, in particular, $G \not\equiv 0$ on the infinity line, hence $\deg F_1 = 2n$. Let Δ denote the zero divisor of the function G. Finally, in our assumptions made without loss of generality one has $F_1 \not\equiv 0$ on every irreducible component of the conic $\mathbb{I}$,

$$\Gamma = \{F_1 = 0\}, \ \deg F_1 = 2n,$$

$$\Delta \text{ is the zero divisor of the polynomial } F_1, \tag{3.5}$$

the intersection $\Gamma \cap \mathbb{I}$*, and hence,* $\gamma \cap \mathbb{I}$ *lie in the affine chart* $\mathbb{C}^2_{(x,y)}$.

3.2 Asymptotics of defining function

Definition 3.3 Let b be a non-linear irreducible germ of analytic curve at a point $C \in \mathbb{CP}^2$. An *adapted system of coordinates* to b is a system of affine coordinates (z,w) centered at C such that the z-axis is tangent to b. In adapted coordinates the germ b can be locally holomorphically and bijectively parametrized by small complex parameter t:

$$t \mapsto (t^q, ct^p(1+o(1))), \quad \text{as } t \to 0; \quad q,p \in \mathbb{N}, \quad 1 \le q < p, \quad c \neq 0, \tag{3.6}$$

$$q = q_b, \ p = p_b, \quad c = c_b,$$

$$q = 1, \quad \text{if and only if } b \text{ is a regular germ.}$$

The *projective Puiseux exponent* [25, p. 250, definition 2.9] of the germ b is the ratio

$$r = r_b = \frac{p_b}{q_b}.$$

The germ b is called *quadratic*, if $r_b = 2$, and is called *subquadratic*, if $r_b \leq 2$, see [27, definition 3.5]. In the case, when b is a germ of line, it is parametrized by $t \mapsto (t, 0)$; then we set $q_b = 1$, $p_b = \infty$, and put the Puiseux exponent r_b to be equal to infinity, as in loc. cit.

Proposition 3.4 *Let a, b be irreducible germs of holomorphic curves at a point $C \in \mathbb{C}^2$. Let f_a, f_b be the irreducible germs of holomorphic functions defining them: $g = \{f_g = 0\}$ for $g = a, b$. Set*

$$\rho_a = \begin{cases} 1, & \text{if } a \text{ is transversal to } b \\ r_a, & \text{if } a \text{ is tangent to } b \end{cases} . \tag{3.7}$$

Let (z, w) be affine coordinates centered at C that are adapted to b. One has

$$f_a(u) = O((z(u))^{q_a \min\{\rho_a, r_b\}}), \text{ as } u \in b \text{ tends to } C. \tag{3.8}$$

The proof of Proposition 3.4 is based on the following property of Newton diagram of irreducible germs of analytic curves.

Proposition 3.5 *Let $b \subset \mathbb{CP}^2$ be an irreducible germ of analytic curve at a point C, and let (z, w) be local affine coordinates adapted to it. Let $t \mapsto (t^q, ct^p(1 + o(1)))$ be its local parametrization: $1 \leq q < p$, $c \neq 0$, see (3.6). Let f be an irreducible germ of analytic function at C defining b: $b = \{f = 0\}$. The Newton diagram of the function f consists of one edge: the segment connecting the points $(p, 0)$ and $(0, q)$. More precisely, the Taylor series of the function $f(z, w)$ contains only monomials $z^\alpha w^\beta$ such that*

$$\nu_{\alpha\beta} = q\alpha + p\beta \geq qp. \tag{3.9}$$

Proof Without loss of generality we will consider that f is a Weierstrass polynomial:

$$f(z, w) = \phi_z(w) = w^d + h_1(z)w^{d-1} + \cdots + h_d(z), \quad h_j(0) = 0, \tag{3.10}$$

since each germ of holomorphic function at 0 that vanishes at 0 and does not vanish identically on the w-axis is the product of a unique polynomial as above (called Weierstrass polynomial) and a non-zero holomorphic function, by Weierstrass Preparatory Theorem [29, chapter 0, section 1]. For every z

small enough the polynomial $\phi_z(w) = f(z,w)$ has q roots $\zeta_l(z)$, $l = 1,\dots,q$: $\zeta_l(z) = ct_l^p(1+o(1))$, $t_l^q = z$, as $z \to 0$; thus, $\zeta_l(z) \simeq cz^{\frac{p}{q}}$. This implies that the Weierstrass polynomial (3.10) is the product of q factors $w - \zeta_l(z)$ with $\zeta_l(z) \simeq cz^{\frac{p}{q}}$, as $z \to 0$. Hence, in formula (3.10) one has $d = q$,

$$h_q(z) = (-1)^q \prod_{l=1}^{q} \zeta_l(z) = (-1)^{q+p(q+1)} c^q z^p (1+o(1)).$$

The latter equality follows from the equality $\prod_{l=1}^q t_l = (-1)^{q+1} z$: the product of q-th roots of unity equals to $(-1)^{q+1}$. One has

$$h_s(z) = O(z^{\frac{p}{q}s}) \text{ for } 1 \leq s < q, \text{ as } z \to 0, \tag{3.11}$$

since $h_s(z) = (-1)^s \sigma_s$, where σ_s is the s-th elementary symmetric polynomial in the roots $\zeta_l(z) \simeq cz^{\frac{p}{q}}$. Formula (3.11) implies that the Taylor series of the Weierstrass polynomial (3.10) contains only the monomials w^q, z^p and those monomials $z^\alpha w^\beta$ for which $\beta < q$ (set $s = q - \beta$) and $\alpha \geq \frac{p}{q}s = \frac{p}{q}(q-\beta)$, i.e., $q\alpha + p\beta \geq pq$. This proves the proposition. □

Proof of Proposition 3.4. Case 1): the curve a is transversal to b. Then $\rho_a = 1 < r = r_b = \frac{p_b}{q_b}$, and we have to show that $f_a|_b = O(z^{q_a})$. To do this, let us take the coordinates (z_a, w_a) adapted to a so that the w_a-axis coincides with the z-axis $T_C b$, $w_a = z$ on $T_C b$ and $z_a = w$: one can do this, by transversality. One has

$$w_a \simeq z, \ z_a = w \simeq c_b z^r \ \text{ along the curve } b. \tag{3.12}$$

Hence, each Taylor monomial $z_a^\alpha w_a^\beta$ of the function f_a has asymptotics $O(z^{\alpha r + \beta})$ along the curve b. Now it suffices to show that $\alpha r + \beta \geq q_a$. Recall that $\alpha q_a + \beta p_a \geq p_a q_a$, by (3.9). Dividing the latter inequality by p_a yields to $\nu = \alpha r_a^{-1} + \beta \geq q_a$. Hence, $\alpha r + \beta \geq \nu \geq q_a$, since $r_a, r > 1$. This proves the proposition.

Case 2): the curve a is tangent to b, thus $\rho_a = r_a$. Then the coordinates (z,w) are adapted for both curves b and a. Each Taylor monomial $z^\alpha w^\beta$ of the function $f_a(z,w)$ is asymptotic to cz^ν, $\nu = \alpha + \beta r$, $c = const$, along the curve b, since $w \simeq c_b z^r$. It suffices to show that $\alpha + \beta r \geq s = q_a \min\{r_a, r\}$.

Subcase 2a): $r_a \leq r$. Thus, $s = q_a r_a = p_a$. One has $\alpha + \beta r \geq \alpha + \beta r_a \geq p_a = s$, by inequality (3.9) divided by q.

Subcase 2b): $r_a > r$. Thus, $\min\{\rho_a, r\} = r$, $s = q_a r$,

$$\frac{r_a}{r}(\alpha + \beta r) = \alpha \frac{r_a}{r} + \beta r_a \geq \alpha + \beta r_a \geq p_a = q_a r_a,$$

by (3.9). Multiplying the latter inequality by $\frac{r}{r_a}$ yields to $\alpha+\beta r \geq q_a r = s$. Proposition 3.4 is proved. $\square$

3.3 Asymptotics of Hessian of local defining function

Proposition 3.6 *Let $b \subset \mathbb{CP}^2$ be an irreducible germ of analytic curve at a point C. Let f be the irreducible germ of its defining function, $b = \{f = 0\}$, and let $H(f)$ be its Hessian defined in (3.3) in some affine chart $\mathbb{C}^2_{(x,y)}$ containing C. Let (z, w) be an affine chart on $\mathbb{CP}^2$ centered at C that is adapted to b: the projective line $T_C b$ is the z-axis. Then*

$$H(f)(u) = O((z(u))^{3q_b r-2(r+1)}),\ r = r_b,\ \text{as } u \in b \text{ tends to } C. \qquad (3.13)$$

Proof The norm of the skew gradient of the function f written in the coordinates (x, y) has the same asymptotics (up to nonzero constant factor), as the norm of its skew gradient written in the coordinates (z, w), since applying the local coordinate change $(x, y) \mapsto (z, w)$ to the function f multiplies its gradient by a holomorphic matrix function with non-zero determinant. Note that both skew gradients are tangent to the level curves of the function f, including its zero locus b. Everywhere below by $\nabla_{skew} f$ we denote the skew gradient in the coordinates (z, w). For every $u \in b$ let $L_u \subset \mathbb{C}^2$ denote the affine line tangent to b at u, and let v denote the extension of the vector $\nabla_{skew} f(u) \in T_u b = T_u L_u$ to a constant vector field on L_u. It suffices to prove formula (3.13) for its left-hand side replaced by the derivative $\frac{d^2 f}{dv^2}(u)$: for $u \in b$ the ratio of the absolute values of the latter second derivative and the expression $H(f)(u)$ equals to the ratio of squared norms of the skew gradients of the function f at u in the coordinate systems (x, y) and (z, w); the latter ratio is bounded from above and below, as was mentioned above.

We consider the Taylor series for the function f and evaluate the Hessian quadratic form of each its Taylor monomial on the skew gradient of the function f. We show that the expression thus obtained has asymptotics given by the right-hand side in (3.13). This will prove the proposition.

Let $z^\alpha w^\beta$ be the Taylor monomials of the function f. The skew gradient $(\nabla_{skew} f)|_b$ is a linear combination of the vector monomials

$$h_{\alpha,\beta} = \widetilde{h}_{\alpha,\beta} \frac{\partial}{\partial w},\ \widetilde{h}_{\alpha,\beta} = z^{\alpha-1} w^\beta \simeq c z^{\alpha+\beta r-1},$$

$$v_{\alpha,\beta} = \widetilde{v}_{\alpha,\beta} \frac{\partial}{\partial z},\ \widetilde{v}_{\alpha,\beta} = z^\alpha w^{\beta-1} \simeq c' z^{\alpha+\beta r-r},\ c, c' \neq 0;$$

both above asymptotics are written along the curve b. The restrictions to the curve b of the second derivatives of a monomial $z^\alpha w^\beta$ are asymptotic to

$$\frac{\partial^2(z^\alpha w^\beta)}{\partial w^2}=\beta(\beta-1)z^\alpha w^{\beta-2}=O(z^{\alpha+\beta r-2r});$$

$$\frac{\partial^2(z^\alpha w^\beta)}{\partial z^2}=\alpha(\alpha-1)z^{\alpha-2}w^\beta=O(z^{\alpha+\beta r-2});$$

$$\frac{\partial^2(z^\alpha w^\beta)}{\partial z\partial w}=\alpha\beta z^{\alpha-1}w^{\beta-1}=O(z^{\alpha+\beta r-r-1}).$$

Therefore, applying the Hessian quadratic form of each monomial $z^\alpha w^\beta$ to a linear combination of the vectors $h_{\alpha',\beta'}$ and $v_{\alpha',\beta'}$ yields to a linear combination of expressions of the three following types:

$$\frac{\partial^2(z^\alpha w^\beta)}{\partial w^2}\widetilde{h}_{\alpha',\beta'}\widetilde{h}_{\alpha'',\beta''}=O(z^\nu),\ \nu=(\alpha'+\beta' r-1)+(\alpha''+\beta'' r-1)$$

$$+\alpha+\beta r-2r=(\alpha'+\beta' r)+(\alpha''+\beta'' r)+(\alpha+\beta r)-2(r+1); \qquad (3.14)$$

$$\frac{\partial^2(z^\alpha w^\beta)}{\partial z^2}\widetilde{v}_{\alpha',\beta'}\widetilde{v}_{\alpha'',\beta''}=O(z^{\nu_2}),\ \nu_2=(\alpha'+\beta' r)+(\alpha''+\beta'' r)-2r+\alpha+\beta r-2=\nu;$$

$$\frac{\partial^2(z^\alpha w^\beta)}{\partial z\partial w}\widetilde{h}_{\alpha',\beta'}\widetilde{v}_{\alpha'',\beta''}=O(z^{\nu_3}),\ \nu_3=(\alpha'+\beta' r)+(\alpha''+\beta'' r)+\alpha+\beta r-2r-2=\nu.$$

Let us now estimate ν from below. Recall that for every Taylor monomial $z^\alpha w^\beta$ of the function f one has

$$\alpha+\beta r=\frac{1}{q_b}(\alpha q_b+\beta p_b)\geq p_b=q_b r,$$

by (3.9), and hence, the same inequality holds for (α',β') and (α'',β''). This together with formula (3.14) for the number ν implies that $\nu\geq 3q_b r-2(r+1)$. This together with the above discussion proves formula (3.13). □

3.4 Asymptotics of Bialy–Mironov Formula

Everywhere below in this subsection $C\in\gamma\cap\mathbb{I}$ is a regular point of the conic $\mathbb{I}$, and b is a local branch of the curve γ at C that is transversal to $\mathbb{I}$; (z,w) are affine coordinates centered at C and adapted to b. Recall that Δ is the zero divisor of the function G, it coincides with the zero divisor of the polynomial F_1, and $\deg F_1=\deg(\Delta)=2n$, see (3.5).

Proposition 3.7 *The right-hand side in (3.4) has the following asymptotics, as $u=(x,y)\in b$ tends to C:*

$$(\mathcal{Q}(u))^{3m-3}\simeq c(z(u))^{3m-3},\ c\neq 0,\ m=\frac{n}{k}=\frac{1}{2k}\deg(\Delta). \tag{3.15}$$

Proof The degree equality in (3.15) follows from definition. The restriction to T_Cb of the differential $d\mathcal{Q}(C)$ does not vanish, since C is a regular point of the conic $\mathbb{I}=\{\mathcal{Q}=0\}$ and b is transversal to $\mathbb{I}$. Recall that the tangent line T_Cb is the z-axis. Therefore, $\mathcal{Q}(u)|_b\simeq cz(u)$, $c\neq 0$, as $u\to C$. This implies the asymptotic formula in (3.15). □

Let $\sum_{j=1}^{l}s_jb_j$ denote the germ at C of the divisor Δ. Here $s_j\in\mathbb{N}$, and b_j are distinct irreducible germs of analytic curves in Δ at C numerated so that $b_1=b$; thus, $s_1=k$. For $j=1,\dots,l$ let f_j denote the germ at C of defining function of the curve b_j. Set

$$k_j=\frac{s_j}{k},\ \widetilde{g}=\prod_{j=2}^{l}f_j^{k_j};\quad k_1=\frac{s_1}{k}=1;\quad k_j=1 \text{ whenever } b_j\subset\gamma,$$

by definition. Let F be the same, as in (3.2).

Proposition 3.8 *Set $r=r_b$. As $u\in b$ tends to C, one has*

$$H(F)(u)=\widetilde{g}^3H(f_1)(u)=O((z(u))^{\eta}),$$
$$\eta=\eta(b)=3\sum_{j=1}^{l}k_jq_{b_j}\min\{\rho_{b_j},r\}-2(r+1). \tag{3.16}$$

Here ρ_{b_j} are the same, as in (3.7); $\rho_{b_1}=\rho_b=r$.

Proof We use [11, formula (17)] valid for every two functions f_1 and β:

$$H(f_1(x,y)\beta(x,y))|_{\{f_1=0\}}=\beta^3(x,y)H(f_1(x,y)). \tag{3.17}$$

One has

$$F(x,y)=h(x,y)f_1(x,y)\widetilde{g}(x,y), \tag{3.18}$$

where h is a germ of holomorphic function at C, $h(C)\neq 0$. Formula (3.18) follows from definition, see (3.2). This together with (3.17) implies that

$$H(F)(u)\simeq c_1(\widetilde{g}^3H(f_1))(u)=c_1(H(f_1)\prod_{j=2}^{l}f_j^{3k_j})(u),\ c_1=(h(C))^3\neq 0.$$

Substituting formula (3.8) with $a=b_j$ and (3.13) to the above right-hand side yields to (3.16), taking into account that $k_1=1$ and $\rho_{b_1}=\rho_b=r$. □

Corollary 3.9 *For every local branch b as at the beginning of the subsection the corresponding exponent $\eta = \eta(b)$ satisfies inequality*

$$\eta = 3\sum_{j=1}^{l} k_j q_{b_j} \min\{\rho_{b_j}, r\} - 2(r+1) \leq 3m - 3 = 3\frac{\deg(\Delta)}{2k} - 3. \quad (3.19)$$

Proof If the contrary inequality were true, then the left-hand side in (3.4) would be asymptotically dominated by the right-hand side along the branch b. This follows from formulas (3.15) and (3.16). Thus obtained contradiction to formula (3.4) proves the corollary. $\square$

4 Local branches and relative $\mathbb{I}$-angular symmetry property

In this section we prove the following theorem.

Theorem 4.1 *Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic (either regular, or a pair of distinct lines), and let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve generating a rationally integrable $\mathbb{I}$-angular billiard. Then every intersection point $C \in \gamma \cap \mathbb{I}$ satisfies the following statements:*

(i) Case, when $\mathbb{I}$ is a union of two distinct lines through C. Let b be a local branch of the curve γ at C that is transversal to both lines forming $\mathbb{I}$. Then b is quadratic.

(ii) Case, when C is a regular point of the conic $\mathbb{I}$. Then

(ii-a) each local branch of the curve γ at C that is tangent to $\mathbb{I}$ is quadratic;

(ii-b) each its branch at C that is transversal to $\mathbb{I}$ is regular and quadratic.

In our assumptions for every $u \in \gamma$ the restriction to $T_u\gamma$ of the rational function G is invariant under the $\mathbb{I}$-angular symmetry with center u, and $\gamma \subset \Gamma = \{G = 0\}$. This implies that the following *relative projective symmetry property* takes place: *for every $u \in \gamma$ the intersection of the projective tangent line $T_u\gamma$ with a bigger algebraic curve $\Gamma \supset \gamma$ (or a divisor) is invariant under a projective involution $T_u\gamma \to T_u\gamma$ fixing u:* the $\mathbb{I}$-angular symmetry in our case.

In Subsection 4.4 we state and prove Theorem 4.17, which unifies and generalizes statements (i) and (ii-a) of Theorem 4.1, and deduce statements (i) and (ii-a). Theorem 4.17 is stated for a germ of analytic curve b at $C \in \mathbb{CP}^2$ (that needs not be algebraic) that has local relative projective symmetry property with respect to a bigger finite collection Γ of germs at points in $T_C b$ (called a local multigerm) and projective involutions $T_u b \to T_u b$ fixing u with appropriate asymptotics, as $u \to C$. The formal definitions of a local multigerm and the latter local symmetry property will be given in Subsections 4.1 and 4.3 respectively.

For the proof of Theorem 4.1 we first describe those points of intersection $T_u b \cap \Gamma$, whose z- (w-) coordinates in the chart (z, w) adapted to b have asymptotics linear, sublinear and superlinear in $z(u)$ (respectively, $w(u)$), as $u \in b$ tends to C. Their description, which mostly follows from results of [25, 27], is presented in Subsection 4.1. Then in Subsection 4.3 we show that for every local branch b as in Theorem 4.1 the $\mathbb{I}$-angular symmetries of the tangent lines $T_u b$ written in appropriate affine coordinate form families of degenerating conformal involutions of two possible asymptotic types A or B. The latter families of involutions are introduced in Subsection 4.2, where we prove general Propositions 4.13 and 4.14 on their asymptotics. In Subsection 4.4 we show that the collection (divisor) of asymptotic factors of points of the intersection $T_u b \cap \Gamma$ with linear asymptotics in $z(u)$ $(w(u))$ is symmetric with respect to appropriate conformal involution $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$, and then deduce Theorem 4.17.

The proof of statement (ii-b) takes the rest of the section: Subsections 4.5–4.8. First in Subsection 4.5 we prove subquadraticity of the branch b under question. In Subsection 4.6 we prove that every local branch of the curve Γ that is tangent to b (if any) has Puiseux exponent no greater than r_b. In Subsection 4.7 we deal with the zero divisor $\widetilde{\Delta} = \frac{1}{k}\Delta$ of the function $F_1^{\frac{1}{k}}$, whose germ at C contains b with multiplicity 1. We prove that its local intersection index with the tangent line to b at its base point C is no less than its half-degree plus 1, and this inequality is strict, unless the germ b is regular and quadratic. The above-mentioned Puiseux exponent and intersection index inequalities will be proved in a general situation, for a germ b having local projective symmetry property, with the projective symmetries forming a family of involutions of type A in the adapted coordinate z.

Afterwards in Subsection 4.8 we prove statement (ii-b). Namely, we show that the above-mentioned Puiseux exponent and intersection index inequalities together would bring a contradiction to upper bound (3.19) of the exponent η in the asymptotics of Bialy–Mironov formula, unless the germ b is regular and quadratic. This will finish the proof of Theorem 4.1.

4.1 Local multigerms and asymptotics of intersections with tangent line

Let a, b be irreducible germs of planar complex analytic curves at the origin in $\mathbb{C}^2$. Let p_g, q_g, c_g, $g = a, b$ be respectively the corresponding exponents and constants from their parametrizations (3.6) in their adapted coordinates. Let t be the corresponding local parameter of the germ b. We identify points of the curve b with the corresponding local parameter values t. We use the following statements on the asymptotics of the points of intersection $T_t b \cap a$.

Proposition 4.2 *[27, proposition 3.8] Let a, b be transversal irreducible germs of holomorphic curves at the origin in $\mathbb{C}^2$. Let (z, w) be affine coordinates centered at 0 and adapted to b: the germ b is tangent to the z-axis. Then for every t small enough the intersection $T_t b \cap a$ consists of q_a points $\xi_1, \dots, \xi_{q_a}$ whose coordinates have the following asymptotics, as $t \to 0$:*

$$z(\xi_j) = O(t^{p_b}) = O(w(t)) = o(z(t)) = o(t^{q_b}),$$

$$w(\xi_j) = (1 - r_b)w(t)(1 + o(1)) = (1 - r_b)c_b t^{p_b}(1 + o(1)). \tag{4.1}$$

Proposition 4.3 *([25, p. 268, proposition 2.50], [27, proposition 3.10]) Let a, b be nonlinear irreducible tangent germs of holomorphic curves at the origin in the plane $\mathbb{C}^2$. Consider their parametrizations (3.6) in common adapted coordinates (z, w). Let c_a and c_b be the corresponding constants from (3.6). Then for every t small enough the intersection $T_t b \cap a$ consists of p_a points $\xi_1, \dots, \xi_{p_a}$ whose coordinates have the following asymptotics, as $t \to 0$.*

Case 1): $r_a > r_b$. One has two types of intersection points ξ_j:

$$\text{for } j \leq q_a: \quad z(\xi_j) = \frac{r_b - 1}{r_b} z(t)(1 + o(1)) = \frac{r_b - 1}{r_b} t^{q_b}(1 + o(1)), \tag{4.2}$$

$$w(\xi_j) = O(t^{q_b r_a}) = o(t^{p_b}) = o(w(t));$$

$$\text{for } j > q_a: \quad z(t) = O((z(\xi_j))^{\frac{r_a - 1}{r_b - 1}}) = o(z(\xi_j)), \tag{4.3}$$

$$w(t) = O(z^{r_b}(t)) = O((z(\xi_j))^{\frac{r_b(r_a - 1)}{r_b - 1}}) = o(z^{r_a}(\xi_j)) = o(w(\xi_j)).$$

Case 2): $r_a = r_b = r$. One has

$$z(\xi_j) = \zeta_j^{q_a} z(t)(1 + o(1)) = \zeta_j^{q_a} t^{q_b}(1 + o(1)), \tag{4.4}$$

$$w(\xi_j) = c_a \zeta_j^{p_a} t^{p_b}(1 + o(1)) = c\zeta_j^{p_a} w(t)(1 + o(1)),$$

where ζ_j are the roots of the polynomial

$$R_{p_a,q_a,c}(\zeta) = c\zeta^{p_a} - r\zeta^{q_a} + r - 1; \;\; r = \frac{p_a}{q_a}, \;\; c = \frac{c_a}{c_b}. \tag{4.5}$$

(In the case, when $b = a$, one has $c = 1$, and the above polynomial has double root 1 corresponding to the tangency point t.)

Case 3): $r_a < r_b$. One has

$$z(\xi_j) = O((z(t))^{\frac{r_b}{r_a}}) = o(z(t)), \tag{4.6}$$

$$w(\xi_j) = (1 - r_b)w(t)(1 + o(1)) = (1 - r_b)c_b t^{p_b}(1 + o(1)).$$

Definition 4.4 [27, definition 3.3] Let $L \subset \mathbb{CP}^2$ be a line, and let $C \in L$. A (L, C)*-local multigerm (divisor)* is respectively a finite union (linear combination $\sum_j k_j b_j$ with $k_j \in \mathbb{R} \setminus \{0\}$) of distinct irreducible germs of analytic curves b_j (called *components*) at base points $C_j \in L$ such that each germ at $C_j \neq C$ is different from the line L. (A germ at C can be arbitrary, in particular, it may coincide with the germ (L, C).) The (L, C)-*localization* of an algebraic curve (divisor) in $\mathbb{CP}^2$ is the corresponding (L, C)-local multigerm (divisor) formed by all its local branches of the above type.

Everywhere below in the present subsection b is an irreducible germ of analytic curve at a point $C \in \mathbb{CP}^2$, Γ is a $(T_C b, C)$-local multigerm (or divisor), and (z, w) is a local affine chart centered at C that is adapted to b: $T_C b$ is the z-axis. For every affine coordinate h, which will be either z, or w, we consider its restriction to the projective lines $T_u b$.

Definition 4.5 Let h be an affine coordinate on a neighborhood of the point C in $\mathbb{CP}^2$ centered at C: $h(C) = 0$. *The points of intersection* $\Gamma \cap T_u b$ *with linear h-asymptotics* are those intersection points whose h-coordinates have asymptotics $\tau_j h(u)(1 + o(1))$, $\tau_j \neq 0$, as $u \to C$; the corresponding constant factors τ_j are called the *asymptotic h-factors.* In the case, when Γ is a divisor, we take each factor τ_j with multiplicity, which is the total multiplicity n_j of all the intersection points with the same asymptotic factor τ_j. The formal linear combination $M_h = \sum_j n_j[\tau_j]$, which is a divisor in $\mathbb{C}^*$, will be called the *asymptotic h-divisor.*

Definition 4.6 We say that a continuous family of points $Q = Q(u)$ of intersection $T_u b \cap \Gamma$ has *sublinear (superlinear) h-asymptotics,* if $h(Q(u)) = o(h(u))$ (respectively, if $h(u) = o(h(Q(u)))$), as $u \to C$.

Remark 4.7 In general, the function $h(Q(u))$ can be multivalued. It can be always written as a Puiseux series (after multiplication by a power u^s, $s \in \mathbb{Q}_{>0}$, if $h(Q(u)) \to \infty$, as $u \to C$). The above notions of family of points with sublinear, linear and superlinear h-asymptotics and the asymptotic factors are well-defined in this general case. For every given affine coordinate h on a neighborhood of the point C in $\mathbb{CP}^2$ with $h(C) = 0$ each (multivalued) continuous family of intersection points $Q(u)$ has one of the three above types.

In what follows for a multigerm (divisor) Γ by Γ_C we will denote its part consisting of the irreducible germs based at C. Recall that for every irreducible germ a in Γ_C we define the number ρ_a by formula (3.7): $\rho_a = 1$, if a is transversal to b; $\rho_a = r_a$, if a is tangent to b. Set

$$\Gamma_{\rho<r_b} = \text{ the collection (divisor) of germs } a \text{ in } \Gamma_C \text{ with } \rho_a < r_b, \tag{4.7}$$

$$\Gamma_{\rho>r_b} = \text{ the collection (divisor) of germs } a \text{ in } \Gamma_C \text{ with } \rho_a > r_b, \tag{4.8}$$

$$\Gamma_{\rho=r_b} = \text{ the collection (divisor) of germs } a \text{ in } \Gamma_C \text{ with } \rho_a = r_b, \tag{4.9}$$

$$\Gamma_{out} = \Gamma \setminus \Gamma_C, \text{ which consists of germs that are not based at } C. \tag{4.10}$$

Thus, $\Gamma_{\rho<r_b}$ consists of exactly those germs a in Γ that are based at C, and such that

- either a is transversal to b,
- or a is tangent to b and $r_a < r_b$.

All the germs in $\Gamma_{\rho>r_b}$ and $\Gamma_{\rho=r_b}$ are tangent to b.

Proposition 4.8 *1) The points of intersection $T_u b \cap \Gamma$ with sublinear z-asymptotics are exactly the points of intersection of the line $T_u b$ with $\Gamma_{\rho<r_b}$.*

2) If $\Gamma_{\rho>r_b} \neq \emptyset$, then $T_u b \cap \Gamma_{\rho>r_b}$ is split into two **non-empty** *parts,*

$$T_u b \cap \Gamma_{\rho>r_b} = \mathcal{L}_u^< \sqcup \mathcal{L}_u^> : \tag{4.11}$$

- the points in $\mathcal{L}_u^<$ have linear z-asymptotics with z-factors equal to $\frac{r_b-1}{r_b}$;

- the points in $\mathcal{L}_u^> \sqcup (T_u b \cap \Gamma_{out})$ have superlinear z-asymptotics.

3) The set of points of intersection $T_u b \cap \Gamma$ with linear z-asymptotics coincides with $(T_u b \cap \Gamma_{\rho=r_b}) \sqcup L_u^<$.

4) Let

$$r = r_b = \frac{p}{q}$$

be the irreducible fraction presentation of the Puiseux exponent r_b. Let $a_1, \dots, a_N$ denote the germs forming $\Gamma_{\rho=r_b}$: they are tangent to b and $r_{a_i} =$

r. Let p_{a_i}, q_{a_i}, c_{a_i} be respectively the asymptotic exponents and coefficients in their parametrizations (3.6):

$$p_{a_i} = s_i p, \ q_{a_i} = s_i q, \ s_i \in \mathbb{N}, \ s_i = G.C.D.(p_{a_i}, q_{a_i}); \ c_{a_i} \in \mathbb{C}^*. \qquad (4.12)$$

Let ζ_{ij} $(i = 1\dots, N, \ j = 1, \dots p)$ be the roots of the polynomials

$$R_{p,q,c(i)}(\zeta) = c(i)\zeta^p - r\zeta^q + r - 1, \ c(i) = \frac{c_{a_i}}{c_b} \in \mathbb{C}^*. \qquad (4.13)$$

The asymptotic z-factors of points of the intersection $T_u b \cap \Gamma_{\rho=r_b}$ are ζ_{ij}^q.

5) One has

$$\zeta_{ij}^q \neq \frac{r-1}{r} \text{ for all } i \text{ and } j. \qquad (4.14)$$

Addendum to Proposition 4.8. *In the conditions of Proposition 4.8 in the case, when Γ is a divisor, let $m_i \in \mathbb{N}$ denote the multiplicities of the germs a_i in Γ. The asymptotic z-divisor of the divisor Γ equals to*

$$M_z = \sum_{i=1}^{N} \sum_{j=1}^{p} \ell_i[\zeta_{ij}^q] + \kappa_z[\frac{r-1}{r}], \ \ell_i = m_i s_i \in \mathbb{N}, \ \kappa_z \in \mathbb{Z}_{\geq 0}, \qquad (4.15)$$

$$\kappa_z = |\mathcal{L}_u^<| > 0 \text{ if and only if } \Gamma_{\rho > r_b} \neq \emptyset. \qquad (4.16)$$

Proof All the statements of Proposition 4.8, except for inequality (4.14), follow from Propositions 4.2 and 4.3, see more details below. Inequality (4.14) is implied by the following general proposition.

Proposition 4.9 *For every $p, q \in \mathbb{N}$, $1 \leq q < p$, $c \in \mathbb{C}^*$, set $r = \frac{p}{q}$, and every root ζ of the polynomial $R_{p,q,c}(z) = cz^p - rz^q + r - 1$ one has*

$$\zeta^q \neq \frac{r-1}{r}, \ c\zeta^p \neq 1 - r. \qquad (4.17)$$

Proof The proof of the first inequality repeats the proof of an equivalent statement from [27, proof of proposition 3.13]. Suppose the contrary: $\zeta^q = \frac{r-1}{r}$ for some root ζ. Then

$$R_{p,q,c}(\zeta) = c\zeta^p - r\zeta^q + r - 1 = c\zeta^p = c(\frac{r-1}{r})^r \neq 0.$$

The contradiction thus obtained proves the first inequality in (4.17). Let us prove the second one. Suppose the contrary: $c\zeta^p = 1 - r$ for some root ζ. Then one has

$$R_{p,q,c}(\zeta) = c\zeta^p - r\zeta^q + r - 1 = -r\zeta^q \neq 0.$$

The contradiction thus obtained proves the second inequality in (4.17) and Proposition 4.9. $\square$

Set $W_i = R_{p,q,c(i)}$, $\widetilde{W}_i = R_{p_{a_i},q_{a_i},c(i)}$. Statement 4) of Proposition 4.8 follows from Proposition 4.3, Case 2) and the relation $\widetilde{W}_i(h) = W_i(h^{s_i})$, which implies that to every root ζ of the polynomial W_i correspond s_i roots $\zeta^{\frac{1}{s_i}}$ of the polynomial $\widetilde{W}_i$ whose q_{a_i}-th powers are equal to ζ^q. Statements (4.15) and (4.16) follow from Statements 3), 4) of Proposition 4.8, the above discussion and inequality (4.14). $\square$

Recall that $a_1, \dots, a_N$ denote the germs forming $\Gamma_{\rho=r_b}$.

Proposition 4.10 *1) The set of points of intersection $T_u b \cap \Gamma$ with sublinear w-asymptotics is exactly the set $\mathcal{L}_u^<$ from (4.11).*

2) The set of points of intersection $T_u b \cap \Gamma$ with superlinear w-asymptotics is the union $\mathcal{L}_u^> \sqcup (T_u b \cap \Gamma_{out})$.

3) The set of points of intersection $T_u b \cap \Gamma$ with linear w-asymptotics is $T_u b \cap (\Gamma_{\rho<r_b} \sqcup \Gamma_{\rho=r_b})$. The asymptotic w-factors of the points in $T_u b \cap \Gamma_{\rho<r_b}$ are all equal to $1-r$, $r = r_b$. The asymptotic w-factors of the points in $T_u b \cap a_i$ are equal to $c(i)\zeta_{ij}^p$, $i = 1, \dots, N$, $j = 1, \dots, p$, where ζ_{ij} are the roots of the polynomials $R_{p,q,c(i)}$, see (4.13). One has

$$c(i)\zeta_{ij}^p \neq 1 - r \text{ for all } i \text{ and } j. \tag{4.18}$$

4) In the case, when Γ is a divisor, let m_i, s_i be the same, as in (4.15). The asymptotic w-divisor of the multigerm Γ equals to

$$M_w = \sum_{i=1}^{N}\sum_{j=1}^{p} \ell_i[c(i)\zeta_{ij}^p] + \kappa_w[(1-r)], \ \ell_i = m_i s_i \in \mathbb{N}, \ \kappa \in \mathbb{Z}_{\geq 0}, \tag{4.19}$$

$$\kappa_w = |T_u b \cap \Gamma_{\rho<r_b}| > 0 \text{ if and only if } \Gamma_{\rho<r_b} \neq \emptyset. \tag{4.20}$$

All the statements of Proposition 4.10 follow from Propositions 4.2 and 4.3, except for inequality (4.18) (which follows from (4.17)) and the part of statement 2) saying that the points in $T_u b \cap \Gamma_{out}$ have superlinear w-asymptotics, which is given by the following proposition.

Proposition 4.11 *For every irreducible germ a of analytic curve at any point $B \in T_C b$, $B \neq C$, the points of intersection $T_u b \cap a$ have superlinear w-asymptotics, as $u \in b$ tends to C.*

Proof For $u \in b$ being close enough to C, let $Q_1 = Q_1(u)$ denote the point of the intersection of the line $T_u b$ with the z-axis. Fix an arbitrary family of points $Q_2(u)$ of the intersection $T_u b \cap a$. Their limits $Q_1(C) = C$ and $Q_2(C) = B$ lie in the z-axis and are distinct, by assumption; $z(C) = 0 \neq z(B)$. Let us show that $w(u) = o(w(Q_2(u)))$, as $u \to C$.

Let $T = T(u)$, $O = O(u)$ denote the respectively the projections of the points u and Q_2 to the z-axis: $z(T) = z(u)$, $z(O) = z(Q_2)$. Consider the triangles TQ_1u and OQ_1Q_2. They are similar in the following complex sense. Their edges Tu and OQ_2 lie in complex lines parallel to the w-axis. Their edges TQ_1, OQ_1 lie in the complex z-axis. Their edges uQ_1 and Q_2Q_1 lie in the same complex line Q_1Q_2. The parallelness of complexified edges of the above triangles implies that

$$\frac{w(u) - w(T)}{w(Q_2) - w(O)} = \frac{z(T) - z(Q_1)}{z(O) - z(Q_1)}. \tag{4.21}$$

Substituting the equalities and asymptotics $w(T) = w(O) = 0$, $z(Q_1(u)) \to 0$, $z(T) = z(u) \to 0$, and $z(O(u)) - z(Q_1(u)) \to z(O(C)) = z(B) \neq 0$ to formula (4.21) yields to $\frac{w(u)}{w(Q_2)} \to 0$. This proves Propositions 4.11 and 4.10. $\square$

4.2 Families of degenerating conformal involutions

In Subsection 4.3 we show that for every local branch b as in Theorem 4.1 the corresponding family of $\mathbb{I}$-angular symmetries $T_u b \to T_u b$ with center u written in appropriate coordinate becomes a degenerating family of conformal involutions $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of one of the following types.

Definition 4.12 Consider a family of conformal involutions $\sigma_u : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Riemann sphere with coordinate z that are parametrized by a small complex parameter u with a given family of fixed points $\zeta(u)$:

$$\sigma_u(\zeta(u)) = \zeta(u); \quad \zeta(u) \to 0, \text{ as } u \to 0.$$

The family σ_u is said to be

- *of type A,* if there exist families of points $\alpha(u), \omega(u) \in \overline{\mathbb{C}}$ such that

$$\sigma_u(\alpha(u)) = \omega(u), \ \alpha(u) = o(\zeta(u)), \ \zeta(u) = o(\omega(u)), \text{ as } u \to 0;$$

- *of type B,* if there exist families of points $\alpha(u), \omega(u) \in \overline{\mathbb{C}}$ such that

$$\sigma_u(\alpha(u)) = \omega(u), \ \alpha(u), \omega(u) = o(\zeta(u)), \text{ as } u \to 0.$$

Proposition 4.13 *Each family of involutions $\sigma_u : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of type A with given fixed points $\zeta(u)$ satisfies the following statements:*

(a) The involutions σ_u converge to the constant mapping $\overline{\mathbb{C}} \mapsto 0$ uniformly on compact subsets in $\overline{\mathbb{C}} \setminus \{0\}$.

(b) Fix a $c \in \mathbb{C}^*$ and a family of points $z_u \in \mathbb{C}$ with the asymptotics $z_u = c\zeta(u)(1 + o(1))$, as $u \to 0$. Then

$$\sigma_u(z_u) = c^{-1}\zeta(u)(1 + o(1)), \text{ as } u \to 0. \tag{4.22}$$

Proof The scalings $\phi_u : z \mapsto \widetilde{z} = \frac{z}{\zeta(u)}$ conjugate the involutions σ_u to the conformal involutions $\Sigma_u = \phi_u \circ \sigma_u \circ \phi_u^{-1} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ fixing 1 and permuting the points $\frac{\alpha(u)}{\zeta(u)}$ and $\frac{\omega(u)}{\zeta(u)}$; $\frac{\alpha(u)}{\zeta(u)} \to 0$, and $\frac{\omega(u)}{\zeta(u)} \to \infty$, as $u \to 0$. Hence, $\Sigma_u(z) \to \frac{1}{z}$ in $Aut(\overline{\mathbb{C}})$ and thus, uniformly on $\overline{\mathbb{C}}$. For every $\delta > 0$ the mapping $\sigma_u = \phi_u^{-1} \circ \Sigma_u \circ \phi_u$ converges to the constant mapping $\overline{\mathbb{C}} \mapsto 0$ uniformly on $\overline{\mathbb{C}} \setminus D_\delta$. Indeed, $\phi_u(z) = \frac{z}{\zeta(u)} \to \infty$ uniformly on $\overline{\mathbb{C}} \setminus D_\delta$, since $\zeta(u) \to 0$. Hence $f_u = \Sigma_u \circ \phi_u \to 0$, $\sigma_u = \phi_u^{-1} \circ f_u = \zeta(u) f_u \to 0$. This proves statement (a). For $z_u = c\zeta(u)(1 + o(1))$ with $c \neq 0$ one has

$$\sigma_u(z_u) = \zeta(u)\Sigma_u((\zeta(u))^{-1}z_u) = \zeta(u)\Sigma_u(c + o(1)) = \zeta(u)(c^{-1} + o(1)).$$

This proves statement (b) and finishes the proof of the proposition. □

Proposition 4.14 *Each family of involutions $\sigma_u : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of type B with given fixed points $\zeta(u)$ satisfies the following statements:*

(a) The coordinate change $\widetilde{z} = \frac{\zeta(u)}{z}$ conjugates the involutions σ_u to conformal involutions $\Sigma_u : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that converge in $Aut(\overline{\mathbb{C}})$ to the central symmetry with respect to one: $\widetilde{z} \mapsto 2 - \widetilde{z}$.

(b) For every $c \in \mathbb{C} \setminus \{0, 2\}$ and every family of points $z_u = c^{-1}\zeta(u)(1 + o(1))$ one has $\sigma_u(z_u) = d^{-1}\zeta(u)(1 + o(1))$, where $d = 2 - c$.

Proof The above change of coordinate $z \mapsto \widetilde{z}$ sends the fixed point $\zeta(u)$ of the involution σ_u to 1, and $\widetilde{z}(\alpha(u)), \widetilde{z}(\omega(u)) \to \infty$, as $u \to 0$, since $\alpha(u), \omega(u) = o(\zeta(u))$. Therefore, the involution σ_u written in the coordinate $\widetilde{z}$ fixes 1 and permutes two distinct points converging to infinity. Its derivative at the fixed point 1 equals to -1, since the involution is nontrivial. Therefore, it converges to the unique non-trivial involution fixing 1 and ∞: the central symmetry with respect to 1. Statement (a) is proved. Statement (a) immediately implies statement (b). The proposition is proved. □

4.3 Relative projective symmetry properties and their types

Definition 4.15 Let b be an irreducible germ of analytic curve at a point $C \in \mathbb{CP}^2$. Let $\Delta = \sum_{j=1}^{l} k_j b_j$ be a $(T_C b, C)$-local divisor containing b: say, $b_1 = b$. We say that the germ b has *relative projective symmetry property* with respect to the divisor Δ, if for every $u \in b \setminus \{C\}$ there exists a projective involution $\sigma_u : T_u b \to T_u b$ with fixed point u such that the intersection $\Delta \cap T_u b$ treated as a divisor on $T_u b$ is σ_u-invariant. For any given affine coordinate h on a neighborhood of the point C in $\mathbb{CP}^2$ with $h(C) = 0$ we say that b has relative projective symmetry property *of type A-h (B-h)*, if the family of involutions σ_u written in the coordinate h on the lines $T_u b$ is of type A (respectively, B), see Definition 4.12.

Proposition 4.16 *Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic: either a regular conic, or a pair of distinct lines. Let an irreducible algebraic curve $\gamma \subset \mathbb{CP}^2$ generate a rationally integrable $\mathbb{I}$-angular billiard with integral G, let $C \in \gamma$. Let Δ denote the zero divisor of the function G. Every local branch b of the curve γ at C has relative projective symmetry property with respect to the $(T_C b, C)$-localization (see Definition 4.4) of each one of the divisors Δ and $\Delta + \mathbb{I}$: the corresponding projective involution from Definition 4.15 is the $\mathbb{I}$-angular symmetry centered at u. In the case, when $C \in \gamma \cap \mathbb{I}$, the following statements hold in the corresponding cases listed below; here (z, w) is a system of affine coordinates centered at C and adapted to b.*

Case 1): C is a regular point of the conic $\mathbb{I}$, and b is transversal to $\mathbb{I}$. Then b has relative projective symmetry property of type A-z.

Case 2): $\mathbb{I}$ is a pair of lines through the point C that are both transversal to b. Then b has relative projective symmetry property of type B-z.

Case 3): C is a regular point of the conic $\mathbb{I}$, and b is tangent to $\mathbb{I}$.

Subcase 3a): $\mathbb{I}$ is a pair of lines. Then b has relative projective symmetry property of type A-w.

Subcase 3b): $\mathbb{I}$ is a regular conic and $r_b < 2$. Then b has relative projective symmetry property of type A-w.

Subcase 3c): $\mathbb{I}$ is a regular conic and $r_b > 2$. Then b has relative projective symmetry property of type B-z.

Proof The first statement of the proposition follows immediately from definition. Let us prove its other statements case by case.

Case 1). Then the line $T_C b$ intersects $\mathbb{I}$ at two points: the point C and a point $B \neq C$. Let $\mathbb{I}_C$ and $\mathbb{I}_B$ denote the germs of the conic $\mathbb{I}$ at C and B respectively. As $u \in b$ tends to C, the $\mathbb{I}$-angular symmetry of the line $T_u b$ with center u permutes its points C_u, B_u of intersection with $\mathbb{I}_C$ and

$\mathbb{I}_B$. The coordinate $z(B_u)$ tends to a non-zero (may be infinite) limit, and $z(C_u)=o(z(u))$, as $u\to C$, by transversality of the germs $\mathbb{I}_C$ and b and Proposition 4.2. Therefore, the $\mathbb{I}$-angular symmetries under question written in the coordinate z form a family of conformal involutions of type A.

Case 2). As $u\to b$, the line T_ub intersects $\mathbb{I}$ at two points permuted by the $\mathbb{I}$-angular symmetry. These intersection points tend to C, and their z-coordinates are $o(z(u))$, by transversality, as in the above case. Hence, the $\mathbb{I}$-angular symmetries of the lines T_ub written in the coordinate z form a family of involutions of type B.

Case 3).

Subcase 3a). Then the conic $\mathbb{I}$ consists of two distinct lines intersecting at some point $B\neq C$: the line $\mathbb{I}_C=T_Cb$ and a line $\mathbb{I}_B$. The (T_Cb,C)-localization of the conic $\mathbb{I}$ consists of two germs: the germ of the line $\mathbb{I}_C$ at C; the germ of the line $\mathbb{I}_B$ at B. As $u\in b$ tends to C, the line T_ub intersects $\mathbb{I}_C$ and $\mathbb{I}_B$ at points C_u and B_u respectively, which are permuted by the $\mathbb{I}$-angular symmetry with center u; $C_u\to C$, $B_u\to B$, as $u\to C$. One has $w(C_u)=0$, since $\mathbb{I}_C=T_Cb$ is the z-axis, and $w(u)=o(w(B_u))$, by Proposition 4.11. Therefore, the $\mathbb{I}$-angular symmetries of the lines T_ub written in the coordinate w form a family of involutions of type A.

Subcase 3b). Then the (T_Cb,C)-localization of the conic $\mathbb{I}$ consists of just one regular germ at C, whose Puiseux exponent 2 is greater than r_b. As $u\in b$ tends to C, the line T_ub intersects $\mathbb{I}$ at two points C_u and B_u tending to C so that $w(C_u)=o(w(u))$ and $w(u)=o(w(B_u))$, by Proposition 4.3, Case 1). The points C_u and B_u are permuted by the $\mathbb{I}$-angular symmetry with center u. Therefore, the $\mathbb{I}$-angular symmetries of the lines T_ub written in the coordinate w form a family of conformal involutions of type A.

Subcase 3c). Then $r_b>2=r_{\mathbb{I}}$. As $u\in b$ tends to C, both points of intersection $T_ub\cap\mathbb{I}$ tend to C so that their z-coordinates are $o(z(u))$, by Proposition 4.3, Case 3). The latter points are permuted by the $\mathbb{I}$-angular symmetry centered at u. Therefore, these $\mathbb{I}$-angular symmetries of the lines T_ub written in the coordinate z form a family of conformal involutions of type B. This proves Proposition 4.16. □

4.4 Symmetry of asymptotic divisors. Proof of statements (i) and (ii-a)

Here we prove the following theorem generalizing statements (i) and (ii-a).

Theorem 4.17 *Let b be an irreducible germ of analytic curve in $\mathbb{CP}^2$ at a point C, and let (z,w) be affine coordinates centered at C that are adapted* *to b. Let b have local relative projective symmetry property of type either A-w, or B-z. Then b is quadratic.*

We will deduce Theorem 4.17 from invariance of asymptotic divisors under appropriate conformal involutions, see the following propositions.

Proposition 4.18 *Let an irreducible germ $b \subset \mathbb{CP}^2$ of analytic curve at a point C have local relative projective symmetry property of type A-h for some affine coordinate h, $h(C) = 0$. Then its asymptotic h-divisor is invariant under the involution $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of taking inverse: $z \mapsto z^{-1}$.*

Proposition 4.18 follows from Proposition 4.13, Statement (b).

Definition 4.19 For a divisor $M = \sum_j k_j[z_j]$ on $\overline{\mathbb{C}}$ its *inverse divisor* is

$$M^{-1} = \sum_j k_j[z_j^{-1}].$$

For every divisor M on $\overline{\mathbb{C}}$ and every subset $K \subset \overline{\mathbb{C}}$ by $M \setminus K$ we denote the divisor obtained from M by deleting those its points that lie in K (taken with their total multiplicities).

Proposition 4.20 *Let an irreducible germ $b \subset \mathbb{CP}^2$ of analytic curve at a point C have local relative projective symmetry property of type B-h for some affine coordinate h, $h(C) = 0$. Let M_h^{-1} denote the inverse to its asymptotic h-divisor M_h. The divisor $M_h^{-1} \setminus \{2\}$ is invariant under the central symmetry $\mathbb{C} \to \mathbb{C}$ with respect to one: $z \mapsto 2 - z$.*

Proposition 4.20 follows from Proposition 4.14, Statement (b).

Proof of Theorem 4.17.

Case 1) of symmetry property of type A-w. The asymptotic w-divisor M_w being invariant under taking inverse (Proposition 4.18), the product of its points equals to one. On the other hand, the latter product equals to the product of natural powers of expressions

$$U_i = \prod_{j=1}^{p}(c(i)\zeta_{ij}^{p}) = (c(i))^p(\prod_{j=1}^{p}\zeta_{ij})^p \tag{4.23}$$

and a non-negative integer power of the number $1 - r$, see (4.19). One has $\prod_{j=1}^{p}\zeta_{ij} = (c(i))^{-1}(r-1)$ up to sign, by Vieta's Formula. Therefore, in formula (4.23) the number $c(i)$ cancels out and $U_i = \pm(1-r)^p$. Finally, the product of points of the divisor M_w, which is equal to one, equals to a natural power of the number $1-r$, up to sign. Hence, $r=2$ and the germ b is quadratic.

Case 2) of symmetry property of type B-z. The divisor $M_z^{-1}\setminus\{2\}$ being invariant under the symmetry with respect to one (Proposition 4.20), the sum of its points equals to its degree. Let us write this equation explicitly and deduce that $r=r_b=2$.

The divisor M_z^{-1} has the form

$$M_z^{-1}=\sum_i \ell_i\sum_{j=1}^{p}[\theta_{ij}^q]+\kappa_z[\frac{r}{r-1}],\ \theta_{ij}=\zeta_{ij}^{-1},\ \kappa_z\in\mathbb{Z}_{\geq 0},$$

$\ell_i\in\mathbb{N}$, see (4.15). The numbers θ_{ij} are the roots of the polynomials

$$H_{p,q,c(i)}(\theta)=\theta^p R_{p,q,c(i)}(\theta^{-1})=(r-1)\theta^p-r\theta^{p-q}+c(i).$$

The points of the divisor M_z^{-1} are distinct from zero. Those of them that are powers θ_{ij}^q are different from the number $\frac{r}{r-1}$, by Proposition 4.9. A priori, M_z^{-1} may contain some of the points 2 and $\frac{r-2}{r-1}=2-\frac{r}{r-1}$, which are symmetric to 0 and $\frac{r}{r-1}$, respectively. Set $M=M_z^{-1}\setminus\{2,\frac{r}{r-1},\frac{r-2}{r-1}\}$:

$$M=\text{ the sum of those terms } \ell_i[\theta_{ij}^q], \text{ for which } \theta_{ij}^q\neq 2,\frac{r-2}{r-1}. \qquad (4.24)$$

The divisor M is symmetric with respect to one, as is $M_z^{-1}\setminus\{2\}$.

Lemma 4.21 *[27, lemma 3.16]. Let $r=\frac{p}{q}>1$; here $p,q\in\mathbb{N}$, $(p,q)=1$. Consider a finite collection of polynomials $H_{p,q,c(i)}(\theta)$, $c(i)\neq 0$ and numbers $\ell_i\in\mathbb{N}$, $i=1,\dots,N$. Let θ_{ij} denote the roots of the polynomials $H_{p,q,c(i)}$. Let the divisor M given by (4.24) be invariant under the symmetry of the line $\mathbb{C}$ with respect to one. Then $r=2$.*

Remark 4.22 In fact, lemma 3.16 in [27] was stated in a slightly different but equivalent form. It dealt with a collection of polynomials $H_{p_i,q_i,c(i)}$, $q_i,p_i\in\mathbb{N}$, $\frac{p_i}{q_i}=r>1$, $c(i)\neq 0$ and the divisor M of those q_i-th powers of their roots that are distinct from the numbers 2 and $\frac{r-2}{r-1}$. Set $s_i=G.C.D(p_i,q_i)$. The latter q_i-th powers of roots coincide with the q-th powers of roots of the corresponding polynomials $H_{p,q,c(i)}$, $p=\frac{p_i}{s_i}$, $q=\frac{q_i}{s_i}$, and the divisor M contains each of them s_i times. Hence, M is given by (4.24) with $\ell_i=s_i$, and this yields to equivalence of the above lemma to [27, lemma 3.16].

Lemma 4.21 together with the symmetry of the divisor M given by (4.24) imply that $r = 2$. Theorem 4.17 is proved. □

Proof of statements (i) and (ii-a) of Theorem 4.1. Every branch b satisfying condition (i) of Theorem 4.1 has local relative projective symmetry property of type B-z, by Proposition 4.16, Case 2). Hence, it is quadratic, by Theorem 4.17. Statement (i) is proved.

Let us prove statement (ii-a). Let b be a branch satisfying condition (ii-a) of Theorem 4.1. Then its base point C is a regular point of the conic $\mathbb{I}$, and b is tangent to $\mathbb{I}$. We treat the two following cases separately.

Case 1): $\mathbb{I}$ is a union of two lines. Then b has local relative projective symmetry property of type A-w, by Proposition 4.16, Subcase 3a). Hence, it is quadratic, by Theorem 4.17.

Case 2): $\mathbb{I}$ is a regular conic. Suppose the contrary: $r = r_b \neq 2$. We treat the two following subcases separately.

Subcase 2a): $r < 2$. Then b has local relative projective symmetry property of type A-w, by Proposition 4.16, Subcase 3b). Hence, it is quadratic, by Theorem 4.17, – a contradiction.

Subcase 2b): $r > 2$. Then b has local relative projective symmetry property of type B-z, by Proposition 4.16, Subcase 3c). Hence, it is quadratic, by Theorem 4.17, – a contradiction. Statements (i) and (ii-a) are proved. □

4.5 Subquadraticity

Here we prove the following theorem implying that every local branch b satisfying the conditions of Statement (ii-b) of Theorem 4.1 is subquadratic. Recall that such a branch has local relative projective symmetry property of type A-z, see Proposition 4.16, Case 1).

In what follows $b \subset \mathbb{CP}^2$ is an irreducible germ of analytic curve at a point C, and (z, w) are affine coordinates centered at C and adapted to b.

Theorem 4.23 *Every germ b having local relative projective symmetry property of type A-z with respect to some $(T_C b, C)$-local divisor Γ is subquadratic.*

Proof In what follows for a given divisor M in $\mathbb{C}$ by $S(M)$ we denote the sum of its points. The asymptotic z-divisor M_z is invariant under taking inverse (Proposition 4.18). Therefore, $S(M_z) = S(M^{-1}(z))$. Let us write down the latter equality explicitly. Let $a_1, \dots, a_N$ be the germs in Γ that are tangent to b and have the same Puiseux exponent $r = r_b$. Let ζ_{ij} be the same, as in (4.15), set $\theta_{ij} = \zeta_{ij}^{-1}$. One has

$$S(M_z) = \sum_{ij} \ell_i \zeta_{ij}^q + \kappa_z \frac{r-1}{r} = S(M_z^{-1}) = \sum_{ij} \ell_i \theta_{ij}^q + \kappa_z \frac{r}{r-1}, \qquad (4.25)$$

by (4.15). Recall that for every fixed i the numbers θ_{ij} are the roots of the polynomial $(r-1)\theta^p - r\theta^{p-q} + c(i)$. Hence, the sum of their q-th powers equals to $\frac{p}{r-1}$, by [27, formula (3.17)], and

$$S(M_z^{-1}) = \frac{\Pi}{r-1} + \kappa_z \frac{r}{r-1}, \ \Pi = p \sum_i \ell_i. \qquad (4.26)$$

Suppose the contrary: $r > 2$, i.e., $p > 2q$. Then $\sum_j \zeta_{ij}^q = 0$ for every $i = 1, \dots, N$. Indeed, the latter sum is expressed as a polynomial in the symmetric polynomials in ζ_{ij} of degrees $1, \dots, q$. All of these symmetric polynomials vanish, as do the coefficients of the polynomial $R_{p,q,c(i)}(\zeta) = c(i)\zeta^p - r\zeta^q + r - 1$ at monomials of degrees $p-1, \dots, p-q > q$. Hence, $S(M_z) = \kappa_z \frac{r-1}{r}$. Substituting the latter equality and (4.26) to (4.25) yields to

$$S(M_z) = \kappa_z \frac{r-1}{r} = S(M_z^{-1}) = \frac{\Pi}{r-1} + \kappa_z \frac{r}{r-1} > \kappa_z \frac{r}{r-1}.$$

The latter inequality is strict, since $\Pi > 0$: the collection of germs a_i contains b, and hence, is non-empty. But its right-hand side is no less than the left-hand side, since $\frac{r}{r-1} > 1 > \frac{r-1}{r}$. The contradiction thus obtained proves the inequality $r \leq 2$. □

Open Problem. *Is it true that every germ b having local relative projective symmetry property of type A-z is a) quadratic? b) regular and quadratic?*

4.6 Puiseux exponents

Here we prove the following theorem implying that for every local branch b of the curve γ satisfying the conditions of Statement (ii-b) one has $\Gamma_{\rho > r_b} = \emptyset$, that is, b has the maximal Puiseux exponent among all the local branches of the curve Γ that are tangent to b.

Theorem 4.24 *Let $b \subset \mathbb{CP}^2$ be an irreducible germ of analytic curve at a point C, and let (z, w) be affine coordinates centered at C and adapted to b. Let b have local relative projective symmetry property of type A-z with respect to a $(T_C b, C)$-local divisor Δ. Then each irreducible germ b_j at C tangent to b in the divisor Δ has Puiseux exponent no greater than $r = r_b$.*

Recall that in our assumptions the asymptotic z-divisor M_z is invariant under taking inverse (Proposition 4.18), and the existence of a germ a at C in Δ with $r_a > r = r_b$ is equivalent to the statement that M_z contains the point $\theta = \frac{r-1}{r}$, see the Addendum to Proposition 4.8. As it is shown below, Theorem 4.24 is implied by the following lemma.

Lemma 4.25 *Let $p, q \in \mathbb{N}$, $1 \leq q < p$. There exist no pair $(\theta, \mathcal{P})$ where $0 < \theta < 1$ and $\mathcal{P}$ is a finite collection of polynomials of type $R_{p,q,c}(\zeta) = c\zeta^p - r\zeta^q + r - 1$ with $r = \frac{p}{q}$ and $c \neq 0$ such that the subset*

$$M = \{\theta\} \cup \{\zeta^q \mid \zeta \text{ is a root of a polynomial from the collection } \mathcal{P}\} \subset \mathbb{C}^*$$

is invariant under the transformation of taking inverse: $z \mapsto z^{-1}$.

The proof of Lemma 4.25 is based on the following proposition.

Proposition 4.26 *Let $p, q \in \mathbb{N}$, $1 \leq q < p$, $r = \frac{p}{q}$. A polynomial $W(z) = R_{p,q,c}(z) = cz^p - rz^q + r - 1$ has a root $z_1 > 1$, if and only if $0 < c < 1$. In this case it has exactly two real positive roots z_0 and z_1, $0 < z_0 < 1 < z_1$, and their product is greater than one.*

Proof For $c \notin \mathbb{R}_+$ one has $W|_{\{z \geq 1\}} \neq 0$, since $-rz^q + r - 1 < 0$ for every $z \geq 1$. Therefore, we consider that $c > 0$. The derivative equals to $W'(z) = cpz^{p-1} - rqz^{q-1} = pz^{q-1}(cz^{p-q} - 1)$. Therefore, $c^{-\frac{1}{p-q}}$ is the unique local extremum of the polynomial W in the positive semiaxis, and it is obviously a local minimum. For $c = 1$ one has $W(1) = 0$, and $z = 1$ is exactly the minimum. Therefore, as c increases, the graph of the polynomial W becomes disjoint from the positive coordinate semiaxis, and it has no positive root, if $c > 1$. As c decreases remaining positive, the graph intersects the coordinate axis on both sides from 1. Thus newly born roots of the polynomial W remain positive: they do not escape from the positive semiaxis through the origin, since $W(0) = r - 1 > 0$ for all c. Thus, for $0 < c < 1$ the polynomial W has exactly two real positive roots z_0 and z_1, $0 < z_0 < 1 < z_1$, and the minimum is between them; $W(z) > 0$ for $z > z_1$. Let us prove that $z_0 z_1 > 1$, or equivalently, $z_0^{-1} < z_1$. The latter inequality would follow from positivity of the polynomial W on the interval $(z_1, +\infty)$ and the inequality

$$W(z_0^{-1}) < 0. \tag{4.27}$$

Let us prove (4.27). By definition, $cz_0^p - rz_0^q + r - 1 = 0$, hence,

$$c = \frac{rz_0^q - r + 1}{z_0^p}.$$

Substituting the latter right-hand side into the polynomial $W(z_0^{-1})$ instead of the coefficient c yields to

$$W(z_0^{-1}) = z_0^{-p}(r(z_0^{q-p} - z_0^{p-q}) - (r-1)(z_0^{-p} - z_0^{p})).$$

Multiplying the right-hand side of the latter formula by qz_0^p and denoting $m = p - q$ yields to the following expression:

$$p(z_0^{-m} - z_0^{m}) - m(z_0^{-p} - z_0^{p}).$$

Let us show that the latter expression is negative. We prove the following stronger inequality:

$$\frac{z_0^{-p} - z_0^{p}}{z_0^{-m} - z_0^{m}} > \frac{p}{m} \text{ whenever } z_0 \in \mathbb{R}_+ \setminus \{1\} \text{ and } p > m, \ p, m \in \mathbb{N}. \quad (4.28)$$

Canceling the common divisor $z_0^{-1} - z_0$ in the left fraction transforms inequality (4.28) to

$$\frac{z_0^{1-p} + z_0^{3-p} + \dots + z_0^{p-1}}{z_0^{1-m} + z_0^{3-m} + \dots + z_0^{m-1}} > \frac{p}{m}. \quad (4.29)$$

Case 1): p and m are of the same parity. Then the difference of the nominator and the denominator in (4.29) is positive and equal to the sum

$$(z_0^{1-p} + z_0^{p-1}) + (z_0^{3-p} + z_0^{p-3}) + \dots + (z_0^{-1-m} + z_0^{m+1}). \quad (4.30)$$

Each sum of inverses in (4.30) is greater than any analogous sum of inverses $z_0^{-j} + z_0^{j}$, $j \leq m-1$, in the above denominator, since the function

$$f_z(s) = z^{-s} + z^{s}$$

in $s > 0$ with fixed $z > 0$, $z \neq 1$ increases. This implies that the average sum of inverses in (4.30) is greater than that in the denominator. Hence, the ratio of expression (4.30) and the denominator is greater than the ratio of the quantities of sums of inverses $f_{z_0}(j)$ in them. (If m is odd, then the denominator contains the half-sum $1 = \frac{f_{z_0}(0)}{2}$; here $f_{z_0}(0)$ is counted with weight $\frac{1}{2}$.) This implies that the ratio in the left-hand side in (4.29) is greater than the ratio $\frac{p}{m}$ of the numbers of powers z^j in its nominator and denominator. This proves (4.29).

Case 2): $p \not\equiv m (\operatorname{mod} 2)$. Then the nominator in (4.29) equals to

$$\nu_m = f_{z_0}(p-1) + f_{z_0}(p-3) + \dots + f_{z_0}(m+2) + \sigma_m, \ \sigma_m = f_{z_0}(m) + f_{z_0}(m-2) + \dots \quad (4.31)$$

The denominator equals to

$$\eta_m = f_{z_0}(m-1) + f_{z_0}(m-3) + \dots.$$

Here each one of the sums σ_m and η_m ends with either $f_{z_0}(1)$, or $1 = \frac{f_{z_0}(0)}{2}$. Note that $f_{z_0}(s) < \frac{1}{2}(f_{z_0}(s-1) + f_{z_0}(s+1))$ for all $s \in \mathbb{R}$, since the function $f_z(s)$ is convex in s for $z > 0$, $z \neq 1$: $f_z''(s) = (\ln z)^2 f_z(s) > 0$. Writing these mean inequalities for $s = m-1, m-3, \dots$ and summing them up yields to

$$\sigma_m > \eta_m + \frac{1}{2} f_{z_0}(m).$$

Substituting this inequality to (4.31) yields to

$$\nu_m > \psi_{pm} + \eta_m, \ \psi_{pm} = f_{z_0}(p-1) + f_{z_0}(p-3) + \dots + f_{z_0}(m+2) + \frac{1}{2} f_{z_0}(m).$$

Note that the sum ψ_{pm} contains only terms $f_{z_0}(j)$ with $j \geq m$, while η_m contains only terms $f_{z_0}(j)$ with $j < m$. Thus, each term $f_{z_0}(j)$ in ψ_{pm} is greater than each term in η_m, as in the previous case (increasing of the function $f_z(s)$). This implies that the ratio $\frac{\psi_{pm}}{\eta_m}$ is greater than the ratio of the numbers of terms in ψ_{pm} and η_m respectively. (Here $\frac{1}{2} f_{z_0}(m)$ and a possible free term $1 = \frac{1}{2} f_{z_0}(0)$ are counted as half-terms, that is, with weight $\frac{1}{2}$.) Therefore, the same statement holds for the ratio $\frac{\psi_{pm}+\eta_m}{\eta_m}$, and hence, for the ratio $\frac{\nu_m}{\eta_m}$, since $\nu_m > \psi_{pm} + \eta_m$ and the number of terms in the expression (4.31) for the value ν_m equals to the number of terms in the sum $\psi_{pm} + \eta_m$. This proves (4.29) and the proposition. □

Corollary 4.27 *Let $p, q \in \mathbb{N}$, $1 \leq q < p$, $r = \frac{p}{q}$. Let a polynomial $R_{p,q,c}$ with $c \in \mathbb{C}^*$ have a root with q-th power $\mu > 1$. Then it has another root with q-th power $\theta \in (0,1)$ such that $\theta\mu > 1$.*

Proof Without loss of generality we assume that the polynomial $W(z) = R_{p,q,c}(z)$ has a real root $\mu^{\frac{1}{q}} > 1$. One can achieve this by rescaling the variable z by multiplication by q-th root of unity: the collection of q-th powers of roots remains unchanged. Then $W(z)$ satisfies the condition of Proposition 4.26, which immediately implies the statement of the corollary. □

Proof of Lemma 4.25. Suppose the contrary: a pair $(\theta, \mathcal{P})$ as in the lemma exists. Then the number $\mu_1 = \theta^{-1} > 1$ is contained in M, by symmetry. Hence, it is the q-th power of a root of a polynomial $W_1 \in \mathcal{P}$.

Therefore, W_1 has another root whose q-th power θ_2 satisfies the inequalities $0 < \theta_2 < 1$ and $\theta_2\mu_1 > 1$, by Corollary 4.27. The number $\mu_2 = \theta_2^{-1} > 1$ lies in M, by symmetry, and hence, it is the q-th power of a root of a polynomial $W_2 \in \mathcal{P}$. One has $\mu_1 > \theta_2^{-1} = \mu_2 > 1$, by the previous inequalities. The polynomial W_2 has another root whose q-th power θ_3 satisfies the inequalities $0 < \theta_3 < 1$ and $\theta_3\mu_2 > 1$, by Corollary 4.27. Proceeding further we obtain an infinite sequence $\mu_1 > \mu_2 > \mu_3 > \dots$ of elements of the finite set M. The contradiction thus obtained proves the lemma. $\square$

Proof of Theorem 4.24. Suppose the contrary: the divisor Δ contains a germ a tangent to b with $r_a > r_b$. Let M denote the set of the asymptotic z-factors (i.e., points of the divisor M_z). It is invariant under taking inverse. It consists of the number $\theta = \frac{r-1}{r}$ and the q-th powers of the roots of a finite collection of polynomials $R_{p,q,c(i)}$, $c(i) \neq 0$, by formula (4.15). Therefore the set M has the properties forbidden by Lemma 4.25. The contradiction thus obtained proves Theorem 4.24. $\square$

4.7 Concentration of intersection index

In the condition of statement (ii-b) of Theorem 4.1 let Δ be the zero divisor of a rational integral of the $\mathbb{I}$-angular billiard generated by γ; we normalize it by positive rational factor so that b is included in Δ with multiplicity one. Here we prove the following theorem implying that more than one half of the intersection index (Δ, T_Cb) is concentrated at the base point C.

Theorem 4.28 *Let $b \subset \mathbb{CP}^2$ be an irreducible germ of analytic curve at a point C. Let (z,w) be affine coordinates centered at C and adapted to b. Let b have local relative projective symmetry property of type A-z with respect to an* **effective** *(T_Cb, C)-local divisor $\Delta = \sum_{j=1}^N k_jb_j$, i.e., $k_j > 0$. Let Δ include the germ b with coefficient 1. Set $D = \deg(\Delta)$: this is the intersection index (Δ, T_Cb). Then the local intersection index of the projective tangent line T_Cb with Δ at C is no less than $\frac{D}{2}+1$. The equality may take place only in the case, when the germ b is quadratic and regular, and Δ contains no other germs tangent to b at C with the same Puiseux exponent, as b.*

Proof Everywhere below for any effective divisor $\mathcal{D} = \sum_j n_j[\tau_j]$ on $\mathbb{C}$, $n_j > 0$, we denote by $|\mathcal{D}|$ its degree: $|\mathcal{D}| = \sum_j n_j$. For every $u \in b$ close to C let $\mathcal{X} = \mathcal{X}(u)$ denote the part of the divisor $T_ub \cap \Delta$ on T_ub consisting of those its points that tend to C, as $u \to C$. Let $\Psi(u)$ denote the remaining part of the divisor $T_ub \cap \Delta$, consisting of those its points that do not tend to C: they tend to the other base points of the germs in Δ. The local intersection index $(T_Cb, \Delta)_C$ at the point C equals to the degree $|\mathcal{X}(u)|$ of the divisor $\mathcal{X}(u)$, whenever u is close enough to C.

Let $\mathcal{X}_1 = \mathcal{X}_1(u)$ and $\mathcal{X}_0 = \mathcal{X}_0(u)$ denote the parts of the divisor $\mathcal{X}(u)$ formed respectively by the points with linear z-asymptotics and the points that do not have linear z-asymptotics.

Recall that the divisors $T_ub\cap\Delta$ are invariant under projective involutions $\sigma_u : T_ub \to T_ub$ fixing u and forming a family of type A in the coordinate z.

Claim 1. *The involution σ_u sends the points of the divisor $\Psi(u)$ to some points in $\mathcal{X}_0(u)$, and $|\mathcal{X}_0(u)| \geq |\Psi(u)|$.*

Proof The involutions σ_u written in the coordinate z converge to the constant mapping $\overline{\mathbb{C}} \mapsto 0$ uniformly on compact subsets in $\overline{\mathbb{C}} \setminus \{0\}$, by Proposition 4.13, statement (a). Therefore, the image of a point converging to a limit distinct from C, as $u \to C$, is a point converging to C. This implies that each point of the divisor $\Psi(u)$ is sent to a point in $\mathcal{X}(u)$. Its image in $\mathcal{X}(u)$ cannot lie in $\mathcal{X}_1(u)$, since the divisor $\mathcal{X}_1(u)$ of points with linear z-asymptotics is σ_u-invariant, by Proposition 4.13, statement (b). Hence, σ_u sends $\Psi(u)$ to a part of the divisor $\mathcal{X}_0(u)$. This proves the claim. □

Thus, one has

$$\Delta\cap T_ub = \mathcal{X}_0(u) + \mathcal{X}_1(u) + \Psi(u), \quad |\mathcal{X}_0(u)| \geq |\Psi(u)|,$$

$$|\mathcal{X}_0(u)| + \frac{1}{2}|\mathcal{X}_1(u)| \geq \frac{|\mathcal{X}_0(u)| + |\mathcal{X}_1(u)| + |\Psi(u)|}{2} = \frac{1}{2}|\Delta\cap T_ub| = \frac{D}{2}.$$

This implies that

$$(T_Cb, \Delta)_C = |\mathcal{X}(u)| = |\mathcal{X}_0(u)| + |\mathcal{X}_1(u)| \geq \frac{D}{2} + \frac{1}{2}|\mathcal{X}_1(u)|. \qquad (4.32)$$

One has $|\mathcal{X}_1(u)| \geq 2$. Indeed, the divisor $\mathcal{X}_1(u)$ of points with linear z-asymptotics includes the intersection $b\cap T_ub$ (which has degree at least two) with coefficient one and the intersections of the line T_ub with those germs in Δ that are tangent to b and have the same Puiseux exponent $r = r_b$ with positive coefficients. The equality may take place only if b is regular and quadratic and there are no additional latter germs. This together with (4.32) implies that $(T_Cb, \Delta)_C \geq \frac{D}{2} + 1$ and proves Theorem 4.28. □

4.8 Exponent in the asymptotics of Bialy–Mironov Formula. Proof of statement (ii-b)

Let b be a local branch of the curve γ at a point $C \in \gamma\cap\mathbb{I}$ that is a regular point of the conic $\mathbb{I}$, and let b be transversal to $\mathbb{I}$. Let $\sum_{j=1}^{l} k_jb_j$, $b_1 = b$,

$k_1 = 1$, be the germ at C of the divisor $\frac{1}{k}\Delta$, see (3.5); here $k_j > 0$ for all j. Let ρ_{b_j} and η be the corresponding constants from formulas (3.7) and (3.16) respectively. Let us show that the upper bound (3.19) on the number η proved in Subsection 3.4 cannot hold, unless b is regular and quadratic. Indeed, let (z, w) be affine coordinates adapted to b. The branch b has local relative projective symmetry property of type A-z, by Proposition 4.16, Case 1). Therefore, one has:

$r = r_b \leq 2$, by Theorem 4.23;

$\rho_{b_j} \leq r$ for all $j = 1, \dots, l$, by Theorem 4.24.

Substituting these inequalities to formula (3.16), one gets

$$\eta = 3\sum_{j=1}^{l} k_j q_{b_j} \min\{\rho_{b_j}, r\} - 2(r+1) \geq 3\sum_{j=1}^{l} k_j q_{b_j}\rho_{b_j} - 6. \tag{4.33}$$

The sum in the right-hand side in (4.33) equals to the local intersection index of the divisor $\frac{1}{k}\Delta$ with $T_C b$ at the point C, by definition. The latter local intersection index is no less than $\frac{\deg(\Delta)}{2k} + 1$, by Theorem 4.28. Therefore,

$$\eta \geq 3(\frac{\deg(\Delta)}{2k} + 1) - 6 = 3\frac{\deg(\Delta)}{2k} - 3.$$

The latter inequality is strict, unless the local branch b is regular and quadratic, as in Theorem 4.28. The strict inequality would obviously contradict inequality (3.19), and hence, b is regular and quadratic. Statement (ii-b) is proved. The proof of Theorem 4.1 is complete.

5 Generalized genus and Plücker formulas. Proof of Theorem 1.24

The proof of Theorem 1.24 is based on generalized Plücker and genus formulas for planar algebraic curves and their corollaries, see, e.g., [27, subsection 4.1]. It is done by a modified version of Eugenii Shustin's arguments from [27, subsection 4.2]. The main observation is that the assumptions of Theorem 4.1 on the Puiseux exponents of local branches of the curve and Plücker formulas yield that the singularity invariants of the considered curve γ must obey a relatively high lower bound. On the other hand, the contribution of its potential singular (inflection) points, which lie in the conic $\mathbb{I}$, appears to be not sufficient to fit that lower bound, unless the curve is a conic.

5.1 Invariants of plane curve singularities

The material of the present subsection is contained in [27, subsection 4.1]. It recalls classical results on invariants of singularities presented in [18, Chapter III], [36, §10], see also a modern exposition in [28, Section I.3]. Let $\gamma \subset \mathbb{CP}^2$ be a non-linear irreducible algebraic curve[5]. Let d denote its degree. The intersection index of the curve γ with its Hessian H_γ equals to $3d(d-2)$, by Bézout Theorem. On the other hand, it is equal to the sum of the contributions $h(\gamma, C)$, which are called the *Hessians of the germs* (γ, C), through all the singular and inflection points C of the curve γ:

$$3d(d-2) = \sum_{C \in \gamma} h(\gamma, C). \tag{5.1}$$

An explicit formula for the Hessians $h(\gamma, C)$ was found in [39, formula (2) and theorem 1]. To recall it, let us introduce the following notations. For every local branch b of the curve γ at C let $s(b)$ denote its multiplicity: its intersection index with a generic line through C. Let $s^*(b)$ denote the analogous multiplicity of the dual germ. Note that

$$s(b) = q, \quad s^*(b) = p - q,$$

where p and q are the exponents in the parametrization $t \mapsto (t^q, c_b t^p(1 + o(1)))$ of the local branch b in adapted coordinates. Thus,

$$s(b) = s^*(b) \text{ if and only if } b \text{ is quadratic}, \tag{5.2}$$

$$s(b) \geq s^*(b) \text{ if and only if } b \text{ is subquadratic}. \tag{5.3}$$

Let $b_{C1}, \dots, b_{Cn(C)}$ denote the local branches of the curve γ at C; here $n(C)$ denotes their number. The above-mentioned formula for $h(\gamma, C)$ from [39] has the form

$$h(\gamma, C) = 3\kappa(\gamma, C) + \sum_{j=1}^{n(C)} (s^*(b_{Cj}) - s(b_{Cj})), \tag{5.4}$$

where $\kappa(\gamma, C)$ is the κ-invariant, the class of the singular point. Namely, consider the germ of function f defining the germ (γ, C); $(\gamma, C) = \{f = 0\}$. Fix a line L through C that is transversal to all the local branches of the curve γ at C. Fix a small ball $U = U(C)$ centered at C and consider a

[5]Everything stated in the present subsection holds for every algebraic curve in $\mathbb{CP}^2$ with no multiple components and no straight-line components, see [39, theorem 1].

level curve $\gamma_\varepsilon = \{f = \varepsilon\} \cap U$ with small $\varepsilon \neq 0$, which is non-singular. The number $\kappa(C) = \kappa(\gamma, C)$ is the number of points of the curve γ_ε where its tangent line is parallel to L. (One has $\kappa(C) = 0$ for nonsingular points C.) It is well-known that

$$\kappa(\gamma, C) = 2\delta(\gamma, C) + \sum_{j=1}^{n(C)} (s(b_{Cj}) - 1), \tag{5.5}$$

see, for example, [28, propositions I.3.35 and I.3.38], where $\delta(\gamma, C) = \delta(C)$ is the δ-invariant (whose definition is recalled in the same subsection). Namely, consider the curve γ_ε, which is a Riemann surface whose boundary is a finite collection of closed curves: their number equals to $n(C)$. Let us take the 2-sphere with $n(C)$ deleted disks. Let us paste it to γ_ε: this yields to a compact surface. By definition, its genus is the δ-invariant $\delta(C)$. One has $\delta(C) \geq 0$, and $\delta(C) = 0$ whenever C is a non-singular point. Hironaka's genus formula [30] implies that

$$\sum_{C \in \mathrm{Sing}(\gamma)} \delta(\gamma, C) \leq \frac{(d-1)(d-2)}{2}. \tag{5.6}$$

Formulas (5.1), (5.4) and (5.5) together imply that

$$3d(d-2) = 6 \sum_C \delta(\gamma, C) + 3 \sum_C \sum_{j=1}^{n(C)} (s(b_{Cj}) - 1) + \sum_C \sum_{j=1}^{n(C)} (s^*(b_{Cj}) - s(b_{Cj})).$$

The first term in the latter right-hand side is no greater than $3(d-1)(d-2)$, by inequality (5.6). This implies that

$$3d(d-2) - 3(d-1)(d-2)$$

$$= 3(d-2) \leq 3 \sum_C \sum_{j=1}^{n(C)} (s(b_{Cj}) - 1) + \sum_C \sum_{j=1}^{n(C)} (s^*(b_{Cj}) - s(b_{Cj})). \tag{5.7}$$

5.2 Proof of Theorem 1.24 for a union $\mathbb{I}$ of two lines

Let $\mathbb{I}$ be a union of two distinct lines Λ_1 and Λ_2 through the point O. We know that all the singular and inflection points of the curve γ (if any) lie in $\mathbb{I} = \Lambda_1 \cup \Lambda_2$. Set

$$\mathcal{B}_{tan} = \{\text{the local branches of } \gamma \text{ at points } C \in \mathbb{I} \setminus \{O\} \text{ tangent to } \mathbb{I}\},$$

$$\mathcal{B}_{O,tr} = \{\text{the branches of the curve } \gamma \text{ at } O \text{ transversal to both } \Lambda_1,\ \Lambda_2\},$$

$$\mathcal{B}_{O,tan,j} = \{\text{the branches of the curve } \gamma \text{ at } O \text{ tangent to } \Lambda_j\},$$

$$\mathcal{B}_{O,tan} = \sqcup_{j=1,2}\mathcal{B}_{O,tan,j},\ \mathcal{B}_O = \mathcal{B}_{O,tr} \sqcup \mathcal{B}_{O,tan}.$$

All the local branches $b \notin \mathcal{B}_{O,tan}$ of the curve γ at points in $\gamma \cap \mathbb{I}$ are subquadratic, by the conditions of Theorem 1.24. Therefore, their contributions $s^*(b) - s(b)$ to the right-hand side in (5.7) are non-positive, by (5.3). Every local branch $b \notin (\mathcal{B}_{tan} \cup \mathcal{B}_O)$ is regular, by assumption, hence its contribution $s(b) - 1$ to (5.7) vanishes. This together with (5.7) implies that

$$d - 2 \leq \sum_{b \in \mathcal{B}_{tan} \cup \mathcal{B}_{O,tr} \cup \mathcal{B}_{O,tan}} (s(b) - 1) + \frac{1}{3} \sum_{b \in \mathcal{B}_{O,tan}} (s^*(b) - s(b))$$

$$= \sum_{b \in \mathcal{B}_{tan} \cup \mathcal{B}_{O,tr} \cup \mathcal{B}_{O,tan}} s(b) - |\mathcal{B}_{tan}| - |\mathcal{B}_{O,tr}| - |\mathcal{B}_{O,tan}| + \frac{1}{3} \sum_{b \in \mathcal{B}_{O,tan}} (s^*(b) - s(b)), \tag{5.8}$$

where $|\mathcal{B}_s|$, $s \in \{tan, (O, tr), (O, tan)\}$ are the cardinalities of the sets $\mathcal{B}_s$.

Let us estimate the right-hand side in (5.8) from above. To do this, we use the next equality, which follows from Bézout Theorem.

In what follows for every $j = 1, 2$ by $\mathcal{B}_{reg,j}$ we denote the collection of the local branches of the curve γ at points in $\Lambda_j \setminus \{O\}$ that are transversal to Λ_j. Recall that they are regular, by assumption. Set

$$\nu_j = |\mathcal{B}_{reg,j}|,$$

$$\mathcal{B}_{tan,j} = \{b \in \mathcal{B}_{tan} \mid b \text{ is tangent to } \Lambda_j\},\ \mathcal{B}_{tan} = \mathcal{B}_{tan,1} \sqcup \mathcal{B}_{tan,2}.$$

Claim 1. *For every $j = 1, 2$ one has*

$$\sum_{b \in \mathcal{B}_{tan,j}} s(b) + \frac{1}{2} \sum_{b \in \mathcal{B}_{O,tan,3-j}} s(b) + \frac{1}{2} \sum_{b \in \mathcal{B}_{O,tr}} s(b)$$

$$+ \frac{\nu_j}{2} + \frac{1}{2} \sum_{b \in \mathcal{B}_{O,tan,j}} (s^*(b) + s(b)) = \frac{d}{2}. \tag{5.9}$$

Proof The intersection index of the curve γ with each line Λ_j equals to d (Bézout Theorem). It is the sum of the intersection indices of the line Λ_j with the branches from the collections $\mathcal{B}_{tan,j}$, $\mathcal{B}_{O,tr}$, $\mathcal{B}_{O,tan}$, $\mathcal{B}_{reg,j}$. Let us calculate the latter indices. The contribution of each branch from $\mathcal{B}_{reg,j}$ equals to one, by regularity and transversality. The intersection index of each branch $b \in \mathcal{B}_{O,tr}$ with Λ_j equals to $s(b)$. The intersection index with Λ_j of each branch $b \in \mathcal{B}_{tan,j}$ equals to $p_b = 2s(b)$, by quadraticity (condition of Theorem 1.24). The intersection index with Λ_j of each branch $b \in \mathcal{B}_{O,tan,j}$ equals to $p_b = s(b) + s^*(b)$. The remaining branches $b \in \mathcal{B}_{O,tan,3-j}$ are transversal to Λ_j, and their intersection indices with Λ_j are equal to $s(b)$. Summing up the above intersection indices, writing that their sum should be equal to d and dividing the equality thus obtained by two yields to (5.9).

□

Summing up equalities (5.9) for both $j = 1, 2$ yields to

$$\sum_{b \in \mathcal{B}_{tan} \cup \mathcal{B}_{O,tr} \cup \mathcal{B}_{O,tan}} s(b) = d - \frac{1}{2} \sum_{b \in \mathcal{B}_{O,tan}} s^*(b) - \frac{\nu_1 + \nu_2}{2}. \tag{5.10}$$

Substituting equality (5.10) to (5.8) together with elementary inequalities yields to

$$d - 2 \leq d - \frac{1}{2} \sum_{b \in \mathcal{B}_{O,tan}} s^*(b) - \frac{\nu_1 + \nu_2}{2} - |\mathcal{B}_{tan}| - |\mathcal{B}_{O,tr}| - |\mathcal{B}_{O,tan}|$$

$$+ \frac{1}{3} \sum_{b \in \mathcal{B}_{O,tan}} (s^*(b) - s(b)) = d - |\mathcal{B}_{tan}| - |\mathcal{B}_{O,tr}| - |\mathcal{B}_{O,tan}|$$

$$- \frac{\nu_1 + \nu_2}{2} - \sum_{b \in \mathcal{B}_{O,tan}} (\frac{1}{6} s^*(b) + \frac{1}{3} s(b)),$$

$$|\mathcal{B}_{tan}| + |\mathcal{B}_{O,tr}| + |\mathcal{B}_{O,tan}| + \frac{\nu_1 + \nu_2}{2} + \sum_{b \in \mathcal{B}_{O,tan}} (\frac{1}{6} s^*(b) + \frac{1}{3} s(b)) \leq 2. \tag{5.11}$$

Claim 2. *The total cardinality of the set of singular and inflection points of the curve γ is at most two. There are two possible cases:*

- either there are no inflection points, and each local branch of the curve γ at every singular point is subquadratic;

- or there is just one special point (singular or inflection point), and γ has one local branch at it.

Proof Let Φ denote the collection of all the local branches of the curve γ at points in $\mathbb{I}$. Recall that $\mathbb{I}$ contains all the singular and inflection points of the curve γ.

Case 1): $\mathcal{B}_{O,tan} = \emptyset$. Then all the local branches in Φ are subquadratic, and there are no inflection points; $|\mathcal{B}_{tan}| + |\mathcal{B}_{O,tr}| \leq 2$, by (5.11).

Subcase 1.1): $\mathcal{B}_{tan} = \mathcal{B}_{O,tr} = \emptyset$. Then all the branches in Φ are regular and quadratic, and there are at most four of them: $\nu_1 + \nu_2 \leq 4$, by (5.11).

Thus, the only possible candidates to be singular points of the curve γ are intersections of branches. Since the total number of branches under question is at most four, the number of singular points is at most two.

Subcase 1.2): $|\mathcal{B}_{tan}| + |\mathcal{B}_{O,tr}| = 1$. The branches from the complement $\Phi \setminus (\mathcal{B}_{tan} \cup \mathcal{B}_{O,tr})$ are transversal to the lines Λ_j, quadratic and regular, and there are at most two of them: $\nu_1 + \nu_2 \leq 2$, by (5.11). Thus, Φ consists of at most three branches, and at most one of them is singular. Thus, the only possible candidates to be singular points of the curve γ are the base point of the unique branch from $\mathcal{B}_{tan} \cup \mathcal{B}_{O,tr}$ and a point of intersection of quadratic regular branches (if it is different from the latter base point). Finally, we have at most two singular points.

Subcase 1.3): $|\mathcal{B}_{tan}| + |\mathcal{B}_{O,tr}| = 2$. Then $\Phi = \mathcal{B}_{tan} \cup \mathcal{B}_{O,tr}$, by (5.11), the number of base points of the branches from the collection Φ is at most 2, and they are the only potential singular points.

Case 2): $|\mathcal{B}_{O,tan}| \geq 1$. Then $|\mathcal{B}_{O,tan}| = 1$, and $\Phi = \mathcal{B}_{O,tan}$. This follows from inequality (5.11) and positivity of the sum in $b \in \mathcal{B}_{O,tan}$ in its left-hand side. Thus, the set Φ consists of just one branch, and we have at most one singular (or inflection) point. The claim is proved. □

Theorem 5.1 *[27, theorem 1.6]. Let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve such that there exists a projective line L satisfying the following statements:*

- all the singular and inflection points of the curve γ (if any) lie in L;

- each local branch of the curve γ at every point of the intersection $\gamma \cap L$ that is transversal to L is subquadratic.

Then γ is a conic.

There exists a line L satisfying the conditions of Theorem 5.1 for the curve γ under consideration. Namely, in the first case of Claim 2 the line L is the line passing though (at most two) singular points of the curve γ. In the second case we choose L to be the tangent line to the unique local branch at the unique special point. This together with Theorem 5.1 implies that γ is a conic. Theorem 1.24 is proved.

5.3 Proof of Theorem 1.24: case, when $\mathbb{I}$ is a regular conic

Let $\mathbb{I} \subset \mathbb{CP}^2$ be a regular conic, and let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve, $\gamma \neq \mathbb{I}$, $d = \deg \gamma$, that satisfies the conditions of Theorem 1.24. Let $\mathcal{B}_{tr}$, $\mathcal{B}_{tan}$ denote respectively the set of those local branches of the curve γ

at base points in $\gamma \cap \mathbb{I}$ that are transversal (respectively, tangent) to $\mathbb{I}$. Let $|\mathcal{B}_{tr}|$, $|\mathcal{B}_{tan}|$ denote their cardinalities.

The proof of Theorem 1.24 in the case under consideration is based on the following inequality.

Proposition 5.2 *Let $\mathbb{I}$, γ, d be as above. Let each local branch in $\mathcal{B}_{tan}$ be quadratic, and each branch in $\mathcal{B}_{tr}$ be regular. Then*

$$\frac{1}{2}|\mathcal{B}_{tr}| + \sum_{b \in \mathcal{B}_{tan}} s(b) \leq d. \tag{5.12}$$

Proof The intersection index of the curves γ and $\mathbb{I}$ equals to $2d$ (Bézout Theorem). On the other hand, it equals to the sum of intersection indices of the conic $\mathbb{I}$ with the local branches from the collections $\mathcal{B}_{tr}$ and $\mathcal{B}_{tan}$. Each branch in $\mathcal{B}_{tr}$ has intersection index one with $\mathbb{I}$, since it is regular and transversal to $\mathbb{I}$, by assumptions. Each branch $b \in \mathcal{B}_{tan}$ has intersection index at least $2s(b)$ with $\mathbb{I}$. Indeed, b is quadratic, as is the branch of the conic $\mathbb{I}$ at the same base point. Therefore, applying coordinate change rectifying the germ of the conic $\mathbb{I}$ transforms b to a branch $\widetilde{b}$ with the same local degree $s(\widetilde{b}) = s(b)$ and Puiseux exponent $r \geq 2$. The intersection index of the branch b and the conic $\mathbb{I}$ equals to the intersection index of the branch $\widetilde{b}$ with its tangent line at the base point, that is, $rs(\widetilde{b}) = rs(b) \geq 2s(b)$. Finally, $2d \geq |\mathcal{B}_{tr}| + 2\sum_{b \in \mathcal{B}_{tan}} s(b)$. This proves (5.12). $\square$

Now let us prove Theorem 1.24. Let γ be a curve, as in Theorem 1.24. Recall that all the singular and inflection points of the curve γ (if any) lie in the conic $\mathbb{I}$, and its local branches in $\mathcal{B}_{tan}$ ($\mathcal{B}_{tr}$) are quadratic (respectively, quadratic and regular). Let us calculate their contributions to the right-hand side of inequality (5.7) and substitute inequality (5.12). The second sum in the right-hand side in (5.7) vanishes, by quadraticity. The contribution of each branch $b \in \mathcal{B}_{tr}$ to the first sum also vanishes, since $s(b) = 1$. The total contribution of the branches from the collection $\mathcal{B}_{tan}$ to the first sum equals to $\sum_{b \in \mathcal{B}_{tan}} s(b) - |\mathcal{B}_{tan}|$. This together with (5.7) implies that

$$d - 2 \leq \sum_{b \in \mathcal{B}_{tan}} s(b) - |\mathcal{B}_{tan}|.$$

The latter right-hand side is no greater than $d - \frac{1}{2}|\mathcal{B}_{tr}| - |\mathcal{B}_{tan}|$, by (5.12). Therefore,

$$\frac{1}{2}|\mathcal{B}_{tr}| + |\mathcal{B}_{tan}| \leq 2. \tag{5.13}$$

Let us show that this together with Theorem 5.1 implies that γ is a conic.

Inequality (5.13) implies that the following three cases are possible.

Case 1): $|\mathcal{B}_{tr}| \leq 4$, $\mathcal{B}_{tan} = \emptyset$. Thus, all the local branches of the curve γ at its intersection points with $\mathbb{I}$ lie in $\mathcal{B}_{tr}$, and hence, they are quadratic and regular. A point of intersection $\gamma \cap \mathbb{I}$ can be singular only in the case, when it is a point of intersection of some two of (at most 4) branches in $\mathcal{B}_{tr}$. Hence, γ has at most two singular points (thus, all of them lie in a line), and all the local branches of the curve γ at them are quadratic. This together with Theorem 5.1 implies that γ is a conic.

Case 2): $|\mathcal{B}_{tan}| = 1$, $|\mathcal{B}_{tr}| \leq 2$. Let C denote the base point of the unique branch in $\mathcal{B}_{tan}$. Each point of intersection $\gamma \cap \mathbb{I}$ distinct from the point C lies in the union of (at most two) branches in $\mathcal{B}_{tr}$. It is singular, if and only if it is the intersection point of two latter branches. Thus, γ has at most two singular points, its local branches at them are quadratic, and hence, γ is a conic, by Theorem 5.1, as in the above case.

Case 3): $|\mathcal{B}_{tan}| = 2$, $\mathcal{B}_{tr} = \emptyset$. Then γ has at most two singular points, and all its branches at them, which lie in $\mathcal{B}_{tan}$, are quadratic. Hence, γ is a conic, as in Case 1). Theorem 1.24 is proved.

6 Proof of main theorems

6.1 Rationally integrable $\mathbb{I}$-angular billiards. Proof of Theorem 1.23

Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic (regular or a pair of distinct lines), and let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve generating a rationally integrable $\mathbb{I}$-angular billiard.

Theorem 6.1 *([10, theorem 1], [11, theorem 1.2]). All the singular and inflection points (if any) of the curve γ lie in $\mathbb{I}$.*

Remark 6.2 The above-cited theorems from [10, 11] are stated for a polynomially integrable billiard Ω. Namely, for every smooth arc $\alpha \subset \partial\Omega$ the statement of Theorem 6.1 is proved there for each non-linear irreducible component γ of Zariski closure of the Σ-dual curve α^*. But the proofs given in [10, 11] remain valid in the general context of Theorem 6.1.

Each local branch of the curve γ that satisfies the conditions of some of the statements (i), (ii-a), or (ii-b) of Theorem 4.1 also satisfies the corresponding statement, by Theorem 4.1. Therefore, γ satisfies the conditions of Theorem 1.24, by Theorem 6.1. Hence, it is a conic, by Theorem 1.24. This proves Theorem 1.23.

6.2 Confocal billiards. Proof of Theorem 1.19

Let $\Omega \subset \Sigma$ be a polynomially integrable billiard with countably piecewise C^2-smooth boundary that contains a smooth non-linear arc α. Let $\Psi(M)$ be its non-trivial homogeneous polynomial integral of even degree $2n$: $M = [r, v]$, and $\Psi([r, v])$ is not a function of the squared norm $||v||^2 =< Av, v >$ in the metric of the surface Σ. One has $\Psi(M) \not\equiv c < AM, M >^n$, since $< AM, M >=< Av, v >$, by Proposition 2.1. Let G be the corresponding rational function (1.6): $G \not\equiv const$. The complex Zariski closure of the Σ-dual curve α^* is an algebraic curve that contains at least one nonlinear irreducible component. Each its non-linear irreducible component generates a rationally integrable $\mathbb{I}$-angular billiard with integral G, by Corollary 2.11. Hence, it is a conic, by Theorem 1.23. Therefore, α contains a nonlinear conical arc. This together with Theorem 1.21 implies that the billiard Ω is countably confocal and proves Theorem 1.19.

6.3 Case of smooth connected boundary. Proof of Theorem 1.6

Let $\Omega \subset \Sigma$ be a polynomially integrable billiard with C^2-smooth non-linear connected boundary. Then the billiard Ω is countably confocal, by Theorem 1.19. This means that its boundary $\partial\Omega$ contains an open dense subset R that is a disjoint union of open arcs of confocal conics and geodesic segments, including at least one non-linear conical arc. Let us fix the latter arc and denote it by c, and let $\mathcal{C} \supset c$ denote the ambient conic. Let us show that $\partial\Omega = \mathcal{C}$. We consider that c is a maximal arc of the conic $\mathcal{C}$ that is contained in the C^2-smooth one-dimensional submanifold $\partial\Omega \subset \Sigma$. Suppose the contrary: c has an endpoint Q. The point Q cannot be an accumulation point of the union of geodesic segments in $\partial\Omega$, by C^2-smoothness and since $\partial\Omega$ has non-zero geodesic curvature at Q, as does $\mathcal{C}$: it has quadratic tangency at Q with the geodesic tangent to $T_Q\partial\Omega$. Therefore, the point Q has a neighborhood U in Σ such that the intersections $I_U = \partial\Omega \cap U$, $c_U = \mathcal{C} \cap U$ are connected, ∂U is transversal to $\partial\Omega$, and $R \cap U \subset I_U$ consists of arcs of conics confocal to $\mathcal{C}$. Their ambient conics intersect U by leaves of an analytic foliation having c_U as a leaf, since each confocal conic pencil is locally given by a pair of orthogonal foliations and all the conics under question are C^1-close to $\mathcal{C}$. Thus, the C^2-smooth connected submanifold I_U contains an open and dense subset $R \cap U$ where it is tangent to the above foliation. Therefore, it is the graph of a C^1-smooth function on I_U with values in a transversal section to the foliation whose derivative vanishes on an open and dense set. Hence, the latter function is constant and thus, I_U lies in a leaf. Therefore, $I_U = c_U$, since I_U contains an arc of the leaf c_U. Thus, $\partial\Omega$ contains the neighborhood c_U of the point Q in $\mathcal{C}$. This contradicts maximality of the conical arc $c \subset \partial\Omega$ and proves Theorem 1.6.

6.4 Proof of complexification: Theorem 1.34

The fact that each polynomially integrable complex billiard admits a homogeneous polynomial integral of the form $\Psi(M)$ is proved by a straightforward complexification of Bolotin's proof of the same statement in the real case in [16, 17]. This implies that the curves Γ_t are algebraic, as in loc. cit., and their Σ-dual curves generate rationally integrable $\mathbb{I}$-angular billiards with a common rational integral, as in the proofs of [10, theorem 3], [11, theorem 1.3] and Theorem 2.8. Afterwards Theorem 1.34 is deduced from Theorem 1.23 in the same way, as in Subsection 6.2, by a straightforward complexification of Theorem 1.21 and its proof.

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