On algebraically integrable Birkhoff and angular billiards

Alexey Glutsyuk

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Abstract

We present a solution of the algebraic version of Birkhoff Conjecture on integrable billiards. Namely we show that every polynomially integrable real bounded convex planar billiard with smooth boundary is an ellipse. We also extend this result to the case of piecewise-smooth and not necessarily convex polynomially integrable billiards: we show that the boundary is a union of confocal conical arcs and straight-line segments lying in some special lines defined by the foci. We also present a complexification of these results. The proof, which is obtained by Mikhail Bialy, Andrey Mironov and the author, is split into two parts. The first part is the paper by Bialy and Mironov, where they prove the following theorems: 1) the polar duality transforms a polynomially integrable planar billiard to a rationally integrable angular billiard; 2) the singularities and inflection points of each irreducible component of the complexified curve polar-dual to the billiard boundary lie in the two complex isotropic lines through the origin; 3) the *Hessian Formula:* appropriately defined Hessian of the integral of the angular billiard being restricted to the curve polar-dual to the boundary is a constant multiple of a power $(x^2+y^2)^s$. The present paper provides the second part of the proof. Namely, we prove that each irreducible component of the polar-dual curve that is not a line is a conic. This together with a theorem of S.V.Bolotin implies the main results: solution of the Algebraic Birkhoff Conjecture in both convex smooth and non-convex piecewise smooth cases.

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Contents

1 Introduction

1.1 Main results

The famous Birkhoff Conjecture deals with convex bounded planar billiards with smooth boundary. Recall that a *caustic* of a planar billiard $\Omega \subset \mathbb{R}^2$ is a curve C such that each tangent line to C reflects from the boundary of the billiard to a line tangent to C. A billiard Ω is called *Birkhoff integrable,* if a neighborhood of its boundary in Ω is foliated by closed caustics. It is well-known that each elliptic billiard is integrable, see [26, section 4]. The **Birkhoff Conjecture** states the converse: *the only integrable convex bounded planar billiard with smooth boundary is an ellipse.*[1]

Note that the above-defined integrability of a billiard is equivalent to the Liouville integrability of the corresponding Hamiltonian flow. Namely, consider a billiard[2] $\Omega \subset \mathbb{R}^2$. The *billiard flow* B_t on the tangent bundle $T\mathbb{R}^2|_{\Omega}$ is defined as follows. A point $(Q, P) \in T\mathbb{R}^2|_{\Omega}$, $Q = (x, y) \in \Omega$, $P = (P_1, P_2) \in T_Q\mathbb{R}^2$ moves with constant speed P, $B_t(Q, P) = (Q + tP, P)$ until its trajectory hits the boundary $\partial\Omega$. Then the trajectory reflects from the boundary according to the usual reflection law: the angle of incidence equals the angle of reflection. Then the point moves in the reflected direction with speed having the same absolute value $|P|$ until the trajectory hits the boundary again. Then the trajectory is reflected etc. The billiard flow thus defined, which can be viewed as a geodesic flow with impacts, has obvious first integral: the absolute value $|P|$ of the speed. A billiard Ω is called *integrable in the Liouville sense,* if its flow has an additional first integral independent with $|P|$ on the intersection with $T\mathbb{R}^2|_{\Omega}$ of a neighborhood of the unit tangent bundle to the boundary. It is well-known that the Birkhoff and Liouville integrabilities of a convex planar billiard with smooth boundary are equivalent.

The particular case of the Birkhoff Conjecture, when the additional first integral is supposed to be polynomial in the speed components, motivated the next definition and conjecture.

Definition 1.1 Let $\Omega \subset \mathbb{R}^2$ be a planar billiard with smooth connected boundary. We say that Ω is *polynomially integrable,* if the restriction of the billiard flow to a neighborhood in $T\mathbb{R}^2|_{\overline{\Omega}}$ of the unit tangent bundle of the boundary has a first integral that is polynomial in the speed P and its restriction to the unit energy level $\{|P| = 1\}$ is non-constant. We say that Ω is *analytically integrable,* if there exists an $\varepsilon > 0$ such that there exists a first integral in a neighborhood in $T\mathbb{R}^2|_{\overline{\Omega}}$ of the zero section of the tangent bundle to the boundary that is analytic in P for $|P| < \varepsilon$ and that is not a function of $|P|$.

The **Algebraic Birkhoff Conjecture** states that *if a bounded convex*

[1]This conjecture, classically attributed to G.Birkhoff, was published in print only in the paper [23] by H. Poritsky, who worked with Birkhoff as a post-doctoral fellow in late 1920-ths.

[2]Everywhere in the paper a billiard is a **connected domain** $\Omega \subset \mathbb{R}^2$.

billiard with smooth boundary is polynomially integrable, then its boundary is an ellipse.

Remark 1.2 The Algebraic Birkhoff Conjecture and its extension to billiards with piecewise-smooth and even non-convex boundaries are important and interesting themselves, *independently on a potential solution of the classical Birkhoff Conjecture.* They lie on the crossing of different domains of mathematics, first of all, dynamical systems, algebraic geometry and singularity theory. *They are not implied by the classical Birkhoff Conjecture.* For the general case of piecewise smooth boundaries this is obvious. Even in the case of smooth convex boundary, while the algebraicity condition is a very strong restriction, the condition of just non-constance of a polynomial integral on the unit energy level hypersurface is topologically weaker than the independence condition in the Liouville integrability, which requires independence of the additional integral and the energy on a whole neighborhood in $T\mathbb{R}^2|_\Omega$ of the unit tangent bundle to the boundary. It is known that analytic integrability implies polynomial integrability (the converse is obvious), and the polynomial integral non-constant on the level hypersurface $\{|P| = 1\}$ can be chosen homogeneous in P. Indeed, each homogeneous part of the Taylor series in P of an analytic (polynomial) integral is a first integral itself, and one can choose a homogeneous part that is non-constant on the latter hypersurface, if the initial (non-homogeneous) integral is not a function of $|P|$. It is known that for every polynomially integrable billiard its boundary lies in an algebraic curve and the integral can be chosen to be a homogeneous polynomial in three variables: $\sigma = xP_2 - yP_1$, P_1, P_2, see [9] and also [21, chapter 5, section 3, proposition 5]. This statement is local and holds for reflection from an arbitrary smooth curve. In particular, the latter integral is well-defined on the whole bundle $T\mathbb{R}^2$, and it is an integral for the flow defined in the whole billiard domain.

Example 1.3 The billiard in a disk has linear first integral σ. The billiard in an ellipse (and in any conic) has a quadratic integral: an integral that is a homogeneous quadratic polynomial in the speed components. It can be written as a homogeneous quadratic polynomial in (σ, P_1, P_2), as in the above remark.

In the present paper we prove the Algebraic Birkhoff Conjecture and its generalization for billiards with piecewise smooth boundary that may be non-convex.

Theorem 1.4 *Let a convex planar billiard with C^1-smooth boundary be polynomially integrable. Then the billiard boundary is a conic. (Thus, if the billiard is bounded, then it is an ellipse.)*

Definition 1.5 A domain $\Omega \subset \mathbb{R}^2$ has *countably piecewise smooth boundary,* if $\partial\Omega$ consists of the two following parts:

- the *regular part:* an open and dense subset $\partial\Omega_{reg} \subset \partial\Omega$, where each point $X \in \partial\Omega_{reg}$ has a neighborhood $U = U(X) \subset \mathbb{R}^2$ such that the intersection $U \cap \partial\Omega$ is a C^1-smooth one-dimensional submanifold in U;

- the *singular part:* the closed subset $\partial\Omega_{sing} = \partial\Omega \setminus \partial\Omega_{reg} \subset \partial\Omega$.

Remark 1.6 In the above definition the regular part of the boundary is always a dense and at most countable disjoint union of C^1-smooth arcs (taken without endpoints). The particular case of domains with piecewise smooth boundaries corresponds to the case, when the above union is finite, the arcs are smooth up to their endpoints and the singular part of the boundary is a finite set (which may be empty). For a general planar billiard with countably piecewise smooth boundary the billiard flow is well-defined on a residual set for all time values. In the case, when the singular part of the boundary has zero one-dimensional Hausdorff measure, the billiard flow is well-defined as a flow of measurable transformations.

Definition 1.7 A billiard Ω with countably piecewise smooth boundary is *polynomially integrable,* if its flow has a first integral on $T\mathbb{R}^2|_{\overline{\Omega}}$ (a function constant on well-defined billiard orbits) that is polynomial in the speed P and whose restriction to the level hypersurface $\{|P| = 1\}$ is non-constant. It is said to be *analytically integrable,* if there exists an $\varepsilon > 0$ such that its flow has a first integral on a neighborhood of the zero section of the bundle $T\mathbb{R}^2|_\Omega$ that is analytic in P for $|P| < \varepsilon$ and that is not a function of $|P|$.

Remark 1.8 Analytic integrability of a billiard with countably piecewise smooth boundary implies polynomial integrability with an integral being a homogeneous polynomial in σ, P_1, P_2, as in Remark 1.2. Thus, in the particular case, when the boundary $\partial\Omega$ is connected and smooth, Definitions 1.1 and 1.7 are equivalent.

Definition 1.9 A billiard with countably piecewise smooth boundary is called *countably confocal,* if the regular part of its boundary consists of arcs of confocal conics and may be some straight-line segments such that

- at least one conical arc is present;

- in the case, when the common foci of the conics are distinct and finite (i.e., the conics are ellipses and (or) hyperbolas), the ambient line of each straight-line segment of the boundary is either the line through the foci, or the middle orthogonal line to the segment connecting the foci, see Fig. 1a);

- in the case, when the conics are concentric circles, the above ambient lines may be any lines through their common center, see Fig. 1b);

- in the case, when the conics are confocal parabolas, the ambient line of each straight-line segment of the boundary is either the common axis of the parabolas, or the line through the focus that is orthogonal to the axis, see Fig. 1 c), d).

Proposition 1.10 *[10, proposition 1 in section 2; the theorem in section 4] Each countably confocal planar billiard is polynomially integrable: it has a non-trivial first integral that is either linear, or quadratic, or a degree 4 polynomial in the speed components (it is a homogeneous polynomial of the same degree in three variables σ, P_1, P_2) that is non-constant on the unit speed hypersurface. In the case of a degree 4 integral that cannot be reduced to a degree 2 integral the billiard boundary contains an arc of parabola and a straight-line segment lying in a line through the focus that is orthogonal to the axis of the parabola, see Fig. 1d).*

Remark 1.11 Proposition 1.10 was proved in loc.cit. for countably confocal billiards with piecewise smooth boundaries, together with its generalization to higher-dimensional spaces of constant curvature. The proof remains valid for general countably confocal billiards, with boundaries having infinitely many smooth pieces. The above-mentioned case of degree 4 integral was first discovered in [24].

Theorem 1.12 *Let a planar billiard with countably piecewise C^1-smooth boundary be polynomially integrable, and let the regular part of its boundary have at least one non-linear arc. Then the billiard is countably confocal.*

Corollary 1.13 *Let a planar billiard with C^1-smooth connected boundary be analytically integrable. Then the billiard boundary is a conic (ellipse, if bounded). If a billiard with countably piecewise C^1-smooth boundary is analytically integrable, then it is countably confocal.*

The corollary follows from Theorems 1.4, 1.12 and Remarks 1.2, 1.8. Theorem 1.4 follows from Theorem 1.12, Remark 1.8 and the fact that every countably confocal billiard with smooth boundary is bounded by a conic.

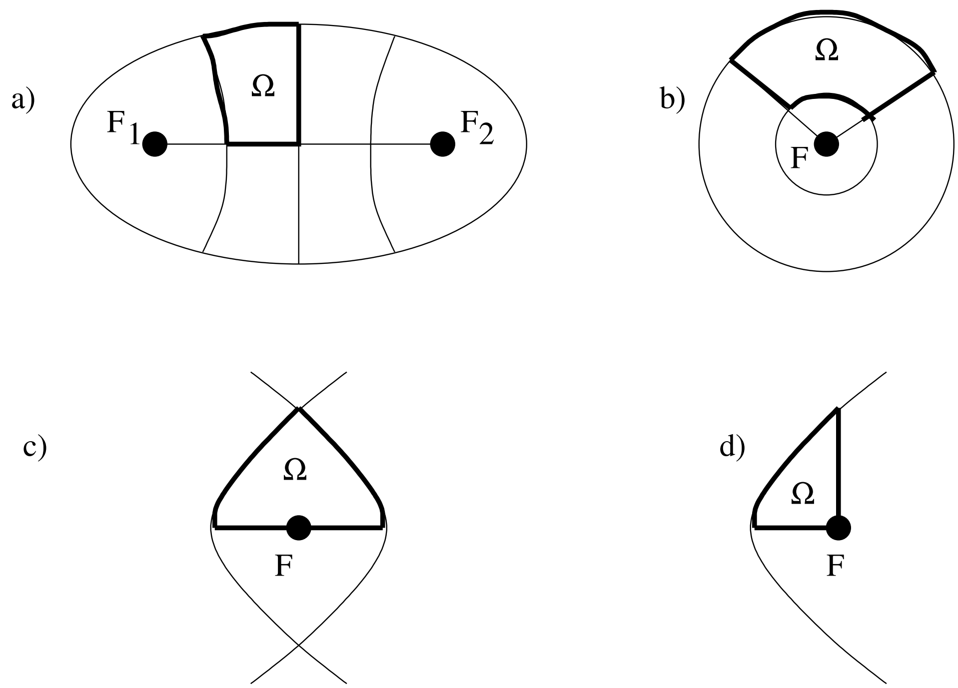

Figure 1: Examples of countably confocal billiards; F_1, F_2, F are the foci. All of them have quadratic integrals, except for the billiard depicted at Fig. 1d), which has a degree 4 integral.

The latter fact follows from the statement saying that two tangent confocal conics coincide: any two intersecting distinct confocal conics are orthogonal.

Theorems 1.4, 1.12 are joint results of M.Bialy, A.E.Mironov and the author. Their proof consists of the two following parts.

The first part, due to Bialy and Mironov, is written in [6]. They have introduced a new type of billiard, the *angular billiard.* They have reduced the Algebraic Birkhoff Conjecture for billiards with smooth convex boundaries to its analogue for the angular billiards via polar duality. They have studied the dual to the minimal complex algebraic curve containing the billiard boundary. For the proof of the Algebraic Birkhoff Conjecture it suffices to show that each irreducible component of the dual curve is a conic. Bialy and Mironov have proved that all the singularities and inflection points of every irreducible component γ of the dual curve lie in the union[3] $\mathbb{I} = \{x^2 + y^2 = 0\} = \Lambda_1 \cup \Lambda_2$ of two special complex projective lines Λ_1 and Λ_2, the isotropic lines through the origin O. They have considered the polynomial f defining γ and proved a remarkable formula expressing its Hessian evaluated at its skew gradient in terms of the polynomial integral and a power $(x^2 + y^2)^l$, see [6, theorem 6.1] and formula (1.8) below.

The second part of the proof is given in the present paper. We prove that each above irreducible component γ is a conic (Theorem 1.31 in Subsection 1.3). This together with S.V.Bolotin's theorem [10, section 4] implies that the billiard under question is countably confocal and proves Theorems 1.4 and 1.12: the implication will be proved at the end of Subsection 1.3. For the proof of Theorem 1.31 we study the asymptotics of the local branches of the curve γ (i.e., irreducible components of its germs) at points of the intersection $\gamma \cap \mathbb{I}$. Each local branch b is a germ of bijectively parametrized curve $t \mapsto (t^q, ct^p(1 + o(1)))$, $1 \leq q < p$, $c \neq 0$, in local affine coordinates (z, w) centered at its base point so that the z-axis is tangent to b: the latter coordinates will be called *adapted.* Its *projective Puiseux exponent* is the ratio $r = r_b = \frac{p}{q}$. We prove the following theorems:

(i) each local branch of the curve γ at O that is tangent to no line Λ_j is quadratic: $r = 2$;

(ii) each its local branch at a point in $\Lambda_j \setminus \{O\}$ that is tangent to Λ_j is quadratic: $r = 2$;

(iii) each its local branch at a point in $\Lambda_j \setminus \{O\}$ that is transverse to Λ_j is regular and quadratic: $q = 1$, $p = 2$.

Afterwards we prove the following general purely algebro-geometric theorem.

Theorem 1.14 *Let $\gamma \subset \mathbb{CP}^2$ be an irreducible complex algebraic curve. Let $\Lambda_1, \Lambda_2 \subset \mathbb{CP}^2$ be two different projective lines, and let O be their intersection point. Let all the singularities and inflection points (if any) of the curve γ be contained in the union $\Lambda_1 \cup \Lambda_2$. Let each local branch of the curve γ at O that is tangent to no line Λ_j be subquadratic: $r \leq 2$. Let all its local branches of types (ii) and (iii) be respectively quadratic (quadratic and regular), see (ii) and (iii). Then γ is a conic.*

Statements (i)–(iii) together with Theorem 1.14 imply that γ is a conic.

The proof of Theorem 1.14 is based on arguments due to E.I.Shustin on curve singularity invariants and generalized Plücker formula from [15, section 3].

[3]Recently Bialy and Mironov have proved an analogous result for billiards on sphere and hyperbolic plane in [8]

The polynomial integrability of the initial Birkhoff billiard implies that the corresponding angular billiard has a rational first integral G expressed in terms of appropriate polynomial integral of the Birkhoff billiard, see [6, theorem 3]. By analytic extension argument, the latter statement is equivalent to the statement that for every $P \in \gamma$ the restriction $G|_{T_P\gamma}$ is invariant under a special conformal involution of the tangent line $T_P\gamma$ fixing the point P: the *angular symmetry*. Note that it is known that $G = const$ on γ (see [6, theorem 3] and Theorem 1.22 in Subsection 1.3 below), and we can normalize G to be identically equal to zero along the curve γ. The angular symmetry permutes the points of intersection of the line $T_P\gamma$ with the zero locus of the integral G, which contains the curve γ.

Statements (i) and (ii) are proved via asymptotic analysis of the latter intersection points and their symmetry. Their proof is based on general asymptotic formulas dealing with an irreducible germ of analytic curve b and another irreducible germ a at the same base point. These are asymptotic formulas for the coordinates of points of the intersection $T_t b \cap a$, as t tends to the base point, see [15, proposition 2.1] and [13, proposition 2.50, p.268].

The main technical part of the paper is the proof of Statement (iii) (Theorem 1.49), which is based on the above-mentioned Bialy–Mironov Hessian formula and asymptotic analysis of local branches and symmetry.

Bialy–Mironov angular billiard construction and their reduction of the Algebraic Birkhoff Conjecture to the analogous Theorem 1.31 on angular billiards are presented in Subsection 1.3. Their Hessian formula is presented in Subsection 1.6: formula (1.8). The plans of the proofs of Theorem 1.31 and statement (iii) (Theorem 1.49) will be presented in Subsections 1.5 and 1.6 respectively. A historical survey will be given in the next subsection.

In Subsection 1.4 we present a complexified version of Theorem 1.12.

1.2 Historical remarks

The Birkhoff Conjecture was studied by many mathematicians. In 1950 H.Poritsky [23] proved it under the additional assumption that the billiard in each closed caustic near the boundary has the same closed caustics, as the initial billiard. Later in 1988 another proof of the same result was obtained by E.Amiran [3]. In 1993 M.Bialy [5] proved the Birkhoff Conjecture under the assumption that the foliation by caustics extends to the whole billiard domain punctured at one point: he proved that then the billiard boundary is a circle. In 2013 D.V.Treschev [28] made a numerical experience indicating that there should exist analytic *locally integrable* billiards, with the billiard reflection map having a two-periodic point where the germ of its second iterate is analytically conjugated to a rigid rotation. Recently Treschev studied the billiards from [28] in more detail in [29] and their multi-dimensional versions in [30]. Recently V.Kaloshin and A.Sorrentino have proved a *local version* of the Birkhoff Conjecture [19]: *an integrable deformation of an ellipse is an ellipse*. (The case of ellipses with small extentricities was treated in the previous paper by A.Avila, J. De Simoi and V.Kaloshin [2].)

In 1988 A.P.Veselov proved that every billiard bounded by confocal quadrics in any dimension has a complete systems of first integrals in involution that are quadratic in P [31, proposition 4]. In 1990 he studied a billiard in an ellipsoid in the sphere and in the Lobachevsky (i.e., hyperbolic) space of any dimension n. He proved its complete integrability and provided an explicit complete list of first integrals [32, the corollary on p. 95]. In the same paper he proved that all the sides of a billiard trajectory are tangent to the same $n-1$ quadrics confocal to the boundary of the ellipsoid and the billiard dynamics corresponds to a shift of the Jacobi variety corresponding to an appropriate hyperelliptic curve [32, theorems 3, 2 on p. 99]. The Algebraic Birkhoff Conjecture was studied by S.V.Bolotin, who proved in 1990 that in its conditions the billiard boundary lies in an algebraic curve [9]. In the same paper he proved the conjecture under the assumption that the complexification of each irreducible component of the ambient algebraic curve is nonsingular. In 1992 he proved integrability of countably confocal billiards with piecewise smooth boundaries in two- and higher-dimensional spaces of constant curvature with integrals of degrees two or four in [10]. Recently M.Bialy and A.E.Mironov proved the Algebraic Birkhoff Conjecture in the case of integrals of degree four [7]. A version of the conjecture for *non-constant continuous families* of billiards sharing the same polynomial integral was proved in [1]. Dynamics in countably confocal billiards with piecewise smooth boundaries in two and higher dimensions was studied in [12]. For further results on the Algebraic Birkhoff Conjecture see the above-mentioned paper [6] by M.Bialy and A.E.Mironov and references therein.

The analogue of the Birkhoff Conjecture for outer billiards was stated by S.L.Tabachnikov [27] in 2008. Its algebraic version was stated by Tabachnikov and proved by himself under genericity assumptions in the same paper and recently solved completely in the joint paper of the author with E.I.Shustin [15].

1.3 Polar duality and angular billiards

Let us recall the following definitions of polar duality and angular billiard.

Definition 1.15 Let $C \subset \mathbb{CP}^2$ be a regular conic (i.e., not a pair of lines), $L \subset \mathbb{CP}^2$ be a line. Let P_1, P_2 denote the points of intersection $L \cap C$. The *point* L^* *polar-dual* to L with respect to the conic C is the intersection point of the tangent lines $T_{P_j}C$, $j = 1, 2$. (If L is tangent to C, then L^* is the tangency point.) For a holomorphic curve $\gamma \subset \mathbb{CP}^2$ its *polar-dual curve* with respect to the conic C is the set of points polar-dual to the tangent lines to γ. For every smooth curve $\gamma \subset \mathbb{R}^2$ its *polar-dual curve* is the set of points in $\mathbb{CP}^2 \supset \mathbb{C}^2 \supset \mathbb{R}^2$ polar-dual to the complexified tangent lines to γ.

Remark 1.16 The polar duality induces a complex projective isomorphism $\mathbb{CP}^{2*} \to \mathbb{CP}^2$ (thus, preserving the incidence relation). In the case, when the conic C is the complexification of a real conic, the polar duality induces a real projective isomorphism $\mathbb{RP}^{2*} \to \mathbb{RP}^2$, and the curve polar-dual to a real curve is also real. In what follows C will be a circle. Everywhere below O denotes the centre of the circle C. Then O is polar-dual to the infinity line. Thus, a curve γ is tangent to the infinity line, if and only if its polar-dual passes through O.

Definition 1.17 Let $O \in \mathbb{R}^2 \subset \mathbb{RP}^2$, $A \in \mathbb{RP}^2$ be two distinct points. Two points $B, C \in \mathbb{RP}^2 \setminus OA$ are called *angular-symmetric* with respect to the point A and center O, if the points B, C, A lie on the same line and the lines OB, OC are symmetric with respect to the line OA. The transformation permuting angular-symmetric points is called the *angular symmetry* with respect to the point A and center O.

Remark 1.18 The angular symmetry and its complexification are projective involutions $\mathbb{RP}^2 \to \mathbb{RP}^2$ (respectively, $\mathbb{CP}^2 \to \mathbb{CP}^2$) that fix each line through A.

Proposition 1.19 *[6, subsection 1.3] Let $O \in \mathbb{R}^2$, $\widetilde{\gamma} \subset \mathbb{R}^2$ be a smooth curve, and let γ be its polar-dual with respect to a circle centered at O. Let $Q \in \widetilde{\gamma}$, and let l_1 and l_2 be two lines through the point Q that are symmetric with respect to the tangent line $l = T_Q\widetilde{\gamma}$. Let l_1^*, l_2^* and l^* be the polar-dual points corresponding to the lines l_1, l_2 and l respectively. The points l_1^* and l_2^* are angular-symmetric with respect to the point l^* and center O.*

Remark 1.20 As it was shown in [6, theorem 2], if Ω is convex, then the polar-dual curve $(\partial\Omega)^*$ is also convex and the Birkhoff billiard map acting on the space of oriented lines intersecting Ω is conjugated on appropriate open subset by polar duality to the angular billiard map associated to $(\partial\Omega)^*$, see the next definition and Figure 2. The latter open subset of conjugacy contains a neighborhood of the family of oriented tangent lines to $\partial\Omega$ in the above space of oriented lines.

Definition 1.21 Let $D \subset \mathbb{R}^2 \subset \mathbb{RP}^2$ be a convex domain, $\gamma = \partial D$, $O \in D$, $U = \mathbb{R}^2 \setminus \overline{D}$. Let $A \in U$. There are two tangent lines through A to γ. We choose the right one (if one looks at γ from A). Let P denote the tangency point. Let B denote the point angular-symmetric to A with respect to the point P and center O. Then B lies in the projective line $AP = T_P\gamma$, and $B \in \mathbb{RP}^2 \setminus \overline{D}$, by convexity. The *angular billiard map* associated to the curve γ and the center O is the map $U \to \mathbb{RP}^2 \setminus \overline{D}$ that sends A to B, see Fig. 2.

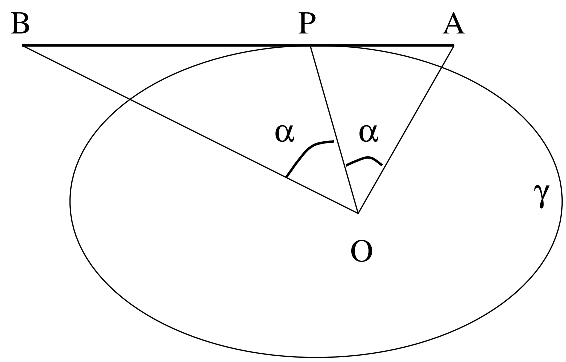

Figure 2: The angular billiard map.

Let now $\Omega \subset \mathbb{R}^2$ be a domain with countably piecewise smooth boundary such that the billiard in Ω is polynomially integrable. Let $Q = (x, y)$ be Euclidean coordinates on the ambient plane. Then there exists a first integral of the billiard flow that is a homogeneous polynomial $\Phi(\sigma, P_1, P_2)$ of even degree in three variables $\sigma(P) = xP_2 - yP_1$, P_1, P_2, whose restriction to the level hypersurface $\{(P, Q) \mid |P| = 1\}$ is non-constant, see [9, 21] and Remarks 1.2, 1.8. (If the homogeneous polynomial integral in σ, P_1, P_2 from the end of Remark 1.2 is of odd degree, we can replace it by its square and get an integral of even degree.) The next theorem translates polynomial integrability of the initial Birkhoff billiard in terms of rational integrability of the corresponding (multivalued) angular billiard map.

Theorem 1.22 *Let $\Omega \subset \mathbb{R}^2$ be a domain with countably piecewise smooth boundary. Let the billiard flow in Ω have a first integral that is a homoge-*

neous polynomial $\Phi(\sigma, P_1, P_2)$ of even degree $deg\Phi = 2s$. Set

$$G(x,y) = \frac{F(x,y)}{(x^2+y^2)^s}, \quad F(x,y) = \Phi(1,-y,x). \tag{1.1}$$

Let $\widetilde{\gamma}$ be a non-linear C^1-smooth arc in $\partial\Omega$, and let γ be its polar-dual. For every $P \in \gamma$ the restriction to the real (and hence, complex) line $T_P\gamma$ of the function G is invariant under the angular symmetry with respect to the point P and center O. The function G is constant along the curve γ; hence $G = const$ along its complex Zariski closure $\Gamma \subset \mathbb{CP}^2$; Γ is an algebraic curve, if $G \not\equiv const$ on $\mathbb{R}^2$.

Proof The statement of the theorem on invariance of the function G under the angular symmetry was stated and proved in [6, theorem 3] for convex domains with smooth boundary. Its proof is purely local and remains valid in the general case: a straightforward calculation presented in loc. cit. shows that if a billiard orbit reflects from a smooth arc $\widetilde{\gamma} \subset \partial\Omega$ at a point $\widetilde{P}$, set $P = (T_{\widetilde{P}}\partial\Omega)^*$, then invariance of the integral Φ under the reflection is translated as invariance of the restriction $G|_{T_P\gamma}$ under the angular symmetry with respect to the point P and center O. For every $P \in \gamma$ the derivative of the function G along a vector tangent to γ at P equals zero: the restriction $G|_{T_P\gamma}$ being invariant under the angular symmetry, it has zero derivative at its fixed point P, similarly to vanishing of derivative at 0 of an even function. Therefore, $G \equiv const$ along the curve γ. The theorem is proved. □

We prove the main theorems by complex methods. To do this, we state a theorem saying that if a complex angular billiard constructed on an irreducible algebraic curve γ has a rational first integral of type (1.1), then γ is a conic. To do this, let us recall the following properties of the complexified Euclidean metric and extend the notion of (rationally integrable) angular billiard to the complex domain.

In what follows we consider the plane $\mathbb{C}^2 \supset \mathbb{R}^2$ equipped with the complexified Euclidean $\mathbb{C}$-bilinear quadratic form $dx^2 + dy^2$. Consider the complex projective plane $\mathbb{CP}^2 \supset \mathbb{C}^2$ equipped with homogeneous coordinates $(z_0 : x : y)$; $\mathbb{C}^2$ being the affine chart $\{z_0 = 1\}$ with coordinates (x, y).

Definition 1.23 ([13, p. 240, definition 1.2]). A complex projective line $L \subset \mathbb{CP}^2$ is *isotropic*, if either the restriction to it of the latter form vanishes, or it coincides with the infinity line. Or equivalently, if it passes through some of the two points with homogeneous coordinates $(0 : 1 : \pm i)$, which lie in the infinity line and are called the *isotropic points at infinity* (also known as *cyclic* (or *circular*) points).

Definition 1.24 ([13, p. 248, definition 2.1]). The *symmetry* with respect to a non-isotropic complex line $L \subset \mathbb{CP}^2$ is the unique non-trivial involution $\mathbb{C}^2 \to \mathbb{C}^2$ that is an isometry of the complex Euclidean form that fixes the points of the line L. It extends to a projective transformation $\mathbb{CP}^2 \to \mathbb{CP}^2$.

Remark 1.25 The symmetry with respect to a real line is the usual symmetry. The symmetry with respect to a non-isotropic line permutes the isotropic directions: the image of an isotropic line through the given isotropic point at infinity passes through the other isotropic point at infinity [14, p. 296, proposition 1].

Definition 1.26 Let $O \in \mathbb{C}^2$, $A \in \mathbb{CP}^2 \setminus \{O\}$ be such that the line OA is not isotropic. The *(complex) angular symmetry* with respect to the point A and center O is the transformation $\sigma : \mathbb{CP}^2 \setminus OA \to \mathbb{CP}^2 \setminus OA$ defined by the following condition: for every $B \in \mathbb{CP}^2 \setminus OA$ the point $\sigma(B)$ lies in the line AB and the lines OB, $O\sigma(B)$ are symmetric with respect to the line OA, see Fig. 3.

Remark 1.27 The angular symmetry is a projective involution $\mathbb{CP}^2 \to \mathbb{CP}^2$ preserving each line L through A. Its restriction to L will be also called the *angular symmetry of the line* L with respect to the point A and center O. The angular symmetry is completely determined by the points O, A and the isotropic lines through the point O. Namely, the angular symmetry of a line L through A is the unique conformal involution $L \to L$ fixing A and permuting the intersection points of the line L with the isotropic lines through O, see Fig. 3. This follows from definition and the above remark. In the case, when the line L and the points O, A are real, the complex angular symmetry is the complexification of the real angular symmetry of the real line.

Definition 1.28 Let $O \in \mathbb{C}^2 \subset \mathbb{CP}^2$, and let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve different from a line. We say that γ generates a *rationally integrable angular billiard,* if there exist an $s \in \mathbb{N}$ and a polynomial $F_1(x, y)$ of degree $degF_1 \leq 2s$, set

$$G(x, y) = \frac{F_1(x, y)}{(x^2 + y^2)^s}, \tag{1.2}$$

such that $G \not\equiv const$, and for every point $P \in \gamma$ lying outside the isotropic lines through the point O the angular symmetry with respect to the point P and center O leaves invariant the restriction $G|_{T_P\gamma}$. The rational function G is called a *first integral* of the angular billiard on γ.

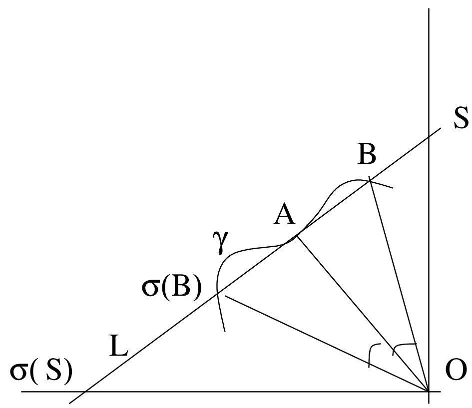

Figure 3: The angular symmetry. The lines OS and $O\sigma(S)$ are isotropic.

Proposition 1.29 *Let an irreducible algebraic curve $\gamma \subset \mathbb{CP}^2$ generate a rationally integrable angular billiard with the first integral G. Then $G|_\gamma \equiv const$.*

Proof For every $P \in \gamma$ as above the derivative of the restriction $G|_{T_P\gamma}$ at the point P equals zero, by invariance of the restriction under the angular symmetry, hence $G|_\gamma \equiv const$, as in the proof of Theorem 1.22. □

Theorem 1.22 implies the following corollary.

Corollary 1.30 *Let the billiard flow in a planar billiard Ω with countably piecewise smooth boundary be polynomially integrable. Let $\widetilde{\alpha}$ be a non-linear C^1-smooth arc in $\partial\Omega$, let α be its polar-dual. Let Γ denote the algebraic curve in $\mathbb{CP}^2$ that is the complex Zariski closure of the curve α. Then each non-linear irreducible component of the curve Γ generates a rationally integrable angular billiard.*

Proof Let $\Phi(\sigma, P_1, P_2)$ be a homogeneous polynomial integral of the billiard flow of even degree $2s$ that is not a function of $|P|$. Let G be the corresponding rational function in (1.1). Let γ be a nonlinear irreducible component of the curve Γ. For every point $B \in \alpha$ the angular symmetry with respect to the point B and center O leaves invariant the restriction to $T_B\alpha$ of the function G (Theorem 1.22). Therefore, the same statement remains valid for the complex angular symmetry of the tangent line $T_B\gamma$ for every point $B \in \gamma$, by uniqueness of analytic extension. One has $G \not\equiv const$ on $\mathbb{R}^2$. Indeed, if, to the contrary, $F(x,y) = \Phi(1, -y, x) \equiv c(x^2+y^2)^s$, then the homogeneous polynomial $\Phi(\sigma, P_1, P_2)$ of degree $2s$ would be identically equal to $c|P|^{2s}$, while it is not a function of $|P|$ by assumption. This means that γ generates a rationally integrable angular billiard, by definition. The corollary is proved. $\Box$

One of the main results of the present paper is the following theorem

Theorem 1.31 *Let an irreducible complex algebraic curve $\gamma \subset \mathbb{CP}^2$ different from a line generate a rationally integrable angular billiard. Then it is a conic.*

Theorem 1.32 *[10, section 4] Let a planar billiard with countably piecewise smooth boundary be polynomially integrable. Let its boundary contain a non-linear conical arc. Then the billiard is countably confocal.*

Remark 1.33 A theorem implying Theorem 1.32 was stated and proved in loc. cit. for piecewise smooth boundary, but its proof does not use finiteness of pieces and remains valid in the countably piecewise smooth case. To make the paper self-contained, we give a proof of Theorem 1.32. The proof follows [10, section 4], but here it is done in terms of angular billiards.

Proof of Theorem 1.32. Let Ω be the billiard under consideration, and let α be a conic whose arc is contained in $\partial\Omega$. Let γ be the conic polar-dual to α. Let $\Phi(\sigma, P_1, P_2)$ be a non-trivial polynomial integral of the billiard Ω of even degree $2s$, and let

$$G(x, y) = \frac{F(x, y)}{(x^2 + y^2)^s}, \ F(x, y) = \Phi(1, -y, x), \ degF \leq 2s,$$

be the corresponding rational integral of angular billiard. The complexified conic γ generates a rationally integrable angular billiard with the integral G, by Corollary 1.30. On the other hand, it is known that the billiard on a conic α admits a non-trivial quadratic first integral $\widetilde{\Phi} = \widetilde{\Phi}(\sigma, P_1, P_2)$ (Proposition 1.10). Moreover, for every conic β confocal to α the quadratic integral $\widetilde{\Phi}$ is also an integral for the billiard on the conic β. This is well-known and follows from the explicit formula [10, formula (12)] for the quadratic integral. Therefore, both dual conics $\gamma = \alpha^*$ and β^* generate rationally integrable angular billiards with a common quadratic rational integral $\widetilde{G}(x, y) = \frac{\widetilde{F}(x,y)}{x^2+y^2}$, here $\widetilde{F}(x, y) = \widetilde{\Phi}(1, -y, x)$, see Corollary 1.30 and its proof. Both G and $\widetilde{G}$ are constant on γ, and $\widetilde{G}$ is constant on β^*, by Proposition 1.29. Let us normalize the integral $\widetilde{G}$ by additive constant (or equivalently, the integral

$\widetilde{\Phi}$ by addition of $c||P||^2$, $c = const$) so that $\widetilde{G}|_\gamma \equiv 0$. After this normalization one has $\widetilde{F}|_\gamma \equiv 0$: that is, the quadratic polynomial $\widetilde{F}$ is the defining polynomial of the conic γ. Therefore,

$$G(x,y) = c_1 + G_1(x,y)\widetilde{G}(x,y),$$

$$G_1(x,y) = \frac{f_1(x,y)}{(x^2+y^2)^{s-1}},\ deg f_1 \leq 2s-2,\ c_1 = G|_\gamma \equiv const.$$

Hence, the fraction G_1 is also a rational integral of the angular billiard generated by γ, as are G and $\widetilde{G}$. Thus, $G_1|_\gamma \equiv const$, by Proposition 1.29. Similarly we get that

$$G_1(x,y) = c_2 + G_2(x,y)\widetilde{G}(x,y),\ G_2(x,y) = \frac{f_2(x,y)}{(x^2+y^2)^{s-2}},\ deg f_2 \leq 2s-4,$$

and G_2 is an integral of the angular billiard generated by γ, as are G_1 and $\widetilde{G}$. Continuing this prodecure we get that G is a polynomial in $\widetilde{G}$. Hence, $G \equiv const$ on the level curves of the function $\widetilde{G}$. The level curves of the quadratic rational function $\widetilde{G}$ are conics whose dual are confocal to $\alpha = \gamma^*$, since $\widetilde{G}$ is constant on these conics, as was mentioned above. Let $\phi \subset \Omega$ be an arbitrary smooth non-linear arc of the boundary. The function G is constant on its polar-dual curve ϕ^*, by Theorem 1.22. Hence, $\widetilde{G}|_{\phi^*} \equiv const$, thus ϕ lies in a conic confocal to α. Finally, the non-linear smooth part of the boundary lies in an at most countable union of conics confocal to α. Let $\mathcal{C}$ denote the whole family of conics confocal to α.

Now it remains to show that the linear part of the boundary $\partial\Omega$ consists of segments lying in some of the lines from the definition of countably confocal billiard. Note that the latter lines are exactly those lines L for which the symmetry with respect to the line L leaves the family $\mathcal{C}$ invariant. This follows from definition. Let us fix an arbitrary line L containing a segment of the boundary $\partial\Omega$, and let us show that it satisfies the latter statement. Its polar-dual is a point L^* such that the function G is invariant under the angular symmetry with respect to the point L^* and center O: invariance of the integral Φ under the billiard flow and reflection from the line L is translated as invariance of the corresponding function G under the angular symmetry, as in the proof of [6, theorem 3]. This implies that the latter angular symmetry permutes the level curves of the function G, and hence, those of the quadratic function $\widetilde{G}$. In other terms, it permutes the conics whose dual lie in the family $\mathcal{C}$. Or equivalently, the symmetry with respect to the line L permutes the conics from the family $\mathcal{C}$. Theorem 1.32 is proved. $\square$

Proof of Theorems 1.4 and 1.12 modulo Theorem 1.31. Consider a polynomially integrable planar billiard with countably piecewise smooth boundary that has at least one non-linear smooth piece α. Then the polar-dual to some smaller arc $\beta \subset \alpha$ lies in a non-linear irreducible complex algebraic curve γ generating a rationally integrable angular billiard, by Corollary 1.30. The curve γ is a conic, by Theorem 1.31. Hence, β lies in the conic polar-dual to γ. This together with Theorem 1.32 implies that the billiard under question is countably confocal and proves Theorems 1.4 and 1.12. $\square$

1.4 Complexification

Consider the plane $\mathbb{C}^2$ with coordinates $Q = (Q_1, Q_2)$ equipped with the complex quadratic Euclidean form $dQ_1^2 + dQ_2^2$. Let (Q, P), $P = (P_1, P_2)$ be the standard coordinates on the tangent bundle $T\mathbb{C}^2$.

Definition 1.34 A *complex billiard* is a collection (finite or infinite, countable or uncountable) of holomorphic curves $\Gamma_\lambda \subset \mathbb{C}^2$ distinct from isotropic lines. A complex billiard is said to be *polynomially integrable,* if there exists a function $\Phi(Q, P)$ on $T\mathbb{C}^2$ polynomial in P with the following properties:

- the restriction of the function Φ to the tangent bundle of every complex line is invariant under the translations of the line;

- for every point Q of each curve Γ_λ such that the line $T_Q\Gamma_\lambda$ is non-isotropic the restriction of the function Φ to $T_Q\mathbb{C}^2$ is invariant under the symmetry with respect to the complex line $T_Q\Gamma_\lambda$.

Example 1.35 Consider a polynomially integrable real planar billiard with countably piecewise smooth boundary. Then the smooth part of the boundary is contained in a union of arcs of conics and straight-line segments (Theorem 1.12). Their complexifications form a complex billiard having a polynomial integral that is the complexification of the real polynomial integral of the real billiard: it can be chosen of degree no greater than four, see Proposition 1.10.

Definition 1.36 [13, p. 260, definition 2.31]. A *focus* of a complex conic is an intersection point of some its two distinct isotropic tangent lines.

Example 1.37 A conic has four finite foci, if it is transverse to the infinity line $\overline{\mathbb{C}}_\infty$ in $\mathbb{CP}^2 \supset \mathbb{C}^2$ and contains no isotropic point at infinity; two or one finite foci, if it is transverse to the infinity line and passes through one (respectively, two) isotropic points at infinity; one finite focus, if it is tangent to $\overline{\mathbb{C}}_\infty$ at a non-isotropic point; no finite foci, if it is tangent to $\overline{\mathbb{C}}_\infty$ an an isotropic point. A complexified real ellipse has four finite complex foci: two usual real foci lying in its bigger axis and two additional complex foci lying in its complexified smaller axis. This observation goes back to Laguerre, see [20, p. 179] and [4, Section 17.4.3.1, p. 334]

Definition 1.38 [13, p. 259, definition 2.24]. A pair of regular complex conics is *confocal,* if it lies in the Zariski closure of the set of confocal pairs of real conics.

Remark 1.39 Two confocal conics are either both transverse to the infinity line $\overline{\mathbb{C}}_\infty$, or tangent to it at a common point. In the second case the tangency contact between the two conics is of order two, if the tangency point is non-isotropic, and of order three if it is isotropic. Any two confocal conics have common isotropic tangent lines, and hence, common foci, and they contain the same isotropic points at infinity (if any). The converse is true in the case, when at least one of the conics under question is transverse to the infinity line. See [13, p. 261, lemma 2.35].

Definition 1.40 A complex billiard Γ_λ is said to be *confocal,* if the set of its curves different from lines is non-empty, all of them are confocal complex conics, and the lines from the family Γ_λ belong to the following lists of admissible lines:

Case 1): the conics are transverse to the infinity line and contain no isotropic point at infinity. Any non-isotropic line through a pair of their common distinct finite foci is admissible.

Case 2): the conics are transverse to the infinity line and pass through exactly one isotropic point at infinity. The set of admissible lines is empty.

Case 3): the conics are transverse to the infinity line and pass through both isotropic points at infinity and thus, have one finite focus B. Any non-isotropic line through B is admissible.

Case 4): the conics are tangent to $\overline{\mathbb{C}}_\infty$ at a common non-isotropic point A and hence, have one common finite focus B. The line AB (called the *axis*) and its orthogonal line through B are both admissible.

Case 5): the conics are tangent to $\overline{\mathbb{C}}_\infty$ at a common isotropic point. The set of admissible lines is empty.

Theorem 1.41 *Every polynomially integrable complex billiard Γ_λ containing at least one curve that is not a line is confocal and has an integral that is a homogeneous polynomial $\Phi(\sigma, P_1, P_2)$, $\sigma = Q_1P_2 - Q_2P_1$, of degree at* *most four. The integral can be chosen quadratic in all the above cases 1)–3), except for the subcase of case 2), when Γ_λ contains the line orthogonal to the axis through the common focus of its conics: in this subcase there is an integral of degree four.*

The fact that each polynomially integrable complex billiard admits a homogeneous polynomial integral of the form $\Phi(\sigma, P_1, P_2)$ is proved by a straightforward complexification of Bolotin's proof of the same real statement in [9]. This implies that the polar-dual to the curves Γ_λ generate angular billiards with a common rational integral, and hence, are algebraic curves, as in the proofs of [6, theorem 3] and Corollary 1.30. Afterwards Theorem 1.41 is deduced from Theorem 1.31 in the same way, as at the end of the previous subsection, by using a straightforward complexification of Theorem 1.32.

1.5 Rationally integrable complex angular billiards. Plan of the proof of Theorem 1.31

We will work in complex Euclidean coordinates (x, y) on $\mathbb{C}^2$ centered at the point O (see the paragraph preceding Definition 1.23). The proof of Theorem 1.31 is based on the following theorem of M.Bialy and A.E.Mironov.

Theorem 1.42 *[6, theorem 1]. Let an irreducible complex algebraic curve $\gamma \subset \mathbb{CP}^2$ generate a rationally integrable angular billiard. Then all the singular and inflection points of the curve γ (if any) lie in its intersection with the isotropic lines through the origin O; the union $\mathbb{I}$ of the latter lines is given by the equation $\mathbb{I} = \{x^2 + y^2 = 0\}$.*

Remark 1.43 Theorem 1 in [6] was stated and proved for a curve γ associated to a polynomially integrable planar billiard: namely an irreducible component of the complex Zariski closure of the dual to a smooth arc of its boundary. (It generates a rationally integrable angular billiard, by Corollary 1.30.) But its proof given in [6, section 6] remains valid for a general irreducible algebraic curve γ generating a rationally integrable angular billiard.

To sketch the proof of Theorem 1.31, let us recall the following definitions.

Definition 1.44 A *local branch* of a germ α of analytic curve at a point $A \in \mathbb{CP}^2$ is an irreducible component of the germ, or equivalently, a germ of analytic curve contained in α and parametrized holomorphically bijectively by some local complex parameter.

Definition 1.45 Consider an irreducible nonlinear germ b of analytic curve in $\mathbb{CP}^2$ at a given point A. Let us choose affine coordinates (z, w) centered at A so that the tangent line $T_A b$ be the z-axis. We will call these coordinates *adapted* to the germ b.

In adapted coordinates one can find a local bijective parametrization of the germ b by a complex parameter $t \in (\mathbb{C}, 0)$ of the type

$$t \mapsto (t^q, c_b t^p(1 + o(1))), \quad q = q_b, \ p = p_b \in \mathbb{N}, \ 1 \leq q < p, \ c_b \neq 0; \tag{1.3}$$

$$q = 1 \text{ if and only if } b \text{ is a regular germ.}$$

Definition 1.46 The *projective Puiseux exponent* [13, p. 250, definition 2.9] of the germ b is the ratio

$$r = r_b = \frac{p}{q}.$$

The germ b is called *quadratic*, if $r_b = 2$, and is called *subquadratic*, if $r_b \leq 2$.

In what follows we will denote the infinity line in $\mathbb{CP}^2$ by $\overline{\mathbb{C}}_\infty$. Let Λ_1 and Λ_2 denote the isotropic lines through the point O. Set

$$\mathbb{I} = \Lambda_1 \cup \Lambda_2.$$

For the proof of Theorem 1.31 we investigate the Puiseux exponents and regularity of the local branches of the curve γ at its points in $\mathbb{I}$. We prove the three following theorems on the local branches; in all these theorems $O \in \mathbb{C}^2$, and $\gamma \subset \mathbb{CP}^2$ is an irreducible algebraic curve that is not a line.

Theorem 1.47 *Let γ generate a rationally integrable angular billiard. Let Λ be an isotropic line through O. Then every local branch of the curve γ tangent to Λ at a point distinct from the point O is quadratic.*

Theorem 1.48 *Let γ generate a rationally integrable angular billiard. Then each its local branch at O (if any) that is transverse to both isotropic lines through O is quadratic.*

Theorem 1.49 *Let γ generate a rationally integrable angular billiard. Let Λ be an isotropic line through O, $A \in \gamma \cap \Lambda$, and let $A \neq O$. Let b be an arbitrary local branch at A of the curve γ that is transverse to Λ. Then b is quadratic and regular.*

The next purely algebro-geometric theorem (reformulating Theorem 1.14 from the introduction) shows that the information about the local branches given by the three above theorems implies that γ is a conic.

Theorem 1.50 *Let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve. Let Λ_1, Λ_2 be two different complex projective lines, and let O be their intersection point. Let all the singularities and inflection points (if any) of the curve γ be contained in the union $\Lambda_1 \cup \Lambda_2$. Let the local branches of the curve γ at points in $\Lambda_1 \cup \Lambda_2$ satisfy the following statements:*

(i) each local branch of the curve γ at O that is tangent to no line Λ_j is subquadratic: $r \leq 2$;

(ii) each its local branch at a point in $\Lambda_j \setminus \{O\}$ that is tangent to Λ_j is quadratic: $r = 2$;

(iii) each its local branch at a point in $\Lambda_j \setminus \{O\}$ that is transverse to Λ_j is regular and quadratic: $q = 1$, $p = 2$.

Then γ is a conic.

Proof of Theorem 1.31 modulo Theorems 1.47–1.50. Let an irreducible algebraic curve $\gamma \subset \mathbb{CP}^2$ generate a rationally integrable angular billiard. Let Λ_1 and Λ_2 denote the isotropic lines through O, set $\mathbb{I} = \Lambda_1 \cup \Lambda_2$. All the singularities and inflection points of the curve γ (if any) are contained in $\mathbb{I}$, by Theorem 1.42. Each local branch of the curve γ at every point $A \in \Lambda_j \setminus \{O\}$ that is transverse to Λ_j is quadratic and regular, by Theorem 1.49, that is, statement (iii) holds. Statements (i) and (ii) follow from Theorems 1.48 and 1.47 respectively. Finally, γ satisfies all the conditions of Theorem 1.50. Hence, it is a conic. Theorem 1.31 is proved. □

Theorems 1.47–1.49 are proved in Section 3. Theorem 1.50 is proved in Section 4. The plan of the proofs of Theorems 1.47 and 1.48 is presented below. The proof of Theorem 1.49 is the main technical part of the paper. Its plan is presented in the next subsection.

The proofs of Theorems 1.47–1.49 consist of the two following ingredients.

A) Asymptotic analysis of the relative angular symmetry property (see the next definition), which is a direct consequence of rational integrability of the angular billiard;

B) Bialy–Mironov formula [6, theorem 6.1] (see formula (1.8) below) for the Hessian of the polynomial defining the curve γ and its asymptotic analysis (used only in the proof of Theorem 1.49).

Definition 1.51 Let $O \in \mathbb{C}^2 \subset \mathbb{CP}^2$. Let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve different from a line. Consider a divisor $\Delta = \sum_{j=1}^{l} k_j \gamma_j$, where $\gamma_j \subset \mathbb{CP}^2$ are distinct irreducible algebraic curves, $k_j \in \mathbb{N}$, and $\gamma_j = \gamma$ for some j: thus Δ contains γ. We say that the curve γ has *relative angular symmetry property* with respect to the divisor Δ, if for every $P \in \gamma \setminus \mathbb{I}$ the intersection $\Delta \cap T_P\gamma$, which is a divisor in the line $T_P\gamma$, is invariant under the angular symmetry of the line $T_P\gamma$ with respect to the point P and center O. Similarly, an irreducible germ of analytic curve b at a point A has *local relative angular symmetry property* with respect to a finite collection Γ of irreducible germs of analytic curves at points in $T_A b$ (or a finite divisor Γ in a neighborhood of the line $T_A b$), if Γ contains b and for every $P \in b$ the intersection $T_P b \cap \Gamma$ (respectively, the divisor $T_P b \cap \Gamma$ in the line $T_P b$) is invariant under the above angular symmetry of the line $T_P b$.

Remark 1.52 Let an irreducible algebraic curve γ generate a rationally integrable angular billiard with the rational first integral $G = \frac{F_1(x,y)}{(x^2+y^2)^s}$, see (1.2). Then $G|_\gamma \equiv c = const$. Without loss of generality we can and will consider that $G|_\gamma \equiv 0$. One can achieve this by replacing its denominator $F_1(x,y)$ by $F_1 - c(x^2+y^2)^s$. The curve γ has relative angular symmetry property with respect to the zero divisor of the function G. In what follows, whenever the contrary is not specified, Γ will denote the zero locus[4] $\{G = 0\} \subset \mathbb{CP}^2$, and Δ will denote the zero divisor of the function G.

Plan of proofs of Theorems 1.47 and 1.48. Let us briefly sketch the proof of Theorem 1.47. Fix a local branch b of the curve γ that is tangent to an isotropic line Λ through O at a point $A \neq O$. We have to prove that b is quadratic. To do this, we study the asymptotics of the intersection points $\Gamma \cap T_P b$, as $P \to A$, using general asymptotic formulas for intersections of a tangent line to an irreducible germ b with another irreducible germ a of analytic curve at the same base point A, see [15, proposition 2.1] and [13, proposition 2.50, p.268]. We introduce local affine coordinates (z,w) adapted to b and consider the so-called *intersection points with moderate w-asymptotics:* those points of the intersection $T_P b \cap \Gamma$, whose w-coordinates are asymptotic to $w(P)$ multiplied by non-zero constant factors, as $P \to A$. The latter factors will be referred to, as *the asymptotic (w-) factors*. We show that the set of the points with moderate w-asymptotics is invariant under the angular symmetry of the tangent line $T_P b$ with respect to the point P and center O. This follows from the angular symmetry property and an

[4]Here and in what follows every curve given by an algebraic equation $F(x,y) = 0$ is treated as a *projective* algebraic curve: its closure in $\mathbb{CP}^2$.

elementary asymptotics for the degenerating angular symmetry, as $P \to A$ (Proposition 3.6). The same proposition implies that the angular symmetry inverses the asymptotic factors of the above intersection points, and hence, the asymptotic factor collection is symmetric under the involution $z \mapsto z^{-1}$. We show that the points with moderate w-asymptotics are exactly the points of intersection of the line $T_P b$ with those local branches of the curve Γ at A that either are transverse to b, or are tangent to b and have Puiseux exponents no greater than r_b. The asymptotic factors corresponding to the intersection points with the tangent branches having the same Puiseux exponent r_b are appropriate powers of the roots of an explicit collection of polynomials of one variable; each polynomial is associated to a tangent branch. The other asymptotic factors are equal to $1 - r_b$. The symmetry of the asymptotic factor collection implies a relation on the roots of the above collection of polynomials, which, in its turn, implies that $r_b = 2$, thus, b is quadratic.

The proof of Theorem 1.48 is analogous to the proof of [15, theorem 1.16], see subsection 2.3 in loc. cit.; both proofs follow arguments similar to the above ones.

1.6 Plan of the proof of Theorem 1.49

Let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve generating a rationally integrable angular billiard. Let $G(x, y) = \frac{F_1(x,y)}{(x^2+y^2)^s}$ be the corresponding rational first integral from (1.2). Set $\Gamma = \{G = 0\}$. Recall that we normalize the function G so that $\gamma \subset \Gamma$, see Remark 1.52. Let Δ denote the zero divisor of the function G. Its denominator vanishes exactly on the lines Λ_j with the same multiplicity s. Without loss of generality we consider that the function G is an irreducible fraction, that is, $F_1|_{\Lambda_j} \not\equiv 0$ for every $j = 1, 2$. Indeed, if $F_1|_{\Lambda_j} \equiv 0$ for some j, then F_1 has zero of the same multiplicity along both lines Λ_1, Λ_2. This follows from the same statement for the function G, which holds by invariance of the intersections $\Delta \cap T_P\gamma$ under the angular symmetries of the lines $T_P\gamma$ (Remark 1.52) and the fact that the angular symmetry permutes the points of intersection of the line $T_P\gamma$ with the lines Λ_j (Remark 1.27). Therefore, if $F_1|_{\Lambda_j} = 0$ for some j, then $F_1(x, y) = h(x, y)(x^2 + y^2)^l$, $l \leq s$, where h is a polynomial, $h|_{\Lambda_j} \not\equiv 0$ for every $j = 1, 2$. Thus, one can cancel $(x^2 + y^2)^l$ in the expression for the function G and obtain an irreducible fraction, with a nominator that does not vanish identically on each line Λ_j. One has $l < s$: otherwise, if $l = s$, then $G \equiv const \neq 0$, which is impossible, since $G|_\gamma \equiv 0$.

Recall that $degF_1 \leq 2s$. Without loss of generality we consider that

$degF_1 = 2s$, or equivalently, the curve Γ does not contain the infinity line, i.e., $\Gamma = \{F_1 = 0\}$, and moreover, Γ contains no isotropic point at infinity: no point of the intersection $\mathbb{I} \cap \overline{\mathbb{C}}_\infty$. One can achieve this by applying a projective transformation Ψ fixing the point O and the isotropic lines Λ_j through it that moves the isotropic points at infinity away from the infinity line. This will not change the rational integrability property: the image $\Psi(\gamma)$ will again generate an integrable angular billiard with the rational first integral $G \circ \Psi^{-1}$. The latter integral has pole of the same degree s, as G, at each line Λ_j, and hence, has the same type, as G. Finally, in our assumptions made without loss of generality one has

$$F_1|_{\Lambda_j} \not\equiv 0 \text{ for every } j = 1, 2, \quad degF_1 = 2s, \quad \Gamma = \{F_1 = 0\}, \tag{1.4}$$

$$\Delta \text{ is the zero divisor of the polynomial } F_1,$$

the intersection $\Gamma \cap \mathbb{I}$, *and hence its subset* $\gamma \cap \mathbb{I}$ *and the base point* A *of the local branch* $b \subset \gamma$ *under consideration lie in the finite affine chart* $\mathbb{C}^2$.

Let $f(x, y)$ be the polynomial defining γ, which is irreducible, as is γ: $\gamma = \{f = 0\}$, the differential df being non-zero on a Zariski open subset in γ. Recall that the polynomial F_1 vanishes on γ. Therefore,

$$F_1 = f^k g_1, \ k \in \mathbb{N}, \ g_1 \text{ is a polynomial coprime with } f. \tag{1.5}$$

Set

$$g = g_1^{\frac{1}{k}}, \ F = F_1^{\frac{1}{k}} = fg, \ m = \frac{s}{k}, \ D = degF_1 = 2s. \tag{1.6}$$

We consider the Hessian of the function $f(x, y)$ evaluated on appropriately normalized tangent vector to $\gamma = \{f = 0\}$ at the point (x, y), namely, the skew gradient $(f_y, -f_x)$:

$$H(f) = f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2. \tag{1.7}$$

Theorem 1.53 *[6, theorem 6.1] The following formula holds for all* $(x, y) \in \gamma$*:*

$$g^3(x, y) H(f)(x, y) = c(x^2 + y^2)^{3m-3}, \ c \equiv const \neq 0. \tag{1.8}$$

Remark 1.54 Theorem 6.1 in [6] deals with a polynomially integrable planar billiard Ω, a curve $\Gamma_1 \subset \mathbb{R}^2$ that is polar-dual to a non-linear smooth arc in $\partial\Omega$ and the polynomial F_1 that is expressed via a homogeneous polynomial integral Φ by formula (1.1). The theorem states that formula (1.8) holds along the curve Γ_1. Then it holds automatically on every irreducible component γ of its complex Zariski closure. Its proof given in [6] remains valid for every irreducible algebraic curve γ generating a rationally integrable angular billiard.

The idea of the proof of Theorem 1.49 is to study the asymptotics of the left- and right-hand sides of formula (1.8), as $(x, y) \to A$ along the local branch b under consideration. We prove it by contradiction. Supposing the contrary, i.e., that the branch b is either not regular, or not quadratic, we show that the right-hand side should asymptotically dominate the left-hand side. The contradiction thus obtained to formula (1.8) will prove Theorem 1.49.

We consider new affine coordinates (z, w) on $\mathbb{C}^2$ adapted to b where b is bijectively parametrized by small complex parameter t via the formula $t \mapsto (t^{q_b}, c_b t^{p_b}(1 + o(1)),\ q_b < p_b,\ c_b \neq 0$. We express the restrictions to the branch b of the left- and right-hand sides in formula (1.8) as Puiseux series in the coordinate $z = t^{q_b}$ lifted to b:

$$g^3(x, y)H(f)(x, y)|_b = O(z^\eta),\ (x^2+y^2)^{3m-3}|_b = c_2 z^\mu + o(z^\mu),\ c_2 \neq 0. \quad (1.9)$$

Our goal is to show that the first asymptotics holds for some $\eta > \mu$, unless the branch b is quadratic and regular.

Step 1. Preparatory asymptotics of Bialy–Mironov formula (1.8).

Let $b_1, b_2, \dots, b_N$ denote the local branches at A of the curve $\Gamma = \{F_1 = 0\} \supset \gamma$,

$$b_1 = b,\ J = \{j \mid b_j \subset \gamma\}; \quad 1 \in J.$$

The germ Δ_A of the divisor Δ at A is a linear combination of the curves b_j:

$$\Delta_A = k \sum_{j \in J} b_j + \sum_{j \notin J} k_j b_j, \quad k_j \in \mathbb{N}. \quad (1.10)$$

Let q_{b_j}, p_{b_j} denote the corresponding exponents from formula (1.3) of the parametrized curves b_j in their adapted coordinates, and let r_{b_j} denote their Puiseux exponents. (For those curves b_j that are tangent to b the corresponding q_{b_j}, p_{b_j} and r_{b_j} can be defined in the same adapted coordinates (z, w), as for the curve b. For the other, transverse branches, the corresponding adapted coordinates defining r_{b_j} will be different.) Set

$$\rho_j = \begin{cases} r_{b_j}, \text{ if } b_j \text{ is tangent to } b \\ 1, \text{ if } b_j \text{ is not tangent to } b \end{cases}, \quad r = r_{b_1} = r_b = \rho_1. \quad (1.11)$$

We show in Proposition 2.1 by direct calculation that

$$g^3 H(f)|_b = O(z^\eta),$$

$$\eta = 3\Big(\sum_{j \in J} q_{b_j} \min\{\rho_j, r\} + \sum_{j \notin J} \frac{k_j}{k} q_{b_j} \min\{\rho_j, r\}\Big) - 2(r + 1), \quad (1.12)$$

$$(x^2+y^2)^{3m-3}|_b \simeq cz^{\mu}, \quad \mu = 3\frac{D}{2k}-3 = 3m-3, \quad c = const \neq 0. \quad (1.13)$$

In the next steps we estimate the above number η from below.

Step 2. We prove that the local branch b is subquadratic: $r = r_b \leq 2$ (Theorem 3.2). This is deduced from the local relative angular symmetry property of the branch b in a similar way, as in the proof of Theorem 1.47 described at the end of the previous subsection.

Step 3. We prove that $r_{b_j} \leq r$ for each local branch b_j of the zero locus Γ of the polynomial F_1 that is tangent to b (Theorem 3.3). This implies that in formula (1.12) one can replace the min-signs by ρ_j. We prove this inequality by contradiction. Suppose the contrary: there exists a local branch b_j tangent to b with $r_{b_j} > r$. The intersection $b_j \cap T_P b$ contains a point Q_1 with $z(Q_1) \simeq \theta_1 z(P)$, $\theta_1 = \frac{r-1}{r}$, as $P \to A$, which follows from [13, proposition 250, p. 268]. Then the point $\hat{Q}_1 \in T_P b$ angular-symmetric to Q_1 with respect to the point P and center O should lie in Γ (by relative angular symmetry property) and $z(\hat{Q}_1) \simeq \mu_1 z(P)$, $\mu_1 = \theta_1^{-1} = \frac{r}{r-1} > 1$, by general Proposition 3.6 implying that a point $Q = Q(P) \in T_P b$ with $z(Q) \simeq \theta z(P)$, $\theta \neq 0$, is angular-symmetric to a point $\hat{Q} = \hat{Q}(P) \in T_P b$ with $z(\hat{Q}) \simeq \theta^{-1} z(P)$. It follows from the inclusion $\hat{Q}_1 \in \Gamma$ and loc. cit. that $\hat{Q}_1$ lies in a local branch, say b_2 of the curve Γ that is tangent to b and has the same Puiseux exponent r. For every irreducible germ a of analytic curve tangent to the curve b and having the same Puiseux exponent r the points of intersection $T_P b \cap a$ have z-coordinates asymptotic to $t_j z(P)$, as $P \to A$, where $t_j \neq 0$ are the q-th powers of roots of appropriate polynomial of the type $R_{p,q,c}(\zeta) = c\zeta^p - r\zeta^q + r - 1$, $r = \frac{p}{q}$, $c \neq 0$, associated to the germ a. This follows from [13, proposition 250, p. 268]. The key Proposition 3.20 on general polynomials $R_{p,q,c}$ states that if $R_{p,q,c}$ has a root $\mu > 1$, then it has another root θ, $0 < \theta < 1$, such that $\theta\mu > 1$. It implies that b_2 intersects the line $T_P b$ at another point Q_2 with z-coordinate asymptotic to $\theta_2 z(P)$, $0 < \theta_2 < 1$, such that $\theta_2 \mu_1 > 1$. Similarly, the point $\hat{Q}_2$ angular-symmetric to Q_2 should lie in a local branch, say b_3 tangent to b with $r_{b_3} = r$ and $z(\hat{Q}_2) \simeq \mu_2 z(P)$, $\mu_2 = \theta_2^{-1} < \mu_1$ etc. We get that $\mu_1 > \mu_2 > \cdots > 1$, and thus, we have an infinite sequence of asymptotic factors μ_j of the points of intersection $\Gamma \cap T_P b$. Hence, Γ contains an infinite number of local branches b_j, since each b_j intersects $T_P b$ at a finite number of points that is equal to its intersection index with the line $T_A b$. But the analytic germ (Γ, A) consists of a finite number of local branches. The contradiction thus obtained will prove absence of branches b_j tangent to b with $r_{b_j} > r = r_b$.

Let $(\Delta, T_A b)_A = (\Delta_A, T_A b)$ denote the local intersection index of the line

$T_A b$ with the divisor Δ at A. The result of Step 3 implies that

$$\rho_j \leq r \text{ for all } j;$$

$$\eta + 2(r+1) = \frac{3}{k}(k \sum_{j \in J} q_{b_j} \rho_j + \sum_{j \notin J} k_j q_{b_j} \rho_j) = \frac{3}{k}(\Delta, T_A b)_A; \tag{1.14}$$

the latter equality is implied by the formula $(b_j, T_A b)_A = q_{b_j} \rho_j$, which follows from the definition of the numbers ρ_j.

Step 4. Recall that $D = degF_1 = (\Delta, T_A b)$ (Bézout Theorem). We show that the local intersection index in (1.14) is no less than the half-degree $\frac{D}{2}$ plus k (i.e., most of the intersection $\Delta \cap T_A b$ is concentrated at A), and the latter inequality is strict, unless the branch b is quadratic and regular (Theorem 3.4). This is also deduced from the local relative angular symmetry property by the following argument. We show that for every $P \in b$ close to A those points of the intersection $\Delta \cap T_P b$ that do not tend to A, as $P \to A$, should be angular-symmetric to some intersection points that tend to A with z-coordinates of order $o(z(P))$. This follows from Proposition 3.6 mentioned in Step 3. The latter intersection points do not coincide with the points of intersection $b \cap T_P b$, which also tend to A but have z-coordinates asymptotic to $z(P)$ times non-zero constants. The contribution of the curve b to the intersection index $(\Delta, T_A b)_A$ equals $k(b, T_A b)$, since b is contained in Δ with multiplicity k. The three last statements together imply that $(\Delta, T_A b)_A \geq \frac{D}{2} + \frac{k}{2}(b, T_A b) > \frac{D}{2} + k$, unless b is regular and quadratic.

Step 5. Substituting inequalities proved in Steps 2–4 to formula (1.14) yields that $\eta \geq \mu$. In more detail, formula (1.14) and inequalities $\frac{1}{k}(\Delta, T_A b)_A \geq \frac{D}{2k} + 1$ (Step 4) and $r \leq 2$ (Step 2) together imply that

$$\eta = \frac{3}{k}(\Delta, T_A b)_A - 2(r+1) \geq 3(\frac{D}{2k} + 1) - 2(r+1) \geq 3\frac{D}{2k} - 3 = \mu. \tag{1.15}$$

The latter inequality is strict, unless the branch b is quadratic and regular, as in Step 4. This proves Theorem 1.49.

2 Preparatory asymptotics of Bialy–Mironov Hessian formula. Proof of formulas (1.12) and (1.13)

Formulas (1.12) and (1.13) are implied by the following more general proposition.

Proposition 2.1 *Consider the affine plane* $\mathbb{C}^2$ *equipped with complex orthogonal coordinates* (x,y) *with respect to the complex Euclidean form* dx^2+dy^2. *Let* $O, A \in \mathbb{C}^2$ *be two distinct points such that the line* OA *is isotropic. Let* $b_1,\dots,b_N$ *be irreducible germs of analytic curves at* A *such that* $b=b_1$ *is transverse to* OA. *Let* $f_1,\dots,f_N$ *be irreducible germs of analytic functions at* A *defining the germs* $b_j=\{f_j=0\}$. *Fix a subset* $J\subset\{1,\dots,N\}$ *such that* $1\in J$, *set* $k_1=1$. *Fix arbitrary numbers* $k_2,\dots,k_N>0$. *Set*

$$f=\prod_{j\in J} f_j^{k_j},\ g=\prod_{j\notin J} f_j^{k_j},\quad r=r_b.$$

Let q_{b_j} *be the exponents from the parametrizations (1.3) of the germs* b_j *in the coordinates adapted to them. Let* ρ_j *be the numbers defined in (1.11). Let* (z,w) *be local affine coordinates centered at* A *that are adapted to the germ* b*; the line* T_Ab *is the* z*-axis. Then the following asymptotic formulas hold along the curve* b[5]*:*

$$g^3H(f)|_b=O(z^\eta),\ \eta=3\sum_{j=1}^{N}k_jq_{b_j}\min\{\rho_j,r\}-2(r+1), \tag{2.1}$$

$$(x^2+y^2)|_b\simeq cz,\quad c=const\neq 0. \tag{2.2}$$

Proof Asymptotic formula (2.2) is obvious. Indeed, one has $x^2+y^2=uv$ in appropriate new affine coordinates u and v centered at O, in which $OA=\{u=0\}$. Then $v(A)\neq 0$, since $A\neq O$. One has $u(P)\simeq cz$, $c\neq 0$, as $P\in b$ tends to A, by transversality. This implies (2.2). Let us prove formula (2.1). In its proof we use the following property of the Hessian $H(f)$.

Proposition 2.2 *Let* F *be a germ of analytic function at a point* $A\in\mathbb{C}^2$, $F(A)=0$. *For every germ of analytic function* g *at* A *one has*

$$H(gF)|_{\{F=0\}}=g^3H(F). \tag{2.3}$$

Proof For every function $h(x,y)$ let us consider its skew gradient

$$\nabla_{skew}h=(-\frac{\partial h}{\partial y},\frac{\partial h}{\partial x})$$

in the coordinates (x,y). It is tangent to the level curves of the function h, by definition. Consider an irreducible component b of the germ at A of

[5]Here and in the proof of Proposition 2.1 all the functions are evaluated at a point $P\in b$ tending to A, and all the written asymptotics are asymptotics of values of functions at $P\in b$, as $P\to A$

the analytic curve $\{F=0\}$. Let $P\in b$. Let us extend the skew gradient vectors $\nabla_{skew}F(P)$ and $\nabla_{skew}(Fg)(P)$ to constant vector fields $v_F=v_{F,P}$ and $v_{Fg}=v_{Fg,P}$ respectively on the line T_Pb. Then the values $H(F)(P)$ and $H(Fg)(P)$ coincide with the second derivative at P of the restriction of the function F (Fg) to the line T_Pb along the field v_F (respectively, v_{Fg}). One has

$$\nabla_{skew}(Fg)|_b=g\nabla_{skew}F,\ v_{Fg}=g(P)v_F, \tag{2.4}$$

by Leibnitz rule and since $F\equiv 0$ on b. One has

$$H(Fg)(P)=\frac{d^2(Fg)}{dv_{Fg}^2}(P)=g^2\frac{d^2(Fg)}{dv_F^2}(P), \tag{2.5}$$

by (2.4). The latter second derivative equals $gF''_{v_F}+2F'_{v_F}g'_{v_F}+Fg''_{v_F}$, by Leibnitz rule. The second and third terms in the latter sum vanish at P, since $F(P)=0$ and $\frac{dF}{dv_F}(P)=0$: the vector $v_F(P)=\nabla_{skew}F(P)$ is tangent to the curve b. The first term equals $gH(F)$, by definition. This together with (2.5) implies that $H(Fg)(P)=g^3H(F)(P)$ and proves the proposition. □

One has

$$g^3H(f)|_b=(\prod_{j\neq 1}f_j^{k_j})^3H(f_1), \tag{2.6}$$

by the above proposition and since $b=b_1=\{f_1=0\}$. In what follows we estimate the asymptotics of the functions f_j and $H(f_1)$ along the curve b.

Proposition 2.3 *Let b be an irreducible germ of analytic curve at 0, and let (z,w) be local affine coordinates adapted to it. Let $t\mapsto(t^q,ct^p(1+o(1))$ be it local parametrization: $1\leq q<p$, $c\neq 0$, see (1.3). Let f be an irreducible germ of analytic function at 0 defining b: $b=\{f=0\}$. The Newton diagram of the function f consists of one edge: the segment connecting the points $(p,0)$ and $(0,q)$. More precisely, the Taylor series of the function $f(z,w)$ consists of monomials $w^\alpha z^\beta$ such that*

$$\nu_{\alpha\beta}=q\beta+p\alpha\geq qp; \tag{2.7}$$

the latter inequality is strict except for $(\alpha,\beta)\in\{(p,0),\ (0,q)\}$.

Proof Without loss of generality we can and will consider that f is a Weierstrass polynomial:

$$f(z,w)=\phi_z(w)=w^d+h_1(z)w^{d-1}+\dots+h_d(z),\quad h_d(0)=0, \tag{2.8}$$

since each germ of holomorphic function at 0 that vanishes at 0 and does not vanish identically on the w-axis is the product of a unique polynomial as above (called Weierstrass polynomial) and a non-zero holomorphic function, by Weierstrass Preparatory Theorem [17, chapter 0, section 1]. For every z small enough the polynomial $\phi_z(w) = f(z,w)$ has q roots $\zeta_l(z)$, $l = 1,\dots,q$: $\zeta_l(z) = ct_l^p(1+o(1))$, $t_l^q = z$, as $z \to 0$. The roots can be written as Puiseux series in z of the type $\zeta_l(z) = cz^{\frac{p}{q}}(1+o(1))$. This implies that the Weierstrass polynomial (2.8) is the product of q factors $w - \zeta_l(z)$ with $\zeta_l(z) \simeq \theta_l = ct_l^p = cz^{\frac{p}{q}}$, as $z \to 0$. Here for every z the points $\theta_1,\dots,\theta_q$ are obtained one from the other by multiplication by q-th roots of unity, or equivalently, form a regular q-gon centered at 0. Hence, in formula (2.8) one has $d = q$, $h_q(z) = (-1)^q \prod_{l=1}^q \zeta_l(z) = (-1)^q c^q z^p(1+o(1))$,

$$h_s(z) = o(z^{\frac{p}{q}s}) \text{ for } 1 \leq s < q, \text{ as } z \to 0. \tag{2.9}$$

Indeed, for $1 \leq s < q$ the s-th elementary symmetric polynomial σ_s in the roots $\zeta_l(z)$ has asymptotics smaller than their s-th powers. Namely, the asymptotic terms of order $z^{\frac{p}{q}s}$ cancel out, since $\zeta_l(z) \simeq \theta_l$ and $\sigma_s(\theta_1,\dots,\theta_q) = 0$: θ_l form a regular q-gon centered at zero. Formula (2.9) implies that the Weierstrass polynomial (2.8) contains only those Taylor monomials $w^\alpha z^\beta$ with $1 \leq \alpha < q$, set $s = q-\alpha$, for which $\beta > \frac{p}{q}s = \frac{p}{q}(q-\alpha)$, i.e., $q\beta + p\alpha > pq$. This proves the proposition. □

Claim 1. *One has*

$$f_j|_b = O(z^{q_{b_j}\min\{\rho_j, r\}}). \tag{2.10}$$

Proof Case 1): the curve b_j is transverse to b. Then $\rho_j = 1 < r$, and we have to show that $f_j|_b = O(z^{q_{b_j}})$. To do this, let us take the coordinates (z_j, w_j) adapted to b_j so that the w_j-axis coincides with the z-axis T_Ab, $w_j = z$ on T_Ab and $z_j = w$: one can do this by transversality. One has

$$w_j \simeq z, \; z_j = w \simeq c_b z^r, \; r = \frac{p_b}{q_b}, \text{ along the curve } b. \tag{2.11}$$

Hence, each Taylor monomial $w_j^\alpha z_j^\beta$ of the function f_j has asymptotics $O(z^{\alpha+\beta r})$ along the curve b. Now it suffices to show that $\alpha + \beta r \geq q_{b_j}$. Recall that $\alpha p_{b_j} + \beta q_{b_j} \geq p_{b_j} q_{b_j}$, by (2.7). Dividing the latter inequality by p_{b_j} yields $\nu = \alpha + \beta r_{b_j}^{-1} \geq q_{b_j}$. Hence, $\alpha + \beta r \geq \nu \geq q_{b_j}$, since $r_{b_j} > 1$. This proves the claim.

Case 2): the curve b_j is tangent to b, thus $\rho_j = r_{b_j}$. Then the coordinates (z,w) are adapted for both curves b and b_j. Each Taylor monomial $w^\alpha z^\beta$ of the function $f_j(z,w)$ is asymptotic to cz^ν, $\nu = \alpha r + \beta$, $c = const$, along the curve b, since $w \simeq c_b z^r$.

Subcase 2a): $r_{b_j} \leq r$. Thus, $\min\{\rho_j, r\} = r_{b_j}$, and it suffices to show that in the above notations $\alpha r + \beta \geq q_{b_j} r_{b_j} = p_{b_j}$. Indeed, $\alpha r + \beta \geq \alpha r_{b_j} + \beta \geq p_{b_j}$, by inequality (2.7) divided by q.

Subcase 2b): $r_{b_j} > r$. Thus, $\min\{\rho_j, r\} = r$, and it suffices to show that $\alpha r + \beta \geq q_{b_j} r$. Indeed,

$$\frac{r_{b_j}}{r}(\alpha r + \beta) = \alpha r_{b_j} + \beta \frac{r_{b_j}}{r} \geq \alpha r_{b_j} + \beta \geq p_{b_j} = q_{b_j} r_{b_j},$$

by (2.7). Multiplying the latter inequality by $\frac{r}{r_{b_j}}$ yields $\alpha r + \beta \geq q_{b_j} r$. The claim is proved. $\square$

Claim 2. *One has*

$$H(f_1)(P) = O(z^{3rq_b - 2(r+1)}), \text{ as } P \to A \text{ along the curve } b; \; r = r_b. \quad (2.12)$$

Proof The norm of the skew gradient of the function f_1 written in the coordinates (x,y) has the same asymptotics (up to nonzero constant factor), as the norm of its skew gradient written in the coordinates (z,w) adapted to the curve b, since applying the local coordinate change $(x,y) \mapsto (z,w)$ to the function f_1 multiplies its gradient by a holomorphic matrix function with non-zero determinant. Note that both skew gradients are tangent to the level curves of the function f_1, including its zero locus b. Everywhere below by $\nabla_{skew} f_1$ we denote the skew gradient in the coordinates (z,w). Let v denote the extension of the vector $\nabla_{skew} f_1(P)$ to a constant vector field on the line $T_P b$. It suffices to prove formula (2.12) for the left-hand side replaced by the second derivative $\frac{d^2 f_1}{dv^2}(P)$: the ratio of the absolute values of the latter second derivative and the expression $H(f_1)(P)$ equals the ratio of squared norms of the skew gradients of the function f_1 at P in the coordinate systems (x,y) and (z,w); the latter ratio is bounded from above and below, as was mentioned above.

We consider the Taylor series for both the function f_1 and its skew gradient and calculate the Hessian form of each Taylor monomial of the function f_1 evaluated on each Taylor monomial of its skew gradient. We show that the expression thus obtained has asymptotics given by the right-hand side in (2.12). This will prove the claim.

Let $w^\alpha z^\beta$ be the Taylor monomials of the function f_1. The skew gradient $(\nabla_{skew} f_1)|_b$ is a linear combination of the monomials

$$u_{\alpha,\beta} = w^\alpha z^{\beta-1} \frac{\partial}{\partial w} \simeq c z^{\alpha r + \beta - 1} \frac{\partial}{\partial w},$$

$$v_{\alpha,\beta} = w^{\alpha-1} z^{\beta} \frac{\partial}{\partial z} \simeq c' z^{\alpha r+\beta-r} \frac{\partial}{\partial z}, \ c, c' \neq 0;$$

both above asymptotics are written along the curve b. The restrictions to the curve b of the second derivatives of a monomial $w^{\alpha} z^{\beta}$ are asymptotic to

$$\frac{\partial^2(w^{\alpha} z^{\beta})}{\partial w^2} = \alpha(\alpha-1) w^{\alpha-2} z^{\beta} \simeq c_1 z^{\alpha r+\beta-2r};$$

$$\frac{\partial^2(w^{\alpha} z^{\beta})}{\partial z^2} = \beta(\beta-1) w^{\alpha} z^{\beta-2} \simeq c_2 z^{\alpha r+\beta-2};$$

$$\frac{\partial^2(w^{\alpha} z^{\beta})}{\partial z \partial w} = \alpha\beta w^{\alpha-1} z^{\beta-1} \simeq c_3 z^{\alpha r+\beta-r-1}, \quad c_1, c_2, c_3 \neq 0.$$

Therefore, applying the Hessian form of each monomial $w^{\alpha} z^{\beta}$ to a linear combination of the vectors $u_{\alpha',\beta'}$ and $v_{\alpha',\beta'}$ (e.g., to the skew gradient of the monomial $w^{\alpha'} z^{\beta'}$) yields a linear combination of expressions of the three following types:

$$\frac{d^2(w^{\alpha} z^{\beta})}{\partial u^2_{\alpha',\beta'}} = O(z^{\nu}), \ \nu = 2(\alpha' r + \beta' - 1) + \alpha r + \beta - 2r$$
$$= 2(\alpha' r + \beta') + (\alpha r + \beta) - 2(r+1); \tag{2.13}$$

$$\frac{d^2(w^{\alpha} z^{\beta})}{\partial v^2_{\alpha',\beta'}} = O(z^{\nu_2}), \ \nu_2 = 2(\alpha' r + \beta' - r) + \alpha r + \beta - 2 = \nu;$$

$$\frac{d^2(w^{\alpha} z^{\beta})}{\partial u_{\alpha',\beta'} \partial v_{\alpha',\beta'}} = O(z^{\nu_3}), \ \nu_3 = 2(\alpha' r + \beta') - r - 1 + \alpha r + \beta - r - 1 = \nu.$$

Let us now estimate ν from below. Recall that for every Taylor monomial $w^{\alpha} z^{\beta}$ of the function f_1 one has

$$\alpha r + \beta = \frac{1}{q_b}(\alpha p_b + \beta q_b) \geq p_b = q_b r,$$

by (2.7), and hence, the same inequality holds for α' and β'. This together with formula (2.13) for the number ν implies that $\nu \geq 3 q_b r - 2(r+1)$. This together with the above discussion proves formula (2.12). □

Substituting asymptotics (2.10) and (2.12) to formula (2.6) yields formula (2.1). Proposition 2.1 is proved. □

Proof of formulas (1.13) and (1.12). Formula (1.13) follows from formula (2.2). Formula (1.12) follows from formula (2.1) applied to $k_j = 1$ for $j \in J$ and k_j replaced by $\frac{k_j}{k}$ for $j \notin J$. This finishes Step 1 of the proof of Theorem 1.49. □

3 Relative angular symmetry property and its corollaries. Proof of Theorems 1.47–1.49

In this section we discuss the relative angular symmetry property in detail and prove its corollaries: Theorems 1.47 and steps 2, 3, 4 of the proof of Theorem 1.49 (see Subsection 1.6). They are stated and proved below in generalized forms as Theorems 3.1–3.4. In Subsection 3.7 we finish the proof of Theorem 1.49. In Subsection 3.8 we prove Theorem 1.48.

Assumptions in Theorems 3.1–3.4 below. These theorems deal with a point $O \in \mathbb{C}^2 \subset \mathbb{CP}^2$, a nonlinear irreducible germ b of analytic curve at a point $A \in (\mathbb{I} \setminus \{O\}) \cap \mathbb{C}^2$, where $\mathbb{I} = \Lambda_1 \cup \Lambda_2$ is the union of the isotropic lines through O. We assume that b has local relative angular symmetry property with respect to a bigger finite collection $\Gamma \supset b$ of irreducible germs of analytic curves at points in $T_A b$. In Theorem 3.4 the collection Γ will be treated as a divisor in a neighborhood of the line $T_A b$: a finite linear combination of irreducible germs of analytic curves at points in $T_A b$ with positive integer coefficients.

Theorem 3.1 *Let b be tangent to the isotropic line $OA = \Lambda_j$. Then b is quadratic.*

Theorem 3.1 implies Theorem 1.47. It will be proved in Subsection 3.3.

Theorem 3.2 *Let b be transverse to the isotropic line OA. Then b is subquadratic.*

Theorem 3.3 *Let b be transverse to the isotropic line OA. Then each germ at A from the collection Γ that is tangent to b has Puiseux exponent no greater than $r = r_b$.*

Theorem 3.4 *Let b be transverse to the isotropic line OA. Let the divisor Γ include the germ b k times, $k \in \mathbb{N}$. Let D denote its degree: the intersection index $(\Gamma, T_A b)$. The local intersection index of the tangent line $T_A b$ with Γ at A is no less than $\frac{D}{2} + k$. The equality may take place only in the case, when the germ b is quadratic and regular, and Γ contains no other germs tangent to b at A with the same Puiseux exponent, as b.*

Remark 3.5 In fact, one can slightly strengthen the last statement of the theorem, with "no other germs in Γ tangent to b," without requiring anything on their Puiseux exponents. This can be obtained via adding a small extra argument to its proof, which is omitted to save the space.

Theorems 3.2–3.4 imply steps 2–4 of the proof of Theorem 1.49. They will be proved in Subsections 3.4–3.6.

The proofs of Theorems 3.1–3.4 and 1.48 are done by similar methods and consist of the two following ingredients:

1) Studying the asymptotics of the family of angular symmetries of the lines T_Pb, as $P \to A$. Let $t = t(P)$ denote the coordinate of the point P in the local parameter of the germ b: $t(A) = 0$, $t(P) \to 0$, as $P \to A$. We identify the projective lines T_Pb with $\overline{\mathbb{C}}$ via appropriate family of affine coordinates (e.g., z in the proof of Theorem 3.2), which will be here also denoted by z. Set $z(t) = z(P)$. In the coordinate z the angular symmetry family becomes a family of conformal involutions $\sigma_t : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with the following properties: $\sigma_t(z(t)) = z(t)$; $z(t) \to 0$, and $\sigma_t(0)$ tends to a non-zero limit, as $t \to 0$. Proposition 3.6 stated and proved in the next subsection says that the mappings σ_t converge to the constant mapping $\overline{\mathbb{C}} \mapsto 0$ uniformly on compact subsets in $\overline{\mathbb{C}} \setminus \{0\}$ and each family of points $z_t \in \mathbb{C}$ with *moderate asymptotics* $z_t \simeq cz(t)$, $c \neq 0$, is sent by σ_t to a family asymptotic to $c^{-1}z(t)$.

2) The asymptotic analysis of the relative angular symmetry property. We consider those points of the intersection $T_Pb \cap \Gamma$ that have moderate asymptotics in the above sense. Their collection is angular-symmetric, and the angular symmetry inverses the corresponding asymptotic factors: $c \mapsto c^{-1}$, by Proposition 3.6. The following description of the asymptotic factors of the intersection points with moderate asymptotics is implied by [15, proposition 2.1] and [13, proposition 2.50, p.268], which are recalled in Subsection 3.2. We consider the local branches $b_1 = b, b_2, b_3, \dots, b_N \in \Gamma$ at the point A that are tangent to b and have the same Puiseux exponent $r_{b_i} = r = r_b$. The points of intersections $b_i \cap T_Pb$ all have moderate asymptotics, and the corresponding asymptotic factors are appropriate powers of roots of appropriate polynomials W_i associated to b_i. The other points of intersection $T_Pb \cap \Gamma$ with moderate asymptotics have known asymptotic factors: all of them are equal to the same number depending on r. The symmetry of the asymptotic factor collection under taking the inverse implies a relation on the collection of polynomials W_i. We show that the latter relation implies the statement of the theorem under question.

The proof of Theorem 3.4 is based on the statement saying that those points of intersection $T_Pb \cap \Gamma$ that do not tend to A, as $P \to A$, are angular-symmetric to points tending to A. The latter statement follows from the uniform convergence statement of Proposition 3.6.

The proof of Theorem 3.3 is more technical. For a detailed sketch of its proof see Subsection 1.6, the description of the step 3 of the proof of Theorem 1.49.

3.1 Asymptotics of degenerating conformal involutions

Here we prove the next proposition on asymptotics of a degenerating family of conformal involutions $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$, which will be applied to families of angular symmetries of the lines $T_P b$.

Proposition 3.6 *Consider a family of conformal involutions $\sigma_t : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Riemann sphere with coordinate z, set $y_t = \sigma_t(0)$. Let there exist a family $z(t)$ of their fixed points, $\sigma_t(z(t)) = z(t)$, such that $z(t) \to 0$, and let $y_t \to y \neq 0$, as $t \to 0$. Then the following statements hold.*

(a) The involutions σ_t converge to the constant mapping $\overline{\mathbb{C}} \mapsto 0$ uniformly on compact subsets in $\overline{\mathbb{C}} \setminus \{0\}$.

(b) Fix a $c \in \mathbb{C}^*$ and a family of points $z_t \in \mathbb{C}$ with the asymptotics $z_t = cz(t)(1 + o(1))$, as $t \to 0$. Then

$$\sigma_t(z_t) = c^{-1} z(t)(1 + o(1)), \ \text{as } t \to 0. \tag{3.1}$$

Proof The transformations σ_t are written in the coordinate z as follows:

$$\sigma_t(z) = \frac{y_t - z}{1 - \lambda_t z}, \ \lambda_t = \frac{2z(t) - y_t}{z^2(t)} : \tag{3.2}$$

the number λ_t is found from the equation $\sigma_t(z(t)) = z(t)$. One has $\lambda_t \to \infty$, since its denominator tends to zero, while the nominator tends to $y \neq 0$. Therefore, the denominator in the formula (3.2) for the involution σ_t is asymptotic to $\lambda_t z$ uniformly on compact subsets in $\overline{\mathbb{C}} \setminus \{0\}$, while the nominator is $O(z) = o(\lambda_t z)$ uniformly on compact subsets. This implies statement (a). Substituting $z_t = cz(t)(1 + o(1))$ to the same formula yields

$$\sigma_t(z_t) \simeq \frac{y}{(cy)/z(t)} = c^{-1} z(t).$$

This proves statement (b) and finishes the proof of the proposition. $\square$

3.2 Preliminaries: asymptotics of intersections with the tangent line

Let a, b be irreducible germs of planar complex analytic curves at the origin in $\mathbb{C}^2$. Let p_g, q_g, c_g, $g = a, b$ be respectively the corresponding exponents and constants from their parametrizations (1.3) in their adapted coordinates. Let t be the corresponding local parameter of the germ b. We identify points of the curve b with the corresponding local parameter values t. We use the following statements on the asymptotics of the points of intersection $T_t b \cap a$.

Proposition 3.7 *[15, proposition 2.1] Let a, b be transverse irreducible germs of holomorphic curves at the origin in $\mathbb{C}^2$. Let (z,w) be affine coordinates centered at 0 and adapted to b: the germ b is tangent to the z-axis. Then for every t small enough the intersection $T_t b \cap a$ consists of q_a points $\xi_1, \dots, \xi_{q_a}$ whose coordinates have the following asymptotics, as $t \to 0$:*

$$z(\xi_j) = O(t^{p_b}) = O(w(t)) = o(z(t)) = o(t^{q_b}),$$

$$w(\xi_j) = (1-r_b)w(t)(1+o(1)) = (1-r_b)c_b t^{p_b}(1+o(1)). \tag{3.3}$$

Proposition 3.8 *([13, p. 268, proposition 2.50], [15, proposition 2.3]) Let a, b be irreducible tangent germs of holomorphic curves at the origin in the plane $\mathbb{C}^2$. Consider their parametrizations (1.3) in common adapted coordinates (z,w). Let c_a and c_b be the corresponding constants from (1.3). Then for every t small enough the intersection $T_t b \cap a$ consists of p_a points $\xi_1, \dots, \xi_{p_a}$ whose coordinates have the following asymptotics, as $t \to 0$.*

Case 1): $r_a > r_b$. One has two types of intersection points ξ_j:

$$\text{for } j \leq q_a: \quad z(\xi_j) = \frac{r_b - 1}{r_b} z(t)(1+o(1)) = \frac{r_b-1}{r_b} t^{q_b}(1+o(1)), \tag{3.4}$$

$$w(\xi_j) = O(t^{q_b r_a}) = o(t^{p_b}) = o(w(t));$$

$$\text{for } j > q_a: \quad z(t) = O((z(\xi_j))^{\frac{r_a-1}{r_b-1}}) = o(z(\xi_j)), \tag{3.5}$$

$$w(t) = O(z^{r_b}(t)) = O((z(\xi_j))^{\frac{r_b(r_a-1)}{r_b-1}}) = o(z^{r_a}(\xi_j)) = o(w(\xi_j)).$$

Case 2): $r_a = r_b = r$. One has

$$z(\xi_j) = \zeta_j^{q_a} z(t)(1+o(1)) = \zeta_j^{q_a} t^{q_b}(1+o(1)), \tag{3.6}$$

$$w(\xi_j) = c_a \zeta_j^{p_a} t^{p_b}(1+o(1)) = c\zeta_j^{p_a} w(t)(1+o(1)),$$

where ζ_j are the roots of the polynomial

$$R_{p_a,q_a,c}(\zeta) = c\zeta^{p_a} - r\zeta^{q_a} + r - 1; \; r = \frac{p_a}{q_a}, \; c = \frac{c_a}{c_b}. \tag{3.7}$$

(In the case, when $b = a$, one has $c = 1$, and the above polynomial has double root 1 corresponding to the tangency point t.)

Case 3): $r_a < r_b$. One has

$$z(\xi_j) = O((z(t))^{\frac{r_b}{r_a}}) = o(z(t)), \tag{3.8}$$

$$w(\xi_j) = (1-r_b)w(t)(1+o(1)) = (1-r_b)c_b t^{p_b}(1+o(1)).$$

3.3 Quadraticity of tangent branches. Proof of Theorem 3.1

The line OA coincides with some of the isotropic lines Λ_j, $j = 1, 2$, say, $OA = \Lambda_1$. Let us introduce affine coordinates (z, w) on $\mathbb{C}^2$ adapted to b: they are centered at A and $\Lambda_1 = OA$ is the z-axis. In addition, we choose them so that the w-axis be parallel to the other isotropic line Λ_2. One has $w(O) = 0$, $z(O) \neq 0$, since $z(A) = 0$ and $A \neq O$. Let $P \in b$ be a point close to A. Let $Q_j = Q_j(P)$ denote the intersection point of the line $T_P b$ with the line Λ_j. In what follows we use the asymptotic relation

$$w(P) = o(w(Q_2)), \text{ as } P \to A. \tag{3.9}$$

Indeed, the points P, Q_1, Q_2 lie in the same line $T_P b$, and

$$P, Q_1 \to A, \text{ as } P \to A. \tag{3.10}$$

Let $T = T(P)$ denote the projection of the point P to the z-axis OA: $z(T) = z(P)$. Consider the triangles TQ_1P and OQ_1Q_2. They are similar in the following complex sense. Their edges TP and Q_2O lie in complex lines parallel to the w-axis: we measure them by the w-coordinate. Their edges TQ_1, Q_1O lie in the complex z-axis: we measure them by the z-coordinate. Their edges PQ_1 and Q_1Q_2 lie in the same complex line Q_1Q_2. The parallelness of complexified edges of the above triangles implies that

$$\frac{w(P) - w(T)}{w(Q_2) - w(O)} = \frac{z(T) - z(Q_1)}{z(O) - z(Q_1)}.$$

Substituting the equalities and asymptotics $w(T) = w(O) = 0$, $z(Q_1), z(P) \to 0$, see (3.10), $z(T) - z(Q_1) = z(P) - z(Q_1) \to 0$, and $z(O) - z(Q_1) \to z(O) \neq 0$ to the latter formula yields $\frac{w(P)}{w(Q_2)} \to 0$ and proves (3.9).

In what follows we consider the subset $M = M(P) \subset T_P b \cap \Gamma$ of *points with moderate w-asymptotics:* those points of the latter intersection, whose w-coordinates are asymptotic to $t_j w(P)(1 + o(1))$, as $P \to A$, with some constants $t_j \neq 0$ called their *asymptotic w-factors.* The intersection being finite, as is Γ, the set M is also finite. Note that the whole intersection $T_P b \cap \Gamma$ is invariant under the angular symmetry of the line $T_P b$ with respect to the point P and center O, by the relative angular symmetry property.

Proposition 3.9 *The set M is invariant under the angular symmetry of the line $T_P b$ with respect to the point P and center O. Each point of the set M with asymptotic w-factor s is sent to a point with the inverse factor s^{-1}.*

Proof Let us equip the line T_Pb with the rescaled affine coordinate

$$\widetilde{w} = \frac{w}{w(Q_2(P))}.$$

One has $\widetilde{w}(Q_1(P)) = 0$, $\widetilde{w}(Q_2(P)) = 1$, $\widetilde{w}(P) \to 0$, by (3.9). The angular symmetry of the projective line T_Pb is its conformal involution fixing P and permuting the points $Q_1(P)$ and $Q_2(P)$. The two last statements together imply that the family of angular symmetries of the lines T_Pb written in the coordinate $\widetilde{w}$ yields a family of conformal involutions of the Riemann sphere satisfying the conditions of Proposition 3.6. This implies that a family of points in T_Pb whose $\widetilde{w}$-coordinates are asymptotic to $s\widetilde{w}(P)(1+o(1))$, $s \neq 0$ (or equivalently, whose w-coordinates are asymptotic to $sw(P)(1+o(1))$) is sent by the angular symmetry to a family of points with $\widetilde{w}$-coordinates asymptotic to $s^{-1}\widetilde{w}(P)(1+o(1))$ (equivalently, with w-coordinates asymptotic to $s^{-1}w(P)(1+o(1))$). This implies the statement of the proposition. □

Corollary 3.10 *The product of the asymptotic w-factors of the points from the set M equals one.*

In what follows we calculate the w-factors of the points of the set M (the two next propositions) and show that their product can be equal to one only if $r = 2$. This will prove Theorem 3.1.

Proposition 3.11 *Let $\mathcal{N} \subset \Gamma$ denote the union of those irreducible germs a (i.e., local branches) of the collection Γ at A that satisfy some of the two following conditions: either a is tangent to b and $r_a < r_b$; or a is transverse to b. Let $\mathcal{R} \subset \Gamma$ denote the union of those local branches at A of the collection Γ that are tangent to b and have the same Puiseux exponent r_b. One has*

$$M = (M_N \sqcup M_R), \quad M_N = T_Pb \cap \mathcal{N}, \; M_R = T_Pb \cap \mathcal{R}. \tag{3.11}$$

The asymptotic w-factors of points of the set M_N are equal to $1 - r_b$.

Proposition 3.12 *Let a be a local branch at A of the collection Γ that is tangent to b, and let $r_a = r_b = r$. Let p_a, q_a, c_a, c_b be the corresponding exponents and coefficients in the parametrization (1.3) of the curves a and b in their common adapted coordinates (z, w). Let $\zeta_1, \dots, \zeta_{p_a}$ denote the roots of the polynomial*

$$R_{p_a,q_a.c}(\zeta) = c\zeta^{p_a} - r\zeta^{q_a} + r - 1, \; c = \frac{c_a}{c_b}.$$

The asymptotic w-factors of the points of intersection $T_Pb \cap a$ are equal to $c\zeta_j^{p_a}$.

Both propositions follow immediately from definition and Propositions 3.7 and 3.8.

Remark 3.13 The product of the above w-factors $c\zeta_j^{p_a}$ through $j = 1, \dots, p_a$ equals $(1-r)^{p_a}$. Indeed, the product of roots ζ_j equals $c^{-1}(-1)^{p_a}(r-1)$. Taking its p_a-th power yields $c^{-p_a}(1-r)^{p_a}$. The product of the w-factors we are looking for is obtained from the latter product by multiplication by c^{p_a}, and thus, c cancels out.

The product of the w-factors of points of the set M should be equal to one (Corollary 3.10). On the other hand, it is equal (up to sign) to a natural power of the number $1-r$, $r = r_b$. This follows from the two above propositions, Remark 3.13 and the fact that the collection of the germs a from Proposition 3.12 is non-empty: it contains b. Therefore, $|1-r| = 1$, hence $r = 2$. Theorems 3.1 and 1.47 are proved.

3.4 Subquadraticity. Proof of Theorem 3.2

Let $O \in \mathbb{C}^2$, b be an irreducible germ of analytic curve at a point $A \in \mathbb{C}^2 \setminus \{O\}$ such that the line OA is isotropic, say, $OA = \Lambda_1$, and let b be transverse to OA. Let b have relative angular symmetry property with respect to a bigger finite collection $\Gamma \supset b$ of irreducible germs of analytic curves with base points in T_Ab. Let (z, w) be affine coordinates centered at A and adapted to b. For every $P \in b$ we equip the tangent line T_Pb with the coordinate z. The intersection $\Gamma \cap T_Pb$ is finite for every P close to A, and the intersection points are multivalued analytic functions of P: their z-coordinates are Puiseux series in $z(P)$.

Definition 3.14 In the above conditions for every $P \in b$ close to A we consider the subset $\Psi_P \subset T_Pb \cap \Gamma$ of *points with moderate z-asymptotics:* those points of the intersection $T_Pb \cap \Gamma$, whose z-coordinates are asymptotic to $t_jz(P)(1+o(1))$, $t_j = const \neq 0$, as $P \to A$. The corresponding constant factors t_j will be called the *asymptotic z-factors.* Some of them may coincide, and we take each asymptotic factor t_j with its multiplicity n_j: the quantity of the corresponding intersection points (taken with their own multiplicities). In other words, we consider the asymptotic z-factor collection as a divisor $\mathcal{D} = \sum_j n_j[t_j]$ in $\mathbb{C}$: a formal finite linear combination of points $t_j \in \mathbb{C}$ with natural coefficients n_j. The divisor $\mathcal{D}$ will be called the *asymptotic z-divisor.*

Definition 3.15 The *sum* of a divisor $D = \sum_j n_j[t_j]$ in $\mathbb{C}$ is the sum $\Sigma(D) = \sum_j n_j t_j \in \mathbb{C}$ of its points. Its *pointwise inverse* is the divisor of inverses: $D^{-1} = \sum_j n_j[t_j^{-1}]$ (provided that $t_j \neq 0$).

Remark 3.16 The angular symmetry family of the lines $T_P b$ (with respect to the points P and center O) written in the coordinate z is a family of conformal involutions $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ satisfying the conditions of Proposition 3.6, as in the proof of Proposition 3.9. Indeed, the fixed point $z(P)$ tends to 0, as $P \to A$. The point of the line $T_P b$ with zero z-coordinate is its intersection point with the line Λ_1, and its image under the angular symmetry is the point of intersection $T_P b \cap \Lambda_2$, which tends to the point of intersection $T_A b \cap \Lambda_2$ with non-zero z-coordinate.

Corollary 3.17 *The above angular symmetries of the lines $T_P b$ written in the coordinate z converge to the constant mapping $\overline{\mathbb{C}} \mapsto 0$ uniformly on compact subsets in $\overline{\mathbb{C}} \setminus \{0\}$. The subset $\Psi_P \subset T_P b$ is invariant under the angular symmetry. The asymptotic z-divisor $\mathcal{D}$ is invariant under the involution $\mathbb{C} \to \mathbb{C}$ of taking inverse $z \mapsto z^{-1}$:*

$$\mathcal{D}^{-1} = \mathcal{D}, \ \Sigma(\mathcal{D}) = \Sigma(\mathcal{D}^{-1}). \tag{3.12}$$

The corollary follows immediately from Remark 3.16 and Proposition 3.6.

We prove Theorem 3.2 by contradiction. Suppose the contrary: $r > 2$. We show that $\Sigma(\mathcal{D}^{-1}) \neq \Sigma(\mathcal{D})$. The contradiction thus obtained to (3.12) will prove the theorem.

Let $b_1 = b, b_2, \dots, b_N$ denote all the germs from the collection Γ that are tangent to b and have the same Puiseux exponent $r_{b_i} = r_b = r$. Let p_{b_i}, q_{b_i}, c_{b_i} denote the corresponding numbers in asymptotics (1.3) written in the coordinates (z, w). Let us represent r as an irreducible fraction $r = \frac{p}{q}$. For every $a = b_1, \dots, b_N$ set

$$s_a := G.C.D(p_a, q_a): \quad p_a = s_a p, \ q_a = s_a q. \tag{3.13}$$

Proposition 3.18 *Consider the collection of the polynomials*

$$W_i(\zeta) = R_{p,q,c(i)}(\zeta) = c(i)\zeta^p - r\zeta^q + r - 1, \ c(i) = \frac{c_{b_i}}{c_b}, \ i = 1, \dots, N.$$

For every polynomial W_i let $\mathcal{R}_i$ denote the collection of q-th powers of its roots considered as a divisor in $\mathbb{C}$: each q-th power being taken with its *multiplicity, the quantity of the corresponding roots. Consider the divisor*

$$\mathcal{R} = \sum_{i=1}^{N} s_{b_i} \mathcal{R}_i.$$

Let $\mathcal{N}_P \subset \Psi_P$ denote the subset lying in the intersection of the line $T_P b$ with those local branches $\phi \in \Gamma$ that are tangent to b and have bigger Puiseux exponents $r_\phi > r$. Let n denote the cardinality of the set $\mathcal{N}_P$ for P close enough to A. One has

$$\mathcal{D} = \mathcal{R} + n[z_r], \ z_r = \frac{r-1}{r}. \tag{3.14}$$

Proof The asymptotic z-factor of each point of the set $\mathcal{N}_P$ equals $\frac{r-1}{r}$, by Proposition 3.8. The complement $\Psi_P \setminus \mathcal{N}_P$ consists of the intersection points of the line $T_P b$ with the local branches b_i, and the divisor of its asymptotic z-factors coincides with $\mathcal{R}$. This follows from Propositions 3.7, 3.8 and the fact that the collection of the q_a-th powers of roots of each polynomial $R_{p_a,q_a,c}$ coincides with the collection of the q-th powers of roots of the polynomial $R_{p,q,c}$, where each one of the latter powers is taken with multiplicity s_a. The latter statement follows from the equality $R_{p_a,q_a,c}(\zeta) = R_{p,q,c}(\zeta^{s_a})$. This proves Proposition 3.18. □

One has $p = rq > 2q$, by assumption. For every $i = 1, \dots, N$ set

$$\Sigma_{1i} = \Sigma(\mathcal{R}_i), \ \Sigma_{2i} = \Sigma(\mathcal{R}_i^{-1}):$$

$$\Sigma(\mathcal{D}) = \sum_{i=1}^{N} s_{b_i}\Sigma_{1i} + n\frac{r-1}{r} = \Sigma(\mathcal{D}^{-1}) = \sum_{i=1}^{N} s_{b_i}\Sigma_{2i} + n\frac{r}{r-1}, \tag{3.15}$$

by (3.12) and (3.14). Let us show that

$$\Sigma_{1i} = 0, \quad \Sigma_{2i} = \frac{p}{r-1}. \tag{3.16}$$

The first equality in (3.16) follows from the inequality $p - q > q$ (which holds by assumption) and the fact that Σ_{1i} is the sum of q-th powers of roots of the polynomial W_i; hence, it is a polynomial in elementary symmetric polynomials of degrees at most q in its roots. The latter symmetric polynomials vanish, as do the corresponding coefficients in W_i at the monomials of degrees $p-1, \dots, p-q > q$. Let us prove the second equality in (3.16). The inverse to the roots of the polynomial $R_{p,q,c(i)}$ are the roots of the polynomial $H_{p,q,c(i)}$, where

$$H_{p,q,c}(\zeta) = (r-1)\zeta^p - r\zeta^{p-q} + c.$$

Proposition 3.19 *(cf. [15, formula (2.16)]). The sum of q-th powers of roots of each polynomial $H_{p,q,c}$ with $r \neq 1$ equals $\frac{qr}{r-1}$.*

Proof The above sum is independent on c, since it is expressed in terms of the elementary symmetric polynomials corresponding to the coefficients in $H_{p,q,c}$ at powers no less than $p-q$. Therefore, it suffices to calculate it for $c = 0$, when zero is a root of multiplicity $p-q$ and the q-th powers of the other q roots are equal to $\frac{r}{r-1}$: it equals $\frac{qr}{r-1}$. The proposition is proved. □

The sum Σ_{2i} is the sum of the q-th powers of roots of the polynomial $H_{p,q,c(i)}$, and hence, equals $\frac{qr}{r-1} = \frac{p}{r-1}$, by the proposition.

Formulas (3.15) and (3.16) together imply that

$$\frac{p}{r-1}\sum_i s_{b_i} = n\left(\frac{r-1}{r} - \frac{r}{r-1}\right) \leq 0.$$

But the latter left-hand side is obviously positive. The contradiction thus obtained proves Theorem 3.2.

3.5 Puiseux exponents. Proof of Theorem 3.3

Let (z,w) be affine coordinates centered at A and adapted to the local branch b under consideration: the z-axis is T_Ab. Let $\Psi_P \subset T_Pb \cap \Gamma$ denote the family of subsets of intersection points with moderate z-asymptotics, and let $\mathcal{D} \subset \mathbb{C} \setminus \{0\}$ denote the collection of the corresponding asymptotic z-factors (now treated as a finite subset in $\mathbb{C}$, not as a divisor), see Definition 3.14. Let $r = \frac{p}{q}$ be the irreducible fraction presenting the rational number $r = r_b$. Let $b_1, \dots, b_N$ be the local branches of the collection Γ that are tangent to b and have the same Puiseux exponent: $r_{b_i} = r = r_b$; $b_1 = b$. Let $W_k = R_{p,q,c(k)}(\zeta)$ be the corresponding polynomials from Proposition 3.18.

We prove Theorem 3.3 by contradiction. Suppose the contrary: there exists a local branch $\phi \in \Gamma$ tangent to b such that $r_\phi > r$. Then the collection $\mathcal{D}$ contains the point $z_r = \frac{r-1}{r} < 1$. Its other points are q-th powers of roots of the polynomials W_k, and

$$\text{the } q-\text{th power of each root of every polynomial } W_k \text{ lies in } \mathcal{D}. \quad (3.17)$$

Both latter statements follow from Proposition 3.18. The collection $\mathcal{D}$ contains the point $\mu_1 = z_r^{-1} > 1$, by its invariance under taking inverse. Therefore, μ_1 is a q-th power of root of a polynomial W_k. We show (using the next proposition and corollary) that W_k has another root, with a real q-th power $\theta_2 \in (0,1)$ such that $\mu_2 = \theta_2^{-1} < \mu_1$; $\mu_2 \in \mathcal{D}$, by (3.17) and invariance. Continuing this procedure will yield an infinite sequence of elements

$\mu_1 > \mu_2 > \cdots > 1$ of the finite set $\mathcal{D}$. The contradiction thus obtained will prove Theorem 3.3.

Proposition 3.20 *Let $p, q \in \mathbb{N}$, $1 \leq q < p$, $r = \frac{p}{q}$. A polynomial $W(z) = R_{p,q,c}(z) = cz^p - rz^q + r - 1$ has a root $z_1 > 1$, if and only if $0 < c < 1$. In this case it has exactly two real positive roots z_0 and z_1, $0 < z_0 < 1 < z_1$, and their product is greater than one.*

Proof For $c \notin \mathbb{R}_+$ one has $W|_{\{z \geq 1\}} \neq 0$, since $-rz^q + r - 1 < 0$ for every $z \geq 1$. Therefore, we consider that $c > 0$. The derivative equals $W'(z) = cpz^{p-1} - rqz^{q-1} = pz^{q-1}(cz^{p-q} - 1)$. Therefore, $c^{-\frac{1}{p-q}}$ is the unique local extremum of the polynomial W in the positive semiaxis, and it is obviously a local minimum. For $c = 1$ one has $W(1) = 0$, and $z = 1$ is exactly the minimum. Therefore, as c increases, the graph of the polynomial W becomes disjoint from the positive coordinate semiaxis, and it has no positive root, if $c > 1$. As c decreases remaining positive, the graph intersects the coordinate axis on both sides from 1. Thus, for $0 < c < 1$ the polynomial W gets exactly two real positive roots z_0 and z_1, $0 < z_0 < 1 < z_1$, and the minimum is between them; $W(z) > 0$ for $z > z_1$. Let us prove that $z_0 z_1 > 1$, or equivalently, $z_0^{-1} < z_1$. The latter inequality would follow from positivity of the polynomial W on the interval $(z_1, +\infty)$ and the inequality

$$W(z_0^{-1}) < 0. \tag{3.18}$$

Let us prove (3.18). By definition, $cz_0^p - rz_0^q + r - 1 = 0$, hence,

$$c = \frac{rz_0^q - r + 1}{z_0^p}.$$

Substituting the latter right-hand side into the polynomial $W(z_0^{-1})$ instead of the coefficient c yields

$$W(z_0^{-1}) = z_0^{-p}(r(z_0^{q-p} - z_0^{p-q}) - (r-1)(z_0^{-p} - z_0^p)).$$

Multiplying this expression by qz_0^p and denoting $m = p - q$ yields

$$p(z_0^{-m} - z_0^m) - m(z_0^{-p} - z_0^p).$$

Let us show that the latter expression is negative. We prove the following stronger inequality:

$$\frac{z_0^{-p} - z_0^p}{z_0^{-m} - z_0^m} > \frac{p}{m} \text{ whenever } z_0 \in \mathbb{R}_+ \setminus \{1\} \text{ and } p > m, \ p, m \in \mathbb{N}. \tag{3.19}$$

Canceling the common divisor $z_0^{-1} - z_0$ in the left fraction transforms inequality (3.19) to

$$\frac{z_0^{1-p} + z_0^{3-p} + \cdots + z_0^{p-1}}{z_0^{1-m} + z_0^{3-m} + \cdots + z_0^{m-1}} > \frac{p}{m}. \tag{3.20}$$

Case 1): p and m are of the same parity. Then the difference of the nominator and the denominator in (3.20) is positive and equal to the sum

$$(z_0^{1-p} + z_0^{p-1}) + (z_0^{3-p} + z_0^{m-3}) + \cdots + (z_0^{-1-m} + z_0^{m+1}). \tag{3.21}$$

Each sum of inverses in (3.21) is greater than any analogous sum of inverses $z_0^{-j} + z_0^{j}$, $j \leq m-1$, in the above denominator, since the function

$$f_z(s) = z^{-s} + z^s$$

in $s > 0$ with fixed $z > 0$, $z \neq 1$ increases. This implies that the average sum of inverses in (3.21) is greater than that in the denominator. Hence, the ratio of expression (3.21) and the denominator is greater than the ratio of the quantities of sums of inverses in them. (If m is odd, then the denominator contains the half-sum $1 = \frac{f_{z_0}(0)}{2}$; here $f_{z_0}(0)$ is counted with weight $\frac{1}{2}$.) This together with the above discussion implies inequality (3.20).

Case 2): p and m are of different parities. Then the nominator in (3.20) equals

$$\nu_m = f_{z_0}(p-1) + \cdots + f_{z_0}(m+2) + \sigma_m, \ \sigma_m = f_{z_0}(m) + f_{z_0}(m-2) + \ldots. \tag{3.22}$$

The denominator equals

$$\eta_m = f_{z_0}(m-1) + f_{z_0}(m-3) + \ldots.$$

Here each one of the sums σ_m and η_m ends with either $f_{z_0}(1)$, or $1 = \frac{f_{z_0}(0)}{2}$. Note that $f_{z_0}(s) \leq \frac{1}{2}(f_{z_0}(s-1) + f_{z_0}(s+1))$ for all $s \geq 0$, since the function $f_z(s)$ is convex in $s > 0$ for $z > 0$, $z \neq 1$: $f_z''(s) = (\ln z)^2 f_z(s) > 0$. Writing the latter mean inequalities for $s = m-1, m-3, \ldots$ and summing them up yields

$$\sigma_m \geq \eta_m + \frac{1}{2} f_{z_0}(m).$$

Substituting this inequality to (3.22) yields

$$\nu_m \geq \psi_{pm} + \eta_m, \ \psi_{pm} = f_{z_0}(p-1) + f_{z_0}(p-3) + \cdots + f_{z_0}(m+2) + \frac{1}{2} f_{z_0}(m).$$

Note that the sum ψ_{pm} contains only terms $f_{z_0}(j)$ with $j \geq m$, while η_m contains only terms $f_{z_0}(j)$ with $j < m$. Thus, each term $f_{z_0}(j)$ in ψ_{pm} is greater than each term in η_m, as in the previous case (increasing of the function $f_z(s)$). This implies that the ratio $\frac{\psi_{pm}}{\eta_m}$ is greater than the ratio of the numbers of terms in ψ_{pm} and η_m respectively. (Here $\frac{1}{2}f_{z_0}(m)$ and a possible free term $1 = \frac{1}{2}f_{z_0}(0)$ are counted as half-terms, that is, with weight $\frac{1}{2}$.) Therefore, the same statement holds for the ratio $\frac{\psi_{pm}+\eta_m}{\eta_m}$, and hence, for the ratio $\frac{\nu_m}{\eta_m}$, since $\nu_m \geq \psi_{pm} + \eta_m$ and the number of terms in the expression (3.22) for the value ν_m equals the number of terms in the sum $\psi_{pm} + \eta_m$. This proves (3.20) and the proposition. □

Corollary 3.21 *Let $p, q \in \mathbb{N}$, $1 \leq q < p$, $r = \frac{p}{q}$. Let a polynomial $R_{p,q,c}$ have a root with q-th power $\mu > 1$. Then it has another root with q-th power $\theta \in (0,1)$ such that $\theta\mu > 1$.*

Proof Without loss of generality we consider that the polynomial $W(z) = R_{p,q,c}(z)$ has a real root $\mu^{\frac{1}{q}} > 1$. One can achieve this rescaling the variable z by multiplication by q-th root of unity: the collection of q-th powers of roots remains unchanged. Then $W(z)$ satisfies the condition of Proposition 3.20, which in its turn immediately implies the statement of the corollary. □

As was already noticed above, the set $\mathcal{D}$ contains the point $\mu_1 = z_r^{-1} = \frac{r}{r-1} > 1$. Hence, the latter is a q-th power of root of a polynomial W_k, say W_2. Then W_2 has a root with q-th power $\theta_2 \in (0,1)$ such that $\theta_2\mu_1 > 1$, by Corollary 3.21. Set $\mu_2 = \theta_2^{-1}$. Thus, $\mu_1 > \mu_2 > 1$, by definition and the previous inequality. One has $\mu_2 \in \mathcal{D}$, by statement (3.17) and invariance of the set $\mathcal{D}$ under taking inverse. Hence, μ_2 is again a q-th power of root of some polynomial W_k, say W_3. Similarly, W_3 has another root with q-th power $\theta_3 \in (0,1)$ such that $\theta_3\mu_2 > 1$. Then $\mu_3 = \theta_3^{-1} \in \mathcal{D}$ is again a q-th power of root of a polynomial W_k, say W_4, and $\mu_1 > \mu_2 > \mu_3 > 1$ etc. Finally we get an infinite sequence $\mu_1 > \mu_2 > \mu_3 > \dots$ of points of a finite set $\mathcal{D}$. The contradiction thus obtained proves Theorem 3.3.

3.6 Concentration of intersection. Proof of Theorem 3.4

The points of intersection $T_Pb \cap \Gamma$ tend to some limits: points of the intersection $T_Ab \cap \Gamma$, as $P \to A$, by analyticity. Let $\mathcal{X}_P$ denote the divisor in T_Pb of those points in $T_Pb \cap \Gamma$ whose limits are distinct from the point A. Consider the divisor Ψ_P formed by the points in $T_Pb \cap \Gamma$ with moderate z-asymptotics, which tend to A by definition. Let $\mathcal{Z}_P$ denote the divisor of the points in $T_Pb \cap \Gamma$ that converge to A but do not have moderate z-asymptotics. The divisor $T_Pb \cap \Gamma$ is the sum $\mathcal{X}_P + \Psi_P + \mathcal{Z}_P$, and the sets of points in the three latter divisors are disjoint. Recall that the *degree* of a divisor $\mathcal{D}$ in T_Pb, which we will denote by $|\mathcal{D}|$, is the total number of its points with multiplicities. The local intersection index $(T_Ab, \Gamma)_A$ under question equals $|\Psi_P| + |\mathcal{Z}_P|$. To estimate the contributions of the latter degrees, we use the following claim.

Claim 1. *The points of the divisor $\mathcal{X}_P$ are angular-symmetric to some points of the divisor $\mathcal{Z}_P$, and $|\mathcal{Z}_P| \geq |\mathcal{X}_P|$.*

Proof A point s of the divisor $T_Pb \cap \Gamma$ whose limit is distinct from the point A is angular-symmetric to a point s^* of the same divisor whose limit is A, and the multiplicities of both points are equal. This follows from angular symmetry of the divisor $T_Pb \cap \Gamma$ and uniform convergence of the angular symmetries of the lines T_Pb written in the coordinate z to the constant mapping $\overline{\mathbb{C}} \mapsto 0$ on compact subsets in $\overline{\mathbb{C}} \setminus \{0\}$, see Corollary 3.17. The point s^* is not contained in the divisor Ψ_P, by its invariance under the angular symmetry, see Corollary 3.17, and since its points tend to A, as $P \to A$. Hence, s^* is contained in $\mathcal{Z}_P$. This proves the claim. $\square$

For every $P \in b$ close enough to A one has

$$D = (T_Ab, \Gamma) = |T_Pb \cap \Gamma| = |\Psi_P| + |\mathcal{Z}_P| + |\mathcal{X}_P| \leq |\Psi_P| + 2|\mathcal{Z}_P|$$

$$= 2(|\Psi_P| + |\mathcal{Z}_P|) - |\Psi_P| = 2(T_Ab, \Gamma)_A - |\Psi_P|,$$

by definition and Claim 1. Therefore,

$$(T_Ab, \Gamma)_A \geq \frac{D}{2} + \frac{1}{2}|\Psi_P|. \tag{3.23}$$

Claim 2. *One has $|\Psi_P| \geq 2k$. The equality may take place only if the germ b is quadratic and regular and there are no local branches $a \in \Gamma$, $a \neq b$, that are tangent to b and have the same Puiseux exponent, as b.*

Proof Recall that the curve Γ includes the germ b with multiplicity k. The intersection points of the line T_Pb with b and with the above branches $a \in \Gamma$ (if any) have moderate z-asymptotics (Proposition 3.8), and hence, are contained in the divisor Ψ_P. For every P close to A the intersection index of the curve b with the line T_Pb equals its intersection index with T_Ab. The latter index is greater or equal to 2, and the equality holds if and only if the germ b is quadratic and regular. Therefore, the total contribution of the germ b and the above branches a to the degree of the ambient divisor

Ψ_P is no less than $2k$, and the equality takes place only if b is quadratic and regular and there are no other branches $a \in \Gamma$ tangent to b with the same Puiseux exponent. This implies the statement of the claim. $\Box$

Claim 2 together with (3.23) implies the statement of Theorem 3.4.

3.7 Quadraticity and regularity. Proof of Theorem 1.49

Let f, k, $F_1 = f^k g_1$, $g = g_1^{\frac{1}{k}}$, $H(f)$, s, m, D be the same, as at the beginning of Subsection 1.6. Recall that $\gamma = \{f = 0\}$. Let $b_1, b_2, \dots, b_N$ be the irreducible components of the germ at A of the zero locus of the polynomial F_1, $b = b_1$,

$$J = \{j \mid b_j \subset \gamma\} \subset \{1, \dots, N\},\ 1 \in J,$$

$$\rho_j = \begin{cases} r_{b_j}, \text{ if } b_j \text{ is tangent to } b \\ 1, \text{ if } b_j \text{ is not tangent to } b \end{cases}, \quad r = r_{b_1} = r_b = \rho_1. \tag{3.24}$$

Let Δ denote the zero divisor of the polynomial $F_1 = f^k g_1$. For every j let k_j denote the multiplicity of the curve b_j in the divisor Δ: $k_j = k$ for $j \in J$. Let (z, w) be adapted affine coordinates to b. Recall that

$$g^3(x,y)H(f)(x,y) = c(x^2+y^2)^{3m-3},\ c = const \neq 0, \tag{3.25}$$

and the values of the latter right- and left-hand sides at $P \in b$ have the following asymptotics, as $P \to A$ along the curve b:

$$(x^2+y^2)^{3m-3}|_b \simeq c' z^{\mu}, \quad \mu = 3\frac{D}{2k} - 3, \quad c' = const \neq 0, \tag{3.26}$$

$$g^3 H(f)|_b = O(z^{\eta}),$$

$$\eta = 3(\sum_{j\in J} q_{b_j} \min\{\rho_j, r\} + \sum_{j\notin J} \frac{k_j}{k} q_{b_j} \min\{\rho_j, r\}) - 2(r+1), \tag{3.27}$$

see (1.8) and already proved formulas (1.13) and (1.12). Let us show that $\eta > \mu$, unless b is quadratic and regular. This together with (3.26) and (3.27) will yield a contradiction to (3.25) and prove Theorem 1.49.

We already know that

$$r \leq 2, \quad \rho_j \leq r. \tag{3.28}$$

The first inequality follows from Theorems 3.2. The second one follows from the inequality $\rho_j = 1 < r$ for branches b_j transverse to b and the inequality

$\rho_j = r_{b_j} \leq r$ for branches b_j tangent to b (Theorem 3.3). Substituting inequalities (3.28) to formula (3.27) for the number η yields

$$\eta \geq 3(\sum_{j\in J} q_{b_j}\rho_j + \sum_{j\notin J} \frac{k_j}{k} q_{b_j}\rho_j) - 6. \tag{3.29}$$

Let $(\Delta, T_A b)_A$ denote the local intersection index of the divisor Δ with the line $T_A b$ at A. One has

$$(\Delta, T_A b)_A = \sum_{j=1}^{N} k_j q_{b_j}\rho_j, \tag{3.30}$$

by definition and since the local intersection index of a branch b_j with $T_A b$ at A equals respectively q_{b_j}, if b_j is transverse to $T_A b$, and $p_{b_j} = q_{b_j} r_{b_j}$, if it is tangent to b. On the other hand,

$$(\Delta, T_A b)_A \geq \frac{D}{2} + k, \ D = degF_1 = (\Delta, T_A b),$$

and the equality may take place only if b is quadratic and regular, by Theorem 3.4. Let b be not quadratic and regular, thus the latter inequality be strict. Substituting it to (3.30) yields

$$\sum_{j=1}^{N} k_j q_{b_j}\rho_j > \frac{D}{2} + k.$$

Substituting this inequality to (3.29) yields

$$\eta > \frac{3}{k}(\frac{D}{2} + k) - 6 = \frac{3D}{2k} - 3 = \mu.$$

This together with the above discussion proves Theorem 1.49.

3.8 Local branches at O: proof of Theorem 1.48

Recall that every local branch of the curve γ has local relative angular symmetry property with respect to a bigger algebraic curve: the zero locus of the integral of the angular billiard. We prove the following generalization of Theorem 1.48.

Theorem 3.22 *Let b be a nonlinear irreducible germ of analytic curve at $O \in \mathbb{C}^2 \subset \mathbb{CP}^2$ that is transverse to both isotropic lines through O. Let b have local relative angular symmetry property with respect to a finite collection Γ containing b of irreducible germs of analytic curves at some points of the projective tangent line $T_A b$. Then the germ b is quadratic.*

Proof The proof of the theorem given below is analogous to the proof of [15, theorem 1.16], see subsection 2.3 in loc. cit. Let us choose coordinates (z, w) adapted to b. Let Λ_1, Λ_2 be the isotropic lines through O. The proof of the theorem is done into the three following steps.

Step 1. We study the angular symmetries of the projective lines $T_P b$ as a family of conformal involutions $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ in the coordinate

$$\widetilde{z} = \frac{z(P)}{z} \tag{3.31}$$

on the lines $T_P b$. We prove the following analogue of Proposition 3.6.

Proposition 3.23 *The angular symmetry $\sigma_P : T_P b \to T_P b$ with respect to the point P and center O written in the coordinate $\widetilde{z}$ as a conformal involution $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ converges to the central symmetry with respect to 1, that is, to $\sigma_0 : \widetilde{z} \mapsto 2 - \widetilde{z}$, as $P \to O$.*

Proof For every $P \in b$ let $Q_j(P)$, $j = 1, 2$ denote the intersection points of the projective tangent line $T_P b$ with the isotropic lines Λ_j. The angular symmetry $\sigma_P : T_P b \to T_P b$ fixes P and permutes the points $Q_j(P)$. One has

$$z(Q_j) = o(z(P)) \text{ for } j = 1, 2, \tag{3.32}$$

which follows from Proposition 3.7 applied to $a = \Lambda_j$, see the first asymptotic formula for the intersection points in (3.3). In the coordinate $\widetilde{z}$ one has

$$\widetilde{z}(P) = 1, \quad \widetilde{z}(Q_j(P)) \to \infty, \text{ as } P \to O,$$

by (3.32). The involution σ_P written in the coordinate $\widetilde{z}$ fixes 1 and permutes the points $\widetilde{z}(Q_j(P))$, which tend to infinity, as $P \to O$. Therefore, σ_P tends to a nontrivial conformal involution $\sigma_0 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ fixing 1 and ∞. The only involution satisfying the latter conditions is the central symmetry with respect to the point 1. This proves the proposition. $\square$

Remark 3.24 The intersection $T_P b \cap \Gamma$ is finite, and the $\widetilde{z}$-coordinates of its points tend to some limits in $\overline{\mathbb{C}}$ (finite or infinite), as $P \to O$, by analyticity. The collection of the latter limits will be treated as a divisor in $\overline{\mathbb{C}}$, denoted by Ψ and called the *limit divisor.* The multiplicity of each its point ζ equals the total multiplicity of the intersection points whose $\widetilde{z}$-coordinates converge to ζ

Corollary 3.25 *The limit divisor Ψ is symmetric with respect to the involution σ_0.*

The corollary follows from definition and angular symmetry of the intersection $T_Pb \cap \Gamma$ (relative angular symmetry property).

Step 2. Explicit description of the limit divisor Ψ and corollaries of its symmetry. It is based on the following proposition.

Proposition 3.26 *(cf. [15, proposition 2.5].) Let a and b be irreducible germs of analytic curves at a point $O \in \mathbb{C}^2$. Set $r = r_b$. Let (z, w) be affine coordinates adapted to b, and let $\widetilde{z}$ be the coordinate (3.31) on the lines T_Pb. The limits of $\widetilde{z}$-coordinates of points of the intersection $a \cap T_Pb$ are the following:*

Case (i): either a is transverse to b, or a is tangent to b and has smaller Puiseux exponent $r_a < r = r_b$. Each limit is ∞.

Case (ii): a is tangent to b and has bigger Puiseux exponent $r_a > r$. The limits are $\frac{r}{r-1}$ and 0.

Case (iii): the germ a is tangent to b and $r_a = r$. Let p_a, q_a, c_a, c_b be respectively the exponents and the constants in the local parametrizations (1.3) of the germs a and b in their common adapted coordinates (z, w). The limits are θ^{q_a}, where θ runs through the roots of the polynomial

$$H_{p_a,q_a,c} = (r-1)\theta^{p_a} - r\theta^{p_a-q_a} + c, \ c = \frac{c_a}{c_b}. \tag{3.33}$$

The proposition follows immediately from Propositions 3.7 and 3.8 and the equality $R_{p_a,q_a,c}(\zeta) = \zeta^{p_a} H_{p_a,q_a,c}(\zeta^{-1})$.

Corollary 3.27 *Let $a_1, \dots, a_N$ denote all the local branches in Γ (including b) that are tangent to b and have the same Puiseux exponent $r = r_b$. Let $M \subset \mathbb{C}$ denote the collection of those q_{a_i}-th powers of roots of the polynomials $H_{p_{a_i},q_{a_i},c(i)}$, $c(i) = \frac{c_{a_i}}{c_b}$, $i = 1, \dots, N$, that are different from 2 and $2 - \frac{r}{r-1} = \frac{r-2}{r-1}$. We treat M as a divisor: we take each its point with multiplicity that equals the sum of multiplicities of the corresponding roots. The limit divisor Ψ equals*

$$\Psi = k_\infty[\infty] + k_1([0]+[2]) + k_2([\frac{r}{r-1}] + [\frac{r-2}{r-1}]) + M; \quad k_\infty, k_1, k_2 \geq 0.$$

The divisor M is invariant under the symmetry $\sigma_0(z) = 2 - z$.

A version of Corollary 3.27 was proved in [15, proposition 2.6].

Proof The corollary follows immediately from the symmetry of the divisor Ψ (Corollary 3.25), Proposition 3.26, symmetry of pairs $(0, 2)$ and $(\frac{r}{r-1}, \frac{r-2}{r-1})$ and the fact that 0 and $\frac{r}{r-1}$ are not q-th powers of roots of polynomials $H_{p,q,c}$

with $\frac{p}{q} = r$ and $c \neq 0$: that is, $H_{p,q,c}(\theta) \neq 0$ for $\theta \in \{0, (\frac{r}{r-1})^{\frac{1}{q}}\}$. For $\theta = 0$ this is obvious. For $\theta = (\frac{r}{r-1})^{\frac{1}{q}}$ one has

$$H_{p,q,c}(\theta) = \theta^{p-q}((r-1)\theta^q - r) + c = c \neq 0.$$

$\square$

Lemma 3.28 *[15, lemma 2.7]. Let $r > 1$. Consider a collection $S_r = \{(p_i, q_i, c_i)\}_{i=1,\dots,N}$ with $q_i, p_i \in \mathbb{N}$, $p_i > q_i$, $\frac{p_i}{q_i} = r$, $c_i \in \mathbb{C} \setminus \{0\}$, set $W_i(\theta) = (r-1)\theta^{p_i} - r\theta^{p_i - q_i} + c_i$. Let θ_{ij} ($j = 1, \dots, p_i$) denote the roots of the polynomials W_i. Let M denote the divisor of those q_i-th powers of roots θ_{ij} that are different from 2 and $\frac{r-2}{r-1}$: each power being taken with the total multiplicity of the corresponding roots. Let M be invariant under the symmetry of the line $\mathbb{C}$ with respect to 1. Then $r = 2$.*

Lemma 3.28 together with the previous corollary imply Theorem 3.22 and hence, Theorem 1.48. $\square$

4 Generalized genus and Plücker formulas. Proof of Theorem 1.50

The proof of Theorem 1.50 is based on generalized Plücker and genus formulas for planar algebraic curves and their corollaries presented in [15, subsection 3.1]. The main observation is that the assumptions of the theorem on the projective Puiseux exponents of local branches of the curve and Plücker formulas yield that the singularity invariants of the considered curve γ must obey a relatively high lower bound. On the other hand, the contribution of the points in the union of two lines $\Lambda_1 \cup \Lambda_2$ appears to be not sufficient to fit that lower bound, unless the curve is a conic.

4.1 Invariants of plane curve singularities

The material of the present subsection is contained in [15, subsection 3.1]. It recalls classical results on invariants of singularities presented in [11, Chapter III], [22, §10], see also a modern exposition in [16, Section I.3]. Let $\gamma \subset \mathbb{CP}^2$ be a non-linear irreducible algebraic curve[6]. Let d denote its degree. The intersection index of the curve γ with its Hessian H_γ equals $3d(d-2)$,

[6]Everything stated in the present subsection holds for every algebraic curve in $\mathbb{CP}^2$ with no multiple components and no straight-line components, see [25, theorem 1].

by Bézout Theorem. On the other hand, it is equal to the sum of the contributions $h(\gamma, A)$, which are called the *Hessians of the germs* (γ, A) through all the singular and inflection points A of the curve γ:

$$3d(d-2) = \sum_{A\in\gamma} h(\gamma, A). \tag{4.1}$$

An explicit formula for the Hessians $h(\gamma, A)$ was found in [25, formula (2) and theorem 1], see also [15, formula (3.4)]. To recall it, let us introduce the following notations. For every local branch b of the curve γ at A let $s(b)$ denote its multiplicity: its intersection index with a generic line through A. Let $s^*(b)$ denote the analogous multiplicity of the dual germ. Note that

$$s(b) = q, \quad s^*(b) = p - q,$$

where p and q are the exponents in the parametrization $t \mapsto (t^q, c_b t^p(1+o(1))$ of the local branch b in adapted coordinates. Thus,

$$s(b) = s^*(b) \text{ if and only if } b \text{ is quadratic,} \tag{4.2}$$

$$s(b) \geq s^*(b) \text{ if and only if } b \text{ is subquadratic.} \tag{4.3}$$

Let $b_{A1}, \dots, b_{An(A)}$ denote the local branches of the curve γ at A; here $n(A)$ denotes their number. The above-mentioned formula for $h(\gamma, A)$ from both loc. cit. has the form

$$h(\gamma, A) = 3\kappa(\gamma, A) + \sum_{j=1}^{n(A)} (s^*(b_{Aj}) - s(b_{Aj})), \tag{4.4}$$

where $\kappa(A)$ is the κ-invariant, the class of the singular point, see [15, subsection 3.1]. Namely, consider the germ of function f defining the germ (γ, A); $(\gamma, A) = \{f = 0\}$. Fix a line L through A that is transverse to all the local branches of the curve γ at A. Fix a small ball $U = U(A)$ centered at A and consider a level curve $\gamma_\varepsilon = \{f = \varepsilon\} \cap U$ with small $\varepsilon \neq 0$, which is non-singular. The number $\kappa(A)$ is the number of points of the curve γ_ε where its tangent line is parallel to L. (One has $\kappa(A) = 0$ for nonsingular points A.) It is well-known that

$$\kappa(\gamma, A) = 2\delta(\gamma, A) + \sum_{j=1}^{n(A)} (s(b_{Aj}) - 1), \tag{4.5}$$

see, for example, [15, subsection 3.1, formula (3.3)], where $\delta(A)$ is the δ-invariant (whose definition is recalled in the same subsection). Namely, consider the curve γ_ε, which is a Riemann surface whose boundary is a finite number of closed curves: their number equals $n(A)$. Let us take the 2-sphere with $n(A)$ deleted disks. Let us paste it to γ_ε: this yields a compact surface. By definition, its genus is the δ-invariant $\delta(A)$. One has $\delta(A) \geq 0$, and $\delta(A) = 0$ whenever A is a non-singular point. Hironaka's genus formula [18] implies that

$$\sum_{A \in \mathrm{Sing}(\gamma)} \delta(\gamma, A) \leq \frac{(d-1)(d-2)}{2}. \tag{4.6}$$

Formulas (4.1), (4.4) and (4.5) together imply that

$$3d(d-2) = 6\sum_A \delta(\gamma, A) + 3\sum_A \sum_{j=1}^{n(A)} (s(b_{Aj}) - 1) + \sum_A \sum_{j=1}^{n(A)} (s^*(b_{Aj}) - s(b_{Aj})).$$

The first term in latter right-hand side is no greater than $3(d-1)(d-2)$, by inequality (4.6). This implies that

$$3d(d-2) - 3(d-1)(d-2)$$

$$= 3(d-2) \leq 3\sum_A \sum_{j=1}^{n(A)} (s(b_{Aj}) - 1) + \sum_A \sum_{j=1}^{n(A)} (s^*(b_{Aj}) - s(b_{Aj})). \tag{4.7}$$

4.2 Proof of Theorem 1.50

The proof of Theorem 1.50 is done by a modified version of Eugenii Shustin's arguments from [15, subsection 3.2]. We know that all the singular and inflection points of the curve γ (if any) lie in $\mathbb{I} = \Lambda_1 \cup \Lambda_2$. Set

$$\mathcal{B}_{tan} = \{\text{the local branches of } \gamma \text{ at points } A \in \mathbb{I} \setminus \{O\} \text{ tangent to } OA\},$$

$$\mathcal{B}_{O,tr} = \{\text{the branches of the curve } \gamma \text{ at } O \text{ transverse to both } \Lambda_1,\ \Lambda_2\},$$

$$\mathcal{B}_{O,tan,j} = \{\text{the branches of the curve } \gamma \text{ at } O \text{ tangent to } \Lambda_j\},$$

$$\mathcal{B}_{O,tan} = \cup_{j=1,2}\mathcal{B}_{O,tan,j},\ \mathcal{B}_O = \mathcal{B}_{O,tr} \cup \mathcal{B}_{O,tan}.$$

All the local branches $b \notin \mathcal{B}_{O,tan}$ of the curve γ at points in $\gamma \cap \mathbb{I}$ are subquadratic, by conditions (i)–(iii). Therefore, their contributions $s^*(b) - s(b)$ to the right-hand side in (4.7) are non-positive, by (4.3). Every local branch $b \notin (\mathcal{B}_{tan} \cup \mathcal{B}_O)$ is regular, by condition (iii), hence its contribution $s(b) - 1$ to (4.7) vanish. This together with (4.7) implies that

$$d - 2 \leq \sum_{b \in \mathcal{B}_{tan} \cup \mathcal{B}_{O,tr} \cup \mathcal{B}_{O,tan}} (s(b) - 1) + \frac{1}{3} \sum_{b \in \mathcal{B}_{O,tan}} (s^*(b) - s(b))$$

$$= \sum_{b \in \mathcal{B}_{tan} \cup \mathcal{B}_{O,tr} \cup \mathcal{B}_{O,tan}} s(b) - |\mathcal{B}_{tan}| - |\mathcal{B}_{O,tr}| - |\mathcal{B}_{O,tan}| + \frac{1}{3} \sum_{b \in \mathcal{B}_{O,tan}} (s^*(b) - s(b)), \tag{4.8}$$

where $|\mathcal{B}_s|$, $s \in \{tan, (O, tr), (O, tan)\}$ denote the cardinalities of the sets $\mathcal{B}_s$.

Let us estimate the right-hand side in (4.8) from above. To do this, we use the next inequality, which follows from Bézout Theorem and conditions (i)–(iii).

In what follows for every $j = 1, 2$ by $\mathcal{B}_{reg,j}$ we denote the collection of the local branches of the curve γ at points in $\Lambda_j \setminus \{O\}$ that are transverse to Λ_j. Recall that they are regular, by condition (iii). Set

$$\nu_j = |\mathcal{B}_{reg,j}|,$$

$$\mathcal{B}_{tan,j} = \{b \in \mathcal{B}_{tan} \mid b \text{ is tangent to } \Lambda_j\}, \ \mathcal{B}_{tan} = \mathcal{B}_{tan,1} \sqcup \mathcal{B}_{tan,2}.$$

Claim 1. *For every $j = 1, 2$ one has*

$$\sum_{b \in \mathcal{B}_{tan,j}} s(b) + \frac{1}{2} \sum_{b \in \mathcal{B}_{O,tan,3-j}} s(b) + \frac{1}{2} \sum_{b \in \mathcal{B}_{O,tr}} s(b)$$

$$+ \frac{\nu_j}{2} + \frac{1}{2} \sum_{b \in \mathcal{B}_{O,tan,j}} (s^*(b) + s(b)) = \frac{d}{2}. \tag{4.9}$$

Proof The intersection index of the curve γ with each line Λ_j equals d (Bézout Theorem). It is the sum of the intersection indices of the line Λ_j with the branches from the collections $\mathcal{B}_{tan,j}$, $\mathcal{B}_{O,tr}$, $\mathcal{B}_{O,tan}$, $\mathcal{B}_{reg,j}$. Let us calculate the latter indices. The contribution of each branch from $\mathcal{B}_{reg,j}$ equals one, by regularity (condition (iii)) and transversality. The intersection index of each branch $b \in \mathcal{B}_{O,tr}$ with Λ_j equals $s(b)$. The intersection index with Λ_j of each branch $b \in \mathcal{B}_{tan,j}$ equals $p_b = 2s(b)$, by quadraticity (condition (ii)). The intersection index with Λ_j of each branch $b \in \mathcal{B}_{O,tan,j}$ equals $p_b = s(b) + s^*(b)$. The remaining branches $b \in \mathcal{B}_{O,tan,3-j}$ are transversal to Λ_j, and their intersection indices with Λ_j are equal to $s(b)$. Summing up the above intersection indices, writing that their sum should be equal to d and dividing the equality thus obtained by two yields (4.9). $\Box$

Summing up equalities (4.9) for both $j = 1, 2$ yields

$$\sum_{b\in\mathcal{B}_{tan}\cup\mathcal{B}_{O,tr}\cup\mathcal{B}_{O,tan}} s(b) = d - \frac{1}{2}\sum_{b\in\mathcal{B}_{O,tan}} s^*(b) - \frac{\nu_1+\nu_2}{2}. \tag{4.10}$$

Substituting equality (4.10) to (4.8) together with elementary inequalities yields

$$d - 2 \leq d - \frac{1}{2}\sum_{b\in\mathcal{B}_{O,tan}} s^*(b) - \frac{\nu_1+\nu_2}{2} - |\mathcal{B}_{tan}| - |\mathcal{B}_{O,tr}| - |\mathcal{B}_{O,tan}|$$

$$+\frac{1}{3}\sum_{b\in\mathcal{B}_{O,tan}} (s^*(b) - s(b)) = d - |\mathcal{B}_{tan}| - |\mathcal{B}_{O,tr}| - |\mathcal{B}_{O,tan}|$$

$$-\frac{\nu_1+\nu_2}{2} - \sum_{b\in\mathcal{B}_{O,tan}} (\frac{1}{6}s^*(b) + \frac{1}{3}s(b)),$$

$$|\mathcal{B}_{tan}| + |\mathcal{B}_{O,tr}| + |\mathcal{B}_{O,tan}| + \frac{\nu_1+\nu_2}{2} + \sum_{b\in\mathcal{B}_{O,tan}} (\frac{1}{6}s^*(b) + \frac{1}{3}s(b)) \leq 2. \tag{4.11}$$

Claim 2. *The total cardinality of the set of singular and inflection points of the curve γ is at most two. There are two possible cases:*

- either there are no inflection points and each local branch at every singular point is subquadratic;

- or there is just one special point (singular or inflection point) and one local branch at it.

Proof Let Φ denote the collection of all the local branches of the curve γ at points in $\mathbb{I}$. Recall that $\mathbb{I}$ contains all the singular and inflection points of the curve γ.

Case 1): $\mathcal{B}_{O,tan} = \emptyset$. Then all the local branches in Φ are subquadratic, by (i)–(iii), and there are no inflection points; $|\mathcal{B}_{tan}| + |\mathcal{B}_{O,tr}| \leq 2$, by (4.11).

Subcase 1.1): $\mathcal{B}_{tan} = \mathcal{B}_{O,tr} = \emptyset$. Then all the branches in Φ are regular and quadratic, by (iii), and there are at most four of them: $\nu_1 + \nu_2 \leq 4$, by (4.11). Thus, the only possible candidates to be singular points of the curve γ are intersections of branches. Since the total number of branches under question is at most four, the number of singular points is at most two.

Subcase 1.2): $|\mathcal{B}_{tan}| + |\mathcal{B}_{O,tr}| = 1$. The branches from the complement $\Phi\setminus(\mathcal{B}_{tan}\cup\mathcal{B}_{O,tr})$ are transverse to the lines Λ_j, quadratic and regular, by (iii), and there are at most two of them: $\nu_1 + \nu_2 \leq 2$, by (4.11). Thus, Φ consists of at most three branches, and at most one of them is singular. Thus, the only possible candidates to be singular points of the curve γ are the base point of the unique branch from $\mathcal{B}_{tan} \cup \mathcal{B}_{O,tr}$ and a point of intersection of quadratic regular branches (if it is different from the latter base point). Finally, we have at most two singular points.

Subcase 1.3): $|\mathcal{B}_{tan}| + |\mathcal{B}_{O,tr}| = 2$. Then $\Phi = \mathcal{B}_{tan} \cup \mathcal{B}_{O,tr}$, by (4.11), the number of base points of the branches from the collection Φ is at most 2, and they are the only potential singular points.

Case 2): $|\mathcal{B}_{O,tan}| \geq 1$. Then $|\mathcal{B}_{O,tan}| = 1$, and $\Phi = \mathcal{B}_{O,tan}$. This follows from inequality (4.11) and positivity of the sum in $b \in \mathcal{B}_{O,tan}$ in its left-hand side. Thus, the set Φ consists of just one branch, and we have at most one singular (or inflection) point. The claim is proved. $\square$

Theorem 4.1 *[15, theorem 1.18]. Let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve such that there exists a projective line L satisfying the following statements:*

- all the singular and inflection points of the curve γ (if any) lie in L;

- each local branch of the curve γ at every point of intersection $\gamma \cap L$ that is transverse to L is subquadratic.

Then γ is a conic.

There exists a line L satisfying the conditions of Theorem 4.1 for the curve γ under consideration. Namely, in the first case of Claim 2 the line L is the line passing though (at most two) singular points of the curve γ. In the second case we choose L to be the tangent line to the unique local branch at the unique special point. This together with Theorem 4.1 implies that γ is a conic. Theorem 1.50 is proved.

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