ON THE COHOMOLOGY OF THE AFFINE SPACE

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by

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Abstract. — We compute the p-adic geometric pro-étale cohomology of the rigid analytic affine space (in any dimension). This cohomology is non-zero, contrary to the étale cohomology, and can be described by means of differential forms.

Introduction

Let K be a complete discrete valuation field of characteristic 0 with perfect residue field of positive characteristic p. Let C be the completion of an algebraic closure $\overline{K}$ of K. We denote by $\mathscr{G}_K$ the absolute Galois group of K (it is also the group of continuous automorphisms of C that fix K).

For $n \geq 1$, let $\mathbf{A}^n_K$ be the rigid analytic affine space over K of dimension n and $\mathbf{A}^n$ be its scalar extension to C. Our main result is the following theorem.

Theorem 1. — *For $r \geq 1$, we have isomorphisms of $\mathscr{G}_K$-Fréchet spaces*

$$H^r_{\text{proét}}(\mathbf{A}^n, \mathbf{Q}_p(r)) \simeq \Omega^{r-1}(\mathbf{A}^n)/\operatorname{Ker} d \simeq \Omega^r(\mathbf{A}^n)_{d=0},$$

where Ω denotes the sheaf of differentials.

Remark 2. — (i) The p-adic pro-étale cohomology behaves in a remarkably different way from other (more classical) cohomologies. For example, for $i \geq 1$, we have :

- $H^i_{\text{dR}}(\mathbf{A}^n) = H^i_{\text{HK}}(\mathbf{A}^n) = 0$,
- $H^i_{\text{ét}}(\mathbf{A}^n, \mathbf{Q}_\ell) = H^i_{\text{proét}}(\mathbf{A}^n, \mathbf{Q}_\ell) = 0$, if $\ell \neq p$,
- $H^i_{\text{ét}}(\mathbf{A}^n, \mathbf{Q}_p) = 0$. (Cf. [**1**] or Remark 12.)

We listed separately the $\ell \neq p$ and $\ell = p$ cases of étale cohomology because, if $\ell \neq p$, the triviality of the cohomology of $\mathbf{A}^n$ is a consequence of the triviality of the cohomology of the closed ball (which explains why the pro-étale cohomology is also trivial), but the p-adic étale cohomology of the ball is highly nontrivial.

(ii) Overconvergent syntomic cohomology allows **[2]** to prove a more general result valid for a Stein space X with a semistable reduction over the ring of integers of K: there exists an exact sequence

$$0 \to \Omega^{r-1}(X)/\mathrm{Ker}\, d \to H^r_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \to (\mathrm{B}^+_{\mathrm{st}} \widehat{\otimes} H^r_{\mathrm{HK}}(X))^{N=0,\varphi=p^r} \to 0.$$

However making syntomic cohomology overconvergent is technically demanding and the simple proof below uses special features of the geometry of the affine space. Another possible approach (cf. **[5]**) is to compute pro-étale cohomology of the relative fundamental exact sequence $0 \to \mathbf{Q}_p(r) \to \mathbb{B}^{\varphi=p^r}_{\mathrm{cris}} \to \mathbb{B}_{\mathrm{dR}}/F^r \to 0$.

Let $\mathring{\mathbf{B}}^n$ be the open unit ball of dimension n. An adaptation of the proof of Theorem 1 shows the following result:

Theorem 3. — *For $r \geq 1$, we have isomorphisms of $\mathscr{G}_K$-Fréchet spaces*

$$H^r_{\mathrm{pro\acute{e}t}}(\mathring{\mathbf{B}}^n, \mathbf{Q}_p(r)) \simeq \Omega^r(\mathring{\mathbf{B}}^n)/\mathrm{Ker}\, d \simeq \Omega^r(\mathring{\mathbf{B}}^n)_{d=0}.$$

1. Syntomic variations

If $r = 1$, one can give an elementary proof of Theorem 1, using Kummer theory, but it does not seem very easy to extend this kind of methods to treat the case $r \geq 2$. Instead we are going to use syntomic methods.

Recall that the étale-syntomic comparison theorem **[6, 3]** reduces the computation of p-adic étale cohomology to that of syntomic cohomology [(1)]. More precisely, if $\mathscr{X}$ is a quasi-compact semistable p-adic formal scheme over $\mathscr{O}_K$, then the Fontaine-Messing period map **[4]**

$$\alpha^{FM} : \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_C}, \mathbf{Z}_p(r)) \to \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(\mathscr{X}_C, \mathbf{Z}_p(r)) \tag{4}$$

is a p^N-isomorphism [(2)] for a constant $N = N(r)$. This generalizes easily to semistable p-adic formal schemes over $\mathscr{O}_C$: the rational étale and pro-étale cohomology of such schemes are computed by the syntomic complexes $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_C}, \mathbf{Z}_p(r))_{\mathbf{Q}}$ and $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_C}, \mathbf{Q}_p(r))$, respectively, where the latter complex is defined by taking $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_C}, \mathbf{Z}_p(r))_{\mathbf{Q}}$ locally and then glueing.

The purpose of this section is to construct a particularly simple complex that, morally, computes the syntomic (hence (pro-)étale as well) cohomology of the (canonical formal model of the) affine space (and the open ball), but does not use a model of the whole space, only of closed balls of increasing radii.

1. The computations in **[3]** are done over K (or over its finite extensions), but working directly over C simplifies a lot the local arguments because there is no need to change the Frobenius and the group Γ becomes commutative (hence so does its Lie algebra, which makes the arguments using Koszul complexes a lot simpler).

2. It means that the kernel and cokernel of the induced map on cohomology are annihilated by p^N.

Period rings. — Let $C^\flat$ be the tilt of C and let $\theta : \mathrm{A}_{\mathrm{cris}} \to \mathscr{O}_C$ be the canonical projection. For $j \in \mathbf{Z}$, let $\mathrm{A}_j = \mathrm{A}_{\mathrm{cris}}/\mathrm{F}^j_\theta$, where $\mathrm{F}^j_\theta \mathrm{A}_{\mathrm{cris}} = \mathrm{A}_{\mathrm{cris}} \cap t^j \mathrm{B}^+_{\mathrm{dR}}$ (hence $\mathrm{A}_j = 0$ for $j \leq 0$ and $\mathrm{A}_1 = \mathscr{O}_C$). We choose a morphism of groups $\alpha \mapsto p^\alpha$ from $\mathbf{Q}$ to C^* compatible with the analogous morphism on $\mathbf{Z}$. We denote by $\tilde{p}^\alpha$ the element $(p^\alpha, p^{\alpha/p}, p^{\alpha/p^2}, \dots)$ of $C^\flat$ and by $[\tilde{p}^\alpha]$ its Teichmüller lift in $\mathrm{A}_{\mathrm{cris}}$.

Closed balls. — For $\alpha \in \mathbf{Q}_+$, let D_α be the closed ball in $\mathbf{A}^n$ $v_p(z_m) \geq -\alpha$, for $1 \leq m \leq n$, and denote by $\mathscr{O}(D_\alpha)$ (resp. $\mathscr{O}^+(D_\alpha)$) the ring of analytic functions (resp. analytic functions with integral values) on D_α. We have

$$\mathscr{O}(D_\alpha) = C\langle p^\alpha T_1, \dots, p^\alpha T_n\rangle \quad \text{et} \quad \mathscr{O}^+(D_\alpha) = \mathscr{O}_C\langle p^\alpha T_1, \dots, p^\alpha T_n\rangle.$$

Consider the lifts

$$R^+_\alpha = \mathrm{A}_{\mathrm{cris}}\langle [\tilde{p}^\alpha]T_1, \dots, [\tilde{p}^\alpha]T_n\rangle \quad \text{and} \quad R_\alpha = R^+_\alpha[\tfrac{1}{p}]$$

of $\mathscr{O}^+(D_\alpha)$ and $\mathscr{O}(D_\alpha)$, respectively. We extend φ on $\mathrm{A}_{\mathrm{cris}}$ to $\varphi : R_\alpha \to R_\alpha$ by setting $\varphi(T_m) = T_m^p$, for $1 \leq m \leq n$.

Definition 5. — Let $r \geq 0$. If $\alpha \in \mathbf{Q}_+$ and $\Lambda = R_\alpha, R^+_\alpha$, we define the complexes

$$\mathrm{Syn}(\Lambda, r) := [\mathrm{HK}_r(\Lambda) \to \mathrm{DR}_r(\Lambda)],$$

where the brackets $[\cdots]$ denote the mapping fiber and [3]

$$\mathrm{HK}_r(\Lambda) := [\Omega^\bullet_\Lambda \xrightarrow{\varphi - p^r} \Omega^\bullet_\Lambda],$$

$$\mathrm{DR}_r(\Lambda) := \Omega^\bullet_\Lambda / F^r = (\cdots \to \mathrm{A}_{r-i} \otimes_{\mathrm{A}_{\mathrm{cris}}} \Omega^i_\Lambda \xrightarrow{1 \otimes d_i} \mathrm{A}_{r-i-1} \otimes_{\mathrm{A}_{\mathrm{cris}}} \Omega^{i+1}_\Lambda \to \cdots).$$

The complex $\mathrm{Syn}(\mathbf{A}^n, r)$. — The above complexes for varying α are closely linked:

- The ring morphism $R_0 \to R_\alpha$, $T_m \to [\tilde{p}^\alpha]T_m$, for $1 \leq m \leq n$, induces an isomorphism of complexes $\mathrm{Syn}(R_0, r) \xrightarrow{\sim} \mathrm{Syn}(R_\alpha, r)$.
- For $\beta \geq \alpha$, the inclusion $\iota_{\beta,\alpha} : R_\beta \hookrightarrow R_\alpha$ induces a morphism of complexes $\mathrm{Syn}(R_\beta, r) \to \mathrm{Syn}(R_\alpha, r)$.

(We have analogous statements, replacing R_α by R^+_α.)

The first point comes just from the fact that two closed balls are isomorphic, but the second point, to the effect that we can find liftings of the $\mathscr{O}(D_\alpha)$'s with compatible Frobenius, is a bit of a miracle, and will simplify greatly the computation of the syntomic cohomology of $\mathbf{A}^n$. In particular, it makes it possible to define the complex $\mathrm{Syn}(\mathbf{A}^n, r) := \mathrm{holim}_\alpha \, \mathrm{Syn}(R_\alpha, r)$ and, similarly, $\mathrm{HK}_r(\mathbf{A}^n)$ and $\mathrm{DR}_r(\mathbf{A}^n)$.

For $i \geq 0$ and $X = \mathbf{A}^n, R_\alpha, R^+_\alpha$, denote by $\mathrm{HK}^i_r(X)$, $\mathrm{DR}^i_r(X)$, and $\mathrm{Syn}^i(X, r)$ the cohomology groups of the corresponding complexes. We have a long exact sequence:

$$\cdots \to \mathrm{DR}^{i-1}_r(X) \to \mathrm{Syn}^i(X, r) \to \mathrm{HK}^i_r(X) \to \mathrm{DR}^i_r(X) \to \cdots$$

3. The differentials are taken relative to $\mathrm{A}_{\mathrm{cris}}$.

Proposition 6. — *If $i \leq r$, we have natural isomorphisms:*
- $H^i_{\mathrm{\acute{e}t}}(D_\alpha, \mathbf{Q}_p(r)) \cong \mathrm{Syn}^i(R_\alpha, r)$, *if* $\alpha \in \mathbf{Q}_+$.
- $H^i_{\mathrm{pro\acute{e}t}}(\mathbf{A}^n, \mathbf{Q}_p(r)) \cong \mathrm{Syn}^i(\mathbf{A}^n, r)$.

Proof. — Take $\alpha \in \mathbf{Q}_+$. By the comparison isomorphism (4), to prove the first claim, it suffices to show that the complex $\mathrm{Syn}(R_\alpha, r)$ computes the rational geometric log-syntomic cohomology of $\mathscr{D}_\alpha := \mathrm{Spf}\, \mathscr{O}^+(D_\alpha)$, the formal affine space over $\mathscr{O}_C$, that is a smooth formal model of D_α. To do this, recall that the latter cohomology is computed by the complex

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{D}_\alpha, \mathbf{Z}_p(r))_{\mathbf{Q}} = [\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{D}_\alpha/\mathrm{A}_{\mathrm{cris}})^{\varphi=p^r}_{\mathbf{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{D}_\alpha/\mathrm{A}_{\mathrm{cris}})_{\mathbf{Q}}/F^r],$$

where $\mathrm{A}_{\mathrm{cris}}$ is equipped with the unique log-structure extending the canonical log-structure on $\mathscr{O}_C/p$. It suffices thus to show that there exists a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{D}_\alpha/\mathrm{A}_{\mathrm{cris}})_{\mathbf{Q}} \simeq \Omega^\bullet_{R_\alpha}$ that is compatible with the Frobenius [(4)] and the filtration. But this is clear since $\mathrm{Spf}\, R^+_\alpha$ is a log-smooth lifting of $\mathscr{D}_\alpha$ from $\mathrm{Spf}\, \mathscr{O}_C$ to $\mathrm{Spf}\, \mathrm{A}_{\mathrm{cris}}$ that is compatible with the Frobenius on $\mathrm{A}_{\mathrm{cris}}$ and $\mathscr{O}^+(D_\alpha)/p$.

To show the second claim, we note that, for $\beta \geq \alpha$, there is a natural map (an injection) of liftings $(R^+_\beta \to \mathscr{O}^+(D_\beta)) \to (R^+_\alpha \to \mathscr{O}^+(D_\alpha))$. This allows us to use the comparison isomorphism (4) to define the second quasi-isomorphism in the sequence of maps

$$\begin{aligned}\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{A}^n, \mathbf{Q}_p(r)) \simeq \tau_{\leq r}\, \mathrm{holim}_k\, \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(D_k, \mathbf{Q}_p(r)) \simeq \tau_{\leq r}\, \mathrm{holim}_k\, \mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{D}_k, \mathbf{Z}_p(r))_{\mathbf{Q}} \\ \simeq \tau_{\leq r}\, \mathrm{holim}_k\, \mathrm{Syn}(R_k, r) = \tau_{\leq r}\mathrm{Syn}(\mathbf{A}^n, r).\end{aligned}$$

Here, the first quasi-isomorphism follows from the fact that $\{D_k\}_{k\in\mathbf{N}}$ is an admissible affinoid covering of $\mathbf{A}^n$ and the third one follows from the first claim. This finishes the proof. $\square$

2. Computation of $\mathrm{HK}^i_r(\mathbf{A}^n)$

The group $\mathrm{HK}^i_r(\mathbf{A}^n)$ is, by construction, obtained from the $\mathrm{HK}^i_r(R_\alpha)$'s, but the latter are, individually, hard to compute and quite nasty: for example, $\mathrm{HK}^1_1(R_\alpha)$ is the quotient of $\mathbf{Q}_p\widehat{\otimes} R^{\boldsymbol{*}}_\alpha$ by the sub $\mathbf{Q}_p$-vector space generated by $R^{\boldsymbol{*}}_\alpha$; hence it is an infinite dimensionnal $\mathbf{Q}_p$-topological vector space in which 0 is dense... Fortunately Lemma 8 below shows that this is not a problem for the computation of $\mathrm{HK}^i_r(\mathbf{A}^n)$.

For $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbf{N}^n$, we set $|\mathbf{k}| = k_1 + \cdots + k_n$ and $T^{\mathbf{k}} = T_1^{k_1}\cdots T_n^{k_n}$. For $1 \leq j \leq n$, let ω_j be the differential form $\frac{dT_j}{T_j}$, and let ∂_j be differential operator defined by $df = \sum_{j=1}^n \partial_j f\, \omega_j$. For $\mathbf{j} = \{j_1, \ldots, j_i\}$, with $j_1 \leq j_2 \leq \cdots \leq j_i$, we set $\omega_{\mathbf{j}} = \omega_{j_1} \wedge \cdots \wedge \omega_{j_i}$. All elements η of $\Omega^i_{R_\alpha}$ can be written, in a unique way, in the form $\sum_{|\mathbf{j}|=i} a_{\mathbf{j}}\omega_{\mathbf{j}}$, where $a_{\mathbf{j}} \in \big(\prod_{j\in\mathbf{j}} T_j\big) R_\alpha$.

4. Recall that the Frobenius on crystalline cohomology is defined via the isomorphism $\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{D}_\alpha/\mathrm{A}_{\mathrm{cris}})_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}((\mathscr{D}_{\alpha,}/p)/\mathrm{A}_{\mathrm{cris}})_{\mathbf{Q}}$ from the canonical Frobenius on the second term.

Lemma 7. — *Let M be a sub-$\mathbf{Z}_p$-module of $\mathrm{A}_{\mathrm{cris}}$ or $\mathscr{O}_C$. Let $\mathbf{k}\in\mathbf{N}^n$. For $\omega = T^{\mathbf{k}}\sum_{|\mathbf{j}|=i} a_{\mathbf{j}}\omega_{\mathbf{j}}$, with $a_{\mathbf{j}}\in M$, such that $d\omega=0$, there exists $\eta = T^{\mathbf{k}}\sum_{|\mathbf{j}|=i-1} b_{\mathbf{j}}\omega_{\mathbf{j}}$, such that $d\eta=\omega$ and $b_{\mathbf{j}}\in p^{-N(\mathbf{k})}M$, with $N(\mathbf{k})=\inf_{j\in\mathbf{j}} v_p(k_j)$.*

Proof. — Permuting the T_m's, we can assume that $v_p(k_1)\leq v_p(k_2)\leq\cdots\leq v_p(k_n)$; in particular, $k_1\neq 0$. Decompose ω as $\omega_1\wedge\sum_{1\in\mathbf{j}} a_{\mathbf{j}}\omega_{\mathbf{j}\setminus\{1\}}+\omega'$, and set $\eta=\frac{1}{k_1}T^{\mathbf{k}}\sum_{1\in\mathbf{j}} a_{\mathbf{j}}\omega_{\mathbf{j}\setminus\{1\}}$; we have $\omega-d\eta = T^{\mathbf{k}}\sum_{1\notin\mathbf{j}} c_{\mathbf{j}}\omega_{\mathbf{j}}$ and it has a trivial differential. But $d(T^{\mathbf{k}}\sum_{1\notin\mathbf{j}} c_{\mathbf{j}}\omega_{\mathbf{j}}) = k_1T^{\mathbf{k}}\sum_{1\notin\mathbf{j}} c_{\mathbf{j}}\omega_{\{1\}\cup\mathbf{j}}+\sum_{1\notin I} c'_I\omega_I$, hence $c_{\mathbf{j}}=0$ for all $\mathbf{j}$, which proves that $d\eta=\omega$ and allows us to conclude. □

Lemma 8. — *Let $\alpha\in\mathbf{Q}_+$ and let $\Lambda_\alpha = R_\alpha^+,\mathscr{O}^+(D_\alpha)$. The following natural maps are identically zero*

$$H^i_{\mathrm{dR}}(\Lambda_{\alpha+1})\to H^i_{\mathrm{dR}}(\Lambda_\alpha),\quad i\geq 1;\quad \mathrm{HK}^i_r(R^+_{\alpha+2})\to\mathrm{HK}^i_r(R^+_\alpha),\quad i\geq 2.$$

The image of the map $\mathrm{HK}^1_r(R^+_{\alpha+2})\to\mathrm{HK}^1_r(R^+_\alpha)$ is annihilated by p^r, $H^0_{\mathrm{dR}}(\Lambda_\alpha)=\mathrm{A}_{\mathrm{cris}},\mathscr{O}_C$, and $\mathrm{HK}^0_r(R^+_\alpha)=\mathrm{A}_{\mathrm{cris}}^{\varphi=p^r}$.

Proof. — The proof for the first map is similar (but easier) to that of the second one, so we are only going to prove the latter. Take $i\geq 2$. Let (ω^i,ω^{i-1}) be a representative of an element of $\mathrm{HK}^i_r(R^+_{\alpha+2})$. That is to say $\omega^i\in\Omega^i_{R^+_{\alpha+2}}$, $\omega^{i-1}\in\Omega^{i-1}_{R^+_{\alpha+2}}$, $d\omega^i=0$ and $d\omega^{i-1}+(\varphi-p^r)\omega^i=0$.

Since $d\omega^i=0$, we deduce from Lemma 7 that there exists $\eta^{i-1}\in\Omega^{i-1}_{R^+_{\alpha+1}}$ such that $\iota_{\alpha+2,\alpha+1}\omega^i=d\eta^{i-1}$ (we used here that $[\tilde{p}]^m\in[\tilde{p}]m\mathrm{A}_{\mathrm{cris}}$). Let $\omega_1^{i-1}=\iota_{\alpha+2,\alpha+1}\omega^{i-1}+(\varphi-p^r)\eta^{i-1}$. Then $d\omega_1^{i-1}=\iota_{\alpha+2,\alpha+1}d\omega^{i-1}+(\varphi-p^r)d\eta^{i-1}=0$; hence there exists $\eta^{i-2}\in\Omega^{i-2}_{R^+_\alpha}$ such that $\iota_{\alpha+1,\alpha}\omega_1^{i-1}=d\eta^{i-2}$. It follows that $\iota_{\alpha+2,\alpha}(\omega^i,\omega^{i-1})=d(\iota_{\alpha+1,\alpha}\eta^{i-1},\eta^{i-2})$, as wanted.

Take now $i=1$ and use the notation from the above computation. Arguing as above we show that (ω^1,ω^0) is in the same class as $(0,\omega^0)$, with $\omega^0\in\mathrm{A}_{\mathrm{cris}}$. But the map $\varphi-p^r:\mathrm{A}_{\mathrm{cris}}\to\mathrm{A}_{\mathrm{cris}}$ is p^r-surjective. This proves the first part of the last statement of the lemma. The remaining part is clear. □

Remark 9. — The same arguments would prove that there exists $C:\mathbf{Q}_+^*\to\mathbf{N}$ such that, if $\beta>\alpha$ and $i\geq 1$, the images of the natural maps $H^i_{\mathrm{dR}}(R^+_\beta)\to H^i_{\mathrm{dR}}(R^+_\alpha)$, $\mathrm{HK}^i_r(R^+_\beta)\to\mathrm{HK}^i_r(R^+_\alpha)$ are killed by $p^{C(\beta-\alpha)}$. However, $C(u)\to+\infty$ when $u\to 0^+$.

Corollary 10. — *If $i\geq 1$ then $\mathrm{HK}^i_r(\mathbf{A}^n)=0$.*

Proof. — Immediate from Lemma 8 and the exact sequence

$$0\to\mathrm{R}^1\varprojlim_k\mathrm{HK}^{i-1}_r(R_k)\to\mathrm{HK}^i_r(\mathbf{A}^n)\to\varprojlim_k\mathrm{HK}^i_r(R_k)\to 0\quad\square$$

3. Computation of $\mathrm{DR}_r^i(\mathbf{A}^n)$

Lemma 11. — *If* $1\leq i\leq r-1$ *then* $\mathrm{DR}_r^i(\mathbf{A}^n)\simeq\big(\Omega^i(\mathbf{A}^n)/\operatorname{Ker} d\big)(r-i-1)$, *if* $i\geq r$ *then* $\mathrm{DR}_r^i(\mathbf{A}^n)=0$, *and, if* $r>0$, *we have an exact sequence*

$$0\to \mathrm{B}^+_{\mathrm{cris}}/F^r_\theta\to \mathrm{DR}^0_r(\mathbf{A}^n)\to\big(\mathscr{O}(\mathbf{A}^n)/C\big)(r-1)\to 0$$

Proof. — We have an exact sequence

$$0\to \mathrm{R}^1\varprojlim_k \mathrm{DR}_r^{i-1}(R_k)\to \mathrm{DR}_r^i(\mathbf{A}^n)\to\varprojlim_k \mathrm{DR}_r^i(R_k)\to 0$$

The $\mathrm{DR}_r^i(R_k)$'s are the cohomology groups of the complex

$$\dots\longrightarrow \mathrm{A}_{r-i}\otimes_{\mathrm{A}_{\mathrm{cris}}}\Omega^i_{R_k}\xrightarrow{1\otimes d_i}\mathrm{A}_{r-i-1}\otimes_{\mathrm{A}_{\mathrm{cris}}}\Omega^{i+1}_{R_k}\longrightarrow\cdots$$

In particular, they are trivially 0 if $i\geq r$, so assume $i\leq r-1$. The kernel of $1\otimes d_i$ is $F_\theta^{r-i-1}\mathrm{A}_{r-i}\otimes_{\mathrm{A}_{\mathrm{cris}}}\Omega^i_{R_k}+\mathrm{A}_{r-i}\otimes_{\mathrm{A}_{\mathrm{cris}}}(\Omega^i_{R_k})_{d=0}$ while the image of $1\otimes d_{i-1}$ is $\mathrm{A}_{r-i}\otimes_{\mathrm{A}_{\mathrm{cris}}}d\Omega^{i-1}_{R_k}$. Since $F_\theta^{r-i-1}\mathrm{A}_{r-i}$ is an $\mathscr{O}_C$-module of rank 1 (generated by the image of $\frac{(p-[\tilde{p}])^{r-i-1}}{(r-i-1)!}$), we have $F_\theta^{r-i-1}\mathrm{A}_{r-i}\otimes_{\mathrm{A}_{\mathrm{cris}}}\Omega^i_{R_k}\simeq\Omega^i(D_k)(r-i-1)$, which gives us the exact sequence

$$0\to \mathrm{A}_{r-i}\otimes_{\mathrm{A}_{\mathrm{cris}}}H^i_{\mathrm{dR}}(R_k)\to\mathrm{DR}_r^i(R_k)\to\big(\Omega^i(D_k)/\operatorname{Ker} d\big)(r-i-1)\to 0.$$

For $i=0$ this gives the sequence in the lemma.

Assume that $i\geq 1$. The natural map $H^i_{\mathrm{dR}}(R_{k+1})\to H^i_{\mathrm{dR}}(R_k)$ is identically zero by Lemma 8. Hence

$$\mathrm{R}^j\varprojlim_k(\Omega^i(D_k)/\operatorname{Ker} d)\simeq\mathrm{R}^j\varprojlim_k\mathrm{DR}_r^i(R_k),\quad j\geq 0.$$

Now, we note that since our systems are indexed by $\mathbf{N}$, $\mathrm{R}^j\varprojlim_k$ is trivial for $j\geq 2$. Since $\mathrm{R}^1\varprojlim_k\Omega^i(D_k)=0$, we have $\mathrm{R}^1\varprojlim_k(\Omega^i(D_k)/\operatorname{Ker} d)=0$ (and $\mathrm{R}^1\varprojlim_k d\Omega^i=0$). It remains to show that $\varprojlim_k(\Omega^i(D_k)/\operatorname{Ker} d)\simeq\Omega^i(\mathbf{A}^n)/\operatorname{Ker} d$. But this amounts to showing that $\mathrm{R}^1\varprojlim_k\Omega^i(D_k)_{d=0}=0$. This is clear for $i=0$ and for $i>0$, since the system $\{H^i_{\mathrm{dR}}(R_k)\}_{k\in\mathbf{N}}$ is trivial (by Lemma 8), this follows from the fact that $\mathrm{R}^1\varprojlim_k d\Omega^{i-1}(D_k)=0$. □

4. Proof of Theorem 1 and Theorem 3

4.1. Algebraic isomorphism. — From Proposition 6 we know that $\tau_{\leq r}\mathrm{Syn}(\mathbf{A}^n,r)\simeq\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(\mathbf{A}^n,\mathbf{Q}_p(r))$. From the long exact sequence

$$\cdots\to\mathrm{DR}_r^{i-1}(\mathbf{A}^n)\to\mathrm{Syn}^i(\mathbf{A}^n,r)\to\mathrm{HK}^i_r(\mathbf{A}_n)\to\mathrm{DR}_r^i(\mathbf{A}^n)\to\cdots$$

and Corollary 10 and Lemma 11, we obtain isomorphisms

$$\big(\Omega^{i-1}(\mathbf{A}^n)/\operatorname{Ker} d\big)(r-i)\xrightarrow{\sim}\mathrm{Syn}^i(\mathbf{A}^n,r),\quad r\geq i\geq 2,$$

and the exact sequence

$$0 \to \mathrm{Syn}^0(\mathbf{A}^n, r) \to \mathrm{B}_{\mathrm{cris}}^{+,\varphi=p^r} \to \mathrm{DR}_r^0(\mathbf{A}^n) \to \mathrm{Syn}^1(\mathbf{A}^n, r) \to 0,$$

which, using the fundamental exact sequence

$$0 \to \mathbf{Q}_p(r) \to \mathrm{B}_{\mathrm{cris}}^{+,\varphi=p^r} \to \mathrm{B}_{\mathrm{cris}}^+/F_\theta^r \to 0,$$

proves the first isomorphism in Theorem 1 (together with $\mathrm{Syn}^0(\mathbf{A}^n, r) \cong \mathbf{Q}_p(r)$). The second isomorphism is an immediate consequence of the fact that $H^i_{\mathrm{dR}}(\mathbf{A}^n) = 0$.

Since an open ball is an increasing union of closed balls, Theorem 3 is proved by the same argument but with Remark 9 in place of Lemma 8 – Corollary 10.

Remark 12. — Let $j \in \mathbf{N}$. We note that, since $[\tilde{p}]^{p-1} \in p\mathrm{A}_{\mathrm{cris}}$, for every $\alpha \in \mathbf{Q}_+$, the maps [5] $\Omega^i(R^+_{\alpha+m})_j \to \Omega^i(R^+_\alpha)_j$, $m \geq (p-1)j$, are the zero maps for $i \geq 1$ and outside the constants for $i = 0$. It follows that

$$\operatorname{holim}_k \mathrm{HK}_r(R^+_k)_j \simeq (\mathrm{A}_{\mathrm{cris},j} \xrightarrow{\varphi - p^r} \mathrm{A}_{\mathrm{cris},j}), \quad \operatorname{holim}_k \mathrm{DR}_r(R^+_k)_j \simeq \mathrm{A}_{\mathrm{cris},j}/F_\theta^r.$$

Computing as above we get $(\operatorname{holim}_{k,\ell} \mathrm{Syn}(R^+_k, r)_j) \otimes \mathbf{Q} \simeq \mathbf{Q}_p(r)$. Hence, by the comparison isomorphism (4), $H^i_{\text{ét}}(\mathbf{A}^n, \mathbf{Q}_p(r)) = 0$, $i \geq 1$, which allows us to recover the result of Berkovich [**1**].

4.2. Topological considerations. — It remains to discuss topology. In what follows, we write $\cong$ for an isomorphism of vector spaces and $\equiv$ for an isomorphism of topological vector spaces.

First, note that all the cohomology groups under consideration are cohomology groups of complexes of Fréchet spaces (and even of finite sums of countable products of Banach spaces), since these complexes can be built out of Čech complexes coming from coverings by affinoids, and the corresponding complexes for affinoids involve finitely many Banach spaces. It follows that, a priori, all the groups we are dealing with are cokernels of maps $F_1 \to F_2$ between Fréchet spaces; if such a group injects continuously into a Fréchet space, then it is a Fréchet space (it is separated hence the image of F_1 in F_2 is closed, and our space is a quotient of a Fréchet space by a closed subspace), and if this injection is a bijection then it is an isomorphism of Fréchet spaces by the Open Mapping Theorem.

Now, we have the following commutative diagram:

$$\begin{tikzcd}
H^r_{\text{proét}}(\mathbf{A}^n, \mathbf{Q}_p(r)) \arrow[r] \arrow[d, "\cong"] & \varprojlim_k H^r_{\text{ét}}(D_k, \mathbf{Q}_p(r)) \arrow[d, "\equiv"] \\
\mathrm{Syn}^r(\mathbf{A}^n, r) \arrow[r, "\cong"] & \varprojlim_k \mathrm{Syn}^r(R_k, r)
\end{tikzcd}$$

The horizontal maps are the natural maps (and are continuous), the bottom one being an isomorphism by the earlier computations. The left vertical arrow is an isomorphism

5. The subscript j refers to moding out by p^j.

by Proposition 6 and the right vertical arrow is a topological isomorphism because the period maps (4) are p^N-quasi-isomorphisms. Thus proving that $\varprojlim_k \mathrm{Syn}^r(R_k, r)$ is Fréchet would imply that so is $H^r_{\mathrm{pro\acute{e}t}}(\mathbf{A}^n, \mathbf{Q}_p(r))$ and that $H^r_{\mathrm{pro\acute{e}t}}(\mathbf{A}^n, \mathbf{Q}_p(r)) \equiv \varprojlim_k \mathrm{Syn}^r(R_k, r)$.

For that, consider the map of distinguished triangles

$$\begin{tikzcd}
\mathrm{Syn}(R_k, r) \arrow[r] \arrow[d, "\theta"] & \mathrm{HK}_r(R_k) \arrow[r] \arrow[d, "\theta"] & \mathrm{DR}_r(R_k) \arrow[d, "\theta"] \\
\Omega^{\geq r}(D_k)[-r] \arrow[r] & \Omega^{\bullet}(D_k) \arrow[r] & \Omega^{\leq r-1}(D_k)
\end{tikzcd}$$

All the maps are continuous (including the boundary maps). For $r \geq 2$, taking cohomology and limits we obtain the commutative diagram

$$\begin{tikzcd}
\varprojlim_k \mathrm{DR}_r^{r-1}(R_k) \arrow[r, "\partial", "\cong"'] \arrow[d, "\cong"] & \varprojlim_k \mathrm{Syn}^r(R_k, r) \arrow[d] \\
\Omega^{r-1}(\mathbf{A}^n)/\operatorname{Ker} d \equiv \varprojlim_k (\Omega^{r-1}(D_k)/\operatorname{Ker} d) \arrow[r, "d", "\cong"'] & \varprojlim_k \Omega^r(D_k)_{d=0} \equiv \Omega^r(\mathbf{A}^n)_{d=0}
\end{tikzcd}$$

The bottom map is an isomorphism because $\varprojlim_k H^r_{\mathrm{dR}}(D_k) \simeq H^r_{\mathrm{dR}}(\mathbf{A}^n) = 0$. The top map is an isomorphism because, on level k, its kernel and cokernel are controlled by $\mathrm{HK}_r^{r-1}(R_k)$ and $\mathrm{HK}_r^r(R_k)$ respectively, which die in R_{k-2} by Lemma 8, and the left vertical map is an isomorphism by the proof of Lemma 11. The space $\Omega^r(\mathbf{A}^n)$ is Fréchet; it follows that all other spaces are also Fréchet (in particular $\varprojlim_k \mathrm{Syn}^r(R_k, r)$) and that all the maps are topological isomorphisms. This concludes the proof of Theorem 1 if $r \geq 2$.

For $r = 1$, the argument is similar, with $\varprojlim_k \mathrm{DR}_r^{r-1}(R_k)$ in the above diagram replaced by $(\varprojlim_k \mathrm{DR}_r^{r-1}(R_k))/C$.

The proof for the open ball is similar.

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