Quantitative aspects of linear and affine closed lambda terms

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Abstract

Affine λ-terms are λ-terms in which each bound variable occurs at most once and linear λ-terms are λ-terms in which each bound variable occurs once. and only once. In this paper we count the number of closed affine λ-terms of size n, closed linear λ-terms of size n, affine β-normal forms of size n and linear β-normal forms of size n, for different ways of measuring the size of λ-terms. From these formulas, we show how we can derive programs for generating all the terms of size n for each class. The foundation of all of this is specific data structures, which are contexts in which one counts all the holes at each level of abstractions by λ's.

Keywords: Lambda calculus, combinatorics, functional programming

1 Introduction

The λ-calculus [1] is a well known formal system designed by Alonzo Church [8] for studying the concept of function. It has three kinds of basic operations: variables, application and abstraction (with an operator λ which is a binder of variables). We assume the reader familiar with the λ-calculus and with de Bruijn indices.[1]

In this paper we are interested in terms in which bound variables occur once. A *closed λ-term* is a λ-term in which there is no free variable, i.e., only free variables. An *affine λ-term* (or BCK term) is a $\lambda-$term in which bound variables occur at most once. A *linear* λ-term (or BCI term) is a λ-term in which bound variables occur once and only once.

In this paper we propose a method for counting and generating (including random generation) linear and affine closed λ-terms based on a data structure which we call *SwissCheese* because of its holes. Actually we count those λ-terms up-to α-conversion. Therefore it is adequate to use de Bruijn indices [11], because a term with de Bruijn indices represents an α-equivalence class. An interesting aspect of these terms is the fact that they are simply typed [17, 16]. For instance, generated by the program of Section 5 6, there are 16 linear terms of natural size 8:

$$(\lambda 0\ (\lambda 0\ \lambda 0))\quad (\lambda 0\ \lambda(\lambda 0\ 0))\quad (\lambda 0\ \lambda(0\ \lambda 0))\quad ((\lambda 0\ \lambda 0)\ \lambda 0)\quad (\lambda(\lambda 0\ 0)\ \lambda 0)\quad (\lambda(0\ \lambda 0)\ \lambda 0)\quad \lambda(\lambda 0\ (\lambda 0\ 0))\quad \lambda(\lambda 0\ (0\ \lambda 0))$$

$$\lambda((\lambda 0\ \lambda 0)\ 0)\quad \lambda(\lambda(\lambda 0\ 0)\ 0)\quad \lambda(\lambda(0\ \lambda 0)\ 0)\quad \lambda(0\ (\lambda 0\ \lambda 0))\quad \lambda(0\ \lambda(\lambda 0\ 0))\quad \lambda(0\ \lambda(0\ \lambda 0))\quad \lambda((\lambda 0\ 0)\ \lambda 0)\quad \lambda((0\ \lambda 0)\ \lambda 0)$$

written with explicit variables

$$\lambda x.x\ (\lambda x.x\ \lambda x.x)\quad \lambda x.x\ \lambda y.(\lambda x.x\ y)\quad \lambda x.x\ \lambda y.(y\ \lambda x.x)\quad (\lambda x.x\ \lambda x.x)\ \lambda x.x$$

[1] If the reader is not familiar with the λ-calculus, we advise him to read the introduction of [15], for instance.

$$\lambda y.(\lambda x.x\ y)\ \lambda x.x \quad \lambda y.(y\ \lambda x.x)\ \lambda x.x \quad \lambda y.(\lambda x.x\ (\lambda x.x\ y)) \quad \lambda y.(\lambda x.x\ (y\ \lambda x.x))$$
$$\lambda y.((\lambda x.x\ \lambda x.x)\ y) \quad \lambda y.(\lambda z.(\lambda x.x\ z)\ y) \quad \lambda y.(\lambda z.(z\ \lambda x.x)\ y) \quad \lambda y.(y\ (\lambda x.x\ \lambda x.x))$$
$$\lambda y.(y\ \lambda z.(\lambda x.x\ z)) \quad \lambda y.(y\ \lambda z.(z\ \lambda x.x)) \quad \lambda y.((\lambda x.x\ y)\ \lambda x.x) \quad \lambda y.((y\ \lambda x.x)\ \lambda x.x)$$

and there are 25 affine terms of natural size 7:

$$(\lambda 0\ \lambda\lambda 1) \quad (\lambda 0\ \lambda\lambda\lambda 0) \quad (\lambda\lambda 0\ \lambda\lambda 0) \quad (\lambda\lambda 1\ \lambda 0) \quad (\lambda\lambda\lambda 0\ \lambda 0) \quad \lambda(\lambda\lambda 1\ 0) \quad \lambda(\lambda\lambda\lambda 0\ 0) \quad \lambda(0\ \lambda\lambda 1)$$
$$\lambda(0\ \lambda\lambda\lambda 0) \quad \lambda(\lambda 0\ \lambda 1) \quad \lambda(\lambda 1\ \lambda 0) \quad \lambda\lambda(\lambda 0\ 1) \quad \lambda\lambda(1\ \lambda 0) \quad \lambda(\lambda 0\ \lambda\lambda 0) \quad \lambda(\lambda\lambda 0\ \lambda 0) \quad \lambda\lambda(\lambda\lambda 0\ 0)$$
$$\lambda\lambda(0\ \lambda\lambda 0) \quad \lambda\lambda\lambda(0\ 1) \quad \lambda\lambda\lambda(1\ 0) \quad \lambda\lambda\lambda\lambda 2 \quad \lambda\lambda(\lambda 0\ \lambda 0) \quad \lambda\lambda\lambda(\lambda 0\ 0) \quad \lambda\lambda\lambda(0\ \lambda 0) \quad \lambda\lambda\lambda\lambda\lambda 1 \quad \lambda\lambda\lambda\lambda\lambda\lambda 0$$

The Haskell programs of this development are on GitHub: `https://github.com/PierreLescanne/CountingGeneratingAfffineLinearClosedLambdaterms`.

Notations

In this paper we use specific notations.

Given a predicate p, the Iverson notation written $[p(x)]$ is the function taking natural values which is 1 if $p(x)$ is true and which is 0 if $p(x)$ is false.

Let $\mathbf{m} \in \mathbb{N}^p$ be the p-tuple $(m_0, ..., m_{p-1})$. In Section 5, we consider also infinite tuples. Thus $\mathbf{m} \in \mathbb{N}^\omega$ is the sequence $(m_0, m_1, ...)$. Notice in the case of infinite tuples, we are only interested in infinite tuples equal to 0 after some index.

- p is the *length* of $\mathbf{m}$, which we write also $\mathsf{length}\ \mathbf{m}$
- The p-tuple $(0, ..., 0)$ is written 0^p. 0^ω is the infinite tuple made of 0's.
- The *increment* of a p-tuple at i is:
$$\mathbf{m}^{\uparrow i} = \mathbf{n} \in \mathbb{N}^p \text{ where } n_j = m_j \text{ if } j \neq i \text{ and } n_i = m_i + 1$$
- Putting an element x as *head* of a tuple is written
$$x : \mathbf{m} = x : (m_0, ...) = (x, m_0, ...)$$
tail removes the head of a tuple:
$$\mathsf{tail}(x : \mathbf{m}) = \mathbf{m}.$$
- $\oplus$ is the componentwise addition on tuples.

2 SwissCheese

The basic concept is this of **m-SwissCheese** or **Swisscheese of characteristic m** or simply **SwissCheese** if there is no ambiguity on $\mathbf{m}$. A $\mathbf{m}$-SwissCheese or a SwissCheese of characteristic $\mathbf{m}$, where $\mathbf{m}$ is of length p) is, a λ-term with holes at p levels, which are all counted, using $\mathbf{m}$. The p levels of holes are $\Box_0$,... $\Box_{p-1}$. A hole $\Box_i$ is meant to be a location for a variable at level i, that is under i λ's. According to the way bound variables are inserted when creating abstractions (see below), we consider linear or affine SwissCheeses. The holes have size 0. An $\mathbf{m}$-SwissCheese or a SwissCheese of characteristic $\mathbf{m}$ has m_0 holes at level 0, m_1 holes at level 1, ... m_{p-1} holes at level $p-1$. Let $l_{n,\mathbf{m}}$ (resp. $a_{n,\mathbf{m}}$) count the linear (resp. the affine) $\mathbf{m}$-SwissCheese of size n. $l_{n,\mathbf{m}} = l_{n,\mathbf{m}'}$ and $a_{n,\mathbf{m}} = a_{n,\mathbf{m}'}$ if $\mathbf{m}$ is finite, $\mathsf{length}\ \mathbf{m} \geq n$, $m_i = m'_i$ for $i \leq \mathsf{length}\ \mathbf{m}$, and $m'_i = 0$ for $i > \mathsf{length}\ \mathbf{m}$. $l_{n,0^n}$ (resp. $a_{n,0^n}$) counts the closed linear (resp. the closed affine) λ-terms.

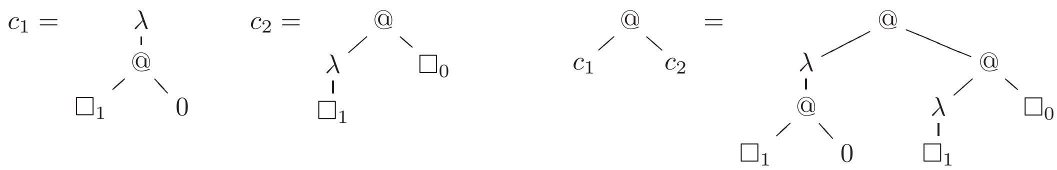

Figure 1: Building a SwissCheese by application

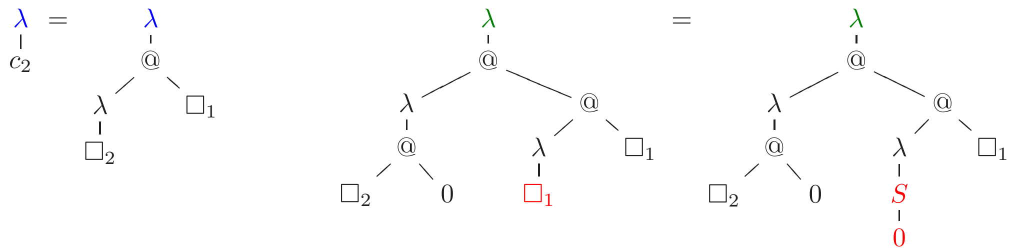

Figure 2: Abstracting SwissCheeses without and with binding

2.1 Growing a SwissCheese

Given two SwissCheeses, we can build a SwissCheese *by application* like in Fig 1. In Fig. 1, c_1 is a $(0,1,0,0,0)$-SwissCheese, c_2 is a $(1,1,0,0,0)$-SwissCheese and $c_1@c_2$ is a $(1,2,0,0,0)$-SwissCheese. Said otherwise, c_1 has characteristic $(0,1,0,0,0)$, c_2 has characteristic $(1,1,0,0,0)$ and $c_1@c_2$ has characteristic $(1,2,0,0,0)$. According to what we said, $c_1@c_2$ has characteristic $(1,2)$ as well as characteristic $(1,2,0,0,...)$ (a tuple starting with 1, followed by 2, followed by infinitely many 0's). We could also say that c_1 has characteristic $(0,1)$ and c_2 has characteristic $(1,1)$ making @ a binary operation on SwissCheeses of length 2 whereas previously we have made @ a binary operation on SwissCheeses of length 5. In other words, when counting SwissCheeses of characteristic $\mathbf{m}$, the trailing 0's are irrelevant. In actual computations, we make the lengths of characteristics consistent by adding trailing 0's to too short ones.

Given a SwissCheese, there are two ways to grow a SwissCheese to make another SwissCheese *by abstraction*.

1. We put a λ on the top of a $\mathbf{m}$-SwissCheese c. This increases the levels of the holes: a hole $\Box_i$ becomes a hole $\Box_{i+1}$. λc is a $(0:\mathbf{m})$-SwissCheese. See Fig 2 on the left. This way, no index is bound by the top λ, therefore this does not preserve linearity (it preserves affinity however). Therefore this construction is only for building affine SwissCheeses, not for building linear SwissCheeses. In Figure 2 (left), we colour the added λ in blue and we call it *abstraction with <u>no</u> binding*.

2. In the second method for growing a SwissCheese by abstraction, we select first a hole $\Box_i$, we top the SwissCheese by a λ, we increment the levels of the other holes and we replace the chosen hole by $S^i 0$. In Figure 2 (right), we colour the added λ in green and we call it *abstraction <u>with</u> binding*.

2.2 Measuring SwissCheese

We considers several ways of measuring the size of a SwissCheese derived from what is done on λ-terms. In all these sizes, applications @ and abstractions λ have size 1 and holes have size 0. The differences are in the way variables are measured.

- Variables have size 0, we call this **variable size** 0.
- Variables have size 1, we call this **variable size** 1 .
- Variables (or de Bruijn indices) $S^i 0$ have size $i+1$, we call this **natural size**.

3 Counting linear closed terms

We start with counting linear terms since they are slightly simpler. We will give recursive formulas first for the numbers $l^{\nu}_{n,\mathbf{m}}$ of linear SwissCheeses of natural size n with holes set by $\mathbf{m}$, then for the numbers $l^0_{n,\mathbf{m}}$ of linear SwissCheeses of size n, for variable size 0, with holes set by $\mathbf{m}$, eventually for the numbers $l^1_{n,\mathbf{m}}$ of linear SwissCheeses of size n, for variable size 1, with holes set by $\mathbf{m}$. When we do not want to specify a chosen size, we write only $l_{n,\mathbf{m}}$ without superscript.

3.1 Natural size

First let us count linear SwissCheeses with natural size. This is given by the coefficient l^{ν} which has two arguments: the size n of the SwissCheese and a tuple $\mathbf{m}$ which specifies the number of holes of each level. In other words we are interested in the quantity $l^{\nu}_{n,\mathbf{m}}$. We assume that the length of $\mathbf{m}$ is p, greater than n.

$\mathbf{n = 0}$ whatever size is considered, there is only one SwissCheese of size 0 namely $\Box_0$. This means that the number of SwissCheeses of size 0 is 1 if and only if $\mathbf{m} = (1, 0, 0, ...)$:

$$l^{\nu}_{0,\mathbf{m}} = l^0_{0,\mathbf{m}} = l^1_{0,\mathbf{m}} = [m_0 = 1 \wedge \bigwedge_{j=1}^{p-1} m_j = 0]$$

$\mathbf{n \neq 0}$ **and application** if a λ-term of size n has holes set by $\mathbf{m}$ and is an application, then it is obtained from a λ term of size k with holes set by $\mathbf{q}$ and a λ term of size $n-k-1$ with holes set by $\mathbf{r}$, with $\mathbf{m} = \mathbf{q} \oplus \mathbf{r}$:

$$\sum_{\mathbf{q}\oplus\mathbf{r}=\mathbf{m}} \sum_{k=0}^{n} l_{k,\mathbf{q}}\, l_{n-1-k,\mathbf{r}}$$

$\mathbf{n \neq 0}$ **and abstraction with binding** consider a level i, that is a level of hole $\Box_i$. In this hole we put a term $S^{i-1}\,0$ of size i. There are m_i ways to choose a hole $\Box_i$. Therefore there are $m_i\, l^{\nu}_{n-i-1,\mathbf{m}}$ SwissCheeses which are abstractions with binding in which a $\Box_i$ has been replaced by the de Bruijn index $S^{i-1}\,0$ among $l^{\nu}_{n,0:\mathbf{m}^{\downarrow \mathbf{i}}}$ SwissCheeses, where $\mathbf{m}^{\downarrow \mathbf{i}}$ is $\mathbf{m}$ in which m_i is decremented. We notice that this refers only to an $\mathbf{m}$ starting with 0. Hence by summing over i and adjusting $\mathbf{m}$, this part contributes as:

$$\sum_{i=0}^{p-1} (m_i + 1)\; l^{\nu}_{n-i,\mathbf{m}^{\uparrow i}}$$

to $l^{\nu}_{n+1,0:\mathbf{m}}$.

We have the following recursive definitions of $l^{\nu}_{n,\mathbf{m}}$:

$$\begin{aligned} l^{\nu}_{n+1,0:\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}\,=\,0:\mathbf{m}} \sum_{k=0}^{n} l^{\nu}_{k,\mathbf{q}}\, l^{\nu}_{n-k,\mathbf{r}} + \sum_{i=0}^{p-1} (m_i + 1)\; l^{\nu}_{n-i,\mathbf{m}^{\uparrow i}} \\ l^{\nu}_{n+1,(h+1):\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=(h+1):\mathbf{m}} \sum_{k=0}^{n} l^{\nu}_{k,\mathbf{q}}\, l^{\nu}_{n-k,\mathbf{r}} \end{aligned}$$

Numbers of closed linear terms with natural size are given in Figure 3.

3.2 Variable size 0

The only difference is that the inserted de Bruijn index has size 0. Therefore we have $m_i\, l^0_{n-1,\mathbf{m}}$ where we had $m_i\, l^\nu_{n-i-1,\mathbf{m}}$ for natural size. Hence the formulas:

$$\begin{aligned}
l^0_{n+1,0:\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}\,=\,0:\mathbf{m}} \sum_{k=0}^{n} l^0_{k,\mathbf{q}}\, l^0_{n-k,\mathbf{r}} + \sum_{i=0}^{p-1}(m_i+1)\ l^0_{n,\mathbf{m}^{\uparrow i}} \\
l^0_{n+1,(h+1):\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=(h+1):\mathbf{m}} \sum_{k=0}^{n} l^0_{k,\mathbf{q}}\, l^0_{n-k,\mathbf{r}}
\end{aligned}$$

The sequence $l^0_{n,0^n}$ of the numbers of closed linear terms is $0, 1, 0, 5, 0, 60, 0, 1105, 0, 27120, 0, 828250$, which is sequence A062980 in the *On-line Encyclopedia of Integer Sequences* with 0's at even indices.

3.3 Variable size 1

The inserted de Bruijn index has size 1. We have $m_i\, l^1_{n-2,\mathbf{m}}$ where we had $m_i\, l^\nu_{n-i-1,\mathbf{m}}$ for natural size.

$$\begin{aligned}
l^1_{n+1,0:\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}\,=\,0:\mathbf{m}} \sum_{k=0}^{n} l^1_{k,\mathbf{q}}\, l^1_{n-k,\mathbf{r}} + \sum_{i=0}^{n-1}(m_i+1)\ l^1_{n-1,\mathbf{m}^{\uparrow i}} \\
l^1_{n+1,(h+1):\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=(h+1):\mathbf{m}} \sum_{k=0}^{n} l^1_{k,\mathbf{q}}\, l^1_{n-k,\mathbf{r}}
\end{aligned}$$

As noticed by Grygiel et al. [13] (§ 6.1) There are no linear closed λ-terms of size $3k$ and $3k+1$. However for the values $3k+2$ we get the sequence: $1, 5, 60, 1105, 27120, \ldots$ which is again sequence A062980 of the *On-line Encyclopedia of Integer Sequences*.

4 Counting affine closed terms

We have just to add the case $n \neq 0$ *and abstraction without binding*. Since no index is added, the size increases by 1. The numbers are written $a^\nu_{n,\mathbf{m}}$, $a^0_{n,\mathbf{m}}$, $a^1_{n,\mathbf{m}}$, and $a_{n,\mathbf{m}}$ when the size does not matter. There are $(0:\mathbf{m})$-SwissCheeses of size n that are abstraction without binding. We get the recursive formulas:

4.1 Natural size

$$\begin{aligned}
a^\nu_{n+1,0:\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}\,=\,0:\mathbf{m}} \sum_{k=0}^{n} a^\nu_{k,\mathbf{q}}\, a^\nu_{n-k,\mathbf{r}} + \sum_{i=0}^{p-1}(m_i+1)\ a^\nu_{n-i,\mathbf{m}^{\uparrow i}} + a^\nu_{n,\mathbf{m}} \\
a^\nu_{n+1,(h+1):\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=(h+1):\mathbf{m}} \sum_{k=0}^{n} a^\nu_{k,\mathbf{q}}\, a^\nu_{n-k,\mathbf{r}}
\end{aligned}$$

The numbers of closed affine size with natural size are given in Figure 4.

4.2 Variable size 0

$$\begin{aligned}
a^0_{n+1,0:\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}\,=\,0:\mathbf{m}} \sum_{k=0}^{n} a^0_{k,\mathbf{q}}\, a^0_{n-k,\mathbf{r}} + \sum_{i=0}^{p-1}(m_i+1)\ a^0_{n,\mathbf{m}^{\uparrow i}} + a^0_{n,\mathbf{m}} \\
a^0_{n+1,(h+1):\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=(h+1):\mathbf{m}} \sum_{k=0}^{n} a^0_{k,\mathbf{q}}\, a^0_{n-k,\mathbf{r}}
\end{aligned}$$

The sequence $a^0_{n,0^\omega}$ of the numbers of affine closed terms for variable size 0 is

$$0, 1, 2, 8, 29, 140, 661, 3622, 19993, 120909, 744890, 4887401, 32795272, ...$$

It does not appear in the *On-line Encyclopedia of Integer Sequences*. It corresponds to the coefficients of the generating function $\mathcal{A}(z, 0)$ where

$$\mathcal{A}(z, u) = u + z(\mathcal{A}(z, u))^2 + z\frac{\partial \mathcal{A}(z, u)}{\partial u} + z\mathcal{A}(z, u).$$

4.3 Variable size 1

$$\begin{aligned}
a^1_{n+1,0:\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}\,=\,0:\mathbf{m}} \sum_{k=0}^{n-1} a^1_{k,\mathbf{q}}\, a^1_{n-k,\mathbf{r}} + \sum_{i=0}^{n-1} (m_i + 1)\ a^1_{n-1,\mathbf{m}\uparrow^i} + a^1_{n,\mathbf{m}}\\
a^1_{n+1,(h+1):\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=(h+1):\mathbf{m}} \sum_{k=0}^{n} a^1_{k,\mathbf{q}}\, a^1_{n-k,\mathbf{r}}
\end{aligned}$$

The sequence $a^1_{n,0^\omega}$ of the numbers of affine closed terms for variable size 1 is

$$0, 0, 1, 2, 3, 9, 30, 81, 242, 838, 2799, 9365, 33616, 122937, 449698, 1696724, 6558855, ...$$

This is sequence A281270 in the *On-line Encyclopedia of Integer Sequences*. However it corresponds to the coefficient of the generating function $\hat{\mathcal{A}}(z, 0)$ where $\hat{\mathcal{A}}(z, u)$ is the solution of the functional equation:

$$\hat{\mathcal{A}}(z, u) = zu + z(\hat{\mathcal{A}}(z, u))^2 + z\frac{\partial \hat{\mathcal{A}}(z, u)}{\partial u} + z\hat{\mathcal{A}}(z, u).$$

Notice that this corrects the wrong assumptions of [13] (Section 6.2).

5 Generating functions

Consider families $F_{\mathbf{m}}(z)$ of generating functions indexed by $\mathbf{m}$, where $\mathbf{m}$ is an infinite tuple of naturals. In fact, we are interested in the infinite tuples $\mathbf{m}$ that are always 0, except a finite number of indices, in order to compute $F_{0^\omega}(z)$, which corresponds to closed λ-terms. Let $\mathbf{u}$ stands for the infinite sequences of variables $(u_0, u_1, ...)$ and $\mathbf{u}^{\mathbf{m}}$ stands for $(u_0^{m_0}, u_1^{m_1}, ..., u_n^{m_n}, ...)$ and $\mathsf{tail}\,(\mathbf{u})$ stand for $(u_1, ...)$. We consider the series of two variables z and $\mathbf{u}$ or double series associated with $F_{\mathbf{m}}(z)$:

$$\mathcal{F}(z, \mathbf{u}) = \sum_{\mathbf{m}\in\mathbb{N}^\omega} F_{\mathbf{m}}(z)\, \mathbf{u}^{\mathbf{m}}.$$

Natural size

$L^\nu_{\mathbf{m}}(z)$ is associated with the numbers of *closed linear SwissCheeses* for natural size:

$$\begin{aligned}
L^\nu_{0:\mathbf{m}}(z) &= z \sum_{\mathbf{m}'\oplus\mathbf{m}''=0:\mathbf{m}} L^\nu_{\mathbf{m}'}(z) L^\nu_{\mathbf{m}''}(z) + z \sum_{i=0}^{\infty} (m_i + 1) z^i L^\nu_{\mathbf{m}\uparrow^i}(z)\\
L^\nu_{(h+1):\mathbf{m}}(z) &= [h = 0 + \bigwedge_{i=0}^{\infty} m_i = 0] + z \sum_{\mathbf{m}'\oplus\mathbf{m}''=(h+1):\mathbf{m}} L^\nu_{\mathbf{m}'}(z) L^\nu_{\mathbf{m}''}(z)
\end{aligned}$$

$L^\nu_{0^\omega}$ is the generating function for the closed linear λ-terms. $\mathcal{L}^\nu(z,\mathbf{u})$ is the double series associated with $L^\nu_{\mathbf{m}}(z)$ and is solution of the equation:

$$\mathcal{L}^\nu(z,\mathbf{u}) \quad = \quad u_0 + z(\mathcal{L}^\nu(z,\mathbf{u}))^2 + \sum_{i=1}^{\infty} z^i \frac{\partial \mathcal{L}^\nu(z,\mathsf{tail}\,(\mathbf{u}))}{\partial u^i}$$

$\mathcal{L}^\nu(z,0^\omega)$ is the generating function of closed linear λ-terms.

For *closed affine SwissCheeses* we get:

$$\begin{aligned}
A^\nu_{0:\mathbf{m}}(z) &= z \sum_{\mathbf{m}'\oplus\mathbf{m}''=0:\mathbf{m}} A^\nu_{\mathbf{m}'}(z)A^\nu_{\mathbf{m}''}(z) + z\sum_{i=0}^{\infty}(m_i+1)z^i A^\nu_{\mathbf{m}\uparrow i}(z) + z\,A^\nu_{\mathbf{m}}(z)\\
A^\nu_{(h+1):\mathbf{m}}(z) &= [h=0+\bigwedge_{i=0}^{\infty} m_i=0] + z\sum_{\mathbf{m}'\oplus\mathbf{m}''=(h+1):\mathbf{m}} A^\nu_{\mathbf{m}'}(z)A^\nu_{\mathbf{m}''}(z)
\end{aligned}$$

$A^\nu_{0^\omega}$ is the generating function for the closed linear λ-terms. $\mathcal{A}^\nu(z,\mathbf{u})$ is the double series associated with $A^\nu_{\mathbf{m}}(z)$ and is solution of the equation:

$$\mathcal{A}^\nu(z,\mathbf{u}) \quad = \quad u_0 + z(\mathcal{A}^\nu(z,\mathbf{u}))^2 + \sum_{i=1}^{\infty} z^i \frac{\partial \mathcal{A}^\nu(z,\mathsf{tail}\,(\mathbf{u}))}{\partial u^i} + z\mathcal{A}^\nu(z,\mathsf{tail}\,(\mathbf{u}))$$

$\mathcal{A}^\nu(z,0^\omega)$ is the generating function of closed affine λ-terms.

Variable size 0

$L^0_{\mathbf{m}}$ is associated with the numbers of *closed linear SwissCheeses* for variable size 0:

$$\begin{aligned}
L^0_{0:\mathbf{m}}(z) &= z \sum_{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}} L^0_{\mathbf{m}'}(z)L^0_{\mathbf{m}''}(z) + z\sum_{i=0}^{\infty}(m_i+1)L^0_{\mathbf{m}\uparrow i}(z)\\
L^0_{(h+1):\mathbf{m}}(z) &= [h=0+\bigwedge_{i=0}^{\infty} m_i=0] + \sum_{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}} zL^0_{\mathbf{m}'}(z)L^0_{\mathbf{m}''}(z)
\end{aligned}$$

$L^0_{0^\omega}$ is the generating function for the closed linear λ-terms. $\mathcal{L}^0(z,\mathbf{u})$ is the double series associated with $L^0_{\mathbf{m}}(z)$ and is solution of the equation:

$$\mathcal{L}^0(z,\mathbf{u}) \quad = \quad u_0 + z(\mathcal{L}^0(z,\mathbf{u}))^2 + \sum_{i=1}^{\infty} \frac{\partial \mathcal{L}^0(z,\mathsf{tail}\,(\mathbf{u})}{\partial u^i}$$

$\mathcal{L}^0(z,0^\omega)$ is the generating function of closed linear λ-terms.

For *closed affine SwissCheeses* we get:

$$\begin{aligned}
A^0_{0:\mathbf{m}}(z) &= z \sum_{\mathbf{m}'\oplus\mathbf{m}''=0:\mathbf{m}} A^0_{\mathbf{m}'}(z)A^0_{\mathbf{m}''}(z) + z\sum_{i=0}^{\infty}(m_i+1)A^0_{\mathbf{m}\uparrow i}(z) + z\,A^0_{\mathbf{m}}(z)\\
A^0_{(h+1):\mathbf{m}}(z) &= [h=0+\bigwedge_{i=0}^{\infty} m_i=0] + \sum_{\mathbf{m}'\oplus\mathbf{m}''=(h+1):\mathbf{m}} zA^0_{\mathbf{m}'}(z)A^0_{\mathbf{m}''}(z)
\end{aligned}$$

$A^0_{0^\omega}$ is the generating function for the affine linear λ-terms. $\mathcal{A}^0(z,\mathbf{u})$ is the double series associated with $A^0_{\mathbf{m}}(z)$ and is solution of the equation:

$$\mathcal{A}^0(z,\mathbf{u}) \quad = \quad u_0 + z(\mathcal{A}^0(z,\mathbf{u}))^2 + \sum_{i=1}^{\infty} \frac{\partial \mathcal{A}^0(z,\mathsf{tail}\,(\mathbf{u}))}{\partial u^i} + z\mathcal{A}^0(z,\mathsf{tail}\,(\mathbf{u}))$$

$\mathcal{A}^0(z,0^\omega)$ is the generating function of closed linear λ-terms.

Variable size 1

The generating functions for $l^1_{n,\mathbf{m}}$ are:

$$\begin{aligned} L^1_{0:\mathbf{m}}(z) &= z \sum_{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}} L^1_{\mathbf{m}'}(z) L^1_{\mathbf{m}''}(z) + z^2 \sum_{i=0}^{\infty} (m_i+1) L^1_{\mathbf{m}^{\uparrow i}}(z) \\ L^1_{(h+1):\mathbf{m}}(z) &= [h = 0 + \bigwedge_{i=0}^{\infty} m_i = 0] + \sum_{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}} z L^1_{\mathbf{m}'}(z) L^1_{\mathbf{m}''}(z) \end{aligned}$$

Then we get as associated double series :

$$\mathcal{L}^1(z,\mathbf{u}) = u_0 + z(\mathcal{L}^1(z,\mathbf{u}))^2 + z^2 \sum_{i=1}^{\infty} \frac{\partial \mathcal{L}^1(z, \mathsf{tail}\,(\mathbf{u}))}{\partial u^i}$$

6 Effective computations

The definition of the coefficients $a^{\nu}_{\mathbf{m}}$ and others is highly recursive and requires a mechanism of memoization. In Haskell, this can be done by using the call by need which is at the core of this language. Assume we want to compute the values of $a^{\nu}_{\mathbf{m}}$ until a value `upBound` for n. We use a recursive data structure:

```
data Mem = Mem [Mem] | Load [Integer]
```

in which we store the computed values of a function

```
a :: Int -> [Int] -> Integer
```

In our implementation the depth of the recursion of `Mem` is limited by `upBound`, which is also the longest tuple $\mathbf{m}$ for which we will compute $a^{\nu}_{\mathbf{m}}$. Associated with `Mem` there is a function

```
access :: Mem -> Int -> [Int] -> Integer access (Load l) n [] = l !! n
access (Mem listM) n (k:m) = access (listM !! k) n m
```

The leaves of the tree memory, corresponding to `Load`, contains the values of the function:

```
memory :: Int -> [Int] -> Mem
memory 0 m = Load [a n (reverse m) | n<-[0..]]
memory k m = Mem [memory (k-1) (j:m) | j<-[0..]]
```

The memory relative to the problem we are interested in is

```
theMemory = memory (bound) []
```

and the access to `theMemory` is given by a specific function:

```
acc :: Int -> [Int] -> Integer acc n m = access theMemory n m
```

Notice that `a` and `acc` have the same signature. This is not a coincidence, since `acc` accesses values of `a` already computed. Now we are ready to express `a`:

```
a 0 m = iv (head m == 1 && all ((==) 0) (tail m))
a n m = aAPP n m + aABSwB n m + aABSnB n m
```

`aAPP` *counts affine terms that are applications:*

```
aAPP n m = sum (map (\((q,r),(k,nk))->(acc k q)*(acc nk r)) (allCombinations m (n-1)))
```

where `allCombinations` returns a list of all the pairs of pairs $(\mathbf{m}', \mathbf{m}'')$ such $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$ and of pairs (k, nk) such that $k + nk = n$. `aABSwB` *counts affine terms that are abstractions with binding.*

```
aABSwB n m
   | head m == 0 = sum [aABSAtD n m i |i<-[1..(n-1)]]
   | otherwise = 0
```

`aABSAtD` *counts affine terms that are abstractions with binding at level i:*

```
aABSAtD n m i = (fromIntegral (1 + m!!i))*(acc (n-i-1) (tail (inc i m) ++ [0]))
```

`aABSnB` *counts affine terms that are abstractions with no binding*:

```
aABSnB n m
      | head m == 0 = (acc (n-1) (tail m ++ [0]))
      | otherwise = 0
```

Anyway the efficiency of this program is limited by the size of the memory, since for computing $a^{\nu}_{n,0^n}$, for instance, we need to compute $a^{\nu}_{\mathbf{r}}$ for about $n!$ values.

7 Generating affine and linear terms

By relatively small changes it is possible to build programs which generate linear and affine terms. For instance for generating affine terms we get.

```
amg :: Int -> [Int] -> [SwissCheese]
amg 0 m =  if (head m == 1 && all ((==) 0) (tail m)) then [Box 0] else []
amg n m = allAPP n m ++ allABSwB n m ++ allABSnB n m

allAPP  :: Int -> [Int] -> [SwissCheese]
allAPP n m = foldr (++) [] (map (\((q,r),(k,nk))-> appSC (cartesian (accAG k q)
                                                                     (accAG nk r))
                            (allCombinations m (n-1)))

allABSAtD :: Int -> [Int] -> Int -> [SwissCheese]
allABSAtD n m i = foldr (++) [] (map (abstract (i-1)) (accAG (n - i - 1)
                                                       (tail (inc i m) ++ [0])))

allABSwB  :: Int -> [Int] -> [SwissCheese]
allABSwB n m
  | head m == 0 = foldr (++) [] [allABSAtD n m i |i<-[1..(n-1)]]
  | otherwise = []

allABSnB  :: Int -> [Int] -> [SwissCheese]
allABSnB n m
  | head m == 0 = map (AbsSC . raise) (accAG (n-1) (tail m ++ [0]))
  | otherwise = []

memoryAG :: Int -> [Int] -> MemSC
memoryAG 0 m = LoadSC [amg n (reverse m) | n<-[0..]]
memoryAG k m = MemSC [memoryAG (k-1) (j:m) | j<-[0..]]

theMemoryAG = memoryAG (upBound) []

accAG :: Int -> [Int] -> [SwissCheese]
accAG n m = accessSC theMemoryAG n m
```

From this, we get programs for generating random affine terms or random linear terms as follows: if we want a random closed linear term of a given size n, we throw a random number, say p, between 1 and $l_{n,0^n}$ and we look for the p^{th} in the list of all the closed linear terms of size n. Haskell laziness mimics the unranking. Due to high requests in space, we cannot go further than the random generation of closed linear terms of size 23 and closed affine terms of size 19 . There are similar programs for generating all the terms of size n for variable size 0 and variable size 1.

8 Normal forms

From the method used for counting affine and linear closed terms, it is easy to deduce method for counting affine and linear closed normal forms. Like before, we use SwissCheeses. In this section we consider only natural size.

8.1 Natural size

Affine closed normal forms

Let us call $anf^{\nu}_{n,\mathbf{m}}$ the numbers of affine SwissCheeses with no β-redex and $ane^{\nu}_{n,\mathbf{m}}$ the numbers of neutral affine SwissCheeses, i.e., affine SwissCheeses with no β-redexes that are sequences of applications starting with a de Bruijn index. In addition we count:

- $anf^{\nu}\lambda w_{n,m}$ the number of affine SwissCheeses with no β-redex which are abstraction with a binding of a de Bruijn index,
- $anf^{\nu}\lambda n_{n,m}$ the number of affine SwissCheeses with no β-redex which are abstraction with no binding.

$$\begin{aligned} anf^{\nu}_{0,\mathbf{m}} &= ane^{\nu}_{0,\mathbf{m}} \\ anf^{\nu}_{n+1,\mathbf{m}} &= ane^{\nu}_{n+1,\mathbf{m}} + anf^{\nu}\lambda w_{n+1,m} + anf^{\nu}\lambda n_{n+1,m} \end{aligned}$$

where

$$\begin{aligned} ane^{\nu}_{0,\mathbf{m}} &= m_0 = 1 \wedge \bigwedge_{j=1}^{p-1} m_j = 0 \\ ane^{\nu}_{n+1,\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}\,=\,0:\mathbf{m}} \sum_{k=0}^{n} ane^{\nu}_{k,\mathbf{q}}\, anf^{\nu}_{n-k,\mathbf{r}} \end{aligned}$$

and

$$anf^{\nu}\lambda w_{n+1,m} = \sum_{i=0}^{n}(m_i+1)\; anf^{\nu}_{n-i,\mathbf{m}^{\uparrow i}}$$

and

$$anf^{\nu}\lambda n_{n+1,m} = anf^{\nu}_{n,m}$$

There are two generating functions, $\mathcal{A}^{nf}$ and $\mathcal{A}^{ne}$, which are associated to $anf^{\nu}_{n,\mathbf{m}}$ and $anf^{\nu}_{n,\mathbf{m}}$:

$$\begin{aligned} \mathcal{A}^{nf}(z,\mathbf{u}) &= \mathcal{A}^{ne}(z,\mathbf{u}) + \sum_{i=1}^{\infty} z^i \frac{\partial \mathcal{A}^{nf}(z,\mathsf{tail}(\mathbf{u}))}{\partial u^i} + z\mathcal{A}^{nf}(z,\mathsf{tail}(\mathbf{u})) \\ \mathcal{A}^{ne}(z,\mathbf{u}) &= u_0 + z\mathcal{A}^{ne}(z,\mathbf{u})\mathcal{A}^{nf}(z,\mathbf{u}) \end{aligned}$$

Linear closed normal forms

Let us call $lnf^{\nu}_{n,\mathbf{m}}$ the numbers of linear SwissCheeses with no β-redex and $lne^{\nu}_{n,\mathbf{m}}$ the numbers of neutral linear SwissCheeses, linear SwissCheeses with no β-redexes that are sequences of applications starting with a de Bruijn index. In addition we count $lnf^{\nu}\lambda w_{n,m}$ the number of linear SwissCheeses with no β-redex which are abstraction with a binding of a de Bruijn index.

$$\begin{aligned} lnf^{\nu}_{0,\mathbf{m}} &= lne^{\nu}_{0,\mathbf{m}} \\ lnf^{\nu}_{n+1,\mathbf{m}} &= lne^{\nu}_{n+1,\mathbf{m}} + lnf^{\nu}\lambda w_{n+1,m} \end{aligned}$$

where

$$
\begin{aligned}
lne^{\nu}_{0,\mathbf{m}} &= m_0 = 1 \wedge \bigwedge_{j=1}^{p-1} m_j = 0\\
lne^{\nu}_{n+1,\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}\,=\,0:\mathbf{m}} \sum_{k=0}^{n} lne^{\nu}_{k,\mathbf{q}}\, lnf^{\nu}_{n-k,\mathbf{r}}
\end{aligned}
$$

and

$$
lnf^{\nu}\lambda w_{n+1,m} = \sum_{i=0}^{p-1} (m_i + 1)\ lnf^{\nu}_{n-i,\mathbf{m}^{\uparrow i}}
$$

with the two generating functions:

$$
\begin{aligned}
\mathcal{L}^{nf,\nu}(z,\mathbf{u}) &= \mathcal{L}^{ne,\nu}(z,\mathbf{u}) + \sum_{i=1}^{\infty} z^i \frac{\partial \mathcal{L}^{nf,\nu}(z, \mathsf{tail}\,(\mathbf{u}))}{\partial u^i}\\
\mathcal{L}^{ne,\nu}(z,\mathbf{u}) &= u_0 + z\mathcal{L}^{ne,\nu}(z,\mathbf{u})\mathcal{L}^{nf,\nu}(z,\mathbf{u})
\end{aligned}
$$

We also deduce programs for generating all the closed affine or linear normal forms of a given size from which we deduce programs for *generating random closed affine or linear normal forms of a given size*. For instance, here are three randoms linear closed normal forms (using de Bruijn indices) of natural size 28:

λλλλ(2 λ((1 2) λ(0 (5 1)))) λ(0 λλ(1 λλ((0 (2 λλ((1 λ0) 0))) 1))) λ((0 λ0) λλ((0((1 λ0)λλ(1 (0 λ0))))λ0))

8.2 Variable size 0

Linear closed normal forms

A little like previously, let us call $lnf^0_{n,\mathbf{m}}$ the numbers of linear SwissCheeses with no β-redex and $lne^0_{n,\mathbf{m}}$ the numbers of neutral linear SwissCheeses, linear SwissCheeses with no β-redexes that are sequences of applications starting with a de Bruijn index. In addition we count $lnf^0\lambda w_{n,m}$ the number of linear SwissCheeses with no β-redex which are abstraction with a binding of a de Bruijn index. We assume that the reader knows now how to proceed.

$$
\begin{aligned}
lnf^0_{0,\mathbf{m}} &= lne^0_{0,\mathbf{m}}\\
lnf^0_{n+1,\mathbf{m}} &= lne^0_{n+1,\mathbf{m}} + lnf^0\lambda w_{n+1,m}
\end{aligned}
$$

where

$$
\begin{aligned}
lne^{0}_{0,\mathbf{m}} &= m_0 = 1 \wedge \bigwedge_{j=1}^{p-1} m_j = 0\\
lne^{0}_{n+1,\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}\,=\,0:\mathbf{m}} \sum_{k=0}^{n} lne^{0}_{k,\mathbf{q}}\, lnf^{0}_{n-k,\mathbf{r}}
\end{aligned}
$$

$$
lnf^{0}\lambda w_{n+1,m} = \sum_{i=0}^{n} (m_i + 1)\ lnf^{0}_{n,\mathbf{m}^{\uparrow i}}
$$

and the two generating functions:

$$\begin{aligned}\mathcal{L}^{nf,0}(z,\mathbf{u}) &= \mathcal{L}^{ne,0}(z,\mathbf{u}) + \sum_{i=1}^{\infty} \frac{\partial \mathcal{L}^{nf,0}(z,\mathsf{tail}\,(\mathbf{u}))}{\partial u^i}\\ \mathcal{L}^{ne,0}(z,\mathbf{u}) &= u_0 + z\mathcal{L}^{ne,0}(z,\mathbf{u})\mathcal{L}^{nf,0}(z,\mathbf{u})\end{aligned}$$

With no surprise we get for $lnf^0_{n,0^n}$ the sequence:

$$0, 1, 0, 3, 0, 26, 0, 367, 0, 7142, 0, 176766, 0, 5304356, \ldots$$

mentioned by Zeilberger in [19] and listing the coefficients of the generating function $\mathcal{L}^{nf,0}(z, 0^\omega)$.

We let the reader deduce how to count closed affine normal forms for variable size 0 and closed linear and affine normal forms for variable size 1 alike. Notice that the Haskell programs are on the GitHub site.

9 Related works and Acknowledgement

There are several works on counting λ-terms, for instance on natural size [3, 2], on variable size 1 [5, 10, 18], on variable size 0 [14], on affine terms with variable size 1 [7, 6], on linear λ-terms [21, 19, 20], also on a size based binary representation of the λ-calculus [15] (see [12] for a synthetic view of both natural size and binary size).

The basic idea of this work comes from a discussion with Maciej Bendkowski, Olivier Bodini, Sergey Dovgal and Katarzyna Grygiel, I thank them as I thank Noam Zeilberger for interactions.

10 Conclusion

This presentation shares similarity with this of [14, 15, 4]. Instead of considering the size n and the bound m of free indices like in expressions of the form:

$$T_{n+1,m} = T_{n,m+1} + \sum_{i=0}^{n} T_{i,m}T_{n-i,m}$$

here we replace m by the characteristic $\mathbf{m}$. We can imagine a common framework. On another hand, as noticed by Paul Tarau, this approach has features of dynamic programming [9], which makes it somewhat efficient.

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0	0
1	0
2	1
3	0
4	0
5	3
6	2
7	0
8	16
9	24
10	8
11	117
12	252
13	180
14	1024
15	2680
16	2952
17	10350
18	29420
19	42776
20	116768
21	335520
22	587424
23	1420053
24	3976424
25	7880376
26	18103936
27	48816576
28	104890704
29	237500826
30	617733708
31	1396750576
32	3171222464
33	8014199360
34	18688490336
35	42840683418
36	106063081288
37	251769197688
38	583690110208
39	1425834260080
40	3417671496432
41	8007221710652
42	19404994897976
43	46747189542384
44	110498345360800
45	266679286291872
46	644021392071840
47	1533054190557133
48	3693823999533360
49	8931109667692464
50	21375091547312128
51	51496022711337536
52	124591137939086496
53	299402908258405410
54	721839933329222924
55	1747307145272084192
56	4211741383777966592
57	10165998012602469888
58	24620618729658655936
59	59482734150603634286
60	143764591607556354344
61	348379929166234350008
62	843169238563254723200
63	2040572920613086128400
64	4948102905207104837424
65	11992521016286173712196
66	29059897435554891991144
67	70516464312280927105392
68	171105110698292441423968
69	415095704639682396539232
70	1008016383720573882885792
71	2448305474519849567597826
72	5945721872300885649415632
73	14449388516068567845838736
74	35125352062243788817753856
75	85382289240293493116120064
76	207650379931166057815603296
77	505172267243918348155299780
78	1229005880128485245247395000
79	2991079243470267667831893408
80	7281852742753184123608419712
81	17729171587798767750815341440
82	43177454620325445122944305984
83	105185452787117035266315446868
84	256273862465425158211948020048
85	624527413292252904584121980208
86	1522355057007327280427270436480
87	3711429775030704772089070886624
88	9050041253711022076275958636128
89	22073150301758857110072042919800
90	53844910909398928990641101351664
91	131371135544173914537076774932576
92	320588677238085642820920910555968
93	782465218885869813183863213231424
94	1910077425906069707804966102543936
95	4663586586924802791117231052636349
96	11388259565942452837717688743953504
97	27813754361897984543467478917223008
98	67941781284113201998645699501746176
99	165989485724048964272023600773271424
100	405588809305168453963137377442321728

Figure 3: *Natural size:* numbers of closed linear terms of size n from 0 to 100

Data

In the appendix, we give the first values of $l^{\nu}_{n,0^n}$, $a^{\nu}_{n,0^n}$, and $anf^{\nu}_{n,0^n}$.

0	0
1	0
2	1
3	1
4	2
5	5
6	12
7	25
8	64
9	166
10	405
11	1050
12	2763
13	7239
14	19190
15	51457
16	138538
17	374972
18	1020943
19	2792183
20	7666358
21	21126905
22	58422650
23	162052566
24	450742451
25	1256974690
26	3513731861
27	9843728012
28	27633400879
29	77721141911
30	218984204904
31	618021576627
32	1746906189740
33	4945026080426
34	14017220713131
35	39784695610433
36	113057573020242
37	321649935953313
38	916096006168770
39	2611847503880831
40	7453859187221508
41	21292177500898858
42	60875851617670699
43	174195916730975850
44	498863759031591507
45	1429753835635525063
46	4100730353324163138
47	11769771167532816128
48	33804054749367200891
49	97151933333668422006
50	279385977720772581435
51	803928779462727941247
52	2314623127904669382002
53	6667810436356967142481
54	19218411059885449257096
55	55421020161661024650870
56	159899218321197381984561
57	461557020400062903560120
58	1332920908954281811200519
59	3851027068336583693412910
60	11131032444503136571789527
61	32186581221116996967632029
62	93108410048006285466998584
63	269446191702411420790402033
64	780043726186403167392453886
65	2259043189995515315930349650
66	6544612955390252336187266873
67	18966737218108971681014445025
68	54985236298270057405776629352
69	159455737350384637847783055311
70	462562848624435724964181323484
71	1342251884451664733064283251627
72	3896065622127200625653134100538
73	11312117748805772104795220337816
74	32853646116456632492645965741531
75	95442534633482460553801961967438
76	277342191547330839640289978813667
77	806125189457291902863848267463755
78	2343682130911232279285707290604156
79	6815564023736534208079367816340359
80	19824812322145727566417303371819466
81	57679033022808238913186144092831856
82	167851787082561392384648248846390041
83	488574368670832093243802790464796207
84	1422426342380883254459783410845365006
85	4142104564089044203901190817275864665
86	12064305885705003967881526911560653106
87	35145647815239737143373764367447378676
88	102406303052123097062053564818109468705
89	298446029598661205216170897850336550644
90	869935452705023302189031644932803990417
91	2536229492704354513309696228592784181158
92	7395518143425160073537967606298755947391
93	21568776408467701927134211542478146593789
94	62915493935623036562559989770249004382816
95	183553775888862113259168150130266362416356
96	535600661621556969155453544692826625532079
97	1563109720672526919899689366626240867515144
98	4562542818801138452310024131223304186909233
99	13319630286623965617386598746472280781972745
100	38890520391341859449843201188612375394153776

Figure 4: *Natural size:* numbers of closed affine terms of size n from 0 to 100

0	0
1	0
2	1
3	1
4	2
5	3
6	7
7	10
8	20
9	40
10	77
11	160
12	318
13	671
14	1405
15	2981
16	6312
17	13672
18	29399
19	63697
20	139104
21	304153
22	667219
23	1469241
24	3247176
25	7184288
26	15949179
27	35480426
28	79083472
29	176607519
30	395119875
31	885450388
32	1987289740
33	4466760570
34	10053371987
35	22656801617
36	51121124910
37	115478296639
38	261139629999
39	591138386440
40	1339447594768
41	3037843646560
42	6895841598615
43	15666498585568
44	35620848278448
45	81052838239593
46	184564847153821
47	420564871255118
48	958975854646984
49	2188068392529104
50	4995528560788451
51	11411921511827547
52	26084524952754538
53	59654682828889245
54	136500653558490261
55	312496493161999851
56	715760763686417314
57	1640194881084692664
58	3760284787917366081
59	8624561382605096780
60	19789639944299656346
61	45427337308377290201
62	104320438668034814453
63	239656248361374562433
64	550769764273325683828
65	1266217774600330829940
66	2912050679107531357883
67	6699418399886008666265
68	15417663698156810292010
69	35492710197462925262295
70	81732521943462960197057
71	188270363628099910161436
72	433807135012774797924026
73	999851681931974600766994
74	2305129188866501774481545
75	5315847675735178072941600
76	12262083079763320881047944
77	28292248892584567512609357
78	65294907440089718078048829
79	150729070403767032817820543
80	348031015577337732605480908

Figure 5: *Natural size:* numbers of closed affine normal forms of size n from 0 to 80