On polynomially integrable planar outer billiards and curves with symmetry property

Alexey Glutsyuk, E Shustin

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A.Glutsyuk[*,†,‡], E.Shustin[§,¶]

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Abstract

We show that every polynomially integrable planar outer convex billiard is elliptic.

Contents

[*]CNRS, France (UMR 5669 (UMPA, ENS de Lyon) and UMI 2615 (Lab. J.-V.Poncelet)), Lyon, France. E-mail: aglutsyu@ens-lyon.fr

[†]National Research University Higher School of Economics (HSE), Moscow, Russia

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[§]School of Math. Sci., Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel. E-mail: shustin@post.tau.ac.il

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1 Introduction, main result and plan of the paper

1.1 Introduction and main result

Let $C \subset \mathbb{R}^2$ be a smooth closed strictly convex curve. Let S denote the complement of the ambient plane $\mathbb{R}^2$ to the closure of the interior of the curve C. The *(planar) outer billiard* is a dynamical system $\mathcal{T} : S \to S$ defined as follows. Pick a point $A \in S$. There are two tangent rays to C issued from the point A. Let R denote the right tangent ray: the other tangent ray is obtained from R by rotation around the point A of angle between zero and π. Let P denote the tangency point of the ray R with the curve C. By definition, *the image $\mathcal{T}(A)$ is the point of the ray R that is symmetric to A with respect to the point P.*

Definition 1.1 A planar outer billiard is *polynomially integrable,* if there exists a polynomial $f(x, y)$ (called a *first integral*) whose level curves are invariant under the outer billiard mapping.

For a survey on outer billiards see [9, 10, 11].

The main result of the paper is the following theorem.

Theorem 1.2 *Let a planar outer billiard on a C^4-smooth strictly convex closed curve C be polynomially integrable. Then C is an ellipse.*

A particular case of Theorem 1.2 under a non-degeneracy assumption on the polynomial integral was proved by S.L.Tabachnikov [12, theorem 1]. Analogous statement for polynomially integrable Birkhoff billiards under assumption that the complexification of the curve C is nonsingular was proved by S.V.Bolotin in [2]. For a survey and recent results on polynomially integrable Birkhoff billiards see [1] and references therein.

We also prove the following more general theorems. First, instead of strictly convex closed (hence, bounded) curves we can consider a (non-closed) *unbounded* convex connected curve $C \subset \mathbb{R}^2$. In general the corresponding outer billiard mapping is well-defined not on the whole exterior domain, but on some smaller domain. The above definition of polynomial integrability also makes sense in this case.

Example 1.3 If C is a parabola, then the corresponding outer billiard mapping is defined in the whole exterior domain. But if C is a branch of a hyperbola, then the set of those points A for which there exist two tangent lines to C through A is the domain bounded by the branch C and its asymptotic lines at infinity. This is the intersection of the definition domains of the corresponding outer billiard mapping and its inverse.

Theorem 1.4 *Let $C \subset \mathbb{R}^2$ be a C^4-smooth strictly convex curve that is either closed or unbounded. Let the corresponding outer billiard be polynomially integrable. Then the curve C lies in a conic.*

The next theorem deals with non-convex curves C, for which the outer billiard mapping is just a multivalued correspondence.

Definition 1.5 A C^k-smoothly immersed curve $C \subset \mathbb{R}^2$ *generates a polynomially integrable multivalued outer billiard,* if there exists a polynomial $f(x, y)$ such that for every $P \in C$ and every $A, B \in T_P C$ symmetric with respect to the point P one has $f(A) = f(B)$. The latter polynomial is called an *integral* of the multivalued outer billiard constructed on the curve C.

Theorem 1.6 *Let $C \subset \mathbb{R}^2$ be a C^4-smoothly immersed image of either an interval, or a circle, and let C have no rectilinear arcs. Let C generate a polynomially integrable multivalued outer billiard. Then the curve C lies in a conic.*

Theorem 1.4 follows from Theorem 1.6. Theorem 1.2 follows from Theorem 1.4. Thus, it suffices to prove Theorem 1.6.

Remark 1.7 Let a smooth connected curve $C \subset \mathbb{R}^2$ generate a polynomially integrable multivalued outer billiard with the integral $f(x, y)$. Then one has $f|_C \equiv const$, analogously to [12]; in particular, C is contained in an algebraic curve. Indeed, let $P \in C$ be a regular point of quadratic tangency. Then for every given k and every point $A_1 \in T_P C$ close enough to P (dependently on k) there exists a piecewise linear curve $A_1 \dots A_k$ whose edges $A_j A_{j+1}$ are small and tangent to C at their middle-points. (This is an orbit of the point A_1 under appropriate branch of the outer billiard mapping.) One has $f(A_1) = \dots = f(A_k)$, by definition. As A_1 becomes close to P, the latter orbits with growing lengths k limit to a strictly convex arc containing

P of the curve C. Passing to limit, one gets $f = const$ along every strictly convex arc. The constance of the polynomial f along each rectilinear arc of the curve C follows immediately from definition. This together with density of the union of the strictly convex part of the curve C and its rectilinear pieces and connectivity implies that $f|_C \equiv const$.

Remark 1.8 Let a curve C generate a polynomially integrable multivalued outer billiard with polynomial integral f. Let C have a bitangent: there exist a line L that is tangent to C at two distinct points P_1 and P_2. Then $f|_L \equiv const$. Indeed, the restriction $f|_L$ should be constant on each orbit of the group generated by the symmetries with respect to the points P_1 and P_2. The product of the latter symmetries is a nontrivial translation, whose orbit is infinite. Thus, the polynomial f is constant on an infinite subset of the line L, and hence, on all of L.

1.2 Complexification and relative symmetry property. Plan of the proof of Theorem 1.6

Consider the complexification $\mathbb{C}^2$ of the real plane and the ambient projective plane $\mathbb{CP}^2 \supset \mathbb{C}^2$. Let $\overline{\mathbb{C}}_\infty$ denote the infinity line:

$$\overline{\mathbb{C}}_\infty = \mathbb{CP}^2 \setminus \mathbb{C}^2.$$

Let a curve $C \subset \mathbb{R}^2$ generate a polynomially integrable multivalued outer billiard with the integral $f(x, y)$. Then $f|_C \equiv const$, by Remark 1.7. We can and will consider that $f|_C \equiv 0$, adding a constant to f. Then for every point $P \in C$ the intersection $T_P C \cap \{f = 0\}$ is symmetric with respect to the point P, since the values of the polynomial f at symmetric points of the tangent line coincide by assumption. This motivates the following definition.

Definition 1.9 A complex algebraic curve $\gamma \subset \mathbb{CP}^2$ has a *relative symmetry property*, if there exists a bigger algebraic curve $\Gamma \supset \gamma$ such that for every $t \in \gamma \cap \mathbb{C}^2$ the intersection $T_t\gamma \cap \Gamma$ is symmetric with respect to the point t as a subset of the affine complex line $T_t\gamma \cap \mathbb{C}^2$: it is invariant under the central symmetry $x \mapsto -x$ in affine coordinate x on $T_t\gamma$ centered at t.

Example 1.10 Let a smoothly immersed curve $C \subset \mathbb{R}^2$ generate a polynomially integrable multivalued outer billiard with the integral f, $f|_C \equiv 0$. Let $\gamma \subset \mathbb{CP}^2$ denote the minimal complex algebraic curve containing C. Let $\Gamma \supset \gamma$ denote the complexification of the algebraic curve $\{f = 0\}$. Then the curve γ has relative symmetry property with respect to the curve Γ, by the analogous statement in the real domain given at the beginning of the subsection.

Remark 1.11 An algebraic curve $\gamma \subset \mathbb{CP}^2$ has the relative symmetry property with respect to an algebraic curve $\Gamma \supset \gamma$, if and only if so does each of its irreducible components. If in the above example the curve C is analytic non-singular, then the curve γ is irreducible.

Theorem 1.12 *Let a smoothly immersed curve $C \subset \mathbb{R}^2$ without rectilinear pieces generate a polynomially integrable multivalued outer billiard. Let $\gamma \subset \mathbb{CP}^2$ be the minimal complex algebraic curve containing C. Then the curve γ has neither singular, nor inflection points in the affine plane $\mathbb{C}^2$.*

A particular case of Theorem 1.12 was proved by S. Tabachnikov in [12]. M. Bialy and A. E. Mironov have extended his proof to the general case by using their ideas from [1]. This proof of Theorem 1.12 due to Tabachnikov, Bialy, and Mironov will be given in Section 1.2.

We will also deal with the local version of the relative symmetry property.

Definition 1.13 Let $A \in \mathbb{CP}^2$, $b \subset \mathbb{CP}^2$ be an irreducible germ of an analytic curve at A, and let b be not a line. The germ b has a *(local) relative symmetry property,* if there exists a bigger finite union $\Gamma \supset b$ of irreducible germs of analytic curves at points of the tangent line $T_A b \subset \mathbb{CP}^2$ such that for every $t \in b$ close enough to A the intersection $T_t b \cap \Gamma$ is symmetric with respect to the point t in the above sense.

Example 1.14 In the conditions of Definition 1.9 each local branch of the curve γ at every point has the local symmetry property.

Consider an irreducible nonlinear germ b of an analytic curve in $\mathbb{CP}^2$ at a given point A. Let us choose affine coordinates (z, w) centered at A so that the tangent line $T_A b$ be the z-axis. Then one can find a local bijective parametrization of the germ b by a complex parameter $t \in (\mathbb{C}, 0)$ of the type

$$t \mapsto (t^q, c_b t^p(1 + o(1))), \quad q = q_b, \ p = p_b \in \mathbb{N}, \ q < p, \ c_b \neq 0; \tag{1.1}$$

$$q = 1 \text{ if and only if } b \text{ is a smooth germ.}$$

Definition 1.15 The *projective Puiseux exponent* [4, p. 250, definition 2.9] of the germ b is the ratio

$$r = r_b = \frac{p}{q}.$$

The germ b is called *quadratic*, if $r_b = 2$, and is called *subquadratic*, if $r_b \leq 2$.

Theorem 1.16 *Let a nonlinear irreducible germ b of an analytic curve in $\mathbb{CP}^2$ at a point $A \in \overline{\mathbb{C}}_\infty$ be transverse to $\overline{\mathbb{C}}_\infty$ and have the local relative symmetry property. Then it is quadratic.*

Theorem 1.16 is proved in Section 2.

Corollary 1.17 *Let a nonlinear irreducible germ b of an analytic curve in $\mathbb{CP}^2$ at a point $A \in \overline{\mathbb{C}}_\infty$ lie in the complexification of a real curve defining a polynomially integrable (multivalued) outer billiard. Then b is quadratic.*

The corollary follows from the statements of Examples 1.10, 1.14 and Theorem 1.16.

Theorem 1.18 *Let an irreducible algebraic curve $\gamma \subset \mathbb{CP}^2$ have neither singular, nor inflection points in an affine chart $\mathbb{C}^2 \subset \mathbb{CP}^2$. Let each of its local branches at every point in $\gamma \cap \overline{\mathbb{C}}_\infty$ that is transverse to $\overline{\mathbb{C}}_\infty$ be subquadratic. Then γ is a conic.*

Theorem 1.18 is proved in Section 3.

Corollary 1.19 *Let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve distinct from a line such that all its singular and inflection points (if any) lie in the infinity line. Let each local branch β of γ at every point in $\gamma \cap \overline{\mathbb{C}}_\infty$ that is transverse to the infinity line $\overline{\mathbb{C}}_\infty$ have the local relative symmetry property with respect to some collection of germs $\Gamma = \Gamma(\beta)$. Then γ is a conic.*

Proof Each local branch β as above is quadratic (Theorem 1.16). Hence, γ is a conic, by Theorem 1.18. □

Proof of Theorem 1.6. Each irreducible component of the complexification γ of the curve C has relative symmetry property and neither singular, nor inflection points in $\mathbb{C}^2$, by Theorem 1.12. Hence, it is a conic, by Corollary 1.19. Thus, γ is a finite union of conics. This implies that in the case, when the curve C is analytic nonsingular, the curve γ is irreducible (just one conic) and hence, C lies in a conic. In the general case the curve C is a union of arcs of a finite number of conics. Any two adjacent arcs are tangent with tangency order at least 5, since the curve C is C^4-smooth. Therefore, they lie in the same conic. Indeed, any two conics tangent to each other with tangency contact of order at least 5 coincide: otherwise, they would be two distinct conics with intersection index at least 5, which is obviously impossible. Hence, C lies in just one conic. Theorem 1.6 is proved. □

1.3 Singularities and inflection points. Proof of Theorem 1.12

Here we repeat S.Tabachnikov's arguments from [12] modified by M.Bialy and A.Mironov using their ideas from [1, section 6].

Let f be a polynomial integral. Let $\alpha \subset \gamma$ be an irreducible component of the curve γ. Let (x, y) be affine coordinates on $\mathbb{C}^2$. Then for every $P \in \alpha$ and every two points $A, B \in T_P\alpha$ symmetric with respect to the point P one has $f(A) = f(B)$: this equality holds in the real domain (by definition) and extends analytically to the complex domain.

Let Ψ be an irreducible polynomial vanishing on α. Then

$$f = g(x, y)\Psi^m(x, y), \ m \in \mathbb{N}, \ g|_\alpha \not\equiv 0.$$

Set

$$F(x, y) = f^{\frac{1}{m}}(x, y) = g^{\frac{1}{m}}(x, y)\Psi(x, y).$$

The function F is multivalued algebraic, and any two of its leaves differ by multiplication by m-th root of unity. Its branching locus is contained in the curve $\{g = 0\}$. For every P, A, B as above one has $F(A) = F(B)$ for appropriate choice of leaves of the multivalued function F. Namely, this holds for every $P \notin \{g = 0\}$, any leaf of the function F analytic in a neighborhood of the point P, and any two points $A, B \in T_P\alpha$ symmetric with respect to P and close enough to it.

Consider the (multivalued) vector field $v = F_y\frac{\partial}{\partial x} - F_x\frac{\partial}{\partial y}$, which is tangent to the level curves of the function F and does not vanish identically on α. The above symmetry is equivalent to the statement that the function

$$U(x, y, \varepsilon) = F(x + \varepsilon F_y, y - \varepsilon F_x) \tag{1.2}$$

is even in ε for all $P = (x, y) \in \alpha$. Equivalently, its Taylor series should contain only even powers of the variable ε. The linear term in ε obviously vanishes. The first nontrivial condition is vanishing of the cubic coefficient, which takes the form

$$W(F) := F_{xxx}F_y^3 - 3F_{xxy}F_y^2F_x + 3F_{xyy}F_yF_x^2 - F_{yyy}F_x^3 = 0. \tag{1.3}$$

We claim that vanishing of the expression $W(F)$ implies the statements of Theorem 1.12. To show that, let us consider the function

$$H(F) = F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2,$$

which is equal to the coefficient at ε^2 in U multiplied by two. One has

$$\frac{dH(F)}{dv} = W(F),$$

as in [12]. (This can be proved by straightforward calculation.) This together with (1.3) implies that

$$H(F)|_\alpha \equiv const. \tag{1.4}$$

Claim. *$H(F)(P) \to 0$, as $P \in \alpha$ tends to either a finite singular point of the curve γ, or a finite inflection point of the curve α.*

Proof The expression $H(F)(P)$ is the second derivative of the function (1.2) in ε. That is, the second derivative of the function F at P along the constant vector field on $T_P\alpha$ given by the vector v. Let us show that the function $U(x, y, \varepsilon)$ from (1.2) has asymptotics $o(\varepsilon^2)$, as $\varepsilon \to 0$ and P tends to a point $Q \in \alpha$ that is either a finite singular point of the curve γ, or a finite inflection point of the curve α. This implies that $H(F)(P) \to 0$, as $P \to Q$ and will prove the claim.

Case 1): Q is a singular point of the curve γ. Then either $g(Q) = 0$, or Q is a cusp of the curve α and $g(Q) \neq 0$. In both cases $v(P) \to 0$. One has $\Psi(P + \varepsilon v) = O(\varepsilon^2 v^2) = o(\varepsilon^2)$, as $P \to Q$ along the curve α and $\varepsilon \to 0$, since v is tangent to the zero level curve α of the polynomial Ψ. Therefore, $U(x, y, \varepsilon) = (g\Psi)((x, y) + \varepsilon v) = o(\varepsilon^2)$.

Case 2): Q is a finite inflection point of the curve α that is nonsingular for the curve γ. Then $\Psi(P + \varepsilon v) = o(\varepsilon^2)$, as $P \to Q$, and the function F and the field v are analytic on a neighborhood of the point Q. Therefore, $U(x, y, \varepsilon) = o(\varepsilon^2)$, as in the above case. The claim is proved. □

Proof of Theorem 1.12. Suppose, by contradiction, that the affine curve $\alpha \cap \mathbb{C}^2$ contains either its inflection point, or a singular point of the curve γ. Then $H(F)|_\alpha \equiv 0$, by (1.4) and the claim. Therefore, α is a straight line. Hence, the real curve C contains a rectilinear segment. The contradiction thus obtained proves Theorem 1.12. □

2 Quadraticity of germ. Proof of Theorem 1.16

2.1 Plan of the proof of Theorem 1.16

Let us introduce affine coordinates (x, y) on $\mathbb{C}^2$ so that the x-axis be tangent to b at A. The symmetry property implies that for every $t \in b$ close enough to

A the union of x-coordinates of the points of intersection $T_t b \cap \Gamma$ is invariant under the symmetry $\mathbb{C} \to \mathbb{C}$ with respect to the point $x(t)$. In Subsection 2.2 we will write explicit formulas for the asymptotics of the x-coordinates of the intersection points of $T_t b$ with the germs at A of the curve Γ (Corollaries 2.2 and 2.4). For intersections with the germs tangent to b, these formulas were basically obtained in [4, p.268, Proposition 2.50]. Most of the next briefly discussed asymptotic results follow from them. Afterwards in Subsection 2.3 we study those points of intersection $T_t b \cap \Gamma$ whose x-coordinates are of order $\nu x(t)$, where ν is a nonzero multiplicative constant depending on the family of intersection points. The above-mentioned corollaries imply that the latter intersection points lie in the intersection of the line $T_t b$ with those irreducible germs $(a, A) \subset \Gamma$ tangent to b, for which $r_a \geq r_b = r$. The intersections with those irreducible tangent germs $(a_i, A) \subset \Gamma$ that have the same projective Puiseux exponent $r_{a_i} = \frac{p_{a_i}}{q_{a_i}} = r$ have x-coordinates asymptotically equivalent to $\theta_{ij}^{q_{a_i}} x(t)$, where θ_{ij} are roots of the polynomial $W_i(\theta) = (r-1)\theta^{p_{a_i}} - r\theta^{p_{a_i}-q_{a_i}} + c_i$, $c_i = \frac{c_{a_i}}{c_b} \neq 0$; $j = 1, \dots, p_{a_i}$. In at least one of these polynomials, one has $c_i = 1$: in the polynomial W_i corresponding to $a_i = b$, i.e., to the intersection $T_t b \cap b$.

We prove Theorem 1.16 by contradiction. The contrary assumption $r \neq 2$ is equivalent to the statement that the q_{a_i}-th powers of roots of each individual polynomial W_i are not all equal to one. Those intersection points with the germs a_i that correspond to $\theta_{ij}^{q_{a_i}} = 2$ (and only them) should be symmetric to those points of intersection $T_t b \cap \Gamma$ whose x-coordinates are $o(x(t))$. For every family $\xi(t)$ of intersection points of the line $T_t b$ with a germ (a, A) tangent to b with $r_a > r_b$ such that $x(\xi(t)) = \nu x(t)(1 + o(1))$, $\nu \neq 0$, one has $\nu = \frac{r}{r-1}$ (Corollary 2.4). Therefore, the latter intersection points should be symmetric to those points of intersection of the line $T_t b$ with the germs a_i that correspond to $\theta_{ij}^{q_{a_i}} = 2 - \frac{r}{r-1} = \frac{r-2}{r-1}$. Let k_1 and k_2 denote the numbers of those roots θ_{ij} whose q_{a_i}-th powers are equal to 2 and $\frac{r-2}{r-1}$ respectively. The collection of the powers $\theta_{ij}^{q_{a_i}} \neq 2, \frac{r-2}{r-1}$ is symmetric with respect to 1, by the above statements. Hence, their sum equals their cardinality. This implies an explicit relation on k_1, k_2 and the sum of q_{a_i}-th powers of all the roots θ_{ij} through all i and j. We show that the latter relation is impossible, whenever $r \neq 2$, by applying an explicit formula for the sum of powers of roots of each individual polynomial W_i (elementary algebra). This will prove Theorem 1.16.

2.2 Asymptotics of intersection points

Proposition 2.1 *Let a, b be transverse irreducible germs of holomorphic curves at the origin in $\mathbb{C}^2$. Let (z,w) be coordinates in a neighborhood of the origin in $\mathbb{C}^2$ centered at 0 such that b is tangent to the z-axis at 0. Let t be the local parameter of the curve b as in (1.1): $z(t) = t^{q_b}$; $w(t) = c_b t^{p_b}(1+o(1))$. Then for every t small enough the intersection $T_t b \cap a$ consists of q_a points $\xi_1, \dots, \xi_{q_a}$ whose coordinates have the following asymptotics, as $t \to 0$:*

$$z(\xi_j) = O(t^{p_b}) = o(t^{q_b}) = o(z(t)), \;\; w(\xi_j) = (1 - r_b)w(t)(1 + o(1)). \quad (2.1)$$

Proof It suffices to prove just the second asymptotic formula in (2.1). Indeed, one has $z(\xi_j) = O(w(\xi_j))$, by transversality. This together with the second formula in (2.1) implies the first one: $z(\xi_j) = O(w(t)) = O(t^{p_b})$.

For every t small enough the tangent line $T_t b$ intersects the z-axis at a point P_t with the coordinate

$$z(P_t) = \nu z(t)(1 + o(1)) = \nu t^{q_b}(1 + o(1)), \;\; \nu = \frac{r_b - 1}{r_b}, \quad (2.2)$$

by [4, Proposition 2.10, p. 250]. Let Q_t denote the intersection point of the line $T_t b$ with the w-axis. One has

$$w(Q_t) = \frac{\nu}{\nu - 1} w(t)(1 + o(1)) = (1 - r_b)w(t)(1 + o(1)). \quad (2.3)$$

Indeed, the triangle with the vertices P_t, t, $(z(t), 0)$ is "complex-similar" to the triangle $P_t Q_t O$ (O is the origin), since their sides opposite to the vertex P_t lie in parallel affine complex lines. That is, in the new affine coordinates centered at P_t the second triangle is obtained from the first one by multiplication by the complex number $\frac{z(P_t)}{z(P_t)-z(t)} = \frac{\nu}{\nu-1}(1 + o(1))$, see (2.2). This implies (2.3). Let now a be an arbitrary irreducible germ of holomorphic curve at the origin that is transverse to b. Every family of points $\zeta(t)$ of the intersection $T_t b \cap a$ has w-coordinate asymptotically equivalent to $w(Q_t)$, by transversality and since the line $T_t b = Q_t \zeta(t)$ tends to the z-axis. This together with (2.3) proves the second equality in (2.1). The proposition is proved. □

Corollary 2.2 *Let $A \in \overline{\mathbb{C}}_\infty$, and let $(b, A) \subset \mathbb{CP}^2$ be an irreducible germ of an analytic curve that is transverse to $\overline{\mathbb{C}}_\infty$. Let (a, A) be another irreducible* *germ transverse to b, and let $\xi_1, \dots, \xi_{q_a}$, $\xi_j = \xi_j(t)$ be the points of intersection $T_t b \cap a$. Let (x, y) be affine coordinates in $\mathbb{C}^2$ such that the x-axis is tangent to b at A. Then*

$$x(t) = o(x(\xi_j(t))) \quad \textit{for all } j, \quad \textit{as } t \to A. \tag{2.4}$$

Proof Take the local coordinates $(z, w) = (\frac{1}{x}, \frac{y}{x})$ centered at A and apply the first formula in (2.1). $\square$

Proposition 2.3 *(cf. [4, p. 268, Proposition 2.50][1]) Let a, b be irreducible germs of holomorphic curves at the origin in the plane $\mathbb{C}^2$ with coordinates (z, w) that are tangent to the z-axis. Let c_a and c_b be the corresponding coefficients in (1.1). Then for every t small enough the intersection $T_t b \cap a$ consists of p_a points $\xi_1, \dots, \xi_{p_a}$ whose z-coordinates have the following asymptotics, as $t \to 0$.*

Case 1): $r_a > r_b$. One has

$$z(\xi_j) = \frac{r_b - 1}{r_b} z(t)(1 + o(1)) = \frac{r_b - 1}{r_b} t^{q_b}(1 + o(1)) \textit{ for } 1 \leq j \leq q_a, \tag{2.5}$$

$$z(t) = O((z(\xi_j))^{\frac{r_a - 1}{r_b - 1}}) = o(z(\xi_j)) \textit{ for } j > q_a. \tag{2.6}$$

Case 2): $r_a = r_b$. One has

$$z(\xi_j) = \zeta_j^{q_a} z(t)(1 + o(1)) = \zeta_j^{q_a} t^{q_b}(1 + o(1)), \tag{2.7}$$

where ζ_j are the roots of the polynomial

$$R_{p_a, q_a, c}(\zeta) = c\zeta^{p_a} - r\zeta^{q_a} + r - 1; \; r = \frac{p_a}{q_a} > 1, \; c = \frac{c_a}{c_b}. \tag{2.8}$$

(In the case, when $b = a$, one has $c = 1$, and the above polynomial has the double root 1 corresponding to the tangency point t.)

Case 3): $r_a < r_b$. One has

$$z(\xi_j) = O((z(t))^{\frac{r_b}{r_a}}) = o(z(t)),$$

Proof All the statements of the proposition were proved in loc. cit. except for the statement saying that in Case 1) one has exactly q_a intersection points with asymptotics (2.5) and exactly $p_a - q_a$ intersection points with

[1] The formulas from loc. cit. provide the inverse expressions, for the coordinate $z(t)$ in terms of $z(\xi_j(t))$. They are equivalent to the formulas given here.

asymptotics (2.6). Let us prove the latter statement. Thus, we consider that $r_a > r_b$. Let τ denote the value of the local parameter of the curve a at a point of intersection $\xi(t) \in T_t b \cap a$. The obvious analytic equality

$$G(t,\tau) = w(t) + \frac{w'(t)}{z'(t)}(z(\xi(t)) - z(t)) - w(\xi(t)) = 0 \tag{2.9}$$

has asymptotic form

$$G(t,\tau) = t^{p_b}(1+o(1)) + r_b t^{p_b - q_b}(\tau^{q_a} - t^{q_b})(1+o(1)) - c\tau^{p_a}(1+o(1)) = 0, \tag{2.10}$$

$c = \frac{c_a}{c_b} \neq 0$, as in [4, p. 269, proof of Proposition 2.50]. The Newton diagram of the germ of analytic function $G(t,\tau)$ is generated by the three monomials: $(1-r_b)t^{p_b}$, $r_b t^{p_b-q_b}\tau^{q_a}$, $-c\tau^{p_a}$. It consists of two edges: the first one with the vertices $(p_b, 0)$ and $(p_b - q_b, q_a)$; the second one with the vertices $(p_b - q_b, q_a)$ and $(0, p_a)$. The latter edges lie on distinct lines. The three above statements on Newton diagram follow from the inequality $r_a > r_b$, as in loc. cit. The germ of analytic curve $\{G = 0\} \subset \mathbb{C}^2_{(t,\tau)}$ at the origin is a union of two germs $\eta_1 \cup \eta_2$ corresponding to the edges of the Newton diagram, as in loc. cit. Namely, the monomials $(1-r_b)t^{p_b}$ and $r_b t^{p_b-q_b}\tau^{q_a}$ generating the first edge are asymptotically opposite (asymptotically "cancel out") along the germ η_1, and all the other Taylor monomials of the function G are of higher order along η_1, as in loc. cit. This implies that for every fixed small t there are exactly q_a parameter values τ for which $(t,\tau) \in \eta_1$, and they satisfy asymptotic equality (2.5). Similarly, the monomials $r_b t^{p_b-q_b}\tau^{q_a}$ and $-c\tau^{p_a}$ are asymptotically opposite along the germ η_2, and for every fixed small t there are exactly $p_a - q_a$ values τ such that $(t,\tau) \in \eta_2$, and they satisfy asymptotic equality (2.6). Proposition 2.3 is proved. □

Corollary 2.4 *Let $a, b \subset \mathbb{CP}^2$ be irreducible germs of holomorphic curves at a point $A \in \overline{\mathbb{C}}_\infty$ that are tangent to each other and transverse to $\overline{\mathbb{C}}_\infty$. Let (x,y) be affine coordinates on $\mathbb{C}^2$ with the x-axis being tangent to b at A. Let $\xi_1, \dots, \xi_{p_a}$, $\xi_j = \xi_j(t)$ be the points of intersection $T_t b \cap a$. Their x-coordinates have the following asymptotics, as $t \to A$:*

Case 1): $r_a > r_b$. One has

$$x(\xi_j) = \frac{r_b}{r_b - 1} x(t)(1+o(1)) \text{ for } 1 \leq j \leq q_a, \tag{2.11}$$

$$x(\xi_j) = o(x(t)) \text{ for } j > q_a. \tag{2.12}$$

Case 2): $r_a = r_b$. One has

$$x(\xi_j) = \theta_j^{q_a} x(t)(1+o(1)), \tag{2.13}$$

where θ_j are the roots of the polynomial[2]

$$Q_{p_a,q_a,c}(\theta) = \theta^{p_a} R_{p_a,q_a,c}(\theta^{-1}) = (r-1)\theta^{p_a} - r\theta^{q_a} + c; \qquad (2.14)$$

$$r = \frac{p_a}{q_a} = \frac{p_b}{q_b} > 1, \;\; c = \frac{c_a}{c_b}.$$

Case 3): $r_a < r_b$. One has

$$x(t) = o(x(\xi_j)).$$

The corollary follows from Proposition 2.3 by writing its asymptotics in the local coordinates $(z, w) = (\frac{1}{x}, \frac{y}{x})$.

2.3 Intersections with the germs having the same projective Puiseux exponents. Proof of Theorem 1.16

Let $A \in \overline{\mathbb{C}}_\infty$, and let b be an irreducible germ of analytic curve at A that is transverse to $\overline{\mathbb{C}}_\infty$. Let $\Gamma \supset b$ be an arbitrary finite union of germs of analytic curves at points of the line $T_A b$ including b.

Proposition 2.5 *Those points of intersection $T_t b \cap \Gamma$ whose x-coordinates are asymptotically equivalent to $x(t)$ times a nonzero multiplicative constant lie in the intersection of the line $T_t b$ with those irreducible germs $(a, A) \subset \Gamma$ tangent to b for which $r_a \geq r_b$. The asymptotics of their x-coordinates are given by either (2.11) if $r_a > r_b$, or (2.13) if $r_a = r_b$.*

The proposition follows immediately from Corollaries 2.2 and 2.4.

In what follows we assume that the curve b has relative symmetry property with respect to the curve Γ. First we prove Theorem 1.16 in the next simplest special case and then in the general case.

Special case: $\Gamma \setminus b$ is a union of germs at A transverse to b. Let us prove quadraticity of the germ b. To do this, we consider those intersection points of the tangent line $T_t b$ with Γ, whose x-coordinates have asymptotics $\nu x(t)(1 + o(1))$, $\nu \neq 0$, as $t \to A$. These are exactly the points of the intersection $T_t b \cap b$ (Proposition 2.5). Their x-coordinates have asymptotics $\theta_j^q x(t)(1 + o(1))$, where $\theta_1, \dots, \theta_p$ are the roots of the polynomial $W(\theta) = Q_{p,q,1}(\theta) = (r-1)\theta^p - r\theta^{p-q} + 1$; here $p = p_b$ and $q = q_b$ are the degrees from the parametrization (1.1) of the germ b (Corollary 2.4). The intersection

[2] In the case, when $b = a$, one has $c = 1$, and the polynomial $Q_{p_a,q_a,1}$ has double root 1 corresponding to the tangency point t. It has roots θ with $\theta^{q_a} \neq 1$, if and only if $r = \frac{p_a}{q_a} \neq 2$.

points of the line $T_t b$ with the other, transverse germs of the curve Γ have x-coordinates with bigger asymptotics, by Corollary 2.2. This together with the relative symmetry property implies that the collection of x-coordinates of the points of intersection $T_t b \cap b$ is invariant under the symmetry with respect to the point $x(t)$. This implies that the collection of powers θ_j^q is invariant under the symmetry with respect to 1. Therefore,

$$\sum_{j=1}^{p} \theta_j^q = p, \tag{2.15}$$

by symmetry. On the other hand,

$$\sum_{j=1}^{p} \theta_j^q = \frac{p}{r-1}. \tag{2.16}$$

Indeed, the latter sum is independent of the free term of the polynomial W, being expressed via elementary symmetric polynomials of degrees at most $q < p$, which are independent of the free term. Therefore, it equals the sum of the q-th powers of nonzero roots of the polynomial $(r-1)\theta^p - r\theta^{p-q}$. All the latter q-th powers are equal to $\frac{r}{r-1}$, hence their sum equals $\frac{qr}{r-1} = \frac{p}{r-1}$. This proves (2.16). Formulas (2.15) and (2.16) together imply that $p = \frac{p}{r-1}$. Hence, $r = 2$. Theorem 1.16 is proved.

General case. Let $a_1, \dots, a_l \subset \Gamma$ be the irreducible germs at A that are tangent to b and have the same projective Puiseux exponent: $r_{a_i} = r_b = r$. Let q_{a_i}, p_{a_i}, c_{a_i} be the corresponding degrees and coefficients from their parametrizations (1.1) in the local chart $(z, w) = (\frac{1}{x}, \frac{y}{x})$. Let

$$c_i = \frac{c_{a_i}}{c_b}, \ W_i = Q_{p_{a_i}, q_{a_i}, c_i}(\theta)$$

be the corresponding constants and polynomials from (2.14). Let θ_{ij}, $i = 1, \dots, l$, $j = 1, \dots, p_{a_i}$ denote the roots of the polynomials W_i.

Let k_1 denote the number of those points $\xi(t)$ of intersection $T_t b \cap \Gamma$, for which $x(\xi(t)) = o(x(t))$, as $t \to A$. Let k_2 denote the number of points of intersection of the line $T_t b$ with the union of those irreducible germs at A in Γ that are tangent to b and have projective Puiseux exponent bigger than $r = r_b$.

Proposition 2.6 *Let $r = r_b \neq 2$. The collection of powers $\theta_{ij}^{q_{a_i}}$ of the above roots contains exactly k_1 powers equal to 2 and exactly k_2 powers equal to $\frac{r-2}{r-1}$. The collection M of the other powers $\theta_{ij}^{q_{a_i}} \neq 2, \frac{r-2}{r-1}$ is invariant under the symmetry of the line $\mathbb{C}$ with respect to 1.*

Proof The points of intersection $T_t b \cap \Gamma$ whose x-coordinates are $o(x(t))$ should be symmetric with respect to t to other intersection points with x-coordinates asymptotically equivalent to $2x(t)$ and vice versa. The latter should be points of intersection with the germs a_i. This follows from Proposition 2.5 and the fact that they cannot be points of intersection with germs a having $r_a > r$. The latter statement follows from (2.11) and the inequality $\frac{r}{r-1} \neq 2$, which follows from the assumption that $r \neq 2$. This together with Corollary 2.4 and Proposition 2.5 implies that exactly k_1 powers $\theta_{ij}^{q_{a_i}}$ are equal to 2. Similarly, the points of intersection of the line $T_t b$ with germs (a, A) tangent to b and having $r_a > r$ have x-coordinates asymptotic to $\frac{r}{r-1}x(t)$, by (2.11). Vice versa, the points of intersection $T_t b \cap \Gamma$ with the latter asymptotics are the intersection points of the line $T_t b$ with the germs (a, A), $r_a > r$, tangent to b. This follows from Corollary 2.4, Proposition 2.5 and the fact that they cannot be points of intersection with the germs a_i: no number $s = (\frac{r}{r-1})^{\frac{1}{q_{a_i}}}$ can be a root of a polynomial W_i with $c_i \neq 0$. Indeed,

$$W_i(s) = (r-1)s^{p_{a_i}} - rs^{p_{a_i}} \left(\frac{r}{r-1}\right)^{-1} + c_i = c_i \neq 0.$$

The above intersection points with the germs (a, A), $r_a > r$ should be symmetric to points of intersection $T_t b \cap \Gamma$ with x-coordinates asymptotically equivalent to $(2 - \frac{r}{r-1})x(t) = \frac{r-2}{r-1}x(t)$. The latter points of intersection lie in the union of the germs a_i, by Proposition 2.5 and due to $\frac{r-2}{r-1} \neq \frac{r}{r-1}$. Therefore, exactly k_2 powers $\theta_{ij}^{q_{a_i}}$ are equal to $\frac{r-2}{r-1}$. The collection M of the powers $\theta_{ij}^{q_{a_i}} \neq 2, \frac{r-2}{r-1}$ is symmetric with respect to 1, by relative symmetry property and the above arguments. The proposition is proved. □

Set

$$\Pi = \sum_i p_{a_i} = \text{ the cardinality of the collection of all the roots } \theta_{ij}.$$

Corollary 2.7 *Let $r = r_b \neq 2$. Then*

$$(r-2)\Pi = k_2 - k_1(r-1). \tag{2.17}$$

Proof The invariance of the collection M under the symmetry with respect to 1 implies that the sum of its elements equals the cardinality $card(M) = \Pi - k_1 - k_2$. On the other hand,

$$\Pi - k_1 - k_2 = \sum_{x \in M} x = \sum_{i,j} \theta_{ij}^{q_{a_i}} - 2k_1 - \frac{r-2}{r-1}k_2, \tag{2.18}$$

by definition. Let us calculate the latter right-hand side. One has

$$\sum_{ij}\theta_{ij}^{q_{a_i}}=\sum_i\frac{p_{a_i}}{r-1}=\frac{\Pi}{r-1}, \tag{2.19}$$

as in (2.16). Substituting (2.19) to (2.18) yields

$$\frac{\Pi}{r-1}-2k_1-\frac{r-2}{r-1}k_2=\Pi-k_1-k_2,$$

which is equivalent to (2.17). □

Proof of Theorem 1.16. Suppose the contrary: the germ b is not quadratic, i.e., $r=r_b\neq 2$. Let us write $r=\frac{p}{q}$ with p and q being coprime. Then

$$p_{a_i}=s_ip,\ q_{a_i}=s_iq,\ s_i=\gcd(p_{a_i},q_{a_i}). \tag{2.20}$$

Case 1): Suppose that $r>2$. Hence, $r-2\geq\frac{1}{q}$. One has

$$k_2\geq(r-2)\Pi\geq\frac{1}{q}\Pi>\frac{1}{p}\Pi,$$

by (2.17). This implies that there exists a polynomial W_i for which more than $\frac{1}{p}$-th part of its roots have q_{a_i}-th powers equal to $\frac{r-2}{r-1}$. Thus, the number of the latter roots is no less than $s_i+1=\frac{p_{a_i}}{p}+1$. We will show that the above W_i cannot exist. Let it exist, and let us fix it. None of its roots θ_{ij} with $\theta_{ij}^{q_{a_i}}=\frac{r-2}{r-1}$ can be a multiple root. Indeed, the derivative of the polynomial $W_i(\theta)$ equals

$$\theta^{p_{a_i}-q_{a_i}-1}(p_{a_i}(r-1)\theta^{q_{a_i}}-r(p_{a_i}-q_{a_i}))=p_{a_i}(r-1)\theta^{p_{a_i}-q_{a_i}-1}(\theta^{q_{a_i}}-1),$$

since $r(p_{a_i}-q_{a_i})=rq_{a_i}(r-1)=p_{a_i}(r-1)$. Therefore, the q_{a_i}-th powers of the roots of the derivative are equal to $0,1\neq\frac{r-2}{r-1}$. Hence, the s_i+1 roots $\theta=\theta_{ij}$ of the polynomial W_i with $\theta_{ij}^{q_{a_i}}=\frac{r-2}{r-1}$ are distinct and satisfy the equality

$$\theta^{p_{a_i}-q_{a_i}}((r-1)\theta^{q_{a_i}}-r)+c_i=\theta^{p_{a_i}-q_{a_i}}((r-2-r)+c_i$$
$$=-2\theta^{p_{a_i}-q_{a_i}}+c_i=0. \tag{2.21}$$

Therefore, equation (2.21) has at least s_i+1 distinct solutions obtained from each other by multiplication with q_{a_i}-th roots of unity. Their p_{a_i}-th powers are equal, by (2.21). Therefore, the ratio of any two distinct solutions is simultaneously a q_{a_i}-th and p_{a_i}-th root of unity, and hence, an s_i-th root of unity, since $p_{a_i} = s_i p$, $q_{a_i} = s_i q$ and p, q are coprime. This implies that equation (2.21) may have at most s_i distinct solutions with equal q_{a_i}-th powers. The contradiction thus obtained proves Theorem 1.16.

Case 2): Suppose that $1 < r < 2$. One has

$$k_1 \geq \Pi \frac{2-r}{r-1} = \Pi \frac{2q-p}{p-q} \geq \frac{1}{p-q} \Pi > \frac{1}{p} \Pi,$$

by (2.17). This implies that the there exists a polynomial W_i that has at least $s_i + 1$ roots whose q_{a_i}-th powers are equal to 2, as in the previous case. We then get a contradiction, as in the above discussion. Theorem 1.16 is proved. □

3 Invariants of singularities, Plücker formulas, and the proof of Theorem 1.18

The proof essentially uses general Plücker and genus formulas for plane algebraic curves. The main observation is that the upper bound 2 to the projective Puiseux exponents of all local branches of the curve and Plücker formulas yield that the singularity invariants of the considered curve must obey a relatively high lower bound. On the other hand, the contribution of the points in the infinite line $\overline{\mathbb{C}}_\infty$ appears to be not sufficient to fit that lower bound unless the curve is a conic.

3.1 Invariants of plane curve singularities

For the reader's convenience, we recall here main definitions and formulas. Almost all this stuff is classically known (see [3, Chapter III], [7, §10], and the modern exposition in [5, Section I.3]).

Let $(x_0 : x_1 : x_2)$ be homogeneous coordinates on $\mathbb{CP}^2$. Let $\gamma \subset \mathbb{CP}^2$ be a reduced, irreducible curve of degree $d > 1$, i.e., given by a homogeneous square-free, irreducible polynomial $F(x_0, x_1, x_2)$ of degree $d > 1$. For any point $A \in \gamma$, denote by (γ, A) the germ of γ at A, or, more precisely, an intersection $\gamma \cap V$, where $V \subset \mathbb{CP}^2$ is a sufficiently small closed ball centered at A. Topologically, (γ, A) is a bouquet of discs b_i, $1 \leq i \leq r$, called local branches of γ at A. In an affine plane in $\mathbb{CP}^2$ containing A and having affine coordinates (x, y) such that $A = (0, 0)$ and the x-axis is tangent to the branch b_i, the branch b_i admits a Puiseux parametrization (1.1).

(1) Multiplicity and dual multiplicity of a local branch. Given a local branch b_i of the curve γ at $A \in \gamma$ and its Puiseux parametrization (1.1), the

number $s(b_i) = q$ is called the *multiplicity*, and the number $s^*(b_i) = p - q$ the *dual multiplicity* of the branch b_i. Note that $s(b_i)$ is the intersection multiplicity of the branch b_i with a transversal line, while $s^*(b_i) + s(b_i)$ is the intersection multiplicity with the tangent line $T_A b_i$. Observe also that the subquadraticity condition for b_i is equivalent to the relation

$$s^*(b_i) \leq s(b_i) \ . \tag{3.1}$$

(2) δ-invariant. Let $f(x, y) = 0$ be an equation of the germ (γ, A) (just $F = 0$ rewritten in the coordinates x, y). Then $\gamma_\varepsilon := \{f(x, y) = \varepsilon\} \cap V$, for $0 < |\varepsilon| \ll 1$, is a smooth surface with r holes (*Milnor fiber*). The δ-invariant of the germ (γ, A) admits several equivalent definitions and topologically can be defined as the genus of the closed surface obtained by attaching a sphere with r holes to the surface γ_ε. The genus formula, originally discovered by Hironaka [6], reads

$$\frac{(d-1)(d-2)}{2} = g(\gamma) + \sum_{A \in \mathrm{Sing}(\gamma)} \delta(\gamma, A) \ ,$$

where $g(\gamma)$ is the geometric genus of γ, i.e., the genus of the Riemann surface obtained by the resolution of singularities of γ. In particular, we have

$$\sum_{A \in \mathrm{Sing}(\gamma)} \delta(\gamma, A) \leq \frac{(d-1)(d-2)}{2} \ . \tag{3.2}$$

(3) Class of the singular point (κ-invariant). Given a germ (γ, A) and local affine coordinates (x, y) as above, suppose that the y-axis is not tangent to any of the local branches b_i, $1 \leq i \leq r$. Denote by γ' the polar curve of γ defined by the equation $\frac{\partial f}{\partial y} = 0$. The class of the germ (γ, A) is defined by

$$\kappa(\gamma, A) = (\gamma \cdot \gamma')_A \ ,$$

the intersection multiplicity of γ and γ' at A. It is well-known that

$$\kappa(\gamma, A) = 2\delta(\gamma, A) + \sum_{i=1}^{r} (s(b_i) - 1) \ . \tag{3.3}$$

(4) Hessian of the singular (or inflection) point. The *Hessian* H_γ of the curve γ is the curve given by the equation $\det \left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{0 \leq i,j \leq 2} = 0$. The *Hessian* of the germ (γ, A) is

$$h(\gamma, A) = (\gamma \cdot H_\gamma)_A \ ,$$

the intersection multiplicity of γ and H_γ at A. It vanishes in all smooth points of γ, where γ quadratically intersects its tangent line. An expression for $h(\gamma, A)$ via the preceding invariants was found in [8, Formula (2)]. It can be written as

$$h(\gamma, A) = 3\kappa(\gamma, A) + \sum_{i=1}^{r} (s^*(b_i) - s(b_i)) \ . \tag{3.4}$$

In view of $\deg H_\gamma = 3(d-2)$, Bézout's theorem yields (a Plücker formula)

$$3d(d-2) = \sum_{A \in \gamma} h(\gamma, A) \ . \tag{3.5}$$

3.2 Proof of Theorem 1.18

Let $\gamma \subset \mathbb{CP}^2$ be a curve of degree $d \geq 2$, satisfying the hypotheses of Theorem 1.18. We will show that $d = 2$.

Observe that $\delta(\gamma, A) = \kappa(\gamma, A) = h(\gamma, A) = 0$ for all points $A \in \gamma \setminus \overline{\mathbb{C}}_\infty$. Denote by $\mathcal{B}_{tr}$, resp. $\mathcal{B}_{tan}$ the set of the local branches of γ centered on $\overline{\mathbb{C}}_\infty$ and transversal, resp. tangent to $\overline{\mathbb{C}}_\infty$. Relation (3.1) holds for all the local branches $b \in \mathcal{B}_{tr}$. Thus, from (3.4) and (3.5), we get

$$3d(d-2) \leq 3 \sum_{A \in \gamma \cap \overline{\mathbb{C}}_\infty} \kappa(\gamma, A) + \sum_{b \in \mathcal{B}_{tan}} (s^*(b) - s(b)) \ . \tag{3.6}$$

Together with (3.2) and (3.3) this yields

$$\begin{aligned} 3d(d-2) &\leq 6 \sum_{A \in \gamma \cap \overline{\mathbb{C}}_\infty} \delta(\gamma, A) + 3 \sum_{b \in \mathcal{B}_{tr} \cup \mathcal{B}_{tan}} (s(b) - 1) + \sum_{b \in \mathcal{B}_{tan}} (s^*(b) - s(b)) \\ &\leq 3(d-1)(d-2) + 3 \sum_{b \in \mathcal{B}_{tr} \cup \mathcal{B}_{tan}} (s(b) - 1) + \sum_{b \in \mathcal{B}_{tan}} (s^*(b) - s(b)) \\ &= 3(d-1)(d-2) + \sum_{b \in \mathcal{B}_{tr} \cup \mathcal{B}_{tan}} (s(b) - 1) + \sum_{b \in \mathcal{B}_{tr}} s(b) \\ &\quad + \sum_{b \in \mathcal{B}_{tr}} (s(b) - 2) + \sum_{b \in \mathcal{B}_{tan}} (s^*(b) + s(b) - 2) \ . \end{aligned} \tag{3.7}$$

Developing $d = (\gamma \cdot \overline{\mathbb{C}}_\infty)$ into contributions of local branches $b \in \mathcal{B}_{tr} \cup \mathcal{B}_{tan}$, we obtain

$$\begin{cases} \sum_{b\in\mathcal{B}_{tr}\cup\mathcal{B}_{tan}}(s(b)-1) = d - |\mathcal{B}_{tr}\cup\mathcal{B}_{tan}| - \sum_{b\in\mathcal{B}_{tan}} s^*(b) \\ \qquad\qquad\qquad\qquad \le d - |\mathcal{B}_{tr}| - 2|\mathcal{B}_{tan}|, \\ \sum_{b\in\mathcal{B}_{tr}} s(b) = d - \sum_{b\in\mathcal{B}_{tan}}(s^*(b)+s(b)) \le d - 2|\mathcal{B}_{tan}|, \\ \sum_{b\in\mathcal{B}_{tr}}(s(b)-2) + \sum_{b\in\mathcal{B}_{tan}}(s^*(b)+s(b)-2) \\ \qquad\qquad = d - 2|\mathcal{B}_{tr}\cup\mathcal{B}_{tan}|, \end{cases} \tag{3.8}$$

and hence the sequence of relations (3.7) reduces to

$$2 \ge |\mathcal{B}_{tr}| + 2|\mathcal{B}_{tan}| \ . \tag{3.9}$$

If $\mathcal{B}_{tan} = \emptyset$ and all the branches $b \in \mathcal{B}_{tr}$ are centered at one point, then $\sum_{b\in\mathcal{B}_{tr}} s(b) = d$, and the intersection multiplicity of γ with the tangent to one of the branches $b \in \mathcal{B}_{tr}$ appears to be greater than d. This implies that the latter tangent line is contained in γ. Thus, γ splits off a line, contrary to the irreducibility assumption.

If $\mathcal{B}_{tan} = \emptyset$, $|\mathcal{B}_{tr}| = 2$, and the two branches $b_1, b_2 \in \mathcal{B}_{tr}$ have distinct centers, we have an equality in (3.9); hence, equalities in all the above relations, in particular, in (3.6). Thus, in view of (3.4), (3.5) and the inequality $s^*(b_i) \le s(b_i)$ (subquadraticity), it means $s^*(b_i) = s(b_i)$, $i = 1, 2$. Intersecting C with the tangent lines to b_1 and b_2, we obtain $s(b_i) \le \frac{d}{2}$, $i = 1, 2$, while the intersection with $\overline{\mathbb{C}}_\infty$ yields $s(b_1) + s(b_2) = d$. It follows that $s(b_1) = s(b_2) = \frac{d}{2}$. Choosing affine coordinates in $\mathbb{CP}^2 \setminus \overline{\mathbb{C}}_\infty$ so that the coordinate axes are tangent to b_1 and b_2 (at infinity) respectively, we obtain that the Newton polygon of the defining polynomial of γ is just the segment $[(0,0),(d/2,d/2)]$. Indeed, in local affine coordinates z_1, w_1 in a neighborhood of the center A of the branch b_1 such that the tangent line $T_A b_1$ is the z_1-axis and $\overline{\mathbb{C}}_\infty$ is the w_1-axis, we have b_1 given by

$$z_1 = t^{d/2}, \quad w_1 = c_1 t^d (1 + o(1)), \quad t \in (\mathbb{C}, 0)$$

which means that the Newton diagram of γ in these coordinates is the segment $[(d,0),(0,d/2)]$, i.e., the coefficients of all the monomials $z_1^i w_1^j$ with (i,j) below this segment vanish. In the coordinates $x = \frac{1}{z_1}$, $y = \frac{w_1}{z_1}$, this yields that the coefficients of all monomials $x^i y^j$ with $j < i$ vanish. The same consideration with the affine coordinates z_2, w_2 in a neighborhood of the center B of the branch b_2 such that $T_B b_2$ is the z_2 -axis and $\overline{\mathbb{C}}_\infty$ is the w_2-axis leads to the conclusion that the coefficients of all monomials $x^i y^j$ with

$j > i$ vanish. This finally leaves the only Newton segment $[(0,0),(d/2,d/2)]$. Note that a polynomial with such a Newton segment factors into the product of $\frac{d}{2}$ binomials of type $xy-\lambda$. Thus, $d=2$ due to the irreducibility of γ.

If $|\mathcal{B}_{tan}|=1$ and $|\mathcal{B}_{tr}|=0$, then we have an equality in (3.9); hence, all the above inequalities turn to be equalities, in particular, the second relation in (3.8), that is, $s^*(b)+s(b)=2$ for the unique branch of γ centered on $\overline{\mathbb{C}}_\infty$, which finally means that $d=2$.

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