On 4-reflective complex analytic planar billiards

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Abstract

The famous conjecture of V.Ya.Ivrii [14] says that *in every billiard with infinitely-smooth boundary in a Euclidean space the set of periodic orbits has measure zero.* In the present paper we study its complex analytic version for quadrilateral orbits in two dimensions, with reflections from holomorphic curves. We present the complete classification of 4-reflective complex analytic counterexamples: billiards formed by four holomorphic curves in the projective plane that have open set of quadrilateral orbits. This extends the previous author's result [7] classifying 4-reflective complex planar algebraic counterexamples. We provide applications to real planar billiards: classification of 4-reflective germs of real planar C^4-smooth pseudo-billiards; solutions of Tabachnikov's Commuting Billiard Conjecture and the 4-reflective case of Plakhov's Invisibility Conjecture (both in two dimensions; the boundary is required to be piecewise C^4-smooth). We provide a survey and a small technical result concerning higher number of complex reflections.

Contents

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1 Introduction

The famous V.Ya.Ivrii's conjecture [14] says that *in every billiard with infinitely-smooth boundary in a Euclidean space of any dimension the set of periodic orbits has measure zero.* As it was shown by V.Ya.Ivrii [14], his conjecture implies the famous H.Weyl's conjecture on the two-term asymptotics of the spectrum of Laplacian [28]. A brief historical survey of both conjectures with references is presented in [9, 10].

For the proof of Ivrii's conjecture it suffices to show that for every $k \in \mathbb{N}$ the set ot k-periodic orbits has measure zero. For $k = 3$ this was proved in [2, 22, 23, 27, 29]. For $k = 4$ in dimension two this was proved in [9, 10].

Remark 1.1 Ivrii's conjecture is open already for billiards with piecewise-analytic boundaries, and we believe that this is its principal case. In the latter case Ivrii's conjecture is equivalent to the statement saying that for every $k \in \mathbb{N}$ the set of k-periodic orbits has empty interior.

Besides the traditional real billiards, where each ray hitting the boundary is reflected back to the same side, it is interesting to study the so-called pseudo-billiards (introduced in Section 5), where some reflections change the side. Pseudo-billiards naturally arise, e.g., in the invisibility theory. One can ask the following question analogous to Ivrii's Conjecture: *classify those pseudo-billiards that have an open (positive measure) set of periodic orbits.* This question is closely related, e.g., to Plakhov's Invisibility Conjecture [17, conjecture 8.2] and Tabachnikov's Commuting Billiard Conjecture [25, p. 58], which is related to the famous Birkhoff Conjecture on integrable billiards [24, p. 95].

It appears that planar Ivrii's conjecture and all its analogues for pseudo-billiards have the same complexification stated and partially studied in [7, 8] and recalled below. This is the problem to classify all the so-called *k-reflective complex planar analytic billiards:* those collections of k complex analytic curves in $\mathbb{CP}^2$ for which the corresponding billiard has an open set of k-periodic orbits. Its studying presents a unifying approach to the original Ivrii's conjecture and all its above-mentioned analogues altogether.

In the present paper we solve the complex classification problem completely for $k = 4$ (Theorem 1.7, the main result). As an application, we provide the complete classification of germs of C^4-smooth real planar pseudo-billiards having open set of quadrilateral orbits (Subsections 5.1, 5.2). As applications of the latter result, we give solutions of Tabachnikov's Commuting Billiard Conjecture and the 4-reflective Plakhov's Invisibility Conjecture, both in two-dimensional piecewise C^4-smooth case (Subsections 5.3, 5.4).

The classification of analytic 4-reflective germs of pseudo-billiards follows almost immediately from the main complex result. The proof of the classification of smooth pseudo-billiards is also done by complex methods and is based on the theory of Cartan's prolongations of Pfaffian systems.

It is analogous to Yu.G.Kudryashov's arguments from [10, section 2] reducing the piecewise C^4-smooth case of 4-reflective Ivrii's Conjecture to the piecewise-analytic case.

The state of art of the classification problem of k-reflective complex billiards in the general case is presented in Section 6 together with small new technical results (Theorem 6.7 and Corollary 6.8).

Basic definitions and statement of main result are given below.

1.1 Main result: classification of 4-reflective complex analytic planar billiards

To recall the complexified Ivrii's conjecture and state the main result, let us recall some basic definitions contained in [7, section 1]. The complex plane $\mathbb{C}^2$ with affine coordinates (z_1, z_2) is equipped with the complexified Euclidean metric. It is the standard complex-bilinear quadratic form $dz_1^2 + dz_2^2$. This defines the notion of symmetry with respect to a complex line, reflections with respect to complex lines and more generally, reflections of complex lines with respect to complex analytic (algebraic) curves. The symmetry is defined by the same formula, as in the real case. More details concerning the complex reflection law are given in Subsection 2.3.

Remark 1.2 The geometry of the complexified Euclidean metric is somewhat similar to that of its another real form: the pseudo-Euclidean metric. Billiards in pseudo-Euclidean spaces were studied, e.g., in [6, 15].

Definition 1.3 A complex projective line $l \subset \mathbb{CP}^2 \supset \mathbb{C}^2$ is *isotropic*, if either it coincides with the infinity line, or the complexified Euclidean quadratic form vanishes on l. Or equivalently, a line is isotropic, if it passes through some of two points with homogeneous coordinates $(1 : \pm i : 0)$: the so-called *isotropic points at infinity* (also known as *cyclic* (or *circular*) points).

Convention 1.4 *Everywhere below by an* **irreducible analytic curve** *in* $\mathbb{CP}^n$ *we mean a non-constant* $\mathbb{CP}^n$*- valued holomorphic function on a connected Riemann surface.*

Definition 1.5 [7, definition 1.3] A *complex analytic (algebraic) planar billiard* is a finite collection of irreducible complex analytic (algebraic) curves $a_1, \dots, a_k \subset \mathbb{CP}^2$ that are not isotropic lines; set $a_{k+1} = a_1$, $a_0 = a_k$. A *k-periodic billiard orbit* is a collection of points $A_j \in a_j$, $A_{k+1} = A_1$, $A_0 = A_k$, such that for every $j = 1, \dots, k$ one has $A_{j+1} \neq A_j$, the tangent line $T_{A_j}a_j$ is not isotropic and the complex lines $A_{j-1}A_j$ and A_jA_{j+1} are symmetric with respect to the line $T_{A_j}a_j$ and are distinct from it. (Properly saying, we have to take points A_j together with prescribed branches of curves a_j at A_j: this specifies the line $T_{A_j}a_j$ in unique way, if A_j is a self-intersection point of the curve a_j.)

Definition 1.6 [7, definition 1.4] A complex analytic (algebraic) billiard $a_1, \dots, a_k$ is *k-reflective,* if it has an open set of k-periodic orbits. In more detail, this means that there exists an open set of pairs $(A_1, A_2) \in a_1 \times a_2$ extendable to k-periodic orbits $A_1 \dots A_k$. (Then the latter property automatically holds for every other pair of neighbor mirrors a_j, a_{j+1}.)

Problem (Complexified version of Ivrii's conjecture). *Classify all the k-reflective complex analytic (algebraic) billiards.*

Theorem 1.7 *A complex planar analytic billiard a, b, c, d is 4-reflective, if and only if it has one of the three following types:*

1) one of the mirrors, say a is a line, $c = a$, the curves b and d are symmetric with respect to the line a and distinct from it, see Section 5, Fig.7;

2) the mirrors are distinct lines through the same point $O \in \mathbb{CP}^2$, the pair of lines (a, b) is transformed to (d, c) by complex rotation around O, i.e., a complex isometry $\mathbb{C}^2 \to \mathbb{C}^2$ fixing O with unit Jacobian, see Section 5, Fig.8;

3) $a = c$, $b = d$, and they are distinct confocal conics, see Section 5, Fig.9–12.

Remark 1.8 Theorem 1.7 in the algebraic case is given by [7, theorem 1.11], which implies the 4-reflectivity of billiards of types 2) and 3). The proof of 4-reflectivity of billiards of type 1) repeats the proof in the algebraic case, see [7, example 1.7].

1.2 The plan of the proof of Theorem 1.7

Theorem 1.7 is obviously implied by the two following theorems.

Theorem 1.9 *Every 4-reflective complex planar analytic billiard with at least one algebraic mirror has one of the above types 1)–3).*

Theorem 1.10 *Let in a complex planar analytic 4-reflective billiard no mirror be a line. Then all the mirrors are algebraic curves.*

Theorems 1.9 and 1.10 are proved in Subsection 3.4 and Section 4 respectively.

Remark 1.11 Theorem 1.7 is local and can be stated for a *germ of 4-reflective analytic billiard:* a collection of irreducible germs of analytic curves (a, A), (b, B), (c, C), (d, D) in $\mathbb{CP}^2$ such that the quadrilateral $ABCD$ lies in an open set of quadrilateral orbits of the corresponding billiard.

For the proof of Theorem 1.7 we study the maximal analytic extensions of the mirrors. These are analytic curves parametrized by abstract connected Riemann surfaces, which we will denote by $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$. The latter are called the *maximal normalizations,* see the corresponding background material in Subsection 2.2. We represent the open set of quadrilateral orbits as a subset in $\hat{a} \times \hat{b} \times \hat{c} \times \hat{d}$ and will denote it by U_0. Its closure

$$U = \overline{U_0} \subset \hat{a} \times \hat{b} \times \hat{c} \times \hat{d}$$

in the usual topology is an analytic subset with only two-dimensional irreducible components. It will be called the *4-reflective set*, see [7, definition 2.13 and proposition 2.14]. The complement $U \setminus U_0$ consists of the so-called degenerate quadrilateral orbits: quadrilaterals $ABCD$ satisfying the reflection law that have either a pair of coinciding neighbor vertices, or a pair of coinciding adjacent edges, e.g., an edge tangent to a mirror through an adjacent vertex, or an isotropic tangency vertex.

One of the main ideas of the proof of Theorem 1.7 is similar to that from [9, 10, 7]: to study the degenerate orbit set $U \setminus U_0$. This idea itself together with basic algebraic geometry allowed to treat the algebraic case in [7]. One of the key facts used in the proof was properness (and hence, epimorphicity) of the projection $U \to \hat{a} \times \hat{b}$ to the position of two neighbor vertices. In the algebraic case the properness is automatic (follows from Remmert's Proper Mapping Theorem [11, p.34]), but in the general analytic case under consideration it isn't. We prove that the above projection is indeed proper in the analytic case. The most part of the proof of Theorem 1.7, and in particular, the proof of properness are based on studying restricted versions of Birkhoff distribution, which was introduced in [2]. All the Birkhoff distributions are briefly described below; more details are given in Subsection 2.7.

Definition 1.12 Let M be an n-dimensional (real or complex) analytic manifold. Let $\mathcal{D}$ be a d-dimensional analytic distribution on M, i.e., $\mathcal{D}(x) \subset T_xM$ is a d-dimensional subspace for every $x \in M$ and the map $x \mapsto \mathcal{D}(x)$ is analytic. Let $l \leq d$. An l-dimensional surface $S \subset M$ is said to be an *integral surface* for the distribution $\mathcal{D}$, if $T_xS \subset \mathcal{D}(x)$ for every $x \in S$.

Consider the projectivization of the tangent bundle $T\mathbb{CP}^2$:

$$\mathcal{P} = \mathbb{P}(T\mathbb{CP}^2).$$

It is the space of pairs (A, L): $A \in \mathbb{CP}^2$, $L \subset T_A\mathbb{CP}^2$ is a one-dimensional subspace. The space $\mathcal{P}$ is three-dimensional and it carries the standard two-dimensional contact distribution $\mathcal{H}$: the plane $\mathcal{H}(A, L) \subset T_{(A,L)}\mathcal{P}$ is the preimage of the line $L \subset T_A\mathbb{CP}^2$ under the derivative of the bundle projection $\mathcal{P} \to \mathbb{CP}^2$. The product $\mathcal{P}^k$ carries the product distribution $\mathcal{H}^k$. Let $\mathcal{R}_{0,k} \subset \mathcal{P}^k$ denote the subset of points $((A_1, L_1), \dots, (A_k, L_k))$ such that for every j one has $A_j \in \mathbb{C}^2 \subset \mathbb{CP}^2$, $A_{j\pm1} \neq A_j$, the lines A_jA_{j-1}, A_jA_{j+1} are symmetric with respect to the line L_j, and the three latter lines are distinct and non-isotropic. The above product distribution induces the so-called *Birkhoff distribution* $\mathcal{D}^k$ on $\mathcal{R}_{0,k}$, see [2]. It is well-known [2] that for every analytic billiard $a_1, \dots, a_k$ *the natural lifting to* $\mathcal{P}^k$ *of any analytic family of its k-periodic orbits* $A_1 \dots A_k$ *with* $L_j = T_{A_j}a_j$ *lies in* $\mathcal{R}_{0,k}$ *and is tangent to Birkhoff distribution.* In particular, if the billiard is k-reflective, then *the lifting to* $\mathcal{R}_{0,k}$ *of an open set of its k-periodic orbits is an integral surface of Birkhoff distribution.*

We will study the following restricted versions $\mathcal{D}_a$ and $\mathcal{D}_{ab}$ of Birkhoff distribution that correspond respectively to 4-reflective billiards a, b, c, d with one given mirror a (or two given mirrors a and b). The products $\hat{a} \times \mathcal{P}^3$ and $\hat{a} \times \hat{b} \times \mathcal{P}^2$ admit natural inclusions to $\mathcal{P}^4$ induced by parametrizations $\hat{a} \to a$, $\hat{b} \to b$. Let $M_a \subset \hat{a} \times \mathcal{P}^3$, $M_{ab} \subset \hat{a} \times \hat{b} \times \mathcal{P}^2$ denote the closures of the corresponding pullbacks of the set $\mathcal{R}_{0,4}$. The distributions $\mathcal{D}_a$, $\mathcal{D}_{ab}$ are the pullbacks of the Birkhoff distribution $\mathcal{D}^4$ on $\mathcal{R}_{0,4}$. They are 3- and 2-dimensional singular analytic distributions on M_a and M_{ab} in the sense of Subsection 2.6. For every billiard as above the natural lifting to $\hat{a} \times \mathcal{P}^3$ ($\hat{a} \times \hat{b} \times \mathcal{P}^2$) of any open set of its quadrilateral orbits lies in M_a (M_{ab}) and is an integral surface of the corresponding distribution $\mathcal{D}_a$ (respectively, $\mathcal{D}_{ab}$).

The proof of Theorem 1.7 is split into the following steps.

Step 1. Case of two neighbor algebraic mirrors. In this case it is easy to show that all the mirrors are algebraic (Proposition 2.1 in Subsection 2.1). This together with [7, theorem 1.11] implies that the billiard under question is of one of the types 1)–3), see Remark 1.8.

From now on we consider that no two neighbor mirrors are algebraic.

Step 2. Preparatory description of the complement $U \setminus U_0$. In Subsection 2.4 we study degenerate quadrilaterals $ABCD \in U \setminus U_0$ with a pair of coinciding neighbor vertices, say $A = D$, analogously to the arguments from [10, p.320]. Under mild additional assumptions, in particular, $B, C \neq A =$

D, we show that the other mirrors b and c are special curves called *triangular spirals centered at A.* Namely, they are phase curves of algebraic line fields on $\mathbb{CP}^2$: the so-called triangular line fields centered at A introduced in the same subsection (Proposition 2.19). One of the key arguments used in the proof of Theorem 1.7 is Proposition 2.21, which says that every triangular spiral with at least two distinct centers is algebraic, provided that the line through the centers is not isotropic. In Subsections 2.3 and 2.5 we recall the results of [7, subsections 2.1, 2.2] on partial description of degenerate quadrilaterals in $U \setminus U_0$ with either an isotropic tangency vertex, or an edge tangent to a mirror through an adjacent vertex.

Step 3. Properness of the projection $U \to \hat{a} \times \hat{b}$ (Section 3, Corollary 3.4). To prove it, we study the Birkhoff distribution $\mathcal{D}_{ab}$ and prove its non-integrability in Subsections 3.1–3.3. Moreover, we show that *the closure in M_{ab} of the union of its integral surfaces (if any) is a two-dimensional analytic subset in M_{ab}* (Lemma 3.1 and Corollary 3.2.) In the proof of the latter statement and in what follows we use Proposition 2.38 from Subsection 2.6. It deals with an m-dimensional singular analytic distribution, a given union of m-dimensional integral surfaces and the minimal analytic set M containing the latter union. Proposition 2.38 states that *the restriction to M of the distribution is m-dimensional and integrable.* Proposition 2.38 is a key tool for the whole paper.

The set U is identified with either the above two-dimensional analytic subset in M_{ab}, or a smaller analytic subset. This together with Proper Mapping Theorem implies properness of the projection $U \to \hat{a} \times \hat{b}$ (Corollary 3.4). The proof of Lemma 3.1 is done by contradiction. The contrary would imply the existence of at least three-dimensional invariant irreducible analytic subset $M \subset M_{ab}$ where the distribution $\mathcal{D}_{ab}$ is integrable. Then a complement $M^0 \subset M$ to a smaller analytic subset is saturated by open sets of quadrilateral orbits of 4-reflective billiards a, b, c, d with variable mirrors $c = c(x)$ and $d = d(x)$, $x \in M^0$. We treat separately two cases:

- some of the projections of the set M to the space of triples $(A(x), B(x), G(x))$, $G = C, D$, is not bimeromorphic.

- both latter projections are bimeromorphic.

The first case will be treated in Subsection 3.2. We show that there exist $x, y \in M^0$ projected to the same vertices A, B, $D = D_0$ but with distinct tangent lines $T_{D_0}d(x) \neq T_{D_0}d(y)$, D_0 being not a cusp[1] of the curves $d(x)$

[1]Everywhere in the paper by *cusp* we mean the singularity of an arbitrary irreducible singular germ of analytic curve, not necessarily the one given by equation $x^2 = y^3 + \dots$ in appropriate coordinates.

and $d(y)$, the projection to (A, B, D) being a local submersion at x, y. We then deduce that the billiard $d(y)$, $d(x)$, $c(x)$, $c(y)$ is 4-reflective (as in [7, proof of lemma 3.1]), and the mirror $c(x)$ is a triangular spiral with center D_0 (Proposition 2.19, Step 2). Then we slightly deform y with fixed vertices A and B to a point y' so that the corresponding mirror $d(y')$ intersects $d(x)$ at a point $D_1 \neq D_0$. We get analogously that the curve $c(x)$ is a triangular spiral with two distinct centers D_0 and D_1. This implies that $c(x)$ is algebraic (Proposition 2.21, Step 2). Similarly, we show that $c(y)$ is algebraic, fixing y and deforming x. Hence, the mirror $d(x)$ of the 4-reflective billiard $d(y)$, $d(x)$, $c(x)$, $c(y)$ is algebraic, as are $c(x)$ and $c(y)$ (Proposition 2.1, Step 1). Similarly, a and b are algebraic, as are $c(x)$ and $d(x)$. The contradiction thus obtained implies that the first case is impossible.

In the second case we show (in Subsection 3.3) that for an open set of points $x \in M^0$ the mirrors $c(x)$ and $d(x)$ are lines. Hence, the curves a and b are algebraic, by Step 1, – a contradiction. Finally, none of the above cases is possible. The contradiction thus obtained will prove Lemma 3.1.

Step 4. Case of one algebraic mirror, say a: proof of Theorem 1.9 (Subsection 3.4). Properness of the projection $U \to \hat{a} \times \hat{b}$ (Step 3) implies properness of the projection $U \to \hat{b}$ (algebraicity). Therefore, the preimage in U of every point $B \in \hat{b}$ is a compact holomorphic curve. This immediately implies that the mirror c is algebraic and there are two possibilities:

- either all the mirrors are algebraic, and we are done;

- or the projection of the above preimage to the position of the point D is constant for every B.

In the latter case we show that $a = c$ is a line and the mirrors $b, d \neq a$ are symmetric with respect to it: the billiard has type 1). This will prove Theorem 1.9.

From now on we consider that no mirror is algebraic. We show that this case is impossible. This will prove Theorem 1.10 and hence, Theorem 1.7.

Step 5. Case of intersected mirrors, say a and b intersect at a point A. Under the additional assumption that A is regular and not an isotropic tangency point for both a and b we show that $a = c$ (Corollary 3.5 proved in Subsection 3.5). The set U contains a non-empty at most one-dimensional compact analytic subset of quadrilaterals $AACD$ (properness of projection, Step 3). For the proof of Corollary 3.5 we show in Subsection 3.5 that this is a discrete subset in U consisting of quadrilaterals with all the vertices coinciding with A. Indeed, otherwise, if the above subset were one-dimensional, this would immediately imply that some of the mirrors c or d is algebraic, – a contradiction.

Step 6. Proof of Theorem 1.10. To do this, we study the three-dimensional Birkhoff distribution $\mathcal{D}_a$ on the 6-dimensional analytic set M_a. We fix a connected component of the open set of quadrilateral orbits of the billiard a, b, c, d. It is an integral surface of the distribution $\mathcal{D}_a$, which we will denote S. We consider the minimal analytic subset $M \subset M_a$ containing S, which is irreducible (easy to show), and study the restriction $\mathcal{D}_M$ to M of the distribution $\mathcal{D}_a$. We treat two separate cases: 1) $dim\mathcal{D}_M = 2$ (Subsection 4.1); 2) $dim\mathcal{D}_M = 3$ (Subsection 4.3). In the two-dimensional Case 1) we show that there exists an open subset $V \subset M$ saturated by integral surfaces of the distribution $\mathcal{D}_M$ that correspond to 4-reflective billiards a, $b(x)$, $c(x)$, $d(x)$ with $b(x)$ intersecting a (easily follows from transcendence of the curve a). We then deduce that either the mirror $b(x)$ is a line for all $x \in V$ (and hence, for all x regular for both M and $\mathcal{D}_M$), or the mirror $c(x)$ coincides with a for all x as above. This basically follows from Corollary 3.5, Step 5. The first subcase is impossible, since then the mirror b of the initial transcendental billiard would be a line, – a contradiction. In the second subcase the projection $\nu_C(M)$ of the whole variety M to the position of the vertex C lies in a. For a generic $A \in \hat{a}$ we consider its preimage $W_A \subset M$ under the projection $\nu_a : M \to \hat{a}$, which is a projective algebraic variety. It follows that the projection $\nu_C(W_A)$ lies in a transcendental curve a, while it should be an algebraic subset in $\mathbb{CP}^2$ (Remmert's Proper Mapping and Chow's Theorems). Hence, $\nu_C(W_A)$ is discrete. On the other hand, it cannot be discrete, whenever b is neither a line, nor a conic, by [7, proposition 2.32]. The contradiction thus obtained shows that Case 1) is impossible. The three-dimensional Case 2) is treated analogously, but it is more technical. The existence of two-dimensional integral surfaces as above of a three-dimensional distribution $\mathcal{D}_M$ is not automatic. Its proof is based on Cartan–Kuranishi–Rashevskii involutivity theory of Pfaffian systems. The corresponding background material is recalled in Subsection 4.2.

2 Preliminaries

2.1 Case of two neighbor algebraic mirrors

Proposition 2.1 *Let in a 4-reflective billiard a, b, c, d the mirrors a and b be algebraic curves. Then all the mirrors are algebraic.*

Proof By symmetry, it suffices to prove algebraicity of the mirror c. Fix a quadrilateral orbit $A_0B_0C_0D_0$. Consider the family of quadrilateral orbits $ABCD$ with fixed $D = D_0$. They are locally parametrized by the line

$l = AD$, which lies in the space $\mathbb{CP}^1$ of lines through D. The point A depends algebraically on l, since a is algebraic. Similarly, the line AB, and hence, the point B depend algebraically on l, since a, b are algebraic and AB is symmetric to l with respect to the line $T_A a$. Analogously, the line BC, which is symmetric to AB with respect to the line $T_B b$, depends algebraically on l. The line DC also depends algebraically on l, being the reflected image of the line l with respect to the fixed line $T_D d$. Finally, the variable intersection point $C = BC \cap DC$ should also depend algebraically on l. Hence, c is algebraic. The proposition is proved. $\square$

2.2 Maximal analytic extension

Recall that a germ $(a, A) \subset \mathbb{CP}^n$ of analytic curve is *irreducible*, if it is the image of a germ of analytic mapping $(\mathbb{C}, 0) \to \mathbb{CP}^n$.

Definition 2.2 [8, definition 5] Consider two holomorphic mappings of connected Riemann surfaces S_1, S_2 with base points $s_1 \in S_1$ and $s_2 \in S_2$ to $\mathbb{CP}^n$, $f_j : S_j \to \mathbb{CP}^n$, $j = 1, 2$, $f_1(s_1) = f_2(s_2)$. We say that $f_1 \leq f_2$, if there exists a holomorphic mapping $h : S_1 \to S_2$, $h(s_1) = s_2$, such that $f_1 = f_2 \circ h$. This defines a partial order on the set of classes of Riemann surface mappings to $\mathbb{CP}^n$ up to conformal reparametrization respecting base points.

The following proposition is classical, see the proof, e.g., in [8].

Proposition 2.3 *[8, proposition 2]. Every irreducible germ of analytic curve in $\mathbb{CP}^n$ has maximal analytic extension. In more detail, let $(a, A) \subset \mathbb{CP}^n$ be an irreducible germ of analytic curve. There exists an abstract connected Riemann surface $\hat{a}$ with base point $\hat{A} \in \hat{a}$ (which we will call the* **maximal normalization** *of the germ a) and a holomorphic mapping $\pi_a : \hat{a} \to \mathbb{CP}^n$, $\pi_a(\hat{A}) = A$ with the following properties:*

- the image of germ at $\hat{A}$ of the mapping π_a is contained in a;

- π_a is the maximal mapping with the above property in the sense of Definition 2.2.

Moreover, the mapping π_a is unique up to composition with conformal isomorphism of Riemann surfaces respecting base points.

Corollary 2.4 *Let M be a complex manifold, and let $f : M \to \mathbb{CP}^n$ be a non-constant holomorphic mapping. Let $U \subset M$ be an irreducible analytic subset, and let the restriction $f|_U$ have rank one on an open subset. Let*

$x \in U$ be a regular point, $A = f(x)$; then the image of the germ $f : (U, x) \to \mathbb{CP}^n$ is an irreducible germ (a, A) of analytic curve. Let $\pi_a : \hat{a} \to a$ be its maximal normalization. Let $\hat{U}$ be the normalization of the analytic set U (see [5, p.78]), $\pi_U : \hat{U} \to U$ be the natural projection (which is invertible on the regular part of U). Then there exists a unique holomorphic lifting $F : \hat{U} \to \hat{a}$ such that $f \circ \pi_U = \pi_a \circ F$.

Proof For every point $y \in \hat{U}$ and any point $z \in \hat{U}$ close enough to y there exists an analytic curve in $\hat{U}$ through y and z. This together with Proposition 2.3 (applied to the latter curves) and Hartogs' and Osgood's Theorems imply the corollary. □

2.3 Complex reflection law

The material presented in this subsection is contained in [7, subsection 2.1].

We fix an Euclidean metric on $\mathbb{R}^2$ and consider its complexification: the complex-bilinear quadratic form $dz_1^2 + dz_2^2$ on the complex affine plane $\mathbb{C}^2 \subset \mathbb{CP}^2$. We denote the infinity line in $\mathbb{CP}^2$ by $\overline{\mathbb{C}}_\infty = \mathbb{CP}^2 \setminus \mathbb{C}^2$.

Definition 2.5 The *symmetry* $\mathbb{C}^2 \to \mathbb{C}^2$ with respect to a non-isotropic complex line $L \subset \mathbb{CP}^2$ is the unique non-trivial complex-isometric involution fixing the points of the line L. It extends to a projective transformation of the ambient plane $\mathbb{CP}^2$. For every $x \in L$ it acts on the space $\mathbb{CP}^1$ of lines through x, and this action is called *symmetry at x*. If L is an isotropic line through a finite point x, then a pair of lines through x is called symmetric with respect to L, if it is a limit of symmetric pairs of lines with respect to non-isotropic lines converging to L.

Lemma 2.6 *[7, lemma 2.3] Let L be an isotropic line through a finite point x. A pair of lines (L_1, L_2) through x is symmetric with respect to L, if and only if some of them coincides with L.*

Convention 2.7 *For every irreducible analytic curve $a \subset \mathbb{CP}^2$ and a point $A \in \hat{a}$ the* **local branch** *a_A of the curve a at A is the germ of curve $\pi_a : (\hat{a}, A) \to \mathbb{CP}^2$, which is contained in a. By $T_A a$ we denote the tangent line to the local branch a_A at $\pi_a(A)$. Sometimes we idendify a point (subset) in a with its preimage in the normalization $\hat{a}$ and denote both subsets by the same symbol. In particular, given a subset in $\mathbb{CP}^2$, say a line l, we set $\hat{a} \cap l = \pi_a^{-1}(a \cap l) \subset \hat{a}$. If $a, b \subset \mathbb{CP}^2$ are two irreducible analytic curves, and $A \in \hat{a}$, $B \in \hat{b}$, $\pi_a(A) \neq \pi_b(B)$, then for simplicity we write $A \neq B$ and the line $\pi_a(A)\pi_b(B)$ will be referred to, as AB.*

Definition 2.8 A triple of points $BAD \in (\mathbb{CP}^2)^3$ satisfies the *complex reflection law* with respect to a given line L through A, if one of the following statements holds:

- either $B, D \neq A$, the line L is non-isotropic and the lines AB, AD are symmetric with respect to L;

- or $B, D \neq A$, the line L is isotropic and some of the lines AB, AD coincides with L;

- or A coincides with some of the points B or D.

Definition 2.9 Let $a_1, \dots, a_k \subset \mathbb{CP}^2$ be an analytic (algebraic) billiard, and let $\hat{a}_1, \dots, \hat{a}_k$ be the maximal normalizations of its mirrors. Let $P_k \subset \hat{a}_1 \times \dots \times \hat{a}_k$ denote the subset corresponding to k-periodic billiard orbits. The set P_k is contained in the subset $Q_k \subset \hat{a}_1 \times \dots \times \hat{a}_k$ of (not necessarily periodic) k-orbits: the k-gons $A_1 \dots A_k$ such that for every $2 \leq j \leq k-1$ one has $A_j \neq A_{j\pm1}$, the line $T_{A_j} a_j$ is not isotropic and the lines $A_j A_{j-1}$, $A_j A_{j+1}$ are symmetric with respect to it and distinct from it. Let $U_0 = \mathrm{Int}(P_k)$ denote the interior of the subset $P_k \subset Q_k$. Set

$$U = \overline{U_0} \subset \hat{a}_1 \times \dots \hat{a}_k : \text{ the closure is taken in the usual product topology.}$$

The set U will be called the *k-reflective set*.

Proposition 2.10 *[7, proposition 2.14]. The k-reflective set U is an analytic (algebraic) subset in $\hat{a}_1 \times \dots \times \hat{a}_k$. The billiard is k-reflective, if and only if the k-reflective set U is non-empty; then each its irreducible component is two-dimensional. If the billiard is k-reflective, then for every point $A_1 \dots A_k \in U$ each triple $A_{j-1}A_jA_{j+1}$ satisfies the complex reflection law from Definition 2.8 with respect to the line $T_{A_j} a_j$, and each projection $U \to \hat{a}_j \times \hat{a}_{j+1}$ is a submersion on an open dense subset in U.*

Addendum. *For every k-reflective billiard the latter projections $U \to \hat{a}_j \times \hat{a}_{j+1}$ are local biholomorphisms on the set of those k-periodic orbits whose vertices are not cusps of the corresponding mirrors.*

The addendum follows from definition.

2.4 Triangular algebraic line fields and spirals

Here we deal with a 4-reflective complex analytic billiard a, b, c, d whose 4-reflective set U contains a quadrilateral $ABCD$ with coinciding vertices $A = D$. We show (Proposition 2.19) that under mild genericity assumptions (implying, e.g., that $ABCD$ is not a single-point quadrilateral) either the mirrors b and c are conics, or they are so-called triangular spirals centered at A: phase curves of algebraic line fields invariant under the rotations around A. We show (Proposition 2.21) that every triangular spiral with two distinct centers is algebraic.

To define triangular spirals and state the above-mentioned results, we introduce yet another restricted Birkhoff distribution on the space of "framed triangles with fixed vertex". Let us fix a point $A \in \mathbb{C}^2$ (take it as the origin) and a non-trivial complex isometry $H \in SO(2, \mathbb{C}) \setminus Id$ fixing A. Recall that $\mathcal{P} = \mathbb{P}(T\mathbb{CP}^2)$, and $\mathcal{H}$ is the standard contact plane field on $\mathcal{P}$, see Subsection 1.2. Namely, for every $x = (B, L) \in \mathcal{P}$, where $B \in \mathbb{CP}^2$, $L \subset T_B\mathbb{CP}^2$ is a one-dimensional subspace, the plane $\mathcal{H}(x) \subset T_x\mathcal{P}$ is the preimage of the line L under the differential of the bundle projection $\mathcal{P} \to \mathbb{CP}^2$. Consider the product $\mathcal{P}^2$ equipped with the four-dimensional algebraic distribution $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$. Let $\mathcal{T}_{A,H} \subset \mathbb{CP}^2 \times \mathbb{CP}^2$ denote the subset of pairs (B, C) such that $B, C \neq A$, $B \neq C$, the lines AB, AC are distinct, non-isotropic and $AC = H(AB)$. Let $M^0_{A,H} \subset \mathcal{P}^2$ denote the subset of those pairs $((B, L_B), (C, L_C))$, for which $(B, C) \in \mathcal{T}_{A,H}$, the lines L_B, L_C are non-isotropic, the lines AB, BC are symmetric with respect to the line L_B; AC, BC are symmetric with respect to the line L_C; $AB \neq L_B$, $AC \neq L_C$. Set

$$M_{A,H} = \overline{M^0_{A,H}} \subset \mathcal{P}^2: \text{ the closure in the usual topology.}$$

This is a three-dimensional projective algebraic variety, and $M^0_{A,H} \subset M_{A,H}$ is its Zariski open and dense subset.

Proposition 2.11 *The variety $M^0_{A,H}$ is smooth and transversal to the distribution $\mathcal{H}^2$.*

Proof The smoothness is obvious. The restriction $\nu : M^0_{A,H} \to \mathcal{T}_{A,H}$ of the bundle projection $\mathcal{P}^2 \to (\mathbb{CP}^2)^2$ is a local diffeomorphism, by construction. For every $x = ((B, L_B), (C, L_C)) \in M^0_{A,H}$ the subspace $\mathcal{H}^2(x) \subset T_x(\mathcal{P}^2)$ is the preimage of the direct sum $L_B \oplus L_C \subset T_{(B,C)}(\mathbb{CP}^2)^2$ under the differential of the bundle projection. Thus, it suffices to show that for every $(B, C) \in \mathcal{T}_{A,H}$ the space $T_{(B,C)}\mathcal{T}_{A,H}$ is transversal to $L_B \oplus L_C$. Here L_B, L_C are arbitrary lines such that AB and BC are symmetric with respect to the line L_B and AC, BC are symmetric with respect to the line L_C.

For every $(B, C) \in \mathcal{T}_{A,H}$ finitely punctured lines $AB \times C$ and $B \times AC$ are contained in $\mathcal{T}_{A,H}$, by definition. We identify AB and AC with the corresponding one-dimensional subspaces in $T_B\mathbb{CP}^2$ and $T_C\mathbb{CP}^2$ respectively. Thus, $AB \oplus AC \subset T_{(B,C)}\mathcal{T}_{A,H}$ and $AB \oplus AC$ is transversal to $L_B \oplus L_C$ in

$T_{(B,C)}(\mathbb{CP}^2 \times \mathbb{CP}^2)$, since $AB \neq L_B$ and $AC \neq L_C$ by definition. This proves the proposition. $\Box$

Corollary 2.12 *For all $x \in M^0_{A,H}$ the intersections*

$$\mathcal{D}_{A,H}(x) = \mathcal{H}^2(x) \cap T_x M^0_{A,H}$$

are one-dimensional and form an algebraic line field on $M^0_{A,H}$.

Proof The transversal variety $M^0_{A,H}$ and distribution $\mathcal{H}^2$ in the ambient six-dimensional space $\mathcal{P}^2$ have dimensions 3 and 4 respectively. Hence, the intersections of their tangent spaces are one-dimensional. The algebraicity of the line field $\mathcal{D}_{A,H}$ is obvious. $\Box$

The next proposition shows that the line field $\mathcal{D}_{A,H}$ has an algebraic first integral: an appropriate holomorphic branch of squared perimeter of triangle. To define the latter branch, let us introduce the following definition.

Definition 2.13 Let $A, B, C \in \mathbb{C}^2$, $B \neq A, C$, and let the lines AB, BC be non-isotropic. Let L be a symmetry line of the pair of lines AB, BC. The symmetry $\sigma : AB \to BC$ with respect to the line L is an isometry with respect to the complexified Euclidean metric. For every line $l = AB, BC$ the complex distance function $l \to \mathbb{C}$, $x \mapsto |Bx| = dist(B, x)$ has two affine branches that differ by sign. Let us choose those affine distance functions on the lines AB, BC for which $|B\sigma(x)| \equiv -|Bx|$. We then say that the distances $|BA| = |AB|$, $|BC| = |CB|$ calculated with respect to the above affine distance functions are *L-concordant.*

Remark 2.14 The L-concordant distances are well-defined up to simultaneous change of sign. Their ratio $\frac{|AB|}{|BC|}$ is uniquely defined.

Example 2.15 Let in the above conditions $A, B, C \in \mathbb{R}^2$, and let $|AB|$, $|BC|$ be the Euclidean distances. Then $|AB|$, $|BC|$ are L-concordant, if L is the exterior bisector of the angle $\angle ABC$. Otherwise L is the interior bisector and $|AB|$, $-|BC|$ are L-concordant, see Figure 1.

Take an arbitrary point $((B, L_B), (C, L_C)) \in M^0_{A,H}$. We identify the lines L_B and L_C with the corresponding projective lines in $\mathbb{CP}^2$. Let us normalize the distances $|AB|$, $|BC|$, $|CA|$ to make $|AB|$ and $|BC|$ L_B-concordant and $|BC|$, $|CA|$ L_C-concordant. The perimeter $P = |AB| + |BC| + |CA|$ thus constructed is well-defined up to sign, and its square P^2 is a well-defined holomorphic function on $M^0_{A,H}$, see Figure 1.

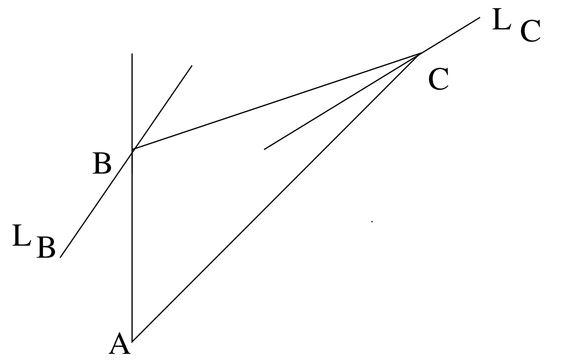

Figure 1: Concordant lengths in the real case: here $P = |AB| + |BC| - |AC|$.

Proposition 2.16 *The above squared perimeter P^2 is a first integral of the line field $\mathcal{D}_{A,H}$. The space $M_{A,H}$, the line field $\mathcal{D}_{A,H}$ and P^2 are invariant with respect to the complex rotation group $SO(2,\mathbb{C})$ fixing A.*

Proof The invariance follows from construction. The statement saying that P^2 is a first integral is a complexification of the classical statement on the real perimeter and the real Birkhoff distribution, see, e.g., [2, section 2]. Its proof is analogous to that in the real case. The proposition is proved. $\square$

Proposition 2.17 *Let $P^2 : M^0_{A,H} \to \mathbb{C}$ be the squared perimeter function from the above proposition. Let $p \in \mathbb{C}$, $\mathcal{S}_p$ be an irreducible component of the level set $\{P^2 = p\}$ in $M^0_{A,H}$. The projections $\nu_G : \mathcal{S}_p \to \mathbb{CP}^2$ to the position of the vertex $G = B, C$ have discrete preimages, and thus, are submersions on Zariski open dense subsets. The restriction to $\mathcal{S}_p$ of the line field $\mathcal{D}_{A,H}$ is sent by each projection to an $SO(2,\mathbb{C})$-invariant (multivalued) algebraic line field on $\mathbb{CP}^2$ depending on the choice of G and called* **triangular line field centered at A with parameters H, p.**

Proof The contrary to the discreteness of preimages of the projection, say ν_B would imply constance of the perimeter on a one-parameter family of triangles ABC with fixed vertices A and B, fixed line $AC = H(AB) = L$ and variable $C \in L$. This is obviously impossible. The algebraicity and invariance of the projected line field obviously follow from the algebraicity and invariance of the surface $\mathcal{S}_p$ and submersivity. $\square$

Definition 2.18 A *triangular spiral centered at A* is a complex orbit of a triangular line field centered at A, see Fig.2a).

Proposition 2.19 [2] *Let (a, A), (b, B), (c, C), (d, D) be irreducible germs of analytic curves in $\mathbb{CP}^2$ forming a 4-reflective analytic planar billiard in the following more general sense than in Remark 1.11: the quadrilateral $ABCD$ is contained in the 4-reflective set U, cf. Remark 1.11. Let the mirror germs a and d intersect: $A = D$. Let $B, C \neq A$, $AB \neq T_A a, T_B b$, $AC \neq T_D d, T_C c$, and let the lines AB, $T_A a$, $T_D d$, $T_B b$, $T_C c$ be not isotropic. If $AB \neq AC$, then the mirrors b and c are triangular spirals centered at A. Otherwise, if $AB = AC$, then the mirrors b and c are conics: complex circles centered at A, see Fig.2.*

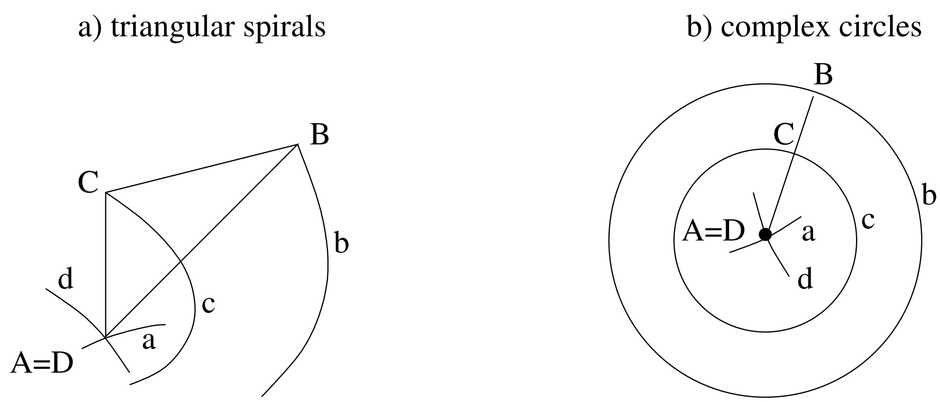

Figure 2: Family of degenerate orbits with $A = D$: the mirrors b and c are either spirals, or conics

Proof There exists an irreducible germ $\Gamma \subset U$ of analytic curve at $ABCD$ parametrized by local small complex parameter t and consisting of quadrilaterals $AB_tC_tD_t$ with fixed vertex A: $AB_0C_0D_0 = ABCD$. Let us fix it. One has $D_t \equiv D = A$. This follows from the fact that D_t is found as a point of intersection of the curve d with the line L_t symmetric to AB_t with respect to the tangent line $T_A a$. Indeed, the line L_0 is transverse to $T_D d$, as is AC, by assumption and since L_0 and AC are symmetric with respect to the line $T_D d$. Therefore, the intersection point $D_t \in L_t \cap d$ identically coincides with $D = D_0$. Let H denote the composition of symmetries with respect to the tangent lines, first $T_A a$, then $T_D d$. Thus, $H \in SO(2, \mathbb{C})$ fixes A and $H(AB_t) = AC_t$ for every $AB_tC_tD \in \Gamma$. We take A as the origin.

[2]Real triangular spirals were introduced in [10, p.320], where a real version of Proposition 2.19 was proved. Our proof of Proposition 2.19 is analogous to arguments from loc. cit.

Case 1): $AB \neq AC$. Then $B \neq C$ and the germ Γ is embedded into $M^0_{A,H}$ via the mapping $t \mapsto ((B_t, T_{B_t}b), (C_t, T_{C_t}c))$, by construction and non-isotropicity condition. Its image is a phase curve of the line field $\mathcal{D}_{A,H}$, analogously to discussions in [2] and [10, p.320]. This together with Proposition 2.17 implies that the projection $\Gamma \to \mathbb{CP}^2$ to the position of each one of the vertices B and C sends Γ to a triangular spiral centered at A, see Fig.2a). Hence, b and c are triangular spirals.

Case 2): $AB = AC$. Then $H = \pm Id$ and $AB_t \equiv AC_t$. Note that at least one of vertices, either B_t, or C_t varies, since Γ is a curve. To treat the case under consideration, we use the following remark.

Remark 2.20 There exist no k-reflective analytic planar billiard such that some its two neighbor mirrors coincide with the same line: such a billiard would have no k-periodic orbits in the sense of Definition 1.5 (cf. [7, proof of corollary 2.19]).

Subcase 2a): $B_t \equiv C_t \not\equiv const$. This implies that $b = c$ and the line AB_t is tangent to b at variable point B_t, as in loc. cit. Therefore, $b = c = AB$, which is impossible by the above remark. Hence, this subcase is impossible.

Subcase 2b): $B_t \not\equiv C_t$. Without loss of generality we consider that $B \neq C$. Thus, for every t small enough the points A, B_t and C_t are distinct and lie on the same line. Note that $T_Bb, T_Cc \neq AB = AC$, by the condition of the proposition. Hence, $T_{B_t}b, T_{C_t}c \perp AB_t$ for all t. This implies that $B_t, C_t \not\equiv const$ and b, c are complex circles centered at A, see Fig.2b). This proves Proposition 2.19. □

Proposition 2.21 *Let a planar analytic curve be a triangular spiral with respect to two distinct centers, and let the line through them be non-isotropic. Then it is algebraic.*

Proof A triangular spiral is a phase curve of a triangular algebraic line field. The latter field is invariant under complex rotations: the isometries fixing the center of the spiral with unit Jacobian. Suppose the contrary: the spiral under consideration is not algebraic. Then the corresponding line field is uniquely defined: two algebraic line fields coinciding on a non-algebraic curve (which is Zariski dense) coincide everywhere. Thus, the latter line field is invariant under complex rotations around two distinct centers, and the line through the centers is not isotropic. The latter rotations generate the whole group of complex isometries of $\mathbb{C}^2$ with unit Jacobian. Thus, the line field is invariant under all of them, which is impossible. The contradiction thus obtained proves the proposition. □

2.5 Tangencies in k-reflective billiards

Here we recall the results of [7, subsection 2.2].

We deal with k-reflective analytic planar billiards $a_1, \dots, a_k$ in $\mathbb{CP}^2$. Let $U \subset \hat{a}_1 \times \dots \times \hat{a}_k$ be the k-reflective set. The results of loc.cit. presented below concern degenerate quadrilaterals in $U \setminus U_0$: limits $A_1 \dots A_k$ of k-periodic orbits such that for a certain j with a_j being not a line the tangent line $T_{A_j} a_j$ and the adjacent edges $A_{j\pm1}A_j$ collide to the same non-isotropic limit. Then the limit vertex A_j will be called a *tangency vertex*. Proposition 2.23 shows that the latter cannot happen to be the only degeneracy of the limit k-gon. Its Corollary 2.26 presented at the end of the subsection concerns the case, when $k = 4$. It says that if the tangency vertex is distinct from its neighbor limit vertices, then its opposite vertex should be either also a tangency vertex, or a cusp with a non-isotropic tangent line. Proposition 2.25 extends Proposition 2.23 to the case, when some subsequent mirrors coincide and the corresponding subsequent vertices of a limiting orbit collide.

Definition 2.22 A point of a planar irreducible analytic curve is *marked*, if it is either a cusp, or an isotropic tangency point. Given a parametrized curve $\pi_a : \hat{a} \to a$, a point $A \in \hat{a}$ is marked, if it corresponds to a marked point of the local branch a_A, see Convention 2.7.

Proposition 2.23 *[7, proposition 2.16] Let $a_1, \dots, a_k$ and U be as above. Then U contains no k-gon $A_1 \dots A_k$ with the following properties:*

- each pair of neighbor vertices correspond to distinct points, and no vertex is a marked point;

- there exists a unique $s \in \{1, \dots, k\}$ such that the line $A_s A_{s+1}$ is tangent to the curve a_s at A_s, and the latter curve is not a line, see Fig.3.

Remark 2.24 A real version of Proposition 2.23 is contained in [10] (lemma 56, p.315 for $k = 4$, and its generalization (lemma 67, p.322) for higher k).

Proposition 2.25 *[7, proposition 2.18] Let $a_1, \dots, a_k$ and U be as at the beginning of the subsection. Then U contains no k-gon $A_1 \dots A_k$ with the following properties:*

1) each its vertex is not a marked point of the corresponding mirror;

2) there exist $s, r \in \{1, \dots, k\}$, $s < r$ such that $a = a_s = a_{s+1} = \dots = a_r$, $A_s = A_{s+1} = \dots = A_r$, and a is not a line;

3) For every $j \notin \mathcal{R} = \{s, \dots, r\}$ one has $A_j \neq A_{j\pm1}$ and the line $A_{j-1}A_j$ is not tangent to a_j at A_j, see Fig.4.

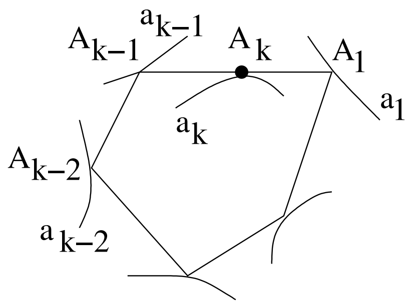

Figure 3: Impossible degeneracy of simple tangency: $s = k$.

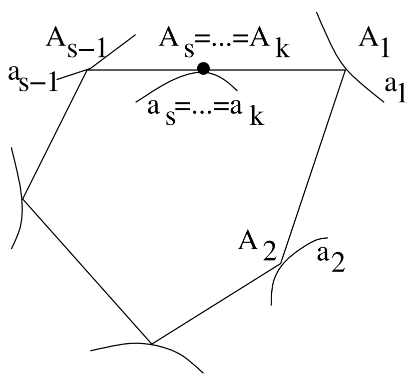

Figure 4: Coincidence of subsequent vertices and mirrors: $r = k$.

Corollary 2.26 *[7, corollary 2.20] Let a, b, c, d be a 4-reflective analytic billiard, and let b be not a line. Let $U \subset \hat{a} \times \hat{b} \times \hat{c} \times \hat{d}$ be the 4-reflective set. Let $ABCD \in U$ be such that $A \neq B$, $B \neq C$, the line $AB = BC$ is tangent to the curve b at B and is not isotropic. Then*

- either $AD = DC$ is tangent to the curve d at D, $\pi_a(A) = \pi_c(C)$, $a = c$ and the corresponding local branches coincide, i.e., $a_A = c_C$ (see Convention 2.7): "opposite tangency connection", see Fig.5a);

- or D is a cusp of the local branch d_D and the tangent line $T_D d$ is not isotropic: "tangency–cusp connection", see Fig.5b).

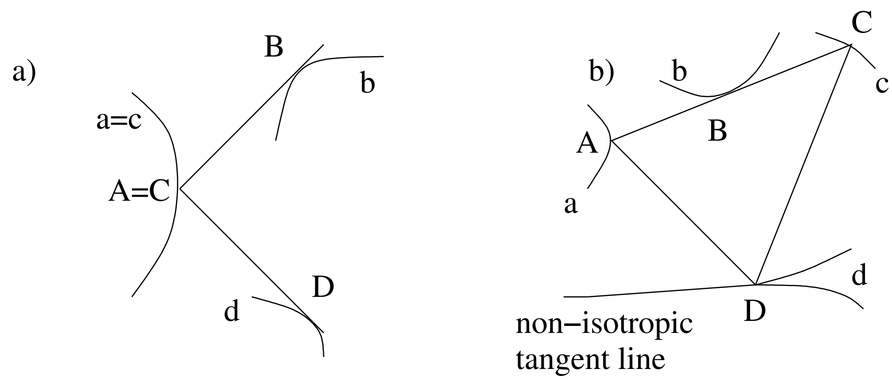

Figure 5: Opposite degeneracy to tangency vertex: tangency or cusp.

2.6 Singular analytic distributions

Here we recall definitions and properties of singular analytic distributions. The author believes that the material of the present subsection is known to specialists, but he did not find it in literature.

Definition 2.27 Let W be a complex manifold, $n = dimW$, $\Sigma \subset W$ be a nowhere dense closed subset, $m < n$. Let $\mathcal{D}$ be an analytic field of codimension m vector subspaces $\mathcal{D}(y) \subset T_yW$, $y \in W \setminus \Sigma$. We say that $\mathcal{D}$ is a *singular analytic distribution* of codimension m (dimension $n-m$) with the singular set $\Sigma = Sing(\mathcal{D})$, if it extends analytically to no point in Σ and each $x \in W$ has a neighborhood U where there exists a finite collection Ω of holomorphic 1-forms such that $\mathcal{D}(y) = \{v \in T_yW \mid \Omega(v) = 0\}$ for every $y \in U \setminus \Sigma$. (This generalizes the definition of a codimension one singular holomorphic foliation [4, p.11]. A similar definition in smooth case can be found in [13, p.8].)

Remark 2.28 Every k-dimensional singular analytic distribution on a complex manifold W is defined by a meromorphic[3] section of the Grassmanian

[3] Recall that a mapping $V \to W$ of complex manifolds (or analytic sets in complex manifolds) is *meromorphic*, if it is well-defined and holomorphic on an open and dense subset in V, and the closure of its graph is an analytic subset in $V \times W$, see Convention 2.31. It is well-known that if W is compact and V is irreducible, then the set of indeterminacies of every meromorphic mapping $V \to W$ is contained in the union of the singular set of V (which has codimension at least two in V, if V is normal) and an analytic subset in V of codimension at least two. A mapping is *bimeromorphic*, if it is meromorphic together with its inverse.

k-subspace bundle $Gr_k(TW)$ of the tangent bundle TW, and vice versa: each meromorphic section defines a k-dimensional singular analytic distribution. Its singular set is an analytic subset in W of codimension at least two, being the indeterminacy locus of a meromorphic section of a bundle with compact fibers.

Example 2.29 Let M be a complex analytic manifold, $N \subset M$ be a connected complex submanifold, $\mathcal{D}$ be a (regular) analytic distribution on M. The intersection $\mathcal{D}|_N(x) = T_xN \cap \mathcal{D}(x)$ with $x \in N$ has constant and minimal dimension on an open and dense subset $N^0 \subset N$. The subspaces $\mathcal{D}|_N(x) \subset T_xN$ form a singular analytic distribution $\mathcal{D}|_N$ on N that is called the *restriction* to N of the distribution $\mathcal{D}$. Its singular set is contained in the complement $N \setminus N^0$: the set of those points x, where the above dimension is not minimal. The restriction to N of a singular analytic distribution $\mathcal{D}$ on M with $N \not\subset Sing(\mathcal{D})$ is defined analogously; it is also a singular analytic distribution on N whose singular set is contained in the union of the intersection $Sing(\mathcal{D}) \cap N$ and the set of those points $x \in N$, where the above dimension $dim(\mathcal{D}|_N(x))$ is not minimal.

Example 2.30 Let $\mathcal{D}$ be a singular analytic distribution on a complex manifold W. Let M be another connected complex manifold, and let $\phi : M \to W$ be a non-constant holomorphic mapping such that $\phi(M) \not\subset Sing(\mathcal{D})$. For every $x \in M$ set

$$\phi^*\mathcal{D}(x) = (d\phi(x))^{-1}(\mathcal{D}(\phi(x)) \cap d\phi(x)(T_xM)) \subset T_xM.$$

The subspaces $\phi^*\mathcal{D}(x)$ form a singular analytic distribution on M called the *pullback distribution.* In the case, when ϕ is an immersion on an open and dense subset, the dimension of the distribution $\phi^*\mathcal{D}$ equals the minimal dimension of the above intersection.

Convention 2.31 *Let W be a complex manifold, $M \subset W$ be an analytic subset. Everywhere below for simplicity we say that a* **subset $N \subset M$ is analytic,** *if it is an analytic subset of the ambient manifold W.*

Definition 2.32 Let W be a complex manifold, $M \subset W$ be an irreducible analytic subset, and let $\mathcal{D}$ be a singular analytic distribution on W, $M \not\subset Sing(\mathcal{D})$. There exists an open and dense subset of those points[4] $x \in M_{reg}$

[4]Everywhere below for an analytic set M by M_{reg} (M_{sing}) we denote the set of its smooth (respectively, singular) points

regular for $\mathcal{D}$, for which the intersection $\mathcal{D}|_M(x) = \mathcal{D}(x) \cap T_x M$ has minimal dimension. Then we say that the subspaces $\mathcal{D}|_M(x)$ form a *singular analytic distribution* $\mathcal{D}|_M$ *on* M. It is regular on an open dense subset $M^0_{reg} \subset M_{reg}$. Its *singular set* $M \setminus M^0_{reg}$ is the union of the set M_{sing} and the set of those points $x \in M_{reg}$ where the distribution $\mathcal{D}|_M$ does not extend analytically. The distribution $\mathcal{D}|_M$ is also called the *restriction* to M of the distribution $\mathcal{D}$. The restriction of a singular analytic distribution $\mathcal{D}|_M$ to an irreducible analytic subset $V \subset M$, $V \not\subset Sing(\mathcal{D}|_M)$ is a singular analytic distribution on V defined analogously: it coincides with $\mathcal{D}|_V$. In the case, when the analytic set M is a union of several irreducible components, the restriction of the distribution $\mathcal{D}$ to each component will be referred to, as a singular distribution on M (which may have different dimensions on different components).

Example 2.33 The Birkhoff distribution $\mathcal{D}^k$ introduced at the end of Section 1 extends to a singular analytic distribution on the closure $\overline{\mathcal{R}_{0,k}} \subset \mathcal{P}^k$.

Definition 2.34 An *integral* l*-surface* of a singular analytic distribution $\mathcal{D}$ on an analytic variety[5] M is a holomorphic connected l-dimensional surface $S \subset M$ lying outside the singular set of $\mathcal{D}$ such that $T_x S \subset \mathcal{D}(x)$ for every $x \in S$. An m-dimensional singular analytic distribution is *integrable,* if there exists an integral m-surface through each its regular point.

Remark 2.35 The singular set of a singular analytic distribution is always an analytic subset in the ambient variety (see Convention 2.31), as in Remark 2.28. In general an integral surface is not an analytic set. Indeed, a generic linear vector field on $\mathbb{CP}^2$ has transcendental orbits. Hence, they are not analytic subsets in $\mathbb{CP}^2$, by Chow's Theorem [11, p.167].

Definition 2.36 Let M be an irreducible analytic subset in a complex manifold V. A p-dimensional *intrinsic* singular analytic distribution on M is a meromorphic section $\mathcal{D} : M \to Gr_p(TV)|_M$ of the Grassmanian bundle (see Footnote 3) such that $\mathcal{D}(x) \subset T_x M$ for regular points $x \in M$ where $\mathcal{D}$ is holomorphic.

Remark 2.37 Each singular analytic distribution is an intrinsic one. Conversely, each intrinsic singular analytic distribution is transformed to a usual

[5] Everywhere below by *analytic variety* we mean an analytic subset in a complex manifold

singular analytic distribution by a bimeromorphic mapping. Namely, consider the bundle projection $\pi : Gr_p(TV) \to V$ and the graph Γ of the section $\mathcal{D}$ (which is an analytic subset in $Gr_p(TV)$). Let $\mathcal{F}$ be the tautological distribution on $Gr_p(TV)$: for every $\lambda \in Gr_p(TV)$ the subspace $\mathcal{F}(\lambda) \subset T_\lambda Gr_p(TV)$ is the preimage of the subspace $\lambda \subset T_{\pi(\lambda)}V$ under the differential $d\pi(\lambda)$. The restriction $\mathcal{F}|_\Gamma$ is a singular holomorphic distribution transformed to $\mathcal{D}$ by the bimeromorphic projection $\pi : \Gamma \to M$.

Proposition 2.38 *Let an m-dimensional (intrinsic) singular analytic distribution $\mathcal{D}$ on an analytic subset N in a complex manifold have at least one m-dimensional integral surface. Given an arbitrary union S of m-dimensional integral surfaces, let $M \subset N$ denote the minimal analytic subset in N containing S. Then the restriction $\mathcal{D}|_M$ is an integrable m-dimensional singular analytic distribution.*

Proof It suffices to prove the proposition for a true singular analytic distribution, locally defined as kernel field of a system of holomorphic 1-forms (Remark 2.37). The set $\{x \in M^0_{reg} \mid \mathcal{D}(x) \subset T_xM\}$ coincides with all of M^0_{reg}, since it contains $S \cap M^0_{reg}$ and its closure is an analytic subset in M. Similarly, the set of those points in M^0_{reg} where the distribution $\mathcal{D}|_M$ satisfies the Frobenius integrability condition coincides with all of M^0_{reg}, since it contains S and its closure is an analytic subset in M. Thus, $\mathcal{D}|_M$ is an m-dimensional integrable distribution. The proposition is proved. $\square$

2.7 Birkhoff distributions and periodic orbits

Here we recall the definition and basic properties of Birkhoff distribution and its restricted versions. Consider the space $\mathcal{P} = \mathbb{P}(T\mathbb{CP}^2)$, which consists of pairs (A, L), $A \in \mathbb{CP}^2$, L being a one-dimensional subspace in $T_A\mathbb{CP}^2$. Its natural projection to $\mathbb{CP}^2$ will be denoted by Π. The *standard contact structure* is the two-dimensional analytic distribution $\mathcal{H}$ on $\mathcal{P}$ given by the $d\Pi$-pullbacks of the lines L:

$$\mathcal{H}(A, L) = (d\Pi(A, L))^{-1}(L) \subset T_{(A,L)}\mathcal{P}.$$

The distribution $\mathcal{H}^k = \oplus_{j=1}^k \mathcal{H}$ is the $2k$-dimensional product distribution on $\mathcal{P}^k$. Recall that $\mathcal{R}_{0,k} \subset \mathcal{P}^k$ is the subset of k-tuples $((A_1, L_1), \dots, (A_k, L_k))$ such that for every $j = 1, \dots, k$ one has $A_j \in \mathbb{C}^2 = \mathbb{CP}^2 \setminus \overline{\mathbb{C}}_\infty$, $A_j \neq A_{j\pm1}$, the lines A_jA_{j-1}, A_jA_{j+1} are symmetric with respect to the line L_j, and the three latter lines are distinct and non-isotropic. This is a $2k$-dimensional smooth quasiprojective variety. The *Birkhoff distribution* $\mathcal{D}^k$ is the restriction to $\mathcal{R}_{0,k}$ of the product distribution $\mathcal{H}^k$:

$$\mathcal{D}^k(x) = T_x\mathcal{R}_{0,k} \cap \mathcal{H}^k(x) \text{ for every } x \in \mathcal{R}_{0,k}. \tag{2.1}$$

This is a k-dimensional analytic distribution. It is the complexification of the real Birkhoff distribution introduced in [2]. For every two irreducible analytic curves $a, b \subset \mathbb{CP}^2$ with maximal normalizations $\pi_a : \hat{a} \to \mathbb{CP}^2$, $\pi_b : \hat{b} \to \mathbb{CP}^2$ we will denote

$$\mathcal{P}_a = \hat{a} \times \mathcal{P}^3; \ \mathcal{P}_{ab} = \hat{a} \times \hat{b} \times \mathcal{P}^2.$$

We consider the natural embeddings $\eta_a : \mathcal{P}_a \to \mathcal{P}^4$, $\eta_{ab} : \mathcal{P}_{ab} \to \mathcal{P}^4$:

$$\eta_a(A, (B, L_B), (C, L_C), (D, L_D)) = ((\pi_a(A), T_A a), (B, L_B), (C, L_C), (D, L_D)),$$

$$\eta_{ab}(A, B, (C, L_C), (D, L_D)) = ((\pi_a(A), T_A a), (\pi_b(B), T_B b), (C, L_C), (D, L_D)).$$

Remark 2.39 The critical points of the mappings η_a (η_{ab}) are contained in the sets $Cusp_a \subset \mathcal{P}_a$, $Cusp_{ab} \subset \mathcal{P}_{ab}$ of those points for which A (A or B) is a cusp of the corresponding curve (see Footnote 1 in Section 1). The mappings η_a and η_{ab} are immersions outside the sets $Cusp_a$ and $Cusp_{ab}$.

Consider the subsets

$$M_a^0 = \eta_a^{-1}(\mathcal{R}_{0,4}) \setminus Cusp_a \subset \mathcal{P}_a, \ M_{ab}^0 = \eta_{ab}^{-1}(\mathcal{R}_{0,4}) \setminus Cusp_{ab} \subset \mathcal{P}_{ab}, \tag{2.2}$$

$$M_a = \overline{M_a^0} \subset \mathcal{P}_a; \ M_{ab} = \overline{M_{ab}^0} \subset \mathcal{P}_{ab} :$$

the closures are taken in the usual topology. The subsets $M_a \subset \mathcal{P}_a$ and $M_{ab} \subset \mathcal{P}_{ab}$ are obviously analytic. The *restricted (pullback) Birkhoff distributions* $\mathcal{D}_a$ on M_a^0 and $\mathcal{D}_{ab}$ on M_{ab}^0 respectively are the pullbacks of the Birkhoff distribution $\mathcal{D}^4$:

$$\mathcal{D}_a(x) = (d\eta_a(x))^{-1}(\mathcal{D}^4(\eta_a(x)) \cap d\eta_a(x)(T_x M_a^0)) \subset T_x M_a^0, \ x \in M_a^0; \tag{2.3}$$

$$\mathcal{D}_{ab}(x) = (d\eta_{ab}(x))^{-1}(\mathcal{D}^4(\eta_{ab}(x)) \cap d\eta_{ab}(x)(T_x M_{ab}^0)) \subset T_x M_{ab}^0, \ x \in M_{ab}^0. \tag{2.4}$$

They extend to singular analytic distributions on M_a and M_{ab} respectively in the sense of Subsection 2.6. For example, $\mathcal{D}_a$ is the restriction to M_a of the distribution $T\hat{a} \oplus \mathcal{H}^3$ on $\mathcal{P}_a = \hat{a} \times \mathcal{P}^3$. One has

$$dim M_a = 6, \ dim \mathcal{D}_a = 3; \ \ dim M_{ab} = 4, \ dim \mathcal{D}_{ab} = 2.$$

Definition 2.40 (complexification of [10, definition 14]) Let $k \in \mathbb{N}$. A k-gon $A_1 \dots A_k \in (\mathbb{CP}^2)^k$ is said to be *non-degenerate,* if for every $j = 1, \dots, k$ (we set $A_{k+1} = A_1$, $A_0 = A_k$) one has $A_j \in \mathbb{C}^2 = \mathbb{CP}^2 \setminus \overline{\mathbb{C}}_\infty$, $A_{j+1} \neq A_j$, $A_{j-1}A_j \neq A_jA_{j+1}$ and the line A_jA_{j+1} is not isotropic. We will call the complex lines $A_jA_{j\pm1}$ the *edges adjacent to* A_j.

Remark 2.41 The above sets $\mathcal{R}_{0,k}$, M_a^0, M_{ab}^0 are projected to the sets of non-degenerate k-gons (quadrilaterals). A periodic billiard orbit in the sense of Definition 1.5 is non-degenerate, provided that its vertices lie in $\mathbb{C}^2$, its edges are non-isotropic and every two adjacent edges are distinct. The k-reflective set U of a k-reflective billiard contains an open and dense subset $U_1 \subset U$ of those non-degenerate orbits whose vertices are not marked points (see Definition 2.22) of the corresponding mirrors.

Definition 2.42 (cf. [10, definition 16]) Consider some of the above Birkhoff distributions, let us denote it $\mathcal{D}$, and let M denote the underlying manifold (e.g., $\mathcal{R}_{0,k}$, M_{ab}^0,...) carrying $\mathcal{D}$. Consider the projections of the manifold M to the positions of vertices of the k-gon (quadrilateral). A subspace $E \subset \mathcal{D}(x)$, $x \in M$, is said to be *non-trivial*, if the restriction to E of the differential of each above projection has positive rank. (Then the rank equals one.) An *integral surface* of the same distribution is *non-trivial*, if its tangent planes are non-trivial.

For every analytic billiard a, b, c, d there exist natural analytic mappings $\Psi_a : \hat{a} \times \hat{b} \times \hat{c} \times \hat{d} \to \mathcal{P}_a$, $\Psi_{ab} : \hat{a} \times \hat{b} \times \hat{c} \times \hat{d} \to \mathcal{P}_{ab}$:

$$\begin{aligned} \Psi_a(ABCD) &= (A, (B, T_B b), (C, T_C c), (D, T_D d)); \\ \Psi_{ab}(ABCD) &= (A, B, (C, T_C c), (D, T_D d)). \end{aligned} \tag{2.5}$$

Proposition 2.43 *Let a, b, c, d be a 4-reflective billiard. The mappings Ψ_a, Ψ_{ab} send the subset $U_1 \subset U$ (see Remark 2.41) to M_a^0, M_{ab}^0, and the images of its connected components are non-trivial integral surfaces of the restricted Birkhoff distributions $\mathcal{D}_a$ and $\mathcal{D}_{ab}$ respectively. Vice versa, each non-trivial integral surface of any of the latter distributions is the image of an open set of quadrilateral orbits of a 4-reflective billiard a, b, c, d with given mirror a (respectively, given mirrors a and b).*

The proposition is the direct complexification of an analogous result from [2] and Yu.G.Kudryashov's lemmas [10, section 2, lemmas 17, 18].

Everywhere below for every $x \in M_{ab}^0 \subset \mathcal{P}_{ab} = \hat{a} \times \hat{b} \times \mathcal{P}^2$ we denote

$$l_a = l_a(x) = A(x)D(x), \ l_b = l_b(x) = B(x)C(x).$$

Remark 2.44 The lines l_a and l_b depend only on $(A, B) = (A(x), B(x))$: these are the lines symmetric to AB with respect to the lines $T_A a$ and $T_B b$ respectively. Sometimes we will write $l_a = l_a(A, B)$, $l_b = l_b(A, B)$.

For every $x \in M^0_{ab}$ ($x \in M^0_a$) the corresponding lines L_G, $G = (B,)C, D$, will be denoted by $L_G(x)$. The projections to the positions of vertices will be denoted by

$$\nu_a : \mathcal{P}_a \to \hat{a},\ \ \nu_{ab} : \mathcal{P}_{ab} \to \hat{a} \times \hat{b},$$

$$\nu_G : \mathcal{P}_a, \mathcal{P}_{ab} \to \mathbb{CP}^2,\ \ x \mapsto G(x) \text{ for } G = B, C, D \text{ (respectively, } G = C, D).$$

Remark 2.45 The above projections ν_a and ν_{ab} are proper and epimorphic. The corresponding preimages of points are compact and naturally identified with projective algebraic varieties.

Proposition 2.46 *The planes of the Birkhoff distribution $\mathcal{D}_{ab}$ on M^0_{ab} are non-trivial, and hence, so is each its integral surface.*

Proof Suppose the contrary: there exists an $x \in M^0_{ab}$ such that the differential of some of the above projections, say ν_a vanishes identically on $\mathcal{D}_{ab}(x)$. We consider only the case of projection ν_a: the cases of other projections are treated analogously. Then the kernel

$$K(x) = \mathrm{Ker}(d\nu_b|_{\mathcal{D}_{ab}(x)}) \subset \mathcal{D}_{ab}(x)$$

is at least one-dimensional subspace. The vertices $A(x) = \nu_a(x) \in \hat{a}$, $B(x) = \nu_b(x) \in \hat{b}$ are not cusps, since $x \in M^0_{ab}$ by assumption. The line functions l_a, l_b have zero derivatives along $K(x)$, as do the vertices A, B. The derivative along $K(x)$ of at least one of the vertices C or D, say D is not identically zero. Then $d\nu_D(K(x)) = l_a(x) \neq L_D(x)$, since $D \in l_a$ and l_a has zero derivative. Thus, $d\nu_D(\mathcal{D}_{ab}(x)) \not\subset L_D(x)$, – a contradiction to the definition of the distribution $\mathcal{D}_{ab}$. The proposition is proved. □

3 Non-integrability of the Birkhoff distribution $\mathcal{D}_{ab}$ and corollaries

3.1 Main lemma, corollaries and plan of the proof

In the present section we prove the following lemma on the non-integrability of the two-dimensional Birkhoff distribution $\mathcal{D}_{ab}$ and corollaries.

Lemma 3.1 *For every pair of analytic curves $a, b \subset \mathbb{CP}^2$ distinct from isotropic lines that are not both lines the corresponding Birkhoff distribution $\mathcal{D}_{ab}$ is non-integrable. Moreover, there is no three-dimensional irreducible analytic subset $M \subset M_{ab}$ (see Convention 2.31) such that $M \cap M^0_{ab} \neq \emptyset$ and the restriction $\mathcal{D}_{ab}|_M$ is two-dimensional and integrable.*

Corollary 3.2 *In the conditions of Lemma 3.1 the union of all the integral surfaces of the Birkhoff distribution $\mathcal{D}_{ab}$ in M^0_{ab} is contained in a two-dimensional analytic subset in M_{ab}.*

Proof The minimal analytic subset $M \subset M_{ab}$ containing all the integral surfaces is tangent to $\mathcal{D}_{ab}$, and the distribution $\mathcal{D}_{ab}$ is integrable there (Proposition 2.38). Hence, $dim M = 2$, by Lemma 3.1. This proves the corollary. $\square$

Remark 3.3 In the case, when a and b are lines, the statements of Lemma 3.1 and the corollary are false. In this case there exists a one-parametric family of 4-reflective billiards a, b, c, d of type 2) from Theorem 1.7. The corresponding open sets of quadrilateral orbits form a one-parametric family of integral surfaces of the distribution $\mathcal{D}_{ab}$. They saturate an open and dense subset in a three-dimensional analytic subset in M_{ab}.

Let $\Psi = \Psi_{ab} : \hat{a} \times \hat{b} \times \hat{c} \times \hat{d} \to \mathcal{P}_{ab}$ be the mapping from (2.5).

Corollary 3.4 *Let a, b, c, d be a 4-reflective complex planar analytic billiard, and let U be the 4-reflective set. Then the image $\Psi(U) \subset \mathcal{P}_{ab}$ is a two-dimensional analytic subset lying in M_{ab}. The natural projection $U \to \hat{a} \times \hat{b}$ is a proper epimorphic mapping.*

Proof In the case, when both a, b are algebraic, the curves c, d are also algebraic (Proposition 2.1), and the statements of the corollary follow immediately. Thus, without loss of generality we consider that some of the curves a, b is not algebraic. It suffices to prove the first statement of the corollary. Then its second statement, which is equivalent to the properness and the epimorphicity of the analytic set projection $\nu_{ab} : \Psi(U) \to \hat{a} \times \hat{b}$, follows from the properness of the projection $\mathcal{P}_{ab} \to \hat{a} \times \hat{b}$ and Remmert's Proper Mapping Theorem [11, p.34]. The image $\Psi(U)$ lies in M_{ab}, which follows from definition. Recall that $U_1 \subset U$ denote the open and dense subset of non-degenerate orbits whose vertices are not marked points. Let $\mathcal{S} \subset \mathcal{P}_{ab}$ denote the minimal analytic subset containing $\Psi(U_1)$, which obviously contains $\Psi(U)$. Each its irreducible component is two-dimensional, as is U_1, by Corollary 3.2 and since $\Psi(U_1)$ is a union of integral surfaces of the distribution $\mathcal{D}_{ab}$, see Proposition 2.43. The projections $\nu_C, \nu_D : \mathcal{S} \to \mathbb{CP}^2$ to the positions of the vertices C and D have rank one on an open dense subset, and $\nu_C(\mathcal{S}) \subset c$, $\nu_D(\mathcal{S}) \subset d$: this holds on $\Psi(U_1)$, and hence, on each irreducible component of the set $\mathcal{S}$. Let $\hat{\mathcal{S}}$ denote the normalization of the analytic set $\mathcal{S}$, and $\pi_{\mathcal{S}} : \hat{\mathcal{S}} \to \mathcal{S}$ denote the natural projection. The above projections lift to holomorphic mappings $\nu_{\hat{g}} : \hat{\mathcal{S}} \to \hat{g}$, $g = c, d$: $\nu_G \circ \pi_{\mathcal{S}} = \pi_g \circ \nu_{\hat{g}}$ on $\hat{\mathcal{S}}$ (Corollary 2.4). This yields an "inverse" mapping $\Psi^{-1} = (\nu_{ab} \circ \pi_S) \times \nu_{\hat{c}} \times \nu_{\hat{d}} : \hat{\mathcal{S}} \to \hat{a} \times \hat{b} \times \hat{c} \times \hat{d}$. Its image is contained in U, by its analyticity (Proposition 2.10) and since the image U_1 of the set $\Psi(U_1)$ (lifted to $\hat{\mathcal{S}}$) lies in U. This together with the inclusion $\Psi(U) \subset \mathcal{S}$ implies that $\Psi(U) = \mathcal{S}$ and proves the corollary. $\square$

Corollary 3.5 *Let a, b, c, d be a 4-reflective planar analytic billiard, and none of the mirrors a, b be a line. Let a and b intersect at a point A represented by some non-marked points in $\hat{a}$ and $\hat{b}$. Then $a = c$ and $a \neq b$.*

At the end of the section we prove Theorem 1.9 and Corollary 3.5. Both of them will be used further on in the proof of Theorem 1.10.

Plan of the proof of Lemma 3.1. Recall that a and b are not both lines. In the case, when they are both algebraic curves, there exist at most unique analytic curves c and d such that the billiard a, b, c, d is 4-reflective, and if they exist, they are algebraic (Proposition 2.1 and Theorem 1.7 in the algebraic case, see Remark 1.8). Thus, the only integral surfaces of the distribution $\mathcal{D}_{ab}$ are given by the open set of its quadrilateral orbits, by Propositions 2.43 and 2.46. Moreover, the latter orbit set is a Zariski open dense subset in a projective algebraic surface. This immediately implies the statement of Lemma 3.1. Everywhere below we consider that some of the curves a or b is transcendental and prove the lemma by contradiction. Suppose the contrary to Lemma 3.1: there exists a three- or four-dimensional irreducible analytic subset $M \subset \mathcal{P}_{ab}$ contained in M_{ab} such that $M \cap M^0_{ab} \neq \emptyset$ and the restriction $\mathcal{D}_M$ to $M^0 = M \cap M^0_{ab}$ of the distribution $\mathcal{D}_{ab}$ is two-dimensional and integrable. (In the second case $M = M_{ab}$.) The complement

$$\Sigma^0 = M \setminus M^0 = M \setminus M^0_{ab} \subset M \tag{3.1}$$

is an analytic subset of positive codimension in M, and M^0 is dense in M. Thus, every $x \in M^0$ where $\mathcal{D}_M$ is regular is contained in an integral surface, and the latter is formed by quadrilateral orbits of a 4-reflective billiard a, b, $c(x)$, $d(x)$ (Propositions 2.43 and 2.46). We show that there exists an $x \in M^0$

such that the corresponding mirrors $c(x)$, $d(x)$ are algebraic. This together with Proposition 2.1 implies that a and b are algebraic. The contradiction thus obtained will prove Lemma 3.1.

For the proof of Lemma 3.1 we study the projections of the set M to the positions of three vertices (A, B, D): set

$$\nu_{ab,D} : x \mapsto (A(x), B(x), D(x)); \ M_D = \nu_{ab,D}(M) \subset \hat{a} \times \hat{b} \times \mathbb{CP}^2.$$

Analogous projections and spaces are defined with D replaced by C.

Remark 3.6 The images $M_C = \nu_{ab,C}(M)$, $M_D = \nu_{ab,D}(M)$ are irreducible analytic subsets in $\hat{a} \times \hat{b} \times \mathbb{CP}^2$, by Remark 2.45, Remmert's Proper Mapping Theorem and irreducibility of the variety M.

In Subsections 3.2 and 3.3 respectively we treat the following cases:

- some of the projections $\nu_{ab,C}$, $\nu_{ab,D}$ is not bimeromorphic (see Footnote 3 in Subsection 2.6);

- both latter projections are bimeromorphic.

In what follows, we denote $\Sigma^1 \subset M^0$ the subset of points x such that

- either x is a singular point of the variety M,

- or it is a singular point of the distribution $\mathcal{D}_M$,

- or the restriction to $\mathcal{D}_M(x)$ of the differential $d\nu_{ab}(x)$ has rank less than two,

- or x is a critical point of the projection $\nu_{ab,D}$: a point where the rank of differential is not maximal,

- or its image under the latter is a singularity of the image,

- or the differential of the projection $\nu_D : M^0_{ab} \to \mathbb{CP}^2$: $y \mapsto D(y)$ vanishes on the distribution plane $\mathcal{D}_M(x)$,

- or one of the three latter statements holds with D replaced by C.

Let Σ^0 be the same, as in (3.1). Set

$$\Sigma = \Sigma^0 \cup \Sigma^1 \subset M. \tag{3.2}$$

This is an analytic subset in $\mathcal{P}_{ab}$ that has positive codimension in M. Its complement in M is contained in M^0 and dense in M.

Remark 3.7 For every point $x \in M \setminus \Sigma$ the corresponding germs $(g, G(x))$ of mirrors $g = a, b, c(x), d(x)$, $G = A, B, C, D$, are regular, and the points $G(x)$ are not isotropic tangency points. This follows from the definition of the set $\Sigma^0 \subset \Sigma$ (for the mirrors a and b) and from the two last conditions in the definition of the set $\Sigma^1 \subset \Sigma$ (for the mirrors $c(x)$ and $d(x)$). The projection $\nu_{ab} : S \to \hat{a} \times \hat{b}$ of each integral surface S of the distribution $\mathcal{D}_M$ in $M \setminus \Sigma$ is a local diffeomorphism, by the Addendum to Proposition 2.10.

3.2 Case of a non-bimeromorphic projection

Here we prove Lemma 3.1 in the case, when some of the projections $\nu_{ab,C}$, $\nu_{ab,D}$, say $\nu_{ab,D}$ is not bimeromorphic. Its proof is based on the following proposition.

Proposition 3.8 *Let Σ be the same, as in (3.2). For every two points $x, y \in M \setminus \Sigma$ projected to the same $(A, B) \in \hat{a} \times \hat{b}$ such that either $(C(x), L_C(x)) \neq (C(y), L_C(y))$, or $(D(x), L_D(x)) \neq (D(y), L_D(y))$, the billiard $c(x)$, $d(x)$, $d(y)$, $c(y)$ is 4-reflective.*

Proof The proof of the proposition repeats the final argument from [7, proof of lemma 3.1].

Case (i): $C(x) \neq C(y)$ and $D(x) \neq D(y)$. Each billiard a, b, $c(z)$, $d(z)$, $z = x, y$ has two-dimensional family of quadrilateral orbits $A'B'C_zD_z$ close to $ABC(z)D(z)$ where $C_z = C_z(A', B')$, $D_z = D_z(A', B')$ depend analytically on parameters $(A', B') \in \hat{a} \times \hat{b}$ (the Addendum to Proposition 2.10). The corresponding quadrilaterals $C_xD_xD_yC_y$ are periodic orbits of the billiard $c(x)$, $d(x)$, $d(y)$, $c(y)$, by definition and reflection law, see Fig.6, and depend analytically on parameters (A', B'). Therefore, the billiard $c(x)$, $d(x)$, $d(y)$, $c(y)$ is 4-reflective.

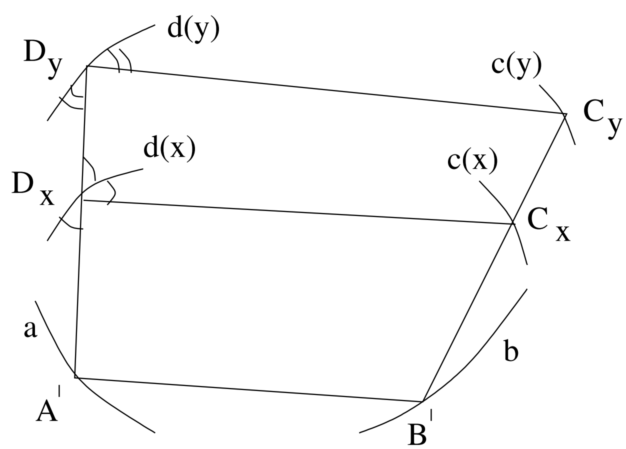

Figure 6: The 4-reflective billiard $c(x)$, $d(x)$, $d(y)$, $c(y)$: open set of quadrilateral orbits $C_xD_xD_yC_y$.

Case (ii): $D(x) = D(y) = D_0$ but $L_D(x) = T_{D(x)}d(x) \neq L_D(y) = T_{D(y)}d(y)$ (the same case with D replaced by C is symmetric). Let us show that one can achieve the inequalities of Case (i) by deforming x and y as orbits of fixed billiards. The mirrors $d(x)$ and $d(y)$ intersect transversely at D_0. Therefore, deforming (A, B), one can achieve that the line $l_a = l_a(A, B)$ intersects $d(x)$ and $d(y)$ at two distinct points close to D_0. This lifts to deformation of x and y along their integral surfaces (the last statement of Remark 3.7), and we get new x, y with $\nu_{ab}(x) = \nu_{ab}(y)$ and $D(x) \neq D(y)$. One has $(C(x), L_C(x)) \neq (C(y), L_C(y))$: otherwise, one would have $l = C(x)D(x) = C(y)D(y)$, hence $D(x) = D(y) = l_a \cap l$, – a contradiction. Hence, one can achieve that $C(x) \neq C(y)$ via small deformation, by the above argument. This reduces us to Case (i) and proves the proposition. $\Box$

Proposition 3.9 *Let the projection $\nu_{ab,D} : M \to M_D$ be not bimeromorphic. Then up to interchanging C and D, there exist an open and dense subset of points $z \in M_D$ for which there exist*

$$x, y \in M \setminus \Sigma \text{ with } \nu_{ab,D}(x) = \nu_{ab,D}(y) = z,\ L_D(x) \neq L_D(y)$$

such that the projection $\nu_{ab,D}$ is a local submersion at both x and y.

Proof The non-bimeromorphicity implies that there exists an open and dense subset of points $z = (A, B, D) \in M_D$ for which there exist $x, y \in \nu_{ab,D}^{-1}(z) \setminus \Sigma$, $x \neq y$ such that the projection $\nu_{ab,D}$ is a local submersion at both x and y. In the case, when $L_D(x) \neq L_D(y)$ for at least some x, y, z as above, the statement of the proposition follows immediately. Now suppose the contrary: there exists an open and dense subset of points $z \in M_D$ such that for every $x, y \in \nu_{ab,D}^{-1}(z) \setminus \Sigma$, $x \neq y$ one has $L_D(x) = L_D(y) = L_D$. Then $C(x) = C(y)$, since the point $C(x) = C(y)$ is found as the intersection point of two lines depending only on z and not on x or y: the line BC symmetric to AB with respect to the line T_Bb; the line DC symmetric to AD with respect to the line L_D. Therefore, $\nu_{ab,C}(x) = \nu_{ab,C}(y)$, but $L_C(x) \neq L_C(y)$, since $x \neq y$. Hence, the statement of the proposition holds with D replaced by C. The proposition is proved. $\Box$

Let us fix arbitrary x, y as in Proposition 3.9. Set

$$D_0 = D(x) = D(y),\ (A, B) = \nu_{ab}(x) = \nu_{ab}(y).$$

Claim 1. *The curves $c(x)$, $c(y)$ are either both triangular spirals centered at D_0, or both conics: complex circles centered at D_0.*

Proof The germ at $C(x)D_0D_0C(y)$ of billiard $c(x)$, $d(x)$, $d(y)$, $c(y)$ is 4-reflective (Proposition 3.8). It satisfies the non-isotropicity and line non-coincidence conditions of Proposition 2.19, since x, y correspond to non-degenerate quadrilaterals. For example, the tangent line to a mirror through each vertex is non-isotropic and distinct from the adjacent edges, by non-degeneracy. This together with Proposition 2.19 implies the claim. $\square$

Corollary 3.10 *The mirrors $c(z)$ and $d(z)$ are algebraic for $z = x, y$.*

Proof It suffices to prove that the mirrors $c(z)$, $z = x, y$, are both algebraic: then so are $d(z)$, by 4-reflectivity of the billiard $c(x)$, $d(x)$, $d(y)$, $c(y)$ and Proposition 2.1. Suppose the contrary: say $c(x)$ is not algebraic. Take an arbitrary $y' \in M \setminus \Sigma$ close to y with $D_1 = D(y') \in d(x)$, $D_1 \neq D_0$, $L_D(y') = T_{D_1}d(y') \neq T_{D_1}d(x)$, so that the line D_0D_1 is non-isotropic (as is $T_{D_0}d(x)$). It exists, since the projection $\nu_{ab,D}$ is a local submersion at y and $L_D(y) \neq L_D(x)$. Deforming x along its integral surface of the distribution $\mathcal{D}_M$ one can achieve that $(A, B) = \nu_{ab}(x) = \nu_{ab}(y')$ (Remark 3.7), and then $D(x) = D_1$. The billiard $c(x)$, $d(x)$, $d(y')$, $c(y')$ is 4-reflective, by Proposition 3.8, and its 4-reflective set contains a degenerate quadrilateral $C_1D_1D_1C_2$ close to $C(x)D_0D_0C(y)$. This together with Proposition 2.19 implies that $c(x)$ is a triangular spiral centered at D_1, as in Claim 1. Thus, $c(x)$ is a triangular spiral with two distinct centers D_0 and D_1. Hence, it is algebraic, by Proposition 2.21. The contradiction thus obtained proves the corollary. $\square$

Thus, the 4-reflective billiard a, b, $c(x)$, $d(x)$ has two neighbor algebraic mirrors $c(x)$ and $d(x)$. Hence, a and b are also algebraic, by Proposition 2.1. This contradicts the non-algebraicity assumption and proves Lemma 3.1.

3.3 Case of bimeromorphic projections

Here we prove Lemma 3.1 in the case, when both projections $\nu_{ab,C}$, $\nu_{ab,D}$ are bimeromorphic. To do this, we show that for every $x \in M \setminus \Sigma$ both mirrors $d(x)$ and $c(x)$ are lines. We then get a contradiction as above.

It suffices to prove the above statement for the mirrors $d(x)$ only. We first show (the next proposition) that the lines $L_D(x) \subset T_{D(x)}\mathbb{CP}^2$ locally depend only on $D(x)$ and form a holomorphic line field. Afterwards we show (Proposition 3.12) that the restriction of the latter line field to each projective line is tangent to a pencil of lines through the same point. This easily implies that its integral curves are lines (Proposition 3.13).

There exists an open and dense subset of points $x \in M \setminus \Sigma$ where the germ of projection $\nu_D : y \mapsto D(y)$ is a submersion. Indeed, the contrary would imply that $d = \nu_D(M \setminus \Sigma)$ is a curve. Therefore, $D(y)$ is locally determined by $(A, B) = (A(y), B(y))$ as a point of intersection $l_a(A, B) \cap d$, hence $dim M_D = 2$. But $dim M_D = dim M \geq 3$, by assumption and bimeromorphicity, – a contradiction. Consider the analytic subset $\Sigma_{ab} \subset \hat{a} \times \hat{b}$ of those pairs (A, B), for which either $\pi_a(A) = \pi_b(B)$, or some of the lines $T_A a$ or $T_B b$ is isotropic. Note that $dim M_D = 3 = dim M$, by the latter inequality and since $D \in l_a(A, B)$ for every $(A, B, D) \in M_D$ with $(A, B) \notin \Sigma_{ab}$. Increasing the "exceptional set" Σ, we will assume that ν_D is a submersion on all of $M \setminus \Sigma$. Its level sets in $M \setminus \Sigma$ are thus holomorphic curves.

Proposition 3.11 *Every $x \in M \setminus \Sigma$ has a neighborhood $V = V(x) \subset M \setminus \Sigma$ such that the lines $L_D(y)$, $y \in V$ depend only on $D(y)$ and thus, form a holomorphic line field Λ_D on a neighborhood $W \subset \mathbb{CP}^2$ of the point $D(x)$.*

Proof Let V be a neighborhood of the point x regularly fibered by local level curves of the submersion ν_D. For every $y \in V$ there exists a holomorphic curve $\gamma(y) \subset V$ that corresponds exactly to the family of quadrilateral orbits of the billiard a, b, $c(y)$, $d(y)$ with fixed vertex $D = D(y)$. The line $L_D = L_D(y) = T_{D(y)} d(y)$ obviously remains constant along the curve $\gamma(y)$. On the other hand, the curve $\gamma(y)$ should obviously coincide with the level curve $\{\nu_D = D(y)\}$. This proves the proposition. □

Proposition 3.12 *The restriction to each projective line of the field Λ_D from Proposition 3.11 is tangent to a pencil of lines through the same point.*

Proof For every $(A, B) \in (\hat{a} \times \hat{b}) \setminus \Sigma_{ab}$ denote by $F^D_{A,B} = A \times B \times l_a(A, B) \simeq \overline{\mathbb{C}}$ the fiber over (A, B) of the set M_D. The fiber $F^C_{A,B}$ of the set M_C is defined analogously. The projections $\nu_{ab,G}$, $G = C, D$ and the mappings $\nu^{-1}_{ab,D} : M_D \to M$, $R_{DC} = \nu_{ab,C} \circ \nu^{-1}_{ab,D} : M_D \to M_C$ are bimeromorphic. Hence, the indeterminacies of the mapping $\nu^{-1}_{ab,D}$ (which include those of R_{DC}) form an analytic subset, whose complement to the projection preimage of the set Σ_{ab} has codimension at least two, i.e., dimension at most one (Footnote 3 in Subsection 2.6). The image of its projection to $\hat{a} \times \hat{b}$ is an analytic subset Ind of dimension at most one (Proper Mapping Theorem). Set $Reg = (\hat{a} \times \hat{b}) \setminus (\Sigma_{ab} \cup Ind)$; it is an open and dense subset in $\hat{a} \times \hat{b}$. Then for every $(A, B) \in Reg$ the following statements hold:

- there exists an invertible holomorphic (hence Möbius) mapping $T_{A,B}$: $l_a(A,B) \to l_b(A,B)$ for which

$$R_{DC}(A,B,D) \equiv (A,B,T_{A,B}(D)); \tag{3.3}$$

- the restriction to $l_a(A,B)$ of the line field Λ_D is holomorphic and hence, the projective lines $\Lambda_D(y)$, $y \in l_a(A,B)$, form a rational curve $\Gamma_{A,B} \subset \mathbb{CP}^{2*}$.

The former statement follows from bimeromorphicity by definition. The global holomorphic family of lines $\Lambda_D(y)$ depending on $y \in l_a(A,B) \simeq F^D_{A,B} \subset M_D$ from the latter statement is given by the holomorphic inverse $\nu^{-1}_{ab,D} : F^D_{A,B} \to M$: $y \mapsto (A,B,(C(y),\Lambda_C(y)),(D(y),\Lambda_D(y)))$.

It suffices to prove the statement of Proposition 3.12 for an open set of projective lines, e.g., for all the lines $l_a(A,B)$, $(A,B) \in Reg$. Suppose the contrary: for some $(A,B) \in Reg$ the restriction to $l_a = l_a(A,B)$ of the line field Λ_D is not tangent to a pencil of lines, i.e., the curve $\Gamma = \Gamma_{A,B}$ has degree at least two. Note that the family of lines

$$\lambda(D) = DT_{A,B}(D), \ D \in l_a = l_a(A,B)$$

is a rational curve $\lambda \subset \mathbb{CP}^{2*}$ parametrized by $D \in l_a$.

Claim 1. *The curve λ has degree at least three.*

Proof Let Q denote the intersection point of the infinity line $\overline{\mathbb{C}}_\infty \subset \mathbb{CP}^2$ with non-isotropic line l_a; thus, $Q \neq I_{1,2}$. Let us take a point $P \in \overline{\mathbb{C}}_\infty \setminus \{Q, I_1, I_2\}$ and a finite line l through P. There are two symmetry lines of the pair of lines l_a, l. They intersect the infinity line at distinct points $E_1, E_2 \in \overline{\mathbb{C}}_\infty$ depending only on P: $E_j = E_j(P)$. For a generic $P \in \overline{\mathbb{C}}_\infty$ for every $j = 1,2$ there are at least two distinct points $D_j, D'_j \in l_a$ such that the projective lines $\Lambda_D(D_j)$, $\Lambda_D(D'_j)$ pass through $E_j = E_j(P)$, since $deg\Gamma \geq 2$. Either the four points D_1, D'_1, D_2, D'_2 are distinct, or at most two of them coincide: the latter happens exactly in the case, when, $\Lambda_D(Q) = \overline{\mathbb{C}}_\infty \supset \{E_1, E_2\}$; then $D_1 = D_2 = Q$. This implies that there are always at least three distinct points $D^1, D^2, D^3 \in l_a$ such that each line $\Lambda_D(D^i)$, $i = 1,2,3$, passes through some of the points E_j; thus, $P \in \lambda(D^i)$. The corresponding points $\lambda(D^i) \in \mathbb{CP}^{2*}$ of the curve λ lie in the projective line P^*, by construction. This proves the claim. $\Box$

Claim 2. *The curve λ has degree at most two.*

Proof Let X denote the point of intersection of the lines l_a and l_b, $Y = T_{A,B}(X) \in l_b$. For every point $C \in l_b \setminus \{Y\}$ the curve λ intersects the dual line C^* in at most two points with multiplicity 1: at the points $\lambda(T^{-1}_{A,B}(C))$ and may be $\lambda(X)$ (if $\lambda(X) = l_b$). The fact that in the latter case the multiplicity of the intersection point $\lambda(X)$ equals one follows from the assumption that $C \neq Y$: the lines $\lambda(D)$ with D close to X asymptotically focus at $Y \neq C$ and are transverse to l_a. This proves the claim. □

Claims 1 and 2 contradict each other. This proves Proposition 3.12. □

Proposition 3.13 *The phase curves of the line field Λ_D form a pencil of projective lines through the same point.*

Proof Suppose the contrary: some phase curve S is not a line. Consider a projective line $L = \Lambda_D(P)$ tangent to S at some point P. Then for every $Q, R \in L$, $Q \neq R$, close to P and distinct from it the projective lines $\Lambda_D(Q)$ and $\Lambda_D(R)$ are distinct from the line $L = \Lambda_D(P)$ and intersect each other at a point outside the line L. Therefore, the lines of the restriction to L of the field Λ_D are not tangent to a pencil of lines through the same point. The contradiction thus obtained to Proposition 3.12 proves Proposition 3.13. □

Proposition 3.13 applied to both Λ_C and Λ_D implies that for every $x \in M \setminus \Sigma$ the mirrors $c(x)$ and $d(x)$ are lines. Hence, a and b are algebraic (Proposition 2.1), – a contradiction. Lemma 3.1 is proved.

3.4 Case of one algebraic mirror. Proof of Theorem 1.9

In the present subsection we consider that a, b, c, d is a 4-reflective analytic planar billiard, and the mirror a is algebraic. Without loss of generality we consider that the curves b and d are transcendental: in the opposite case Theorem 1.9 follows immediately from Proposition 2.1 and [7, theorem 1.11]. As it is shown below, Theorem 1.9 is implied by the following proposition.

Proposition 3.14 *In the above conditions the mirror c is also algebraic.*

Proof The projection $\nu_b : U \to \hat{b}$ of the 4-reflective set U is proper and epimorphic, by Corollary 3.4 and since a is algebraic. This implies that for an open and dense set of points $B \in \hat{b}$ the preimage $\nu_b^{-1}(B) \subset U$ is a compact analytic curve with non-constant holomorphic projection to $\hat{c}$. Hence, $\hat{c}$ is compact and c is algebraic. The proposition is proved. □

Now let us prove Theorem 1.9. Fix a non-marked point $B \in \hat{b}$ as in the above proof and a one-dimensional irreducible component Γ_B of the compact analytic curve $\nu_b^{-1}(B) \subset U$. Its image $\nu_D(\Gamma_B) \subset d \subset \mathbb{CP}^2$ is either an algebraic curve, or a single point (Proper Mapping and Chow Theorems). The former case is impossible, since d is non-algebraic. Hence, for an open and dense set of points $B \in \hat{b}$ the projection of the curve Γ_B to the position of the vertex D is constant and is determined by B. Thus, there exists a mapping $\hat{b} \to \hat{d}$, $B \mapsto D_B$, defined on an open set $V \subset \hat{b}$ such that for every fixed $B \in V$ and variable $A \in \hat{a}$ the lines AB and AD_B are symmetric with respect to the tangent line $T_A a$. This implies that either a is a line and B, D_B are symmetric with respect to a for every $B \in \hat{b}$, or a is a conic with one-dimensional family of foci pairs (B, D_B), see [7, proposition 2.32]. The latter case being obviously impossible, the curves b, d are symmetric with respect to the line a. Applying the above argument to the algebraic mirror c instead of a, we get that B and D_B are symmetric with respect to the line c. Thus, the above pairs (B, D_B) are symmetric with respect to both lines a and c, by construction. Therefore, $a = c \neq b, d$, and the billiard is of type 1) from Theorem 1.7. Theorem 1.9 is proved.

3.5 Intersected neighbor mirrors. Proof of Corollary 3.5

In the conditions of Corollary 3.5 no mirror is a line, by Theorem 1.9 and since a, b are not lines. Without loss of generality we consider that each mirror is transcendental, since otherwise, $a = c$ and $a \neq b$, by Theorem 1.9. Let U be the 4-reflective set. Its projection to $\hat{a} \times \hat{b}$ is proper and epimorphic, by Corollary 3.4.

Claim 1. *$a \neq b$.*

Proof Suppose the contrary: $a = b$. Then U contains a one-parametric analytic family $\mathcal{T}$ of quadrilaterals $AACD$ with variable A, C, D, by the above epimorphicity statement. A generic quadrilateral in $\mathcal{T}$ is forbidden by Proposition 2.25. The contradiction thus obtained proves the claim. $\square$

The projection preimage in U of the pair $(A, A) \in \hat{a} \times \hat{b}$ is a non-empty compact analytic subset $\Gamma \subset U$ of dimension at most one.

Case 1): $dim\Gamma = 1$. Then at least one of the curves $\hat{c}$, $\hat{d}$, say $\hat{c}$ is a compact Riemann surface, – a contradiction to our non-algebraicity assumption. Thus, this case is impossible.

Case 2): $dim\Gamma = 0$: Γ is a finite set.

Claim 2. *Every quadrilateral $AACD \in \Gamma$ is a single-point quadrilateral: the mirrors c and d pass through the same point A; $\pi_c(C) = \pi_d(D) = \pi_a(A)$.*

Proof Suppose the contrary: say, $\pi_c(C) \neq \pi_a(A)$. The projection $U \to \hat{a} \times \hat{b}$ is open on a neighborhood of the point $AACD$, since it contracts no curve to (A, A) by assumption. Therefore, each converging sequence $(A^k, B^k) \to (A, A)$ lifts to a converging sequence $A^k B^k C^k D^k \to AACD$ in U. Let us take two sequences $(A^k_j, B^k_j) \to (A, A)$, $j = 1, 2$, with lines

$A_j^k B_j^k$ converging to different limits for $j = 1, 2$; this is possible, since $a \neq b$. We get two sequences of quadrilaterals $A_j^k B_j^k C_j^k D_j^k$ converging to the same quadrilateral $AACD$. On the other hand, the lines $B_j^k C_j^k$ symmetric to $A_j^k B_j^k$ with respect to the tangent lines $T_{B_j^k} b$ converge to two distinct limits H_j, $j = 1, 2$, by assumption and since A is not a marked point of the curve b. The lines $H_1 \neq H_2$ pass through the same two points $\pi_a(A) \neq \pi_c(C)$, by construction. The contradiction thus obtained proves the claim. $\Box$

Thus, Γ is a finite set of points corresponding to the single-point quadrilateral $AAAA$. Fix one of them and denote it $AAAA$: the corresponding vertices $C \in \pi_c^{-1}(A)$ and $D \in \pi_d^{-1}(A)$ will be denoted by A. One has $c \neq d$, as in Claim 1. The projection $U \to \hat{c} \times \hat{d}$ is open on a neighborhood of the point $AAAA$, as in the above discussion. Let $\gamma \subset \hat{c} \times \hat{d}$ be an irreducible germ at (A, A) of analytic curve consisting of pairs (C', D') with variable C' and D' for which $C' \in T_{D'} d$. The germ γ lifts to an irreducible germ $\widetilde{\gamma}$ of analytic curve through $AAAA$ in U. The curve $\widetilde{\gamma}$ consists of quadrilaterals $A'B'C'D' \in U$ for which $B', D' \not\equiv C'$ (Proposition 2.25). Therefore, $A' \equiv C'$, by Corollary 2.26 and since A is not a marked point of the mirror b. Hence, $a = c$. Corollary 3.5 is proved.

4 Algebraicity: proof of Theorem 1.10

Theorem 1.7, and thus, Theorem 1.10 are already proved in the case, when at least one mirror is algebraic (Theorem 1.9). Here we prove Theorem 1.10 in the general case by contradiction. Suppose the contrary: there exists a 4-reflective billiard a, b, c, d with no algebraic mirrors. We study Birkhoff distribution $\mathcal{D}_a$ on the space M_a and consider its non-trivial integral surface S formed by a connected open set of quadrilateral orbits of the billiard. Recall that $M_a \subset \mathcal{P}_a$ is a six-dimensional analytic subset, and $\mathcal{D}_a$ is a singular three-dimensional distribution on M_a, see Subsection 2.7. Set

$$M = \text{ the minimal analytic subset in } M_a \text{ containing } S.$$

This is an irreducible analytic subset in $\mathcal{P}_a$, by definition, see Convention 2.31. The intersections

$$\mathcal{D}_M(x) = \mathcal{D}_a(x) \cap T_x M, \ x \text{ being a smooth point of the variety } M,$$

induce a singular analytic distribution $\mathcal{D}_M$ on M, for which S is an integral surface. This is either two- or three-dimensional distribution, since $dimS =$

2 and $dim\mathcal{D}_a = 3$. The cases, when $dim\mathcal{D}_M = 2, 3$, will be treated separately in Subsections 4.1 and 4.3 respectively.

The methods of proof in both cases are similar. We show that an open set of points $x \in M$ lie in integral surfaces corresponding to 4-reflective billiards a, $b(x)$, $c(x)$, $d(x)$ with regularly intersected mirrors a and $b(x)$. Then either $b(x)$ is a line, or $c(x) = a$, by Corollary 3.5. We then show that either $\nu_C(M) \subset a$, or the mirror b of the initial billiard is a line. We get a contradiction in both subcases. In the case, when $dim\mathcal{D}_M = 3$, the proof uses Cartan–Kuranishi–Rashevskii involutivity theory of Pfaffian systems. The corresponding background material will be recalled in Subsection 4.2.

The proof of the existence of the above integral surfaces is based on the following key proposition and corollary. They deal with the natural projection $\nu_a : \mathcal{P}_a \to \hat{a}$ and its restriction to M, which are proper and epimorphic. For every $A \in \hat{a}$ the projection preimage

$$W_A = \nu_a^{-1}(A) \cap M$$

is a projective algebraic set.

Proposition 4.1 *There exists a complement $\hat{a}_0 \subset \hat{a}$ to a discrete subset in $\hat{a}$ such that for every $A \in \hat{a}_0$ the projection $\nu_B : W_A \to \mathbb{CP}^2$ is epimorphic.*

Proof For every $A \in \hat{a}$ the image of the projection $\nu_B : W_A \to \mathbb{CP}^2$ is either the whole projective plane, or an algebraic subset of dimension at most one (Remmert's Proper Mapping and Chow's Theorems). Either $\nu_B(W_A) = \mathbb{CP}^2$ for all but a discrete set of points A (and then the statement of the proposition obviously holds), or it is at most one-dimensional algebraic set for an open and dense set Q of points $A \in a$, by analyticity. The latter case cannot happen, since otherwise for every non-marked $A \in \nu_a(S) \cap Q$ the set $\nu_B(W_A \cap S)$ would be an open subset in the non-algebraic curve b that simultaneously lies in at most one-dimensional algebraic set $\nu_B(W_A)$, – a contradiction. This proves the proposition. □

Corollary 4.2 *For every open dense subset $N \subset M$ whose complement $M \setminus N \subset M$ is an analytic subset there exists an open dense subset $\hat{a}_N \subset \hat{a}$ such that for every $A \in \hat{a}_N$ the intersection $\nu_B(W_A \cap N) \cap a$ contains the π_a- image of an open dense subset in $\hat{a}$: a complement to a discrete subset.*

Proof Let $\hat{a}_0$ be the same, as in Proposition 4.1. There exists an open dense subset $\hat{a}_N \subset \hat{a}_0$ such that for every $A \in \hat{a}_N$ the subset $W_A^N = W_A \cap N \subset W_A$ is open and dense, since the complement $M \setminus N$ is an analytic subset and all the W_A are hypersurfaces. Then for the same A the subset $W_A \setminus W_A^N \subset W_A$ is algebraic, since it is analytic (as is $M \setminus N$) and by Chow's Theorem. Therefore, the complement $\mathbb{CP}^2 \setminus \nu_B(W_A^N)$ is contained in an algebraic subset in $\mathbb{CP}^2$ of positive codimension (Proposition 4.1 and Chevalley–Remmert and Chow's Theorems). Its π_a-preimage is at most discrete, since a is non-algebraic. This implies the statement of the corollary. $\square$

4.1 Case of two-dimensional distribution

In the present section we consider that $dim\mathcal{D}_M = 2$. Then the distribution $\mathcal{D}_M$ is integrable, by Proposition 2.38.

We keep the previous notations ν_a, ν_G, $G = B, C, D$ for the projections to the positions of vertices. Let $M_{reg}^0 \subset M \cap M_a^0$ denote the subset of points regular for both M and $\mathcal{D}_M$, cf. Definition 2.32. Set

$$M' = \{x \in M_{reg}^0 \mid d\nu_a(x), d\nu_G(x) \not\equiv 0 \text{ on } \mathcal{D}_M(x) \text{ for } G = B, C, D\}. \quad (4.1)$$

The set M' contains the integral surface S, since S is non-trivial. It is an open and dense subset in M, and its complement $\Sigma = M \setminus M'$ is analytic.

Remark 4.3 The integral surface of the distribution $\mathcal{D}_M$ through each point $x \in M'$ is non-trivial, by (4.1). The germ of its image under each one of the projections ν_G, $G = B, C, D$, is a germ of analytic curve at its non-marked point. The regularity of germ follows by definition from the inequalities in (4.1). The non-isotropicity of tangent line to the germ follows from the definition of the distribution $\mathcal{D}_M$ and non-degeneracy of the quadrilateral corresponding to x. Thus, the above integral surface corresponds to an open set of quadrilateral orbits of a 4-reflective billiard a, $b(x)$, $c(x)$, $d(x)$ (Proposition 2.46), and the above germs are germs of mirrors at non-marked points.

For every $A \in \hat{a}$ set $W_A^0 = W_A \cap M'$.

Proposition 4.4 *There exists an open subset $V \subset M'$ of those x for which the mirrors a and $b(x)$ intersect at some point non-marked for both their local branches.*

Proof There exists an open dense subset $\hat{a}' \subset \hat{a}$ such that for every $A \in \hat{a}'$ the intersection $\nu_B(W_A^0) \cap a$ contains a regular disk $\alpha \subset a$ without marked points for α (Corollary 4.2). Fix α and an $x_0 \in M' \cap \nu_B^{-1}(\alpha)$. The point $\nu_B(x_0) \in b(x_0) \cap \alpha$ is a non-marked point of the corresponding local branches of the curves $b(x_0)$ and α (Remark 4.3). The latter branches are distinct, by Corollary 3.5. Therefore, there exists a neighborhood $V = V(x_0) \subset M'$ such that for every $x \in V$ a regular branch of the curve $b(x)$ intersects α at a non-marked point for both curves (Remark 4.3 and analyticity of the foliation by integral surfaces). This proves the proposition. $\square$

In Proposition 4.4 for every $x \in V$ either $c(x) = a$, or $b(x)$ is a line, by Corollary 3.5. Hence some of the latter statements holds for all $x \in M'$, by analyticity. The mirror $b(x)$ cannot be a line for all x, since the mirror b of the initial billiard is not algebraic by assumption. Hence, $\nu_C(M) \subset a$. Let us show that this is impossible. To do this, we use the next proposition.

Proposition 4.5 *For every analytic billiard a, b, c, d with a non-algebraic mirror b in every one-parametric family of quadrilateral orbits $ABCD$ with fixed non-isotropic vertex $A \neq I_1, I_2$ the vertex C is non-constant.*

Proof If $C \equiv$ *const*, then b would be either a line, or a conic, by [7, proposition 2.32], – a contradiction. $\square$

Suppose, by contradiction, that $\nu_C(M) \subset a$. Fix an $A \in \hat{a}$ such that $\pi_a(A)$ is not an isotropic point at infinity and there exists a one-parametric family of quadrilateral orbits $ABCD$ of the initial billiard a, b, c, d with the given vertex A. The subset $\nu_C(W_A) \subset \mathbb{CP}^2$ is non-discrete, by Proposition 4.5. On the other hand, it is an algebraic subset in $\mathbb{CP}^2$ (Remmert's Proper Mapping and Chow's Theorems). It lies in a transcendental curve a. Hence, it is discrete. The contradiction thus obtained proves Theorem 1.10.

4.2 Background material: Phaffian systems and involutivity

Everywhere below in the present subsection whenever the contrary is not specified, $\mathcal{F}$ is a k-dimensional analytic distribution on an analytic manifold M, $\mathcal{F}(x) \subset T_xM$ are the corresponding subspaces.

Definition 4.6 [21, p.290] Let $k, l \in \mathbb{N}$, $k \geq l$, and let $\mathcal{F}$ be as above. A *Pfaffian system* $\mathcal{F}_{k,l}$ is the problem to find l-dimensional analytic integral surfaces of the k-dimensional distribution $\mathcal{F}$.

Definition 4.7 [21, p.298] An m-dimensional *integral element* of the distribution $\mathcal{F}$ is an m-dimensional vector subspace $E_m(x) \subset \mathcal{F}(x)$ such that for every 1-form ω on the ambient manifold vanishing on the subspaces of the distribution $\mathcal{F}$ its differential $d\omega$ vanishes on $E_m(x)$.

Definition 4.8 [21, p.300] A Pfaffian system $\mathcal{F}_{k,l}$ is *in involution* (or briefly, involutive), if for every $x \in M$, $p < l$ each p-dimensional integral element in T_xM is contained in some $(p+1)$-dimensional integral element.

Example 4.9 A tangent subspace to an integral surface is an integral subspace. Every Pfaffian system defined by a Frobenius integrable distribution is involutive.

Remark 4.10 For every p-dimensional integral element $E_p(x)$ the set of ambient $(p+1)$-dimensional integral elements $E_{p+1}(x) \supset E_p(x)$ either is empty, or consists of a unique integral element, or is a projective space: a pencil of $(p+1)$-dimensional subspaces through $E_p(x)$, which saturate a linear subspace containing $E_p(x)$, see [21, formula (58.13), p.299].

Definition 4.11 [21, p.306] Let $\mathcal{F}$ be an analytic distribution on a connected manifold. A p-dimensional integral element $E_p(x)$ is said to be *nonsingular*, if the space of ambient $(p+1)$-dimensional integral elements $E_{p+1}(x) \supset E_p(x)$ has minimal dimension.

Theorem 4.12 *(a version of Cauchy–Kovalevskaya Theorem; implicitly contained in [21, section 60]). Let an analytic Pfaffian system $\mathcal{F}_{k,l}$ on a manifold M be involutive. Let $p \leq l$, $\Gamma \subset M$ be a $(p-1)$-dimensional analytic integral surface such that all its tangent spaces be nonsingular $(p-1)$-dimensional integral elements. Then for every $x \in \Gamma$ there exists a germ of p-dimensional integral surface through x that contains the germ of Γ at x.*

Definition 4.13 [12, p.188]. A subset N of a complex manifold V is called *analytically constructible,* if each point of the manifold V has a neighborhood U such that the intersection $N \cap U$ is a finite union of subsets defined by finite systems of equations $f_j = 0$ and inequalities $g_i \neq 0$; f_j and g_i are holomorphic functions on U.

Recall that for a singular analytic distribution $\mathcal{D}_M$ on an irreducible analytic subset M in a complex manifold V by $M^0_{reg} \subset M$ we denote the open and dense subset of points regular both for M and $\mathcal{D}_M$; the complement $M \setminus M^0_{reg} \subset V$ is an analytic subset.

Proposition 4.14 *Let $\mathcal{D}_M$ be a singular analytic distribution on an irreducible analytic subset M in a complex manifold V. Its nonsingular one-dimensional integral elements form an open dense subset in the projectivized bundle $\mathbb{P}(TM_{reg})$ that is an analytically constructible subset in $\mathbb{P}(TV)$.*

Proof Fix a point $x_0 \in M$. Let $f_1, \dots, f_r$ be holomorphic functions on a neighborhood $U = U(x_0) \subset V$ that define M: $M \cap U = \{f_1 = \dots = f_r = 0\}$, $T_x M = \mathrm{Ker}(df_1(x), \dots, df_r(x))$ for every $x \in M_{reg} \cap U$. Let $\omega_1, \dots, \omega_s$ be holomorphic 1-forms on U that define a singular analytic distribution whose restriction to $M \cap U$ coincides with $\mathcal{D}_M$. A non-zero vector $v \in TV$ generates a non-singular integral element, if and only if its base point $x = x(v)$ lies in M^0_{reg}, $(df_i)(v) = 0$ for $i = 1, \dots, r$, $\omega_j(v) = 0$ for $j = 1, \dots, s$ and the system of $(r + s)$ 1-forms $df_i(x)$, $i_v(d\omega_j)(x)$, $i = 1, \dots, r$, $j = 1, \dots, s$ has maximal rank. This implies the statements of the proposition. $\square$

Remark 4.15 For every singular analytic distribution $\mathcal{F}$ on an analytic set M in a complex manifold V the set of its two-dimensional integral elements (integral planes) is an analytically constructible subset in $Gr_2(TV)|_M$. The proof of this statement is analogous to the above proof of Proposition 4.14. Therefore, its image under the projection to M is also analytically constructible (Chevalley–Remmert Theorem).

Proposition 4.16 *Every three-dimensional singular analytic distribution $\mathcal{F}$ on an irreducible analytic variety M satisfies one of the two following incompatible statements:*

1) either the corresponding Pfaffian system $\mathcal{F}_{3,2}$ on M^0_{reg} is involutive;

2) or there exists a proper analytic subset $\Sigma \subset M$ such that

a) either for every $x \in M \setminus \Sigma$ the space $\mathcal{F}(x)$ contains no integral plane;

b) or for each $x \in M \setminus \Sigma$ the space $\mathcal{F}(x)$ contains a unique integral plane.

Proof If $\mathcal{F}(x)$ contains some two distinct integral planes P_1, P_2, then it contains a pencil of integral planes through the line $L = P_1 \cap P_2$, which saturate the whole three-dimensional space $\mathcal{F}(x)$, by Remark 4.10. Thus, each $\mathcal{F}(x)$ either is a union of integral planes, or contains a unique integral plane, or contains no integral planes. This together with Remark 4.15 and Chevalley–Remmert Theorem easily implies that M^0_{reg} is a disjoint union of three analytically constructible subsets of points $x \in M$ where $\mathcal{F}(x)$ satisfy respectively one of the three latter statements. One of them contains a complement to a proper analytic subset $\Sigma \subset M$. Therefore, either $\mathcal{F}(x)$ is a union of integral planes for all $x \in M \setminus \Sigma$ (and hence, for all $x \in M^0_{reg}$ by analyticity) and the Pfaffian system $\mathcal{F}_{3,2}$ is involutive, or some of statements a) or b) of Proposition 4.16 holds. This proves Proposition 4.16. $\square$

4.3 Case of three-dimensional distribution

Here we consider that $dim\mathcal{D}_M = 3$. The subcase, when the Pfaffian system $(\mathcal{D}_M)_{3,2}$ is non-involutive, is reduced to the two-dimensional case by Proposition 4.16. Indeed, the image of the projection to M of the set of integral planes is analytically constructible, by Remark 4.15, and contains the integral surface S. Therefore, it contains an open dense complement to an analytic subset in M, since M is the minimal analytic set containing S. Hence, if the system is not involutive, then there exists a proper analytic subset $\Sigma \subset M$ such that for every $x \in M \setminus \Sigma$ the space $\mathcal{F}(x)$ contains a unique integral plane (Proposition 4.16). The latter integral planes form a two-dimensional intrinsic singular analytic distribution on M, by Definition 2.36 and Remark 4.15. Then we apply the arguments from Subsection 4.1 to the latter two-dimensional distribution instead of $\mathcal{D}_M$. Thus, without loss of generality we consider that *the Pfaffian system* $(\mathcal{D}_M)_{3,2}$ *is involutive.*

Corollary 4.2 easily implies the next proposition and corollary, which state that there exist a connected open subset $V \subset M^0_{reg}$, a regular analytic hypersurface $V_a \subset V$, $\nu_B(V_a) \subset a$, and an analytic field $\mathcal{L}$ of one-dimensional nonsingular integral elements in $\mathcal{D}_M$ on V whose all complex orbits intersect V_a transversely (plus a mild genericity condition (4.2)). The germ through every $x \in V$ of complex orbit is included into an integral surface of the distribution $\mathcal{D}_M$, by Theorem 4.12. Condition (4.2) implies that the integral surface is non-trivial, and hence, corresponds to an open set of quadrilateral orbits of a 4-reflective billiard a, $b(x)$, $c(x)$, $d(x)$. If $x \in V_a$, then the mirrors a and $b(x)$ intersect at $\nu_B(x)$, and we deduce that either $c(x) = a$, or $b(x)$ is a line (Corollary 3.5). This easily implies that either $\nu_C(M) \subset a$, or the image under the projection ν_B of every analytic curve tangent to $\mathcal{D}_M$ is a line. We show that none of the latter cases is possible. The contradiction thus obtained will prove Theorem 1.10.

Let $M' \subset M^0_{reg}$ be the subset from (4.1) defined for our three-dimensional distribution $\mathcal{D}_M$. It is open, dense and the complement $M \setminus M'$ is analytic, as at the same place. By definition, $d\nu_a, d\nu_G \not\equiv 0$ on $\mathcal{D}_M(x)$ for every $x \in M'$.

Proposition 4.17 *There exist an $x \in M'$ and a one-dimensional nonsingular integral element $\mathcal{L}_x \subset \mathcal{D}_M(x)$ such that $\nu_B(x) \in a$ and*

$$(d\nu_a(x))(\mathcal{L}_x) \neq 0, \ (d\nu_G(x))(\mathcal{L}_x) \neq 0 \text{ for every } G = B, C, D; \tag{4.2}$$

$$\text{the line } (d\nu_B(x))(\mathcal{L}_x) \text{ is transverse to } T_{\nu_B(x)}a. \tag{4.3}$$

Proof Let $\widetilde{Q}$ denote the set of one-dimensional nonsingular integral elements $\mathcal{L}_x \subset \mathcal{D}_M(x)$, $x \in M'$, satisfying (4.2). Let $Q \subset M'$ denote its projection to M. The sets $\widetilde{Q}$ and Q are analytically constructible subsets in $\mathbb{P}(T\mathcal{P}_a)$ and M. They are open and dense subsets in $\mathbb{P}(\mathcal{D}_M)$ and M respectively. The two latter statements follow from definition, Proposition 4.14 and Chevalley–Remmert Theorem. The intersection $\nu_B(Q) \cap a$ contains a regularly embedded disk $\alpha \subset a$ without isotropic tangent lines, and $V_a = \nu_B^{-1}(\alpha) \cap Q$ is a hypersurface (Corollary 4.2).

Claim. *$\mathcal{D}_M(x) \not\subset T_x V_a$ for an open dense set of points $x \in V_a$.*

Proof Suppose the contrary: each germ of integral curve (and hence, surface) of the distribution $\mathcal{D}_M$ through each point in V_a lies in V_a. Fix an $x \in V_a$, a nonsingular integral element $\mathcal{L}_x \subset \mathcal{D}_M(x)$ satisfying (4.2), a germ of integral curve Γ tangent to $\mathcal{L}_x$ and a germ of integral surface $\hat{S}$ containing Γ (given by Theorem 4.12). The surface $\hat{S}$ lies in V_a, is non-trivial by (4.2), and hence, represents an open set of quadrilateral orbits of a 4-reflective billiard with two coinciding non-linear mirrors $a = b(x)$. But the latter billiard cannot exist by Corollary 3.5. This proves the claim. □

For every $x \in V_a$ such that $\mathcal{D}_M(x) \not\subset T_x V_a$ a generic one-dimensional integral element $\mathcal{L}_x \subset \mathcal{D}_M(x)$ is nonsingular and satisfies conditions (4.2), (4.3). This proves the proposition. □

Corollary 4.18 *There exist an open subset $V \subset M'$, a regularly embedded disk $\alpha \subset a$ without isotropic tangent lines, an analytic hypersurface $V_a \subset V$ with $\nu_B(V_a) \subset \alpha$ and an analytic line field $\mathcal{L}$ on V contained in $\mathcal{D}_M$ and transverse to V_a such that the lines of the field $\mathcal{L}$ are nonsingular integral elements satisfying inequalities (4.2) and each its complex orbit in V intersects V_a.*

The corollary follows immediately from the proposition and openness of the set of nonsingular integral elements satisfying (4.2).

Proposition 4.19 *Let V, V_a and $\mathcal{L}$ be as in the above corollary. Then there are two possible cases:*

Case 1): $\nu_C(V) \subset a$;

Case 2): the projection ν_B sends complex orbits of the field $\mathcal{L}$ to lines.

Proof For every $x \in V$ the germ of the orbit of the field $\mathcal{L}$ through x lies in a germ of integral surface of the distribution $\mathcal{D}_M$ (Theorem 4.12). The latter surface is non-trivial by the inequalities from (4.2), and hence, is given by an open set of quadrilateral orbits of a 4-reflective billiard a, $b(x)$, $c(x)$, $d(x)$. If $x \in V_a$, then the mirrors a and $b(x)$ intersect at the point $B(x) = \nu_B(x)$, and the latter is not marked for their corresponding local branches. This follows from construction and the inequalities from (4.2). Hence, for every $x \in V_a$ either $c(x) = a$, or $b(x)$ is a line (Corollary 3.5). Each orbit of the field $\mathcal{L}$ intersects V_a, by assumption. Hence, either the projection ν_C sends it to a, or the projection ν_B sends it to a line. One of the two latter statements holds for all the orbits, by analyticity. This proves the proposition. □

Now for the proof of Theorem 1.10 it suffices to show that none of the cases from the above proposition is possible.

Case 1): $\nu_C(V) \subset a$. Then $\nu_C(M) \subset a$, and we get a contradiction, as at the end of Subsection 4.1. Hence, Case 1) is impossible.

Case 2): $\nu_C(M) \not\subset a$. Let V and $\mathcal{L}$ be the same, as in the above corollary. Let us deform $\mathcal{L}$. The set of line fields $\mathcal{L}$ satisfying the conditions of the corollary is open in the space of line fields contained in the distribution $\mathcal{D}_M$. This together with the corollary implies that for every line field $\mathcal{L}$ contained in $\mathcal{D}_M$ the projection ν_B sends each its complex orbit to a line. This is equivalent to say that each analytic curve in M^0_{reg} tangent to $\mathcal{D}_M$ is sent to a line by ν_B. In particular, this holds for every one-parametric family of quadrilateral orbits lifted to M of the initial billiard a, b, c, d with variable $B \in b$. This implies that the curve b is a line. The contradiction thus obtained proves Theorem 1.10. The proof of Theorem 1.7 is complete.

5 Applications to real pseudo-billiards

In Subsection 5.1 we introduce and classify the germs of 4-reflective C^4-smooth real planar pseudo-billiards. The proof of the classification Theorem 5.6 is presented in the same subsection (analytic case) and in Subsection 5.2 (smooth case). In the same Subsection 5.2 we prove Theorem 5.8 showing that there are no C^4-smooth pseudo-billiards with only two skew reflection laws at some neighbor mirrors and a positive measure set of 4-periodic orbits. In Subsections 5.3, 5.4 we present applications of Theorems 5.6 and 5.8 respectively to Tabachnikov's Commuting Billiard Conjecture and 4-reflective Plakhov's Invisibility Conjecture.

5.1 Classification of real planar 4-reflective pseudo-billiards

Here by *real smooth (analytic) curve* in $\mathbb{R}^2$ or $\mathbb{RP}^2$ we mean the image of either $\mathbb{R}$, or S^1 under a locally non-constant smooth (analytic) mapping to $\mathbb{R}^2$ (respectively, $\mathbb{RP}^2$). A smooth (analytic) *germ* of curve is given by a smooth (analytic) germ of immersion $(\mathbb{R}, 0) \to \mathbb{R}^2$ $((\mathbb{R}, 0) \to \mathbb{RP}^2)$.

Definition 5.1 [7, remark 1.6] Let a line $L \subset \mathbb{R}^2$ and a triple of points $A, B, C \in \mathbb{R}^2$ be such that $B \in L$, $A, C \notin L$ and the lines AB, BC are symmetric with respect to the line L. We say that the triple A, B, C and the line L satisfy the *usual reflection law*, if the points A and C lie on the same side from the line L. Otherwise, if they are on different sides from the line L, we say that the *skew reflection law* is satisfied.

Example 5.2 In every planar billiard orbit each triple of consequent vertices satisfies the usual reflection law with respect to the tangent line to the boundary of the billiard at the middle vertex.

Definition 5.3 (cf. [7, definition 6.1]) A *real planar pseudo-billiard* is a collection of k curves $a_1, \dots, a_k \subset \mathbb{R}^2$ called *mirrors* with a prescribed reflection law on each curve a_j: either usual, or skew. Its *k-periodic orbit* is a k-gon $A_1 \dots A_k$, $A_j \in a_j$, such that for every $j = 1, \dots, k$ one has $A_j \neq A_{j\pm1}$, $A_jA_{j\pm1} \neq T_{A_j}a_j$ and the lines A_jA_{j-1}, A_jA_{j+1} are symmetric with respect to the tangent line $T_{A_j}a_j$ so that the triple A_{j-1}, A_j, A_{j+1} and the line $T_{A_j}a_j$ satisfy the reflection law corresponding to a_j. Here we set $a_{k+1} = a_1$, $A_{k+1} = A_1$, $a_0 = a_k$, $A_0 = A_k$. A real pseudo-billiard is called *(piecewise) analytic/smooth*, if so are its curves. A *germ* of real pseudo-billiard is a collection of k *germs* of curves (a_j, A_j) with prescribed reflection laws for which the marked k-gon $A_1 \dots A_k$ is a k-periodic orbit. A germ of pseudo-billiard is called *k-reflective,* if the set of its k-periodic orbits has non-empty interior: contains a two-parameter family including $A_1 \dots A_k$. A (germ of) pseudo-billiard is called *measure k-reflective,* if the set of its k-periodic orbits has positive two-dimensional Lebesgue measure. This means that in the set of k-orbits $A_1 \dots A_k$ that satisfy the corresponding reflection laws at A_j for all $j \neq 1, k$ the set of k-periodic ones (i.e., those satisfying the reflection laws at A_1, A_k) has positive two-dimensional Lebesgue measure.

Remark 5.4 A k-reflective pseudo-billiard is automatically measure k-reflective. In the analytic case measure k-reflectivity is equivalent to k-reflectivity. The interior points of the set of k-periodic orbits will be called *k-reflective orbits*, cf. definition 6.1 in loc. cit. The complexification of each k-reflective planar analytic pseudo-billiard is a k-reflective complex billiard.

Convention 5.5 *Given two germs of smooth curves (a, A), (c, C) in $\mathbb{R}^2 \subset \mathbb{RP}^2$, we say that $a = c$, if they lie in the same* **analytic** *curve in $\mathbb{RP}^2$.*

Theorem 5.6 *A germ of C^4-smooth real planar pseudo-billiard (a, A), (b, B), (c, C), (d, D) is 4-reflective, if and only if it has one of the following types:*

1) $a=c$ is a line, the curves $b, d \neq a$ are symmetric with respect to it;

2) a, b, c, d are distinct lines through the same point $O \in \mathbb{RP}^2$, the line pairs (a,b), (d,c) are transformed one into the other by rotation around O (translation, if O is an infinite point), see Fig.8;

3) $a=c$, $b=d$, and they are distinct confocal conics: either ellipses, or hyperbolas, or ellipse and hyperbola, or parabolas.

In every 4-reflective orbit the reflection law at each pair of opposite vertices is the same; it is skew for at least one opposite vertex pair.

Addendum 1. *In every pseudo-billiard of type 1) from Theorem 5.6 each quadrilateral orbit $ABCD$ has the same type, as at Fig.7. It is symmetric with respect to the line a, and the reflection law at A, C is skew. The reflection law at B, D is either usual at both, or skew at both.*

Addendum 2 [7, addendums 2, 3 to theorem 6.3]. *In pseudo-billiards of types 2), 3) the 4-reflective orbits have the same types, as at Fig.8–12.*

Remark 5.7 The main result of paper [10] (theorem 2) concerns usual real planar billiards with piecewise C^4 boundary; the reflection law is usual. It implies that the quadrilateral orbit set has empty interior. This statement also follows from the last statement of Theorem 5.6.

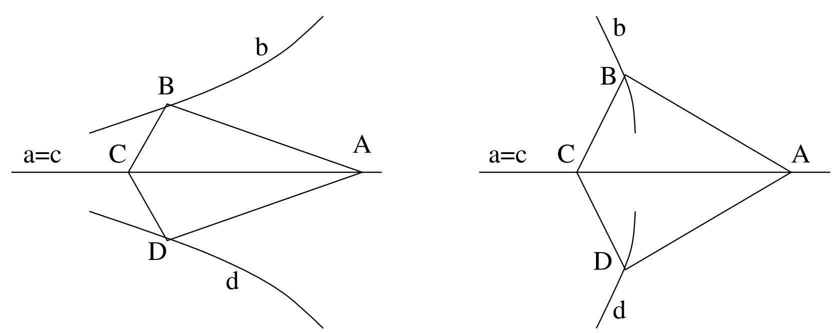

Figure 7: 4-reflective pseudo-billiards symmetric with respect to a line mirror

Here we prove Theorem 5.6 for analytic pseudo-billiards. The general smooth case will be treated in the next subsection. We also prove the following theorem there, which will be applied to Plakhov's Invisibility Conjecture.

Theorem 5.8 *There exist no measure 4-reflective C^4-smooth planar pseudo-billiard with exactly two skew reflection laws at a pair of neighbor mirrors.*

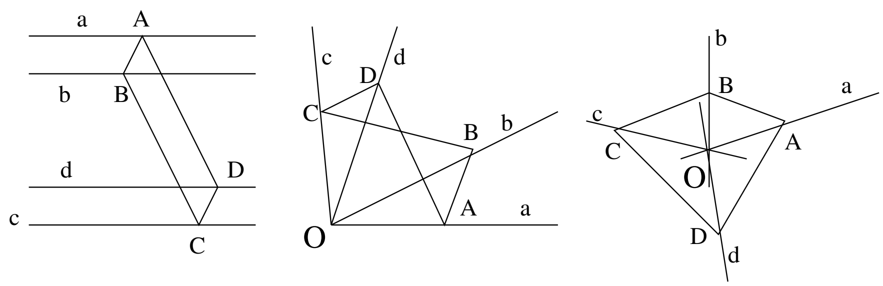

Figure 8: 4-reflective pseudo-billiards on two positively-isometric line pairs

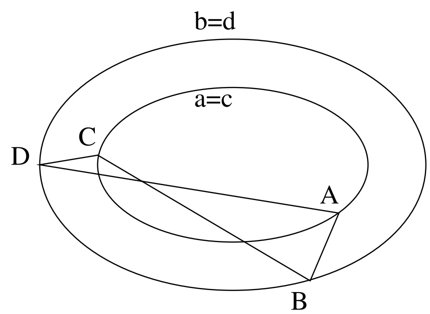

Figure 9: A 4-reflective pseudo-billiard on confocal ellipses

Proof of Theorem 5.6 (analytic case) and Addendum 1 (smooth case). The analytic pseudo-billiard under question being 4-reflective, its complexification is obviously 4-reflective, by Remark 5.4. This together with Theorem 1.7 implies that it has one of the above types 1)–3) (up to cyclic renaming of the mirrors). The 4-reflectivity of pseudo-billiards of types 2), 3) and the classification of open sets of their quadrilateral orbits and reflection law configurations was proved in [7, section 6]. The 4-reflectivity of pseudo-billiards of type 1) is obvious. Addendum 1 (symmetry of the vertices B and D) follows from the fact that they are intersection points of symmetric pairs of lines, by definition. □

5.2 Analytic versus smooth: proofs of Theorems 5.6 and 5.8

The proofs of Theorems 5.6 and 5.8 in the smooth case given here are analogous to the arguments from [10, section 2] due to Yu.G.Kudryashov. We start them with the following simple fact.

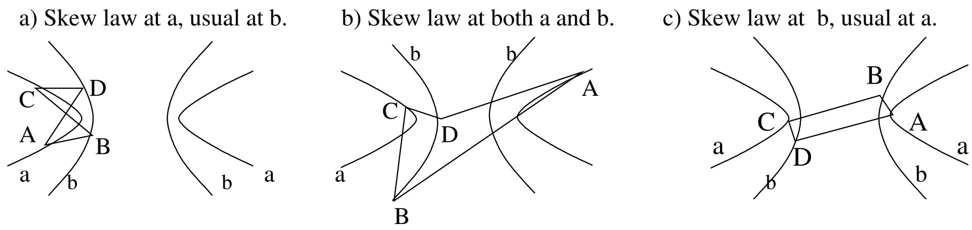

Figure 10: Open sets of orbits on confocal hyperbolas: three types

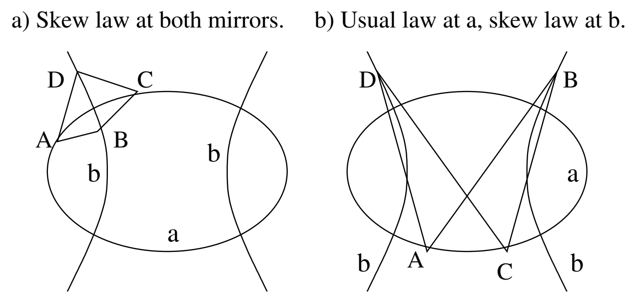

Figure 11: Open sets of orbits on confocal ellipse and hyperbola: two types

Proposition 5.9 *For every $k \in \mathbb{N}$ there exist no measure k-reflective real C^3-germ of pseudo-billiard with odd number of skew reflection laws.*

Proof Reflections from curves act on the space of oriented lines in $\mathbb{R}^2$, which is a two-dimensional oriented manifold. Skew reflections change the orientation, and the usual ones don't. Therefore, a composition of odd number of skew reflections and a number of usual ones (in any order) is a local diffeomorphism changing the orientation. Hence, it cannot be equal to the identity on a set of positive measure. This proves the proposition. $\square$

The proposition implies that the only possible reflection law configurations for a potential measure 4-reflective C^4 pseudo-billiard are the following:
1) all the reflection laws are usual;
2) the reflection laws are skew only at some pair of neighbor mirrors;
3) all the reflection laws are skew;
4) the reflection laws are skew only at some pair of opposite mirrors.

Remark 5.10 Configuration 1) is already forbidden: the proof of its impossibility is implicitly contained in [10]. The proof of Theorem 5.6 given below does not use this result. Configuration 2) is already forbidden in the analytic case, by the last statement of Theorem 5.6 proved in this case.

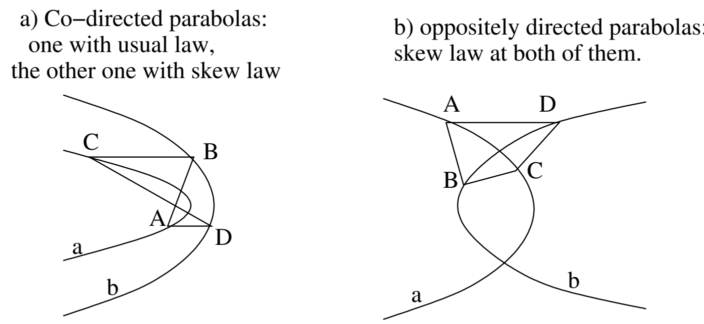

Figure 12: Open set of orbits on confocal parabolas: one type

Definition 5.11 ([10, definition 14], Yu.G.Kudryashov). Let $k \geq 3$. A k-gon $A_1 \dots A_k$ in $\mathbb{R}^2$ is *non-degenerate,* if for every $j = 1, \dots, k$ (we set $A_{k+1} = A_1$, $A_0 = A_k$) one has $A_{j+1} \neq A_j$ and $A_{j-1}A_j \neq A_jA_{j+1}$. The subset of degenerate k-gons in $\mathbb{R}^{2k} = (\mathbb{R}^2)^k$ will be denoted by $\Sigma = \Sigma_k$.

Remark 5.12 A real non-degenerate k-gon is a complex non-degenerate k-gon in the sense of Definition 2.40.

To each configuration 1)–4) we put into correspondence a distribution $\mathcal{D}^4_\alpha$ on $\mathbb{R}^8 \setminus \Sigma$ constructed below: the Birkhoff distribution corresponding to the chosen reflection laws. To define it, set

$$\Psi_k = \{\pm 1\}^k, \ \Psi^{\pm}_k = \{\alpha = (\alpha_1, \dots, \alpha_k) \in \Psi_k \mid \prod_j \alpha_j = \pm 1\}.$$

For every $k \in \mathbb{N}$, $\alpha \in \Psi_k$ and every non-degenerate k-gon $A_1 \dots A_k$ in $\mathbb{R}^2$ we consider the following collection of lines $L_{A_j} = L_{A_j}(\alpha)$ through A_j:

- the line L_{A_j} is the exterior bisector of the angle $\angle A_{j-1}A_jA_{j+1}$ if $\alpha_j = 1$;
- the line L_{A_j} is its interior bisector if $\alpha_j = -1$.

We will briefly call the configuration $((A_1, L_{A_1}), \dots, (A_k, L_{A_k}))$ *α-framed.* We identify each line L_{A_j} with the corresponding one-dimensional subspace in $T_{A_j}\mathbb{R}^2$. For every $x = A_1 \dots A_k \in \mathbb{R}^{2k} \setminus \Sigma$ set

$$\mathcal{D}^k_\alpha(x) = \oplus^k_{j=1} L_{A_j}(\alpha) \subset T_x\mathbb{R}^{2k}.$$

The latter subspaces define an analytic distribution $\mathcal{D}^k_\alpha$ on $\mathbb{R}^{2k} \setminus \Sigma$. The above reflection law configurations 1)–4) correspond to $\mathcal{D}^4_\alpha$ with $\alpha \in \Psi^+_4$.

We show that all $\mathcal{D}^k_\alpha$ with $\alpha \in \Psi^+_4$ naturally embed to the complex Birkhoff distribution $\mathcal{D}^k$ on one and the same irreducible component $\mathcal{R}^+_{0,k}$

of the variety $\mathcal{R}_{0,k} \subset \mathcal{P}^k = (\mathbb{P}(T\mathbb{CP}^2))^k$ (the next proposition). We study the two-dimensional Pfaffian problem for the complex distribution $\mathcal{D}^4$ on $\mathcal{R}^+_{0,4}$: to find its two-dimensional integral surfaces. We study its first and second Cartan jet prolongations. (Basic theory of Cartan prolongations may be found in [3, 16, 21]; the background material used in the proofs will be recalled below.) We show (the next two lemmas) that each second prolongation is a two-dimensional distribution outside a union of two subvarieties Λ_0, Λ_1. The latter union consists of the quadrilaterals symmetric with respect to some of the diagonals A_1A_3 or A_2A_4 with $L_{A_1} = L_{A_3} = A_1A_3$ (respectively, $L_{A_2} = L_{A_4} = A_2A_4$). We then deduce that each C^4-smooth non-trivial integral surface of the distribution $\mathcal{D}^4_\alpha$ should be either analytic or contained in the above set of symmetric quadrilaterals. Therefore, the pseudo-billiard under question should have one of the types 1)–3), by the analytic case of Theorem 5.6 proved in the previous subsection. This will prove Theorem 5.6 in the C^4 case. For the proof of Theorem 5.8 we show that the existence of a measure 4-reflective pseudo-billiard with reflection law configuration 2) implies the existence of a 4-reflective analytic one. This will be deduced from the above-mentioned lemmas and Kudryashov's results from [10]. But 4-reflective analytic pseudo-billiards with reflection law configuration 2) are forbidden by the last statement of Theorem 5.6, analytic case. The contradiction thus obtained will prove Theorem 5.8.

Now let us pass to the proofs.

Consider the k-dimensional complex Birkhoff distribution $\mathcal{D}^k$ on the $2k$-dimensional smooth quasiprojective variety $\mathcal{R}_{0,k} \subset \mathcal{P}^k = (\mathbb{P}(T\mathbb{CP}^2))^k$ introduced at the beginning of Subsection 2.7. For every $\alpha \in \Psi_k$ there is a natural embedding

$$j_\alpha : \mathbb{R}^{2k} \setminus \Sigma \to \mathcal{R}_{0,k} : \ j_\alpha(A_1 \dots A_k) = ((A_1, L_{A_1}(\alpha)), \dots, (A_k, L_{A_k}(\alpha))).$$

One has $(j_\alpha)_* \mathcal{D}^k_\alpha \subset \mathcal{D}^k$, and the latter image is a field of totally real subspaces in the corresponding spaces of the complex distribution $\mathcal{D}^k$.

Proposition 5.13 *For every $k \geq 3$ the variety $\mathcal{R}_{0,k}$ consists of two irreducible components $\mathcal{R}^\pm_{0,k}$. Each $\mathcal{R}^\pm_{0,k}$ contains the union of all the images $j_\alpha(\mathbb{R}^{2k} \setminus \Sigma)$ with $\alpha \in \Psi^\pm_k$. The component $\mathcal{R}^+_{0,k}$ consists exactly of those configurations $((A_1, L_{A_1}), \dots, (A_k, L_{A_k}))$ for which the complex lengths $|A_jA_{j+1}|$, $j = 1, \dots, k$ can be simultaneously normalized so that for every j the lengths $|A_{j-1}A_j|$, $|A_jA_{j+1}|$ are L_{A_j}-concordant, see Definition 2.13.*

Remark 5.14 The above collection of lengths $|A_1A_2|, \dots, |A_kA_1|$ (if exists) will be called *concordant.* It is unique up to simultaneous change of sign.

Proof The projection of the variety $\mathcal{R}_{0,k}$ to the space of k-gons has 2^k preimages: at each vertex A_j we can choose a symmetry line L_{A_j} between the lines $A_jA_{j\pm1}$ in two ways; the two symmetry lines are orthogonal.

Claim. *For every $j = 1, \dots, k$ every configuration $((A_1, L_{A_1}), \dots, (A_k, L_{A_k})) \in \mathcal{R}_{0,k}$ is connected by path in $\mathcal{R}_{0,k}$ to the same configuration with simultaneously changed lines L_{A_j}, $L_{A_{j+1}}$.*

Proof Let us identify the complex infinity line $\overline{\mathbb{C}}_\infty$ with the space of complex lines through the point A_j. Its complement $\mathbb{C}^* = \overline{\mathbb{C}}_\infty \setminus \{I_1, I_2\}$ to the isotropic points has fundamental group $\mathbb{Z}$. Consider a closed path $\phi : [0,1] \to \mathbb{C}^*$ starting at the line A_jA_{j+1} and disjoint from the line A_jA_{j-1} that represents its generator. This yields a closed path $\psi(t) = A_1 \dots A_j A^t_{j+1} A_{j+2} \dots A_k$ in the space of non-degenerate complex k-gons (in the sense of Definition 2.40): $A^t_{j+1} = \phi(t) \cap A_{j+1}A_{j+2}$, $A^0_{j+1} = A_{j+1}$. The line $L_{A_j} = L_{A_j}(0)$ deforms into a family $L_{A_j}(t)$ of symmetry lines between the lines A_jA_{j-1} and $\phi(t)$ that is analytic along the path ψ. One has $L_{A_j}(1) \neq L_{A_j}(0)$. Indeed, let z_{j-1}, $z_j(t)$, $z_{j+1}(t)$ denote the points of intersection of the infinity line with the lines A_jA_{j-1}, $L_{A_j}(t)$, $A_jA^t_{j+1} = \phi(t)$ written in the standard affine coordinate z on $\overline{\mathbb{C}}_\infty$: $z(I_1) = 0$, $z(I_2) = \infty$. One has $z_{j+1}(t) = \frac{z_j^2(t)}{z_{j-1}}$, by symmetry and [7, proposition 2.4, p.249]. The point $z_j(t)$ makes half-turn around zero, by the latter formula and since $z_{j+1}(t)$ makes one turn by assumption. Hence, $L_{A_j}(1) \neq L_{A_j}(0)$. We similarly consider the deformation $L_{A_{j+1}}(t) = L_{A^t_{j+1}}$ of the line $L_{A_{j+1}}$ and get $L_{A_{j+1}}(1) \neq L_{A_{j+1}}$. The claim is proved. $\square$

The claim implies that any two configurations $((A_1, L_{A_1}), \dots, (A_k, L_{A_k}))$ and $((A_1, L'_{A_1}), \dots, (A_k, L'_{A_k}))$ in $\mathcal{R}_{0,k}$ with even number of pairs of different lines $L_{A_j} \neq L'_{A_j}$ are connected by path in $\mathcal{R}_{0,k}$. Therefore, the variety $\mathcal{R}_{0,k}$ consists of at most two irreducible components $\mathcal{R}^\pm_{0,k}$, each of them contains the images $j_\alpha(\mathbb{R}^{2k} \setminus \Sigma)$ for all $\alpha \in \Psi^\pm_k$. Each real $(1, \dots, 1)$-framed configuration has a concordant collection of real lengths (Example 2.15). The existence of concordant complex length collection is invariant under continuous deformations in $\mathcal{R}_{0,k}$, which follows from definition. Therefore, each configuration from the component $\mathcal{R}^+_{0,k}$ has a concordant length collection. On the other hand, each real $(1, \dots, 1, -1)$-framed configuration has no concordant length collection, which follows from Example 2.15. Hence, this is true for every configuration from the component $\mathcal{R}^-_{0,k}$, and $\mathcal{R}^+_{0,k} \neq \mathcal{R}^-_{0,k}$. Proposition 5.13 is proved. $\square$

The arguments below are analogous to Yu.G.Kudryashov's arguments from [10, section 2] on the Cartan prolongations of the Birkhoff distributions.

In what follows the restriction of the distribution $\mathcal{D}^4$ to the component $\mathcal{R}^+_{0,4}$ will be denoted $\mathcal{D}^4_+$. Consider the complex Pfaffian system $\mathcal{D}^{4,2}_+$: the problem to find two-dimensional integral surfaces of the distribution $\mathcal{D}^4_+$. For every $x \in \mathcal{R}^+_{0,4}$ let $l_1(x), \dots, l_4(x)$ denote the corresponding concordant collection of lengths $l_j = |A_j A_{j+1}|$. The lengths being defined up to simultaneous change of sign, their ratios are single-valued holomorphic functions on $\mathcal{R}^+_{0,4}$. Set

$$\Lambda = \{l_1 l_3 = l_2 l_4\} \subset \mathcal{R}^+_{0,4},$$

$$\Lambda_0 = \{x \in \mathcal{R}^+_{0,4} \mid A_1(x)A_2(x)A_3(x)A_4(x) \text{ is symmetric with respect to the line } A_1(x)A_3(x) = L_{A_1}(x) = L_{A_3}(x)\},$$

$$\Lambda_1 = \{x \in \mathcal{R}^+_{0,4} \mid A_1(x)A_2(x)A_3(x)A_4(x) \text{ is symmetric with respect to the line } A_2(x)A_4(x) = L_{A_2}(x) = L_{A_4}(x)\}.$$

Remark 5.15 One has $\Lambda_0, \Lambda_1 \subset \Lambda$. Indeed, if $x \in \Lambda_0$, then $l_4(x) = -l_1(x)$, $l_2(x) = -l_3(x)$, by symmetry and length concordance (Definition 2.13). Hence, $l_1(x)l_3(x) = l_2(x)l_4(x)$ and $x \in \Lambda$. For every $x \in \Lambda_0$ the lines $L_{A_2}(x)$, $L_{A_4}(x)$ are symmetric with respect to the line $L_{A_1}(x)$, since the symmetry respects concordance of lengths and transforms the $L_{A_2}(x)$-concordant lengths $l_1(x)$, $l_2(x)$ to the $L_{A_4}(x)$-concordant lengths $-l_4(x)$, $-l_3(x)$. Similar statement holds for $x \in \Lambda_1$ and $L_{A_1}(x)$, $L_{A_3}(x)$.

Recall that an integral plane E (see Definition 4.7) of the distribution $\mathcal{D}^4_+$ is *non-trivial*, if for every j the restriction $dA_j|_E$ is not identically zero.

Let $\mathcal{I}_2 = \mathcal{I}_2(\mathcal{D}^4_+) \subset Gr_2(\mathcal{D}^4_+)$ denote the subset of integral planes; it is algebraic, since so is $\mathcal{D}^4_+$. By $\mathcal{I}^0_2 \subset \mathcal{I}_2$ we denote the Zariski open subset of the non-trivial integral planes. A point of the space $\mathcal{I}^0_2$ is a pair (x, E), where $x \in \mathcal{R}^+_{0,4}$, $E \subset \mathcal{D}^4_+(x)$ is a non-trivial integral plane. Let $\pi_{gr} : \mathcal{I}^0_2 \to \mathcal{R}^+_{0,4}$ denote the standard projection. The subspaces

$$\mathcal{F}^3(x, E) = (d\pi_{gr})^{-1}(E) \subset T_{(x,E)}\mathcal{I}^0_2$$

form a (singular) analytic distribution $\mathcal{F}^3$ on $\mathcal{I}^0_2$. Set

$$\mathcal{J}_2 = \mathcal{I}^0_2 \setminus \pi^{-1}_{gr}(\Lambda) \subset \mathcal{I}^0_2.$$

Lemma 5.16 *The projection $\pi_{gr}:\mathcal{J}_2\to\mathcal{R}^+_{0,4}\setminus\Lambda$ is a regular fibration by holomorphic curves. The restriction to $\mathcal{J}_2$ of the distribution $\mathcal{F}^3$ is regular and three-dimensional. For every $y\in\mathcal{J}_2$ the corresponding subspace $\mathcal{F}^3(y)$ contains a unique integral 2-plane $\widetilde{E}(y)$ of the distribution $\mathcal{F}^3$. The planes $\widetilde{E}(y)$ form a two-dimensional analytic distribution on $\mathcal{J}_2$.*

Lemma 5.17 *The restriction to $\Lambda\setminus(\Lambda_0\cap\Lambda_1)$ of the distribution $\mathcal{D}^4_+$ has at most one non-trivial integral 2-plane at each point.*

The two lemmas are proved below. In their proofs we use the notations introduced in [10, subsection 2.5.1, p.297][6]. The analytic extensions to $\mathcal{R}^+_{0,4}$ of the complexified 1-forms θ_j, ν_j from loc. cit. are regular and double-valued: well-defined up to sign. The forms $\nu_1,\dots,\nu_4$ yield a coordinate system on the subspaces of the distribution $\mathcal{D}^4_+$, as in loc. cit. The tangent functions t_j from loc. cit. are holomorphic single-valued on $\mathcal{R}^+_{0,4}$. Let us show this in more detail. Recall that for a real convex quadrilateral $A_1\dots A_4$ equipped with its exterior bisectors L_{A_j} one has $t_j=\tan\angle(L^\perp_{A_j},A_jA_{j+1})$. Let z be the above standard affine coordinate on $\overline{\mathbb{C}}_\infty$: $z(I_1)=0$, $z(I_2)=\infty$. Let z_j, w_j denote respectively the z- coordinates of the intersection points $L^\perp_{A_j}\cap\overline{\mathbb{C}}_\infty$, $A_jA_{j+1}\cap\overline{\mathbb{C}}_\infty$. It follows from definition that

$$t_j=\frac{i(z_j-w_j)}{z_j+w_j}.$$

The latter formula defines the analytic extension of the functions t_j to $\mathcal{R}^+_{0,4}$. One has $z_j\neq\pm w_j$ on $\mathcal{R}^+_{0,4}$. Indeed, otherwise either $A_jA_{j+1}=L^\perp_{A_j}$, or $A_jA_{j+1}=L_{A_j}$; in both cases $A_jA_{j-1}=A_jA_{j+1}$, which is impossible by non-degeneracy. Thus, t_j *are non-vanishing holomorphic functions on* $\mathcal{R}^+_{0,4}$.

Proof of Lemma 5.16. The real Pfaffian system $\mathcal{D}^{4,2}_\alpha$ corresponding to the usual real Birkhoff distribution $\mathcal{D}^4_\alpha$, $\alpha=(1,\dots,1)$, was studied by Yu.G.Kudryashov in [10, subsection 2.5]. The proof of the statement of the lemma for the integral planes of the system $\mathcal{D}^{4,2}_\alpha$ is presented in [10, subsection 2.5.4, pp. 299-300]. The calculations presented there extend analytically to all of $\mathcal{R}^+_{0,4}\setminus\Lambda$ and apply without changes. □

Proof of Lemma 5.17. Kudryashov's calculations in [10, p.298] extended analytically to complex domain show that for every $x\in\mathcal{R}^+_{0,4}$ each non-trivial integral plane $E_2\subset\mathcal{D}^4_+(x)$ has a basis of the following type:

$$\begin{pmatrix} 0 & l_1 & \eta & -l_4\\ l_1 & 0 & -l_2 & \eta'\end{pmatrix};\ \eta\eta'=l_2l_4-l_1l_3;$$

[6]The edge lengths denoted by L_j in [10] are denoted here by l_j.

the two raws represent vectors in $\mathcal{D}^4_+(x)$ written in the coordinates given by the forms ν_j from loc. cit. Let $x \in \Lambda \setminus (\Lambda_0 \cap \Lambda_1)$. Then $\eta\eta' = l_2 l_4 - l_1 l_3 = 0$. Thus, either $\eta = 0$, or $\eta' = 0$.

Case 1): $\eta' = 0$. Then the inclusion $E_2 \subset T_x\Lambda$ implies that

$$t_3(l_1 + l_4)\eta = l_1(t_2 - t_4)(l_3 + l_4), \tag{5.1}$$

see [10, p.299, subsection 2.5.3]. Les us suppose the contrary: there exist at least two different non-trivial integral planes in $T_x\Lambda$. Or equivalently, equation (5.1) in η has more than one solution. Recall that $t_j, l_j \neq 0$. Therefore, $l_4 = -l_1$. Hence, $l_3 = -l_2$, since $l_1 l_3 = l_2 l_4$. The first equality $l_4 = |A_1A_4| = -l_1 = -|A_1A_2|$ implies that the points A_2, A_4 are symmetric with respect to the line L_{A_1}, by length concordance, see Definition 2.13. Similarly, the points A_2, A_4 are symmetric with respect to the line L_{A_3}. Finally, the quadrilateral $A_1A_2A_3A_4$ is symmetric with respect to the line $L_{A_1} = L_{A_3}$ and hence, $x \in \Lambda_0$, – a contradiction.

Case 2): $\eta = 0$. We similarly get that $x \in \Lambda_1$, – a contradiction. Lemma 5.17 is proved. □

Remark 5.18 As it was shown in loc.cit., an integral plane $E_2 \subset T_x\Lambda$ may exist only for x from an algebraic subset in Λ. For example, in the case, when $\eta' = 0$, it exists only if $t_1(x) = t_3(x)$.

Remark 5.19 The distributions $\mathcal{D}^4_+|_{\Lambda_j}$, $j = 0, 1$, are three-dimensional. One can easily show that the restriction to Λ_j of the Pfaffian system $\mathcal{D}^{4,2}_+$ is involutive. Each its complex integral surface is an open set of quadrilateral orbits of a 4-reflective complex billiard of type 1): if, say, $j = 0$, then $a_1 = a_3$ is a line, the curves a_2 and a_4 are symmetric with respect to it. Similar statement holds for smooth integral surfaces.

Corollary 5.20 *Let $\alpha \in \Psi^+_4$, $S \subset \mathbb{R}^8 \setminus \Sigma$ be a non-trivial C^3-smooth integral surface of the distribution $\mathcal{D}^4_\alpha$. Let in addition $S \cap j^{-1}_\alpha(\Lambda_0 \cap \Lambda_1) = \emptyset$. Then the set of analyticity points of the surface S is open and dense in S.*

Proof The image $j_\alpha(S) \subset \mathcal{R}^+_{0,4}$ is a totally real integral surface of the complex distribution $\mathcal{D}^4_+$. The subset $(S \setminus j^{-1}_\alpha(\Lambda)) \cup Int(S \cap j^{-1}_\alpha(\Lambda)) \subset S$ is open and dense. Hence, it suffices to prove the corollary in each one of the two following separate cases: $j_\alpha(S) \cap \Lambda = \emptyset$; $j_\alpha(S) \subset \Lambda \setminus (\Lambda_0 \cup \Lambda_1)$.

Case 1): $j_\alpha(S) \cap \Lambda = \emptyset$. The complex span of each tangent plane to $j_\alpha(S)$ is a complex integral plane of the distribution $\mathcal{D}^4_+$. Therefore, $j_\alpha(S)$

lifts to a totally real C^2-smooth integral surface $\widetilde{S} \subset \mathcal{J}_2$ of the distribution $\widetilde{E}$ from Lemma 5.16. Let $M \subset \mathcal{J}_2$ denote the minimal complex analytic subset containing $\widetilde{S}$. The restriction to M of the distribution $\widetilde{E}$ is two-dimensional and integrable (cf. Proposition 2.38). Without loss of generality we can and will assume that $\widetilde{S}$ is contained in the regular part of the analytic set M. Then $\widetilde{S}$ is contained in a complex analytic integral surface of the distribution $\widetilde{E}$ on M and coincides with its intersection with the real part of the variety $\mathcal{I}_2$. Therefore, $\widetilde{S}$ is real analytic, and hence, so is S.

Case 2): $j_\alpha(S) \subset \Lambda \setminus (\Lambda_0 \cap \Lambda_1)$. Let $M \subset \Lambda \setminus (\Lambda_0 \cup \Lambda_1)$ denote the minimal complex analytic subset that contains $j_\alpha(S)$. Recall that for every $x \in \Lambda \setminus (\Lambda_0 \cup \Lambda_1)$ there exists at most one non-trivial integral plane at x of the restriction $\mathcal{D}^4_+|_\Lambda$ (Lemma 5.17). The union of the latter integral planes is an analytic subset in $Gr_2(T\mathcal{R}^+_{0,4})|_{\Lambda\setminus(\Lambda_0\cup\Lambda_1)}$ with proper projection to $\Lambda \setminus (\Lambda_0 \cup \Lambda_1)$. Its image under the latter projection contains M, since it contains $j_\alpha(S)$ and is analytic (Proper Mapping Theorem). Therefore, for every $y \in M$ there exists a unique integral plane in $\mathcal{D}^4_+|_\Lambda(y)$. The restriction to M of the singular distribution formed by the latter planes is two-dimensional and integrable (cf. Proposition 2.38). Afterwards without loss of generality we assume that $j_\alpha(S)$ lies in the regular part of the set M and deduce that S is analytic, as in the above case. Corollary 5.20 is proved. □

Proof of Theorem 5.6, C^4-smooth case. Fix a germ of C^4-smooth 4-reflective pseudo-billiard a_1, a_2, a_3, a_4. It has an open set S of quadrilateral orbits, which is a germ of C^3-smooth non-trivial integral surface of a real distribution $\mathcal{D}^4_\alpha$ at a point $p \in \mathbb{R}^8 \setminus \Sigma$. One has $\alpha \in \Psi^+_4$, by Proposition 5.9.

Case 1): the complement $S \setminus j^{-1}_\alpha(\Lambda_0 \cup \Lambda_1)$ is non-empty. Let us show that S is analytic at p. This will imply that the above pseudo-billiard is of type either 2), or 3) (Theorem 5.6, analytic case). Suppose the contrary. Then the open subset of analyticity points in S contains a connected component $W \not\subset j^{-1}_\alpha(\Lambda_0 \cup \Lambda_1)$ with non-empty boundary, by Corollary 5.20. Fix a boundary point $x \in \partial W$ and a path $\psi \subset W$ going to x such that for every j one has $A_j(y) \neq A_j(x)$ for $y \in \psi$ arbitrarily close to x. The surface S is not analytic at x. The subset W is an open set of quadrilateral orbits of an analytic pseudo-billiard. The latter is not of type 1), since $W \not\subset j^{-1}_\alpha(\Lambda_0 \cup \Lambda_1)$. Hence, it is of type either 2) or 3): its mirrors are either all lines, or all confocal conics. Thus, the mirrors a_j are analytic at $A_j(y)$, $y \in \psi$ and some of them, say a_2 is not analytic at $A_2(x) = \lim_{y\to x; y\in\psi} A_2(y)$. Let us show that this is impossible. To do this, let us fix a $y \in \psi$ close to x with $A_1(y) \neq A_1(x)$. Consider the smooth deformation of a quadrilateral orbit

$A_1(y)A_2(y)A_3(y)A_4(y)$ with fixed A_1. That is, the family of quadrilateral orbits $Q(s) = A_1^s A_2^s A_3^s A_4^s$ depending on the angle parameter $s = \angle A_2^s A_1^s A_4^s$,

$$A_1^s = A_1(y),\ s_0 = \angle A_2(y)A_1(y)A_4(y),\ A_j^{s_0} = A_j(y).$$

Let (s_-, s_+) be the maximal interval of analyticity of the family of quadrilaterals $Q(s)$ as a function of s. For some of $s_\pm$, say s_+, some vertices A_j^{s+} should be singular points of the corresponding mirrors and coincide with $A_j(x)$, by definition. At least two points A_j^{s+} should be singular, see [10, lemma 41 and its proof, pp. 305–306]. If two neighbor vertices are singular, e.g., $A_2^{s+} = A_2(x)$, $A_3^{s+} = A_3(x)$, then $A_1(y) = A_1(x)$: $A_1(y)$ is the point of intersection close to $A_1(x)$ of the curve a_1 with the line $A_1(x)A_2(x)$ symmetric to $A_2(x)A_3(x)$ with respect to the line $T_{A_2(x)}a_2$. This contradicts the assumption $A_1(y) \neq A_1(x)$. The contradiction thus obtained shows that $A_2^{s+} = A_2(x)$ and $A_4^{s+} = A_4(x)$ are singular.

Let $a_j^0 \subset a_j$ denote the arcs saturated by the vertices $A_j(y)$, $y \in \psi$. Recall that the arcs a_j^0 are analytic, and they are either all lines, or all confocal conics, see the above discussion. The above statement implies that for $A = A(y) \in a_1^0$ the lines $AA_2(x)$, $AA_4(x)$ are symmetric with respect to the line $T_A a_1^0$. Hence, this is true for every $A \in a_1^0$, by analyticity. Therefore, a_1^0 is either the symmetry line between the points $A_2(x)$, $A_4(x)$, or a conic with foci at them [7, proposition 2.32]. In the case, when a_j^0 are confocal conics, the conic a_2^0 would contain its own focus $A_2(x)$, – a contradiction. In the case, when they are lines, they should intersect at one point (the pseudo-billiard is of type 2)). This implies that the lines a_2^0, a_4^0 are symmetric with respect to the line a_1^0, as are $A_2(x)$, $A_4(x)$, and hence, the pseudo-billiard is of type 1), – a contradiction. Hence, S is analytic.

Case 2): $j_\alpha(S) \subset \Lambda_j$ for some $j = 0, 1$. Then we obviously get a pseudo-billiard of type 1) (see Remark 5.19).

Case 3): $j_\alpha(S) \subset \Lambda_0 \cup \Lambda_1$, $j_\alpha(p) \in \Lambda_0 \cap \Lambda_1$ and both complements $S \setminus j_\alpha^{-1}(\Lambda_j)$, $j = 0, 1$, are non-empty. Let us show that this case is impossible. Suppose the contrary: the latter assumptions hold. Then p corresponds to a rhombus with interior bisectors. Some of germs of mirrors a_j, $j = 2, 4$ at the points $A_j(p)$ is not a line, since otherwise, $a_2 = a_4$ and we obviously get that $j_\alpha(S) \in \Lambda_1$, – a contradiction. The similar statement holds for the mirrors a_1 and a_3. Finally, some two germs of neighbor mirrors, say a_1 and a_2 are not lines. The billiard under question being 4-reflective, the surface S contains a point q corresponding to a 4-reflective orbit $A_1(q)A_2(q)A_3(q)A_4(q)$ with $A_1(q) \in a_1$, $A_2(q) \in a_2$ being points of non-zero curvature with non-orthogonal tangent lines. Hence, the latter quadrilateral is not a rhombus and thus, $j_\alpha(q)$ is contained in only one set Λ_j, $j = 0, 1$. Hence, the germ at q of the surface S should consist of quadrilateral orbits of a pseudo-billiard of type 1) (see Case 2)). Thus, at least one of the neighbor mirror germs $(a_1, A_1(q))$, $(a_2, A_2(q))$ should be a line, while they both have non-zero curvature. The contradiction thus obtained proves Theorem 5.6. $\square$

Proof of Theorem 5.8. Suppose the contrary: there exists a C^4-smooth pseudo-billiard where only some two neighbor mirrors, say a_1, a_2 have skew reflection law and the set of 4-periodic orbits has positive Lebesgue measure. In more detail, consider the set S of its 3-edge orbits $A_1A_2A_3A_4$: the reflection law is required only at A_2, A_3 but not necessarily at A_1, A_4. This is a two-dimensional non-trivial C^3-smooth surface, which contains a positive measure set of 4-periodic orbits. Thus, S is a C^3-smooth pseudo-integral surface (in terms of [10, definition 13, p.291]) of the analytic distribution $\mathcal{D}^4_\alpha$, $\alpha = (-1, -1, 1, 1)$. Note that $j_\alpha^{-1}(\Lambda) = \emptyset$. Indeed, for every quadrilateral $A_1A_2A_3A_4 \in \mathbb{R}^8 \setminus \Sigma$ consider the concordant lengths $l_j = |A_jA_{j+1}|$ with respect to the interior bisectors at A_1, A_2 and the exterior bisectors at A_3, A_4. The lengths l_j, $j = 2, 3, 4$ have the same signs, while l_1 has opposite sign, see Example 2.15. Therefore, the equality $l_1l_3 = l_2l_4$ defining Λ is impossible. Hence, $S \cap j_\alpha^{-1}(\Lambda) = \emptyset$. This together with Lemma 5.16 and [10, theorem 28, p.295] implies that $\mathcal{D}^4_\alpha$ has an analytic non-trivial integral surface. Therefore, there exists an analytic 4-reflective pseudo-billiard with exactly two skew reflection laws at some neighbor mirrors, – a contradiction to the last statement of Theorem 5.6. Theorem 5.8 is proved. $\square$

5.3 Application 1: Tabachnikov's Commuting Billiard Conjecture

The following theorem solves the piecewise C^4-smooth case of S.Tabachnikov's conjecture on commuting convex planar billiards [25, p.58]. It deals with two billiards in nested convex compact domains $\Omega_1 \Subset \Omega_2 \Subset \mathbb{R}^2$, set $a = \partial\Omega_1$, $b = \partial\Omega_2$. We consider that both a and b are piecewise C^4-smooth. For every Ω_j consider the corresponding billiard transformation acting on the space of oriented lines in the plane. It acts as identity on the lines disjoint from Ω_j. Each oriented line l intersecting Ω_j is sent to its image under the reflection from the boundary $\partial\Omega_j$ at its last intersection point x with $\partial\Omega_j$ in the sense of orientation: the orienting arrow of the line l at x is directed outside Ω_j. The reflected line is oriented by a tangent vector at x directed inside Ω_j. This is a continuous dynamical system if the boundary $\partial\Omega_j$ is smooth and piecewise-continuous (measurable) otherwise.

It is known that confocal elliptic billiards commute [24, p.49, corollary 4.6]. The next theorem shows that the converse is also true.

Theorem 5.21 *Let two nested planar convex piecewise C^4-smooth Jordan curves be such that the corresponding billiard transformations commute. Then they are confocal ellipses.*

In the proof of Theorem 5.21 we use the next commutativity criterion.

Proposition 5.22 *Let $a, b \subset \mathbb{R}^2$ be nested convex Jordan curves, as at the beginning of the subsection. The corresponding billiard transformations commute, if and only if each pair $(A, B) \in a \times b$ extends to a quadrilateral orbit $ABCD$ of the pseudo-billiard a, b, a, b as at Fig.9: the reflection law is usual at b and skew at a; only one of the segments AB, BC intersects the domain bounded by the curve a, if both ambient lines intersect it.*

The proposition follows from definition.

Proof of Theorem 5.21. The interior of the set of quadrilateral orbits from the proposition contains an open and dense subset of those quadrilaterals $ABCD$ for which the germs of curves a, b at the vertices are C^4-smooth and form a 4-reflective pseudo-billiard. The latter has thus some of types 1)–3) from Theorem 5.6. It cannot be of type 1) with germs (b, B), (b, D) lying on the same line and (a, A), (a, C) being symmetric with respect to it (see Addendum 1). This is impossible by convexity and the obvious inclusion $ABCD \subset \overline{\Omega_2}$. Therefore, either (a, A), (a, C) lie in the same line and the pseudo-billiard is of type 1), or its has type 2) or 3). This implies that the curve a contains an open dense union of analytic pieces, each of them being either a line segment, or an arc of conic. In what follows the reflections from the curves a, b acting on the space of oriented lines will be denoted by σ_a and σ_b respectively.

Case 1): some analytic arc a' of the curve a is a conic. Then the above germs of pseudo-billiards with $A \in a'$ have type 3). Thus, the curve b contains an open dense union of analytic arcs lying in conics confocal to a'.

Subcase 1a): the curve $a = a'$ is analytic, and hence, is an ellipse. Note that the tangent lines to b are disjoint from a and hence, from the focal segment, by convexity. This implies that each conical piece of the curve b is not a hyperbola, and thus, is an ellipse. Recall that b is a finite union of C^4-smooth arcs. Each smooth arc b' is elliptic. Indeed, consider an auxiliary function on $\mathbb{R}^2$: the sum of distances of a variable point to the foci of a'. It is constant on b': it is smooth on b' and locally constant on an open dense union of confocal elliptic subarcs; hence it has zero derivative on b', and b' is elliptic. Thus, b is a finite chain of adjacent confocal elliptic arcs. Therefore b is an ellipse: the above sum of distances is continuous on b and constant on each arc, and hence, on all of b.

Subcase 1b): some boundary point A of the analytic arc a' is singular, the curve a is not analytic at A. Fix A and a subarc b' of a conical arc of the curve b such that for every $B \in b'$ the interval (A, B) intersects a, the line AB is distinct from the lines $T_A a'$, $T_A^{\perp} a'$, and the line $\Lambda = \Lambda(B)$ symmetric to AB with respect to $T_A a$ intersects the curve b at its analyticity points. Let us orient the line $L = AB$ from B to A. Consider the one-parametric analytic family of lines $L^* = L^*(B) = \sigma_b^{-1}(L)$. We orient the line Λ so that $\Lambda = \sigma_a(L)$. We claim that all the lines L^* pass through the same point of the curve a. Indeed, the mapping $\sigma_a \circ \sigma_b = \sigma_b \circ \sigma_a$ is singular at each L^*, since $\Lambda = \sigma_a(L) = \sigma_a \circ \sigma_b(L^*)$, σ_b is regular (that is, a local analytic diffeomorphism) at L^* and σ_a is singular at $L = \sigma_b(L^*)$, as is A. On the other hand, σ_b is regular at $\sigma_a(L^*)$, since $\Lambda = \sigma_b \circ \sigma_a(L^*)$ and σ_b^{-1} is a local analytic diffeomorphism at Λ: the points of intersection $\Lambda \cap b$ are regular and transversal (by convexity). Therefore, σ_a is singular at L^*, as is $\sigma_b \circ \sigma_a$. Hence, either all the lines L^* pass through the same singular point A' of the curve a (the set of its singular points is totally disconnected), or they are tangent to a. In the latter case one has $\sigma_b \circ \sigma_a(L^*) = \sigma_b(L^*) = L$, while $\sigma_a \circ \sigma_b(L^*) = \sigma_a(L) \neq L$, since $L \neq T_A a', (T_A a')^{\perp}$, – a contradiction to commutativity. Hence, the tangency case is impossible, and the lines L^* pass through the same point A'. Therefore, b' is a conical arc confocal to a' with foci A, A'. Thus, the conic a' contains its own focus A, – a contradiction.

Case 2): the curve a is a convex polygon. For every its vertex A each line through A intersecting a C^4-smooth arc b' of the curve b is reflected from b' to a line through another vertex A' of the polygon a, as in the above discussion. This implies that each C^4-smooth arc b' is a conic with foci A, A'. Finally, each vertex of the polygon a is a focus of each C^4-smooth arc of the curve b. Therefore, a has at most two vertices and cannot be a convex polygon. The contradiction thus obtained proves Theorem 5.21. □

5.4 Application 2: planar Plakhov's Invisibility Conjecture with four reflections

This subsection is devoted to Plakhov's Invisibility Conjecture: the analogue of Ivrii's conjecture in the invisibility theory [17, conjecture 8.2]. We recall it below and show that in planar case it follows from a conjecture saying that no finite collection of germs of smooth curves can form a measure k-reflective billiard for appropriate "invisibility" reflection law. Both Plakhov's and

Ivrii's conjectures have the same complexification, see [8, subsection 5.2, proposition 8]. We prove the C^4-smooth case of planar Plakhov's Invisibility Conjecture for four reflections as an immediate corollary of Theorem 5.8.

Definition 5.23 [17, chapter 8] Consider a perfectly reflecting (may be disconnected) closed bounded body B in a Euclidean space. For every oriented line (ray) R take its first intersection point A_1 with the boundary ∂B and reflect R from the tangent hyperplane $T_{A_1}\partial B$. The reflected ray goes from the point A_1 and defines a new oriented line, as in the previous subsection. Then we repeat this procedure. Let us assume that after a finite number k of reflections the output oriented line coincides with the input line R and will not hit the body any more. Then we say that the body B *is invisible for the ray R,* see Fig.13. We call R a *ray of invisibility* with k reflections.

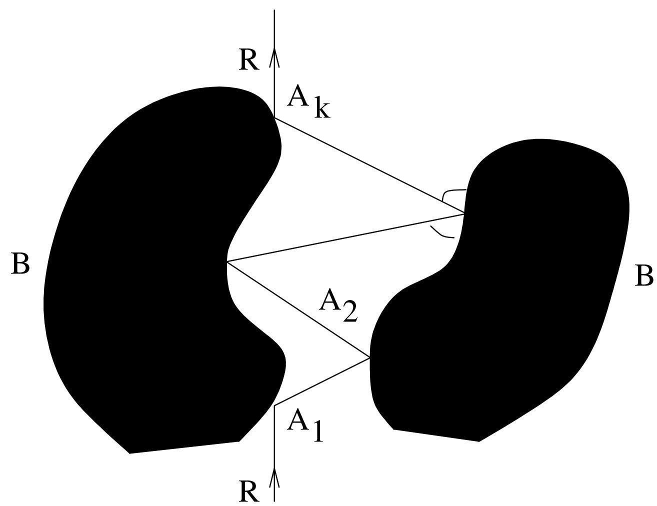

Figure 13: A body invisible for one ray.

Invisibility Conjecture (A.Plakhov, [17, conjecture 8.2, p.274].) *There exist no body with piecewise C^∞ boundary for which the set of rays of invisibility has positive measure.*

Remark 5.24 As is shown by A.Plakhov in his book [17, section 8], there exist no body invisible for all rays. The same book contains a very nice survey on invisibility, including examples of bodies invisible in a finite number of (one-dimensional families of) rays. See also papers [1, 18, 19, 20] for more results. The Invisibility Conjecture is equivalent to the statement saying that for every $k \in \mathbb{N}$ there are no measure k-reflective bodies, see the next definition. It is open even in dimension 2.

Definition 5.25 (cf. [8, subsection 5.2, definition 12]) A body B with piecewise-smooth boundary is called *measure k-reflective,* if the set of invisibility rays with k reflections has positive measure.

Definition 5.26 (cf. [8, subsection 5.2, definition 13]) A (germ of) real planar smooth pseudo-billiard $a_1, \dots, a_k$ is called *measure k-invisible*, if it is measure k-reflective for skew reflection law at a_1, a_k and usual law at the other mirrors a_j: the set of its k-periodic orbits for the above reflection law (called *k-invisible* orbits, see Fig.14) has positive Lebesgue measure.

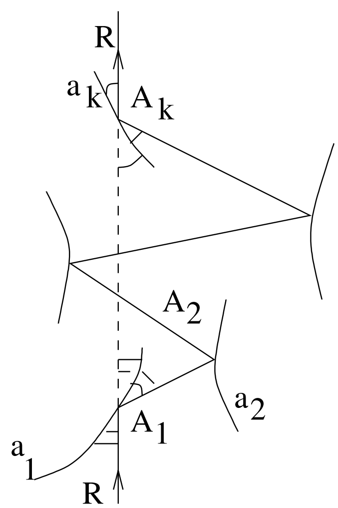

Figure 14: A k-invisible k-gon: skew reflection law at A_1 and A_k.

Proposition 5.27 *Let $k \in \mathbb{N}$ and $B \subset \mathbb{R}^2$ be a body with piecewise-smooth boundary, and no collection of k germs of its boundary form a measure k-invisible smooth pseudo-billiard. Then B is not measure k-reflective.*

Proposition 5.27 is implicitly contained in [17, section 8].

Theorem 5.28 *There are no measure 4-reflective bodies in $\mathbb{R}^2$ with piecewise C^4-smooth boundary.*

Proof The existence of a measure 4-reflective body as above implies the existence of a measure 4-invisible planar C^4-smooth pseudo-billiard (Proposition 5.27). This is a measure 4-reflective planar C^4-smooth pseudo-billiard with skew reflection law at some pair of neighbor vertices and usual reflection law at the other vertices. This contradicts Theorem 5.8. $\square$

6 General case of complex k-reflective billiards: state of art

First let us recall the next conjecture and partial positive results from [8].

Conjecture A [8, p.295]. *There are no k-reflective complex analytic (algebraic) planar billiards for odd k.*

Theorem 6.1 *[8, p.295]. There are no 3-reflective complex analytic planar billiards.*

Theorem 6.2 *[8, p.295]. For every odd k there are no k-reflective complex algebraic planar billiards whose mirrors avoid isotropic points at infinity.*

Conjecture B. *For every $k \geq 3$ there are no k-reflective complex analytic planar billiards $a, \dots, a$ with all the mirrors coinciding with the same irreducible analytic curve $a \subset \mathbb{CP}^2$.*

Recently a positive result was proved by the author (paper in preparation) for every irreducible algebraic curve a that either is smooth, or satisfies a mild condition on either singularities, or tangential correspondence.

Definition 6.3 A *combination* of complex analytic billiards $\alpha = (a_1, \dots, a_l)$, $\beta = (b_1, \dots, b_m)$ is a billiard $\alpha \circ_s \beta = (a_1, \dots, a_s, b_1, \dots, b_m, a_{s+1}, \dots, a_l)$, $s \in \{1, \dots, l\}$. For every collection of analytic curves $\delta = (d_1, \dots, d_t)$ in $\mathbb{CP}^2$ distinct from isotropic lines the billiard

$$\alpha \circ_{s,\delta} \beta = (a_1, \dots, a_s, d_1, \dots, d_t, b_1, \dots, b_m, d_t, \dots, d_1, a_{s+1}, \dots, a_l)$$

will be called a *combination with mirror adding* of the billiards α and β. (The previous combination corresponds to $\delta = \emptyset$.)

Definition 6.4 Let $\alpha = (a_1, \dots, a_l)$, $\beta = (b_1, \dots, b_m)$ be complex planar billiards such that for some $1 \leq s < \min\{l, m\}$, one has $a_j = b_{m-j+1}$ for $j = 1, \dots, s$. Then the billiard $\alpha \circ_{[1,s]} \beta = (a_{s+1}, \dots, a_l, b_1, \dots, b_{m-s})$ is called a *combination with mirror erasing* of the billiards α, β.

Remark 6.5 For every l- and m-reflective billiards α, β and s, t, δ as in Definition 6.3 the billiard $\alpha \circ_{s,\delta} \beta$ is $(l+m+2t)$-reflective, provided that

(E) it has a periodic orbit $A_1 \dots A_s D_1 \dots D_t B_1 \dots B_m D_t \dots D_1 A_{s+1} \dots A_l$ for which $A_1 \dots A_l$ and $B_1 \dots B_m$ are respectively l- and m-reflective orbits of the billiards α and β: interior points of the set of l- (m-) periodic orbits.

For every l- and m-reflective billiards $\alpha = (a_1, \dots, a_l)$, $\beta = (b_1, \dots, b_m)$ and $1 \leq s < \min\{l, m\}$ the billiard $\alpha \circ_{[1,s]} \beta$ is $(l+m-2s)$-reflective, if

(E') there exist l- and m-reflective orbits $A_1 \dots A_l$, $B_1 \dots B_m$ of billiards α and β respectively such that $A_j = B_{m-j+1}$ for $1 \leq j \leq s$ and $A_{s+1} \dots A_l B_1 \dots B_{m-s}$ is a periodic orbit of the combination $\alpha \circ_{[1,s]} \beta$.

Known k-reflective complex analytic planar billiards.

I. 4-reflective billiards of types 1)–3) from Theorem 1.7.

II. Billiards that are obtained from them by subsequent combinations (with mirror adding or erasing); each subsequent combination should satisfy the above condition (E) (respectively, (E')).

Example 6.6 The billiards of type II include the following ones:

- Every billiard $\alpha_l = (a, b_{l-1}, \dots, b_1, a, b_1^*, \dots, b_{l-1}^*)$, where a is a line, b_j, b_j^* are symmetric with respect to the line a, that has at least one symmetric $2l$-periodic orbit. Its $2l$-reflectivity is obvious. It is obtained by subsequent combinations with mirror erasing of 4-reflective billiards of type 1): on each step we combine billiards α_j and $\beta_j = (b_j^*, a, b_j, a)$ erasing one mirror a, which is the last mirror a in β_j identified with the first one in α_j.

- Every billiard formed by an even number of complex confocal conics, some of them coincide and each conic is taken even number of times; no two neighbor mirrors coincide. It is obtained by subsequent combinations (usual ones and those with mirror erasing) of 4-reflective billiards of type 3).

- Every billiard formed by an even number of non-isotropic complex lines such that the product of the corresponding symmetries is the identity. It is obtained by subsequent combinations (usual ones and those with mirror erasing) of 4-reflective billiards of type 2). This easily follows from the complexification of [26, theorem 1.B, p.3].

The next small technical results, which generalize Corollary 3.4, might be useful in studying the general case. They are immediate consequence of the results of Section 3. To state them, consider a complex k-reflective billiard $a_1, \dots, a_k$. For every $j = 1, \dots, k$ let us introduce the corresponding space of $(k-2)$- orbits: collections $A_1 \dots A_{j-1} A_{j+2} \dots A_k \in \prod_{i \neq j, j+1} \hat{a}_i$, $A_i \in \hat{a}_i$, such that for every $i \neq j-1, j, j+1, j+2$ one has $A_i \neq A_{i\pm1}$, the lines $A_i A_{i\pm1}$ are symmetric with respect to the line $T_{A_i} a_i$ and the three latter lines are distinct and non-isotropic. (In the case, when $j = k$, we replace $j+s$ by s, $s = 1, 2$.) The closure of set of the latter $(k-2)$- orbits is an analytic subset $V_j \subset \prod_{i \neq j, j+1} \hat{a}_i$ that has only two-dimensional irreducible components. Let $U \subset \hat{a}_1 \times \cdots \times \hat{a}_k$ denote the k-reflective set, see Subsection 2.3. For every $j = 1, \dots, k$ let P_j denote the product projection

$$P_j : \hat{a}_1 \times \cdots \times \hat{a}_k \to \prod_{r \neq j, j+1} \hat{a}_r, \ U_j = P_j(U) \subset V_j.$$

Theorem 6.7 *Let $a_1, \dots, a_k$ be a k-reflective complex analytic planar billiard. Let $U \subset \hat{a}_1 \times \cdots \times \hat{a}_k$ be its k-reflective set. For every $j = 1, \dots, k$ the set $U_j = P_j(U)$ is analytic: a union of irreducible components of the set V_j. The projection $P_j : U \to U_j$ is proper and bimeromorphic.*

Proof Without loss of generality we prove the theorem for the pair of neighbor indices 1, k: $j = k$.

1) Properness and analyticity. Let us show that the mapping $P_k : U \to V_k$ is proper: then U_k is analytic by Proper Mapping Theorem.

1a) Case, when a_1, a_k are algebraic: the above statements are obvious.

1b) Case, when some of them, say a_k is not algebraic. The properness will be deduced from Corollary 3.2. To do this, we consider the space

$$\mathcal{P}_{1,k} = \hat{a}_2 \times \dots \hat{a}_{k-1} \times \mathcal{P}^2$$

equipped with the distribution

$$\mathcal{H}_{1,k} = T\hat{a}_2 \oplus \cdots \oplus T\hat{a}_{k-1} \oplus \mathcal{H} \oplus \mathcal{H},$$

where $\mathcal{P} = \mathbb{P}(T\mathbb{CP}^2)$, $\mathcal{H}$ is the standard contact plane field on $\mathcal{P}$, see Subsection 2.7. A point of the space $\mathcal{P}_{1,k}$ is a triple $A_2 \dots A_{k-1}, (A_1, L_{A_1}), (A_k, L_{A_k})$, where $L_{A_j} \subset T_{A_j}\mathbb{CP}^2$ is a one-dimensional subspace. Let $\mathcal{R}_{1,k} \subset \mathcal{P}_{1,k}$ denote the analytic variety defined by the conditions that for every $j = 1, \dots, k$ one has $A_j \in \mathbb{C}^2 = \mathbb{CP}^2 \setminus \overline{\mathbb{C}}_\infty$, $A_j \neq A_{j\pm1}$, the lines $A_jA_{j\pm1}$ are symmetric with respect to the line $T_{A_j}a_j$ if $2 \leq j \leq k-1$ (the line L_{A_j} if $j \in \{1, k\}$), the three lines $A_jA_{j\pm1}$, $T_{A_j}a_j$ (respectively, L_{A_j}) are distinct and non-isotropic. In addition, it is required that A_j be not cusps of the curves a_j for $j \neq 1, k$. The set $\mathcal{R}_{1,k}$ is an analytically constructible smooth variety, and its closure $\overline{\mathcal{R}_{1,k}}$ is an analytic set. The distribution $\mathcal{H}_{1,k}$ induces a two-dimensional analytic distribution $\mathcal{D}_{1,k}$ on $\mathcal{R}_{1,k}$ that extends to a singular analytic distribution on $\overline{\mathcal{R}_{1,k}}$. An open dense subset $U_1 \subset U$ lifts to the union of non-trivial integral surfaces of the distribution $\mathcal{D}_{1,k}$, as in Proposition 2.43. Let $M \subset \mathcal{P}_{1,k}$

denote the minimal analytic subset containing all the non-trivial integral surfaces.

Suppose the contrary: the mapping $P_k : U \to V_k$ is not proper. Then $dimM \geq 3$: if $dimM = 2$, then P_k is proper, as in the proof of Corollary 3.4. In what follows we fix some at least three-dimensional irreducible component of the set M and denote by M the latter component. The restriction $\mathcal{D}_M$ to M of the distribution $\mathcal{D}_{1,k}$ is two-dimensional and integrable, by Proposition 2.38. As in Subsection 3.1, there exists an analytic subset $\Sigma \subset M$ with dense complement $M^0 = M \setminus \Sigma \subset M$ such that $\mathcal{D}_M$ is analytic on M^0 and its integral surface through each $x \in M^0$ represents an open set of k-periodic orbits of a k-reflective billiard $a_1(x)$, $a_2, \dots, a_{k-1}$, $a_k(x)$. Some integral surface, which we will denote by S, represents a family of k-periodic orbits of the initial billiard. The projection $\mu_{1,k} : M \to \hat{a}_2 \times \dots \times \hat{a}_{k-1}$ is proper, and the image $N = \mu_{1,k}(M^0) \subset V_k$ is a purely two-dimensional analytically constructible subset, by Chevalley–Remmert Theorem.

Claim 1. *There exists an open subset of points $x \in M^0$ such that the billiard a_1, $a_1(x)$, $a_k(x)$, a_k is 4-reflective.*

Proof Fix $x_1 \in S$ and $x_2 \in M^0 \setminus \{x_1\}$ with $p = \mu_{1,k}(x_1) = \mu_{1,k}(x_2)$. There exist neighborhoods $Y = Y(p) \subset N$, $X_1 = X_1(p) \subset S$, $X_2 = X_2(x_2) \subset M^0 \setminus X_1$ such that $\mu_{1,k}$ projects X_1 and each integral surface of the distribution $\mathcal{D}_M|_{X_2}$ diffeomorphically onto Y (let us fix them). Fix an arbitrary $x \in X_2$, let $\widetilde{S}(x)$ denote the integral surface of the distribution $\mathcal{D}_M|_{X_2}$ through x. Each $y = A_2 \dots A_{k-1} \in Y$ lifts to two points in X_1 and $\widetilde{S}(x)$, which correspond to k-periodic orbits $A_1 \dots A_k$ and $A_1' A_2 \dots A_{k-1} A_k'$ of the billiards $a_1, \dots, a_k$ and $a_1(x), a_2, \dots, a_{k-1}, a_k(x)$ respectively. The quadrilaterals $A_1 A_1' A_k' A_k$ corresponding to generic $y \in Y$ form a two-parametric family of 4-periodic orbits of the billiard $a_1, a_1(x), a_k(x), a_k$, and the latter is 4-reflective, as in the proof of Proposition 3.8. □

The above claim yields at least one-dimensional family of 4-reflective billiards with two fixed neighbor mirrors a_1, a_k, since $dimM \geq 3$. The curves a_1, a_k are not both algebraic, by assumption. This contradicts Corollary 3.2 (or Theorem 1.7) and thus, proves properness of the projection $P_k : U \to V_k$.

2) Bimeromorphicity. Suppose the contrary: the proper analytic set projection $P_k : U \to U_k = P_k(U)$ is not bimeromorphic. This means that its inverse has at least two distinct holomorphic branches on an open subset in the analytic set U_k. In other words, each $A_2 \dots A_{k-1}$ from an open subset in V_k extends to two distinct k-periodic orbits $A_1 A_2 \dots A_{k-1} A_k$, $A_1' A_2 \dots A_{k-1} A_k'$. Then the two-dimensional family of quadrilaterals $A_1 A_1' A_k' A_k$ (depending on generic $A_2 \dots A_{k-1}$) are 4-periodic orbits of the billiard a_1,

a_1, a_k, a_k with coinciding neighbor mirrors, as in [7, proof of lemma 3.1]. Hence, the latter billiard is 4-reflective, – a contradiction to Corollary 3.5 (or Theorem 1.7) forbidding 4-reflective billiards with coinciding neighbor mirrors. Theorem 6.7 is proved. □

Corollary 6.8 *Let $a_1, \dots, a_k$ be a k-reflective complex analytic planar billiard. The subsequent $(k-2)$- orbit correspondence $P_{j+1} \circ P_j^{-1} : U_j \to U_{j+1}$, $A_1 \dots A_{j-1} A_{j+2} \dots A_k \mapsto A_1 \dots A_j A_{j+3} \dots A_k$ is bimeromorphic. Its graph is projected epimorphically onto both U_j and U_{j+1}. If it contracts a curve, then the latter is compact and the mirror a_{j+2} is algebraic.*

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