Gibbs partitions: the convergent case

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Abstract

We study Gibbs partitions that typically form a unique giant component. The remainder is shown to converge in total variation toward a Boltzmann-distributed limit structure. We demonstrate how this setting encompasses arbitrary weighted assemblies of tree-like combinatorial structures. As an application, we establish smooth growth along lattices for small block-stable classes of graphs. Random graphs with n vertices from such classes are shown to form a giant connected component. The small fragments may converge toward different Poisson Boltzmann limit graphs, depending along which lattice we let n tend to infinity. Since proper addable minor-closed classes of graphs belong to the more general family of small block-stable classes, this recovers and generalizes results by McDiarmid (2009).

1 Introduction

The motivation for the present work stems from various areas, starting with enumerative combinatorics. It was conjectured by Bernardi, Noy and Welsh [5] that proper minor-closed addable classes have smooth growth. Such a condition crops up in related contexts, for example, in McDiarmid, Steger and Welsh [24, 25]. The conjecture was confirmed by McDiarmid [23]. One of the methods used is an approach used in Bender, Canfield and Richmond [4], who proved smoothness for classes of graphs embeddable on any fixed surface. It was established in [23] furthermore that random graphs from proper minor-closed addable classes typically admit a giant component, and that the remaining fragments converge in total variation toward a limit called the Boltzmann Poisson random graph of the class. Such a behaviour had previously been observed for random planar graphs by McDiarmid [22]. The enumerative study of minor-closed classes and related classes of graphs has since then received growing attention in the literature, see Noy [28] for a comprehensive survey.

A link from this topic to general models of random partitions can be found in the work by Barbour and Granovsky [2]. The authors use a perturbed Stein recursion approach to study the asymptotic behaviour of random partitions satisfying a conditioning relation. Various regimes with differing behaviour are known for this general model of partitions [1], and the "convergent case" setting of [2] is characterized by exhibiting a giant component, whose remainder converges toward an almost surely finite limit. It is an interesting observation, that the distribution of the remainder in [2] belongs to a

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family encompassing the components of the Boltzmann Poisson random graph constructed by McDiarmid [23], and only natural, to check whether these results on random partitions can be applied to random graphs. Clearly great care was taken in [2] to use only a minimum set of requirements, but too little is known apart from smoothness about the asymptotic number of graphs in an arbitrary proper addable minor-closed classes.

As it is not clear whether these results apply, other options have to be considered. Apart from random partitions that are characterized by a conditioning relation, there is another well-known model that encompasses the component distribution of random graphs from proper addable minor-closed classes: Gibbs partitions. Both families of random structures are quite general and have a non-trivial intersection, but neither contains the other. The term was coined by Pitman [29] in his comprehensive survey on combinatorial stochastic processes, and since then further important additions to the theory were made [11]. Gourdon [15, Thm. 1] gave results in a specific setting, where a giant component emerges, and the size of the remainder converges in distribution. He required the exponential generating function of the structures on the components to be amendable to singularity analysis [12, Thm. 1], such that its coefficients are asymptotically close to $c(\log n)^{\beta} n^{-\alpha}$ for some constants c, β and $1 < \alpha < 2$. Whenever methods from analytic combinatorics apply, they yield results of great precision, which is impressively demonstrated in the tail-bounds [12, Thm. 2] for the size of the remainder. However, these requirements are much more specific than in the mentioned work [2] for partitions with a conditioning relation, and there are known examples of minor-closed addable classes such as random planar graphs [14], for which $\alpha = 7/2$ lies outside of the considered interval.

For these reasons, it is desirable to establish a "convergent case" regime for Gibbs partitions, that is as general as possible, and in which a similar behaviour as in Barbour and Granovsky's setting [2] may be observed. In the present work, we consider Gibbs partitions with a subcritical composition scheme, such that the generating series of the structures on the components belongs to the family of subexponential sequences studied in [7, 9, 10]. The elements of this family correspond up to tilting and normalizing to subexponential densities of lattice distributed random variables, and hence may be put in the general context of heavy-tailed and subexponential distributions [13]. Our first main result establishes that Gibbs partitions in this setting exhibit a giant component, and the small rest converges in total variation toward a limit structure following a weighted Boltzmann distribution. In order to demonstrate its broad scope and relevance for combinatorial questions, we use the strong ratio property and a number of results related to simply generated trees [17], to show how analytic assemblies of arbitrary tree-like combinatorial structures belong to this regime.

We apply our results to small block-stable classes of graphs. For any such class $\mathcal{A}$, we partition the integers into a finite set of shifted lattices of the form $a + d\mathbb{Z}$ for $0 \leqslant a < d$, along which the class $\mathcal{A}$ grows smoothly. This allows us to characterize precisely when $\mathcal{A}$ belongs to the family of smooth graph classes. The uniform n-sized random graph from $\mathcal{A}$ is shown to form a giant component with a stochastically bounded remainder. The fragments not contained in the giant component converge to different Boltzmann Poisson random graphs, depending along which lattice we let n tend to infinity. Any proper addable minor-closed class of graphs is small and block-stable, but the converse does not hold. Hence this recovers and generalizes corresponding results by McDiarmid [23, Theorems 1.2, 1.7], who established smooth growth and convergence of the small fragments for proper addable minor-closed classes. Our approach also works in various other settings, for which we provide some examples, including random graphs drawn with probability proportional to weights assigned to their blocks.

Plan of the paper

In Section 2 we fix notations and recall necessary background related to Gibbs partitions, graph classes and subexponential sequences. Section 3 presents our results on Gibbs partitions in the convergent case, and Section 4 discusses their applications to small block-stable graph classes. Section 5 discusses extensions to similar settings. In Section 6 we collect all proofs.

2 Preliminaries

2.1 Notation

Throughout, we set

$$\mathbb{N} = \{1, 2, \ldots\}, \qquad \mathbb{N}_0 = \{0\} \cup \mathbb{N}, \qquad [n] = \{1, 2, \ldots, n\}, \qquad n \in \mathbb{N}_0.$$

We usually assume that all considered random variables are defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All unspecified limits are taken as n becomes large, possibly along an infinite subset of $\mathbb{N}$. The *total variation distance* between two random variables X and Y with values in a countable state space S is defined by

$$d_{TV}(X, Y) = \sup_{\mathcal{E} \subset S} |\mathbb{P}(X \in \mathcal{E}) - \mathbb{P}(Y \in \mathcal{E})|.$$

A sequence of $\mathbb{R}$-valued random variables $(X_n)_{n \geqslant 1}$ is *stochastically bounded*, if for each $\epsilon > 0$ there is a constant $M > 0$ with

$$\limsup_{n \to \infty} \mathbb{P}(|X_n| \geqslant M) \leqslant \epsilon.$$

We let $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geqslant 0}$ denote the sets of positive and non-negative real numbers, respectively. A function $h : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is called *slowly varying*, if

$$\lim_{x \to \infty} \frac{h(tx)}{h(x)} = 1$$

for all fixed $t > 0$. For any power series $f(z)$, we let $[z^n]f(z)$ denote the coefficient of z^n.

2.2 Weighted combinatorial species and generating functions

Let $\mathcal{F}^\omega$ denote a *species of combinatorial structures* with non-negative weights in the sense of Joyal [19]. That is, for each finite set U we are given a finite set $\mathcal{F}[U]$ of *$\mathcal{F}$-structures* and a map

$$\omega_U : \mathcal{F}[U] \to \mathbb{R}_{\geqslant 0}.$$

Moreover, for each bijection $\sigma : U \to V$ the species $\mathcal{F}$ produces a corresponding bijection

$$\mathcal{F}[\sigma] : \mathcal{F}[U] \to \mathcal{F}[V]$$

that preserves the ω-weights. This may be expressed by requiring that the diagram

$$\begin{tikzcd} \mathcal{F}[U] \arrow[r, "\mathcal{F}[\sigma]"] \arrow[dr, "\omega_U"'] & \mathcal{F}[V] \arrow[d, "\omega_V"] \\ & \mathbb{R}_{\geqslant 0} \end{tikzcd}$$

commutes. Species are also subject to the usual functoriality requirements: the identity map id_U on U gets mapped to the identity map $\mathcal{F}[\mathrm{id}_U] = \mathrm{id}_{\mathcal{F}[U]}$ on the set $\mathcal{F}[U]$. For any bijections $\sigma : U \to V$ and $\tau : V \to W$ the diagram

$$\begin{tikzcd} \mathcal{F}[U] \arrow[r, "\mathcal{F}[\sigma]"] \arrow[dr, "\mathcal{F}[\tau\sigma]"'] & \mathcal{F}[V] \arrow[d, "\mathcal{F}[\tau]"] \\ & \mathcal{F}[W] \end{tikzcd}$$

commutes. As a last requirement, we also assume that $\mathcal{F}[U] \cap \mathcal{F}[V] = \emptyset$ whenever $U \neq V$. This is not much of a restriction, as we may always replace $\mathcal{F}[U]$ by $\{U\} \times \mathcal{F}[U]$ for all sets U, to make sure that it is satisfied.

Two weighted species $\mathcal{F}^\omega$ and $\mathcal{H}^\gamma$ are *structurally equivalent* or *isomorphic*, denoted by $\mathcal{F}^\omega \simeq \mathcal{H}^\gamma$, if there is a family of weight-preserving bijections $(\alpha_U : \mathcal{F}[U] \to \mathcal{H}[U])_U$ with U ranging over all finite sets, such the following diagram commutes for each bijection $\sigma : U \to V$ of finite sets.

$$\begin{tikzcd} \mathcal{F}[U] \arrow[r, "\mathcal{F}[\sigma]"] \arrow[d, "\alpha_U"] & \mathcal{F}[V] \arrow[d, "\alpha_V"] \\ \mathcal{H}[U] \arrow[r, "\mathcal{G}[\sigma]"] & \mathcal{H}[V] \end{tikzcd}$$

We will often write $\omega(F)$ instead of $\omega_U(F)$ for the weight of a structure $F \in \mathcal{F}[U]$. For any $\mathcal{F}$-object $F \in \mathcal{F}[U]$ we let

$$|F| := |U| \in \mathbb{N}_0$$

denote its *size*. It will be convenient to use the notation

$$\mathscr{U}(\mathcal{F}) = \bigcup_{n \geqslant 0} \mathcal{F}[n].$$

Here we write $\mathcal{F}[n] = \mathcal{F}[\{1, \ldots, n\}]$ for all non-negative integers n. This allows us to define the *exponential generating series*

$$\mathcal{F}^\omega(z) = \sum_{F \in \mathscr{U}(\mathcal{F})} \omega(F) \frac{z^{|F|}}{|F|!}$$

as a formal power series with non-negative coefficients. A simple example is the species SET where $\mathrm{SET}[U] = \{U\}$ for each finite set U and each object receives weight 1. Hence $\mathrm{SET}(z) = \exp(z)$.

Two $\mathcal{F}$-objects $F_1 \in \mathcal{F}[U]$ and $F_2 \in \mathcal{F}[V]$ are termed *isomorphic*, denoted by $F_1 \simeq F_2$, if there is a bijection $\sigma : U \to V$ such that $\mathcal{F}[\sigma](F_1) = F_2$. An *unlabelled* $\mathcal{F}$-object is formally defined as a maximal class of pairwise isomorphic objects. The unlabelled object corresponding to a given $\mathcal{F}$-object F is also termed its *isomorphism type* and denoted by $\tilde{F}$.

We are going to consider probability measures on the collection of all unlabelled $\mathcal{F}$-objects. From a formal viewpoint, this may be slightly problematic, because infinite collections of proper classes are not well-defined objects. But this more of a notational issue that could easily be resolved by working with a fixed set of representatives instead.

2.3 Composite and derived structures

Let $\mathcal{F}^\omega$ and $\mathcal{G}^\nu$ be combinatorial species with non-negative weights, such that $\mathcal{G}^\nu[\emptyset] = \emptyset$. The *composition* $\mathcal{F}^\omega \circ \mathcal{G}^\nu = (\mathcal{F} \circ \mathcal{G})^\mu$ of the two species describes partitions of finite sets where each partition class is endowed with a $\mathcal{G}$-structure, and the collection of partition classes carries an $\mathcal{F}$-structure. Formally, it is defined by setting for each finite set U

$$(\mathcal{F} \circ \mathcal{G})[U] = \bigcup_{\pi} \mathcal{F}[\pi] \times \prod_{Q \in \pi} \mathcal{G}[Q]$$

with the index π ranging over all unordered partitions of U with non-empty partition classes. That is, π is a set of non-empty subsets of U such that $U = \bigcup_{Q \in \pi} Q$ and $Q \cap Q' = \emptyset$ for all $Q, Q' \in \pi$ with $Q \neq Q'$. The weight of a composite structure $(F, (G_Q)_{Q \in \pi})$ is defined by

$$\mu(F, (G_Q)_{Q \in \pi}) = \omega(F) \prod_{Q \in \pi} \nu(G_Q).$$

For any bijection $\sigma : U \to V$, the corresponding function

$$(\mathcal{F} \circ \mathcal{G})[\sigma] : (\mathcal{F} \circ \mathcal{G})[U] \to (\mathcal{F} \circ \mathcal{G})[V]$$

is defined as follows. For each element $(F, (G_Q)_{Q \in \pi}) \in (\mathcal{F} \circ \mathcal{G})[U]$ we let $\bar{\pi} = \{\sigma(Q) \mid Q \in \pi\}$ denote a partition of V and set

$$\bar{\sigma} : \pi \to \bar{\pi}, Q \mapsto \pi(Q).$$

For each $Q \in \pi$ we let

$$\sigma|_Q : Q \to \sigma(Q), x \mapsto \sigma(x)$$

denote the restriction of σ to the class Q. We set

$$(\mathcal{F} \circ \mathcal{G})[\sigma](F, (G_Q)_{Q \in \pi}) = (\mathcal{F}[\bar{\sigma}](F), (\mathcal{G}[\sigma|_Q](G_{\sigma^{-1}(P)}))_{P \in \bar{\pi}}).$$

The generating series of the composition satisfies [19, Prop. 24]

$$(\mathcal{F}^\omega \circ \mathcal{G}^\nu)(z) = \mathcal{F}^\omega(\mathcal{G}^\nu(z)).$$

A further construction that we are going to use is the *derived* species $(\mathcal{F}')^\omega$ defined as follows. For each set U we let $*_U$ denote a placeholder object not contained in U. For example, we could define $*_U = U$, as no set is allowed to be an element of itself. We set

$$\mathcal{F}'[U] = \mathcal{F}[U \cup \{*_U\}].$$

The weight of an element $F' \in \mathcal{F}'[U]$ is its ω-weight as $\mathcal{F}$-structure. Any bijection $\sigma : U \to V$ may canonically be extended to a bijection

$$\sigma' : U \cup \{*_U\} \to V \cup \{*_V\},$$

and we set

$$\mathcal{F}'[\sigma] = \mathcal{F}[\sigma'].$$

Thus, an $\mathcal{F}'$-object with size n is an $\mathcal{F}$-object with size $n + 1$, since we do not count the $*$-placeholder. The exponential generating series of $(\mathcal{F}')^\omega$ is given by the formal derivative

$$(\mathcal{F}')^\omega(z) = \frac{d}{dz}\mathcal{F}^\omega(z).$$

2.4 Kolchin's representation theorem and Boltzmann distributions

Given a weighted species $\mathcal{F}^\omega$ and a parameter $y > 0$ with $0 < \mathcal{F}^\omega(y) < \infty$, we may consider the corresponding Boltzmann probability measure

$$\mathbb{P}_{\mathcal{F}^\omega,y}(\mathsf{F}) = \mathcal{F}^\omega(y)^{-1} y^{|\mathsf{F}|}\omega(\mathsf{F})/|\mathsf{F}|!, \qquad \mathsf{F} \in \mathscr{U}(\mathcal{F}).$$

In a certain sense, Boltzmann measures are invariant under relabelling:

Proposition 2.1. *Let* F *follow a* $\mathbb{P}_{\mathcal{F}^\omega,y}$ *distribution and let* $\mathsf{N} = |\mathsf{F}|$ *denote its random size. If we draw a permutation* $\sigma : [\mathsf{N}] \to [\mathsf{N}]$ *uniformly at random, then the corresponding relabelled object* $\mathcal{F}[\sigma](\mathsf{F})$ *is also* $\mathbb{P}_{\mathcal{F}^\omega,y}$ *distributed.*

The Boltzmann distribution for composite structures admits a useful canonical coupling, which is a combinatorial interpretation of Kolchin's representation theorem [29, Thm. 1.2], and also known as the substitution rule for Boltzmann samplers. It is constructed in [6] for species without weights, and the generalization to the weighted setting is straight-forward.

Lemma 2.2 ([6]). *Let* $\mathcal{F}^\omega$ *and* $\mathcal{G}^\nu$ *be weighted species with* $\mathcal{G}[\emptyset] = \emptyset$*. Let* $x > 0$ *be a parameter with* $0 < \mathcal{F}^\omega(\mathcal{G}^\nu(x)) < \infty$ *and* $y := \mathcal{G}^\nu(x) < \infty$*. If we sample a* $\mathbb{P}_{\mathcal{F}^\omega,y}$*-distributed* $\mathcal{F}$*-object* F*, and for each* $1 \leqslant i \leqslant |\mathsf{F}|$ *and independent* $\mathbb{P}_{\mathcal{G}^\nu,x}$*-distributed* $\mathcal{G}$*-object* G_i*, then the tupel* $(\mathsf{F}, \mathsf{G}_1, \ldots, \mathsf{G}_{|\mathsf{F}|})$ *may be interpreted as an* $\mathcal{F} \circ \mathcal{G}$*-object* S *on the disjoint union*

$$\mathsf{V} = \bigsqcup_{1 \leqslant i \leqslant |\mathsf{F}_i|} [|\mathsf{G}_i|].$$

Let $\sigma : \mathsf{V} \to [|\mathsf{V}|]$ *be a uniformly at random sampled bijection. Then the relabelled object*

$$(\mathcal{F} \circ \mathcal{G})[\sigma](\mathsf{S}) \in \mathscr{U}(\mathcal{F} \circ \mathcal{G})$$

follows a $\mathbb{P}_{\mathcal{F}^\omega \circ \mathcal{G}^\nu,x}$*-distribution.*

2.5 Graph classes

A *simple finite graphs* is a pair $G = (V, E)$ of a finite set $V = V(G)$ of *vertices* or *labels* together with a set $E = E(G)$ of *edges*

$$E \subset \{\{x, y\} \mid x, y \in V, x \neq y\}.$$

To avoid notational ambiguities we assume additionally that $V \cap E = \emptyset$. Two vertices $x, y \in V$ are *adjacent*, if $\{x, y\} \in E$. We also say that y is a *neighbour* of x. A *subgraph* of G is a graph H with $V(H) \subset V(G)$. We say H is a *proper subgraph*, if additionally $H \neq G$.

We term G *connected*, if its vertex set is non-empty, and for all $x, y \in V$ we can reach y by starting at x and traversing edges. A *connected component* of G is a connected subgraph H that is maximal with this property. That is, no other connected subgraph of G exists that contains H as a proper subgraph. We say G is 2-connected, if G is connected, has at least 3 vertices, and deleting an arbitrary single vertex does not disconnect the graph. A subgraph H of G is termed a *block*, if it is either an isolated vertex with no neighbours, or two vertices joined by a single edges whose deletion would increase the number of connected components, or a 2-connected subgraph that is maximal with this property.

For each bijection $\sigma : V \to U$ between the vertex set of G and an arbitrary finite set U we may form the *relabelled graph*

$$\sigma.G := (U, \{\{\sigma(x), \sigma(y)\} \mid \{x, y\} \in E(G)\}.$$

Two graphs are termed *isomorphic* if one is a relabelled version of the other. For any edge $e = \{x, y\} \in E(G)$ we may form a new graph G/e by *contracting* e. The graph G/e is formed by replacing x and y with a single vertex $v_{x,y}$ that is adjacent to all former neighbours of x and y. A graph H is a *minor* of a graph G, if there are graphs $G_0, \ldots, G_t$ such that $G_0 \simeq G$ and $G_t \simeq H$ and for each G_{i+1} arises from G_i by deleting an edge, contracting an edge, or deleting a vertex.

A collection $\mathcal{A}$ of graphs that is closed under relabelling is termed a *graph class*. For notational convenience, we will always assume that $\mathcal{A}$ contains the trivial graph whose vertex-set is the empty set. We may interpret $\mathcal{A}$ as a combinatorial species by letting, for each finite set V, $\mathcal{A}[V] \subset \mathcal{A}$ denote the finite subset of all graphs in $\mathcal{A}$ with vertex set V, and defining

$$\mathcal{A}[\sigma] : \mathcal{A}[V] \to \mathcal{A}[U], G \mapsto \sigma.G$$

for each finite set U and bijection $\sigma : V \to U$. Moreover, we assign weight 1 to each $\mathcal{A}$-object.

We say a graph class $\mathcal{A}$ is

1. *proper*, if there exists a graph that is not contained in $\mathcal{A}$.
2. *small*, if the radius of convergence of the exponential generating series $\mathcal{A}(z)$ is positive.
3. *decomposable*, if any graph lies in $\mathcal{A}$ if and only if all its connected components do.
4. *bridge-addable*, if, for each graph $G \in \mathcal{A}$ and each pair of vertices $x, y \in V(G)$ contained in different connected components of G, the graph obtained by adding the edge $\{x, y\}$ to G also belongs to $\mathcal{A}$.
5. *addable*, if it is both decomposable and bridge-addable.
6. *minor-closed*, if for each $G \in \mathcal{A}$ and each minor H of G it also holds that $H \in A$.
7. *block-stable*, if it contains the graph consisting of a single vertex, and any graph lies in $\mathcal{A}$ if and only if all its blocks do.
8. *smooth*, if its generating series $\mathcal{A}(z)$ has a finite positive radius of convergence and satisfies the ratio test.

If $\mathcal{A}$ is decomposable and $\mathcal{C} \subset \mathcal{A}$ denotes the subclass of all connected graphs in $\mathcal{A}$, then the two species are related by a canonical isomorphism

$$\mathcal{A} \simeq \mathrm{SET} \circ \mathcal{C}. \tag{2.1}$$

This expresses the fact the connected components of a graph form a partition of the vertex set of the graph, and any combination of connected graphs from $\mathcal{C}$ must lie in $\mathcal{A}$, since $\mathcal{A}$ is decomposable.

If the graph class $\mathcal{A}$ is block-stable, then it is also decomposable. We may consider the subclass $\mathcal{B} \subset \mathcal{C}$ of all graphs in $\mathcal{C}$ that are 2-connected or consist of two vertices joined by a single edge. It was noted by Harary and Palmer [16, 1.3.3, 8.7.1], Robinson [30, Thm. 4], and Labelle [21, 2.10] that

$$z\mathcal{C}'(z) = z\phi(z\mathcal{C}'(z)) \qquad \text{with} \qquad \phi(z) = \exp(\mathcal{B}'(z)). \tag{2.2}$$

It is well-known that all minor-closed classes are block-stable. To see this, suppose that $\mathcal{A}$ is a minor-closed graph class. An *excluded minor* of $\mathcal{A}$ is a graph that does not belong to $\mathcal{A}$, but all its proper minors do. If $\mathcal{M}$ denotes the collection of excluded minors of $\mathcal{A}$, then a graph lies in $\mathcal{A}$ if and only if none of its minors belongs to $\mathcal{M}$. A moment's though verifies that $\mathcal{A}$ is decomposable, if and only if all excluded minors are connected, and addable, if and only if all excluded minors are 2-connected. Graph classes defined by excluding 2-connected minors must be block-stable, because every 2-connected subgraph of a graph G is a subgraph of one of its blocks.

It was shown by Norine, Seymour, Thomas and Wollan [27] that proper minor-closed classes are small. Thus proper addable minor-closed classes are examples of small block-stable classes containing all trees.

2.6 Subexponential sequences

We consider power series whose coefficients belong to the family of subexponential sequences studied by Chover, Ney and Wainger [7] and Embrechts [9]. Up to tilting and rescaling, these sequences correspond to subexponential densities of random variables with values in a lattice. Hence they may be put into the more general context of heavy-tailed and subexponential distributions, for which a comprehensive treatment is given in the book by Foss, Korshunov, and Zachary [13].

Definition 2.3. *Let $d \geqslant 1$ be an integer. A power series $g(z) = \sum_{n=0}^\infty g_n z^n$ with non-negative coefficients and radius of convergence $\rho > 0$ belongs to the class $\mathscr{S}_d$, if $g_n = 0$ whenever n is not divisible by d, and*

$$\frac{g_n}{g_{n+d}} \sim \rho^d, \qquad \frac{1}{g_n} \sum_{i+j=n} g_i g_j \sim 2g(\rho) < \infty \tag{2.3}$$

as $n \equiv 0 \mod d$ becomes large.

The following theorem describes the behaviour of randomly stopped sums.

Theorem 2.4 ([13, Thm. 4.8, 4.30])**.** *If $g(z)$ belongs to $\mathscr{S}_d$ with radius of convergence ρ, and $f(z)$ is a non-constant power series with non-negative coefficients that is analytic at ρ, then $f(g(z))$ belongs to $\mathscr{S}_d$ and*

$$[z^n]f(g(z)) \sim f'(g(\rho))[z^n]g(z), \qquad n \to \infty, \qquad n \equiv 0 \mod d.$$

The broad scope of this setting is illustrated by the following easy observation, which has been noted in various places, see for example [10].

Proposition 2.5. *If $g_n = h(n)n^{-\beta}\rho^{-n}$ for some constants $\rho > 0$, $\beta > 1$ and a slowly varying function h, then the series $\sum_{n \in d\mathbb{N}} g_n z^n$ belongs to the class $\mathscr{S}_d$.*

We will make use of the following criterion related to sums of random variables.

Lemma 2.6 ([13, Thm. 4.9])**.** *Let $f(z)$ belong to $\mathscr{S}_1$ with radius of convergence ρ, and $g_1(z), g_2(z)$ be power-series with non-negative coefficients. If*

$$\frac{[z^n]g_1(z)}{[z^n]f(z)} \to c_1, \qquad \textit{and} \qquad \frac{[z^n]g_2(z)}{[z^n]f(z)} \to c_2$$

as $n \to \infty$ with $c_1, c_2 \geqslant 0$, then

$$\frac{[z^n]g_1(z)g_2(z)}{[z^n]f(z)} \to c_1 g_2(\rho) + c_2 g_1(\rho).$$

If additionally $c_1 g_2(\rho) + c_2 g_1(\rho) > 0$, then $g_1(z)g_2(z)$ belongs to $\mathscr{S}_1$.

3 Convergent Gibbs partitions

Suppose that we are given combinatorial species $\mathcal{F}^\omega$ and $\mathcal{G}^\nu$ with $\mathcal{G}[\emptyset] = \emptyset$ and $[z^k]\mathcal{F}^\omega(z) > 0$ for at least one $k \geqslant 1$. For each integer $n \geqslant 0$ with $[z^n](\mathcal{F}^\omega \circ \mathcal{G}^\nu)(z) > 0$ we may sample a random composite structure

$$\mathsf{S}_n = (\mathsf{F}_n, (\mathsf{G}_Q)_{Q \in \pi_n})$$

from the set $(\mathcal{F} \circ \mathcal{G})[n]$ with probability proportional to its weight. The corresponding random partition π_n of the set $[n]$ is termed a *Gibbs partition*. We assume throughout that $(\mathcal{F}^\omega \circ \mathcal{G}^\nu)(z)$ is not a polynomial, so that we may study S_n as n tends to infinity.

We are interested in the behaviour of the remainder R_n when deleting "the" largest component from S_n. More specifically, we construct R_n as follows. We make a uniform choice of a component $Q_0 \in \pi_n$ having maximal size, and let F'_n denote the $\mathcal{F}'$-object obtained from the $\mathcal{F}$-object F_n by relabeling the Q_0 atom of F_n to a $*$-placeholder. In more formal words, we set

$$\mathsf{F}'_n = \mathcal{F}[\gamma](\mathsf{F}_n) \in \mathcal{F}'[\pi_n \setminus \{Q_0\}]$$

for the bijection $\gamma : \pi_n \to (\pi_n \setminus \{Q_0\}) \cup \{*\}$ with $\gamma(Q_0) = *$ and $\gamma(Q) = Q$ for $Q \neq Q_0$. This yields an $\mathcal{F}' \circ \mathcal{G}$-object

$$(\mathsf{F}'_n, (\mathsf{G}_Q)_{Q \in \pi_n \setminus \{Q_0\}}) \in (\mathcal{F}' \circ \mathcal{G})[[n] \setminus Q_0].$$

We are not interested in the precise content of the underlying set $[n] \setminus Q_0$. Any $(n - |Q_0|)$-sized subset of $[n]$ is equally likely. Hence we define the unique order-preserving map

$$\sigma : [n] \setminus \{Q_0\} \to [n - |Q_0|]$$

and set

$$\mathsf{R}_n = (\mathcal{F}' \circ \mathcal{G})[\sigma](\mathsf{F}'_n, (\mathsf{G}_Q)_{Q \in \pi_n \setminus \{Q_0\}}) \in \mathscr{U}(\mathcal{F}' \circ \mathcal{G}).$$

Alternatively, we could have chosen σ uniformly at random, it wouldn't have changed the *distribution* of the outcome. We let μ denote the weighting on $(\mathcal{F}')^\omega \circ \mathcal{G}^\nu$, that is,

$$\mu(\mathsf{F}, (\mathsf{G}_Q)_Q) = \omega(\mathsf{F}) \prod_Q \nu(\mathsf{G}_Q), \qquad (\mathsf{F}, (\mathsf{G}_Q)_Q) \in \mathscr{U}(\mathcal{F}' \circ \mathcal{G}).$$

In a very general setting the remainder R_n converges in total variation toward a limit object following a Boltzmann distribution.

Theorem 3.1. *Suppose that the power series $\mathcal{G}^\nu(z)$ belongs to the class $\mathscr{S}_d$ with radius of convergence ρ, and that $\mathcal{F}^\omega(z)$ has radius of convergence strictly larger than $\mathcal{G}^\nu(\rho)$. Let R be a random element of the set $\mathscr{U}(\mathcal{F}' \circ \mathcal{G})$ that follows a Boltzmann distribution*

$$\mathbb{P}(\mathsf{R} = R) = \mu(R) \frac{\rho^{|R|}}{|R|!} ((\mathcal{F}')^\omega \circ \mathcal{G}^\nu)(\rho))^{-1}, \qquad R \in \mathscr{U}(\mathcal{F}' \circ \mathcal{G}).$$

Then

$$d_{TV}(\mathsf{R}_n, \mathsf{R}) \to 0, \qquad n \to \infty, \qquad n \equiv 0 \mod d. \tag{3.1}$$

This implies convergence in total variation of the number of components, which has also been studied in [26, 3] for the case $\mathcal{F}^\omega = \mathrm{SET}$. We may also verify convergence of moments.

Proposition 3.2. *Suppose that the assumptions of Theorem 3.1 hold. Let $c(\cdot)$ denote the number of components in a composite structure. Then $c(S_n)$ converges towards $1 + c(R)$ in total variation and arbitrarily high moments.*

Roughly speaking, the following lemma shows that Theorem 3.1 applies, whenever the species $\mathcal{G}^\nu$ is related to structures admitting a tree-like decomposition. This encompasses important families of enumerative series in combinatorics, for which it is not known whether all members fall into the setting of Proposition 2.5. We demonstrate its usefulness in Section 4 with a novel application to small block-stable graph classes. The proof of Lemma 3.3 uses the strong ratio property and a variety of results related to simply generated trees.

Lemma 3.3. *Let $\mathcal{Z}(z) = \sum_{n \geqslant 1} Z_n z^n$ and $\phi(z) = \sum_{k \geqslant 0} \omega_k z^k$ be power series with non-negative coefficients that are related by the equation*

$$\mathcal{Z}(z) = z\phi(\mathcal{Z}(z)).$$

Suppose that $\omega_0 > 0$, $\omega_k > 0$ for at least one $k \geqslant 2$, and let d denote the greatest common divisor of all k with $\omega_k > 0$. By [18, Lem. 13.3], we know that $Z_n = 0$ if $n - 1$ is not a multiple of d, and $Z_n > 0$ if $n \equiv 1 \mod d$ is large enough. Suppose that $\mathcal{Z}(z)$ has non-zero radius of convergence $\rho_{\mathcal{Z}}$. Then

$$Z_n^{-1} \sum_{i+j=n+1} Z_i Z_j \sim 2\mathcal{Z}(\rho_{\mathcal{Z}})/\rho_{\mathcal{Z}}, \qquad n \to \infty, \qquad n \equiv 1 \mod d.$$

This implies that the shifted series $\mathcal{Z}(z)/z$ belongs to $\mathscr{S}_d$, since it was shown in [18, Rem. 7.5] that $\mathcal{Z}(\rho_{\mathcal{Z}}) < \infty$, and in [18, Thm. 18.6, 18.10], that

$$Z_n/Z_{n+d} \to \rho_{\mathcal{Z}}^d \qquad \text{and} \qquad Z_n^{1/n} \to 1/\rho_{\mathcal{Z}}$$

as $n \equiv 1 \mod d$ becomes large.

If the series $\mathcal{G}^\nu(z)$ is periodic with a shift, then different behaviour may occur depending along which lattice we let n tend to infinity.

Theorem 3.4. *Suppose that there is an integer $0 \leqslant m < d$ such that $\mathcal{G}^\nu(z)/z^m$ belongs to the class $\mathscr{S}_d$. Let $D = d/\gcd(m, d)$ and for each $0 \leqslant a < D$, let $\mathcal{F}^\omega_a$ denote the restriction of $\mathcal{F}^\omega$ to objects whose size lies in $a + D\mathbb{Z}$. If the exponential generating series $\mathcal{F}^\omega_a(z)$ is not constant, then*

$$d_{TV}(R_n, R(a)) \to 0, \qquad n \to \infty, \qquad n \equiv am \mod d$$

with the limit object $R(a)$ following a $\mathbb{P}_{(\mathcal{F}'_a)^\omega \circ \mathcal{G}^\nu, \rho}$ Boltzmann distribution.

4 Applications to random graphs

In the following, we let $\mathcal{A}$ denote a small block-stable class of graphs, $\mathcal{C} \subset \mathcal{A}$ its subclass of connected graphs, and $\mathcal{B} \subset \mathcal{A}$ the subclass of all graphs that are 2-connected or consist of two vertices joined by a single edge. To exclude the case where $\mathcal{A}$ is the trivial class of all graphs consisting of isolated points, we assume that $\mathcal{B}$ is non-empty. Hence we may let d denote the greatest common divisor of all integers k with $[z^k]\exp(\mathcal{B}'(z)) > 0$. We let $\rho > 0$ denote the radius of convergence of the exponential generating series $\mathcal{A}(z) = \exp(\mathcal{C}(z))$. Clearly ρ is also the radius of convergence $\mathcal{C}(z)$.

Using Lemma 3.3 and the robustness of subexponential sequences against perturbation, we deduce the following enumerative result.

Theorem 4.1. *The series $\mathcal{C}'(z)$ and $\mathcal{C}(z)/z$ both lie in the class $\mathscr{S}_d$ of subexponential sequences with span d. That is $\mathcal{C}'(\rho), \mathcal{C}(\rho) < \infty$, and the coefficients $c_n = [z^n]\mathcal{C}(z)$ satisfy*

$$\frac{c_n}{c_{n+d}} \sim \rho^d, \qquad \frac{1}{c_n}\sum_{i+j=n+1} c_i c_j \sim 2\mathcal{C}(\rho)/\rho$$

as $n \equiv 1 \mod d$ becomes large.

For each integer i we let SET_i denote the restriction of the species SET to objects whose size lies in the lattice $i + d\mathbb{Z}$. For each $0 \leqslant a < d$ we let G_a denote a random graph from the class $\mathcal{A}$ following a $\mathbb{P}_{\mathrm{SET}'_a \circ \mathcal{C}, \rho}$ Boltzmann distribution. That is, R_a is a random graph from the class $\mathcal{A}$ whose number of components lie in the shifted lattice $a - 1 + d\mathbb{Z}$, and its distribution is given by

$$\mathbb{P}(G_a = G) = \frac{\rho^{|G|}}{|G|!}\left(\sum_{\substack{k \geqslant 0 \\ k \in a-1+\mathbb{Z}}} \frac{\mathcal{C}(\rho)^k}{k!}\right)^{-1}, \qquad G \in \mathscr{U}(\mathrm{SET}_{a-1} \circ \mathcal{C}).$$

For each integer $n \in \mathbb{N}_0$ let A_n denote the random graph sampled uniformly from the set $\mathcal{A}[n]$ of graphs in $\mathcal{A}$ with vertex set $[n]$. We let $\mathrm{frag}(A_n)$ denote the graph obtained by deleting a uniformly at random drawn largest component of A_n, and relabelling the rest in a canonical order-preserving way to a set of the form $[k]$ for some $k \geqslant 0$. Alternatively, we may relabel by choosing a bijection uniformly at random, it makes no difference for the resulting *distribution*. Theorem 3.4 yields our main application.

Theorem 4.2. *For each $0 \leqslant a < d$, it holds that*

$$d_{TV}(\mathrm{frag}(A_n), G_a) \to 0$$

as $n \equiv a \mod d$ becomes large. The coefficients of $\mathcal{A}(z)$ along the lattice $a + d\mathbb{Z}$ belong, after a shift by $-a$, to the class $\mathscr{S}_d$ of subexponential sequences with span d. As $n \equiv a \mod d$ becomes large, it holds that

$$[z^n]\mathcal{A}(z) \sim C_{a-1}[z^{n+1-a}]\mathcal{C}(z) \qquad \textit{with} \qquad C_{a-1} = \sum_{\substack{k \geqslant 0 \\ k \equiv a-1 \mod d}} \mathcal{C}(\rho)^k/k!.$$

This allows us to precisely describe under which conditions the graph class $\mathcal{A}$ is smooth.

Theorem 4.3. *The graph class $\mathcal{A}$ is smooth, if and only if $d = 1$.*

Indeed, for $d = 1$ the coefficients of $\mathcal{A}(z)$ behave asymptotically up to a constant factor like those of $\mathcal{C}(z)$, and hence grow smoothly. For $d \geqslant 2$, the only way for $\mathcal{A}$ to be smooth is when $C_0 = \ldots = C_{d-1}$. There is a beautiful reason, why this may never happen. If $C_i = C_{i+1}$ would hold for all i, then we could select a d-th root of unity $\zeta \neq 1$ and deduce the contradiction

$$\exp(\zeta\mathcal{C}(\rho)) = C_0 + C_1\zeta + \ldots + C_{d-1}\zeta^{d-1} = C_0(1 + \zeta + \ldots + \zeta^{d-1}) = 0.$$

Hence $\mathcal{A}$ cannot be smooth for $d \geqslant 2$.

5 Extensions

Many classes of weighted combinatorial composite structures may be expressed by a subcritical substitution scheme $\mathcal{F}^\omega \circ \mathcal{G}^\nu$, such that Lemma 3.3 may be applied either directly, or similarly as in the proof of Theorem 4.1, to show that $\mathcal{G}^\nu$ belongs up to a constant shift to the class $\mathscr{S}_d$ for some $d \geqslant 1$. We illustrate this with some examples. There are of course many more, but we do not aim to provide an exhaustive list.

Random graphs with block-weights. Random graphs from block-stable classes have a natural generalization to the weighted setting. Suppose that we are given a weighting γ on the species $\mathcal{B}$ of all graphs that are two-connected or consist of two vertices joined by a single edge. This yields a weighting on the species of connected graphs $\mathcal{C}$, given by

$$\nu(C) = \prod_B \gamma(B),$$

with the index B ranging over all blocks of the connected graph C. Here we set $\nu(\bullet) = 1$ for the graph "$\bullet$" consisting of a single vertex. Likewise we may define a weighting μ on the species $\mathcal{G}$ of all graphs in the same way, such that

$$\mathcal{G}^\mu \simeq \mathrm{SET} \circ \mathcal{C}^\nu.$$

We may consider a random n-vertex graph G_n^μ drawn from $\mathcal{G}^\mu[n]$ with probability proportional to its weight. This encompasses uniform random graphs from block-stable classes, which correspond precisely to the case where $\gamma(B) \in \{0, 1\}$ for all blocks B.

The block-decomposition of connected graphs into 2-connected components described for example in Labelle [21, 2.10] can easily be seen to preserve the weights, yielding a weighted version of Equation (2.2):

$$z(\mathcal{C}')^\nu(z) = z \exp((\mathcal{B}')^\gamma(z(\mathcal{C}')^\nu(z)).$$

Hence a straight-forward analogon of Theorem 4.2 also holds for the random graph G_n^μ. The corresponding proof requires no modification at all.

Forests of Galton–Watson trees with a random number of trees. Let $(\mathcal{T}_i)_{i \geqslant 1}$ be a family of independent copies of a subcritical or critical Galton–Watson tree $\mathcal{T}$ with offspring distribution ξ. Let K denote an independent random non-negative integer having finite exponential moments. We may consider a Galton–Watson forest F with a random number of trees

$$\mathsf{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_\mathsf{K}),$$

and let

$$|\mathsf{F}| = \sum_{i=1}^{\mathsf{K}} |\mathcal{T}|_i$$

denote its size. The probability generating functions

$$f(z) = \mathbb{E}[z^{|\mathsf{F}|}], \qquad \psi(z) = \mathbb{E}[z^{\mathsf{K}}], \qquad \phi(z) = \mathbb{E}[z^{\xi}], \qquad \text{and} \qquad \mathcal{Z}(z) = \mathbb{E}[z^{|\mathcal{T}|}]$$

are related by

$$f(z) = \psi(\mathcal{Z}(z)) \qquad \text{and} \qquad \mathcal{Z}(z) = z\phi(\mathcal{Z}(z)).$$

Obviously Lemma 3.3 applies to $\mathcal{Z}(z)$, so we obtain an analogon of Theorem 4.2 that describes the asymptotic behaviour if we condition the forest F to be large and cut down the largest tree.

6 Proofs

We list the proofs of our results in order of their appearance.

6.1 Proofs from Section 3

Proof of Theorem 3.1. In the following, we assume tacitly that n is divisible by d, and large enough such that $[z^n](\mathcal{F}^\omega \circ \mathcal{G}^\nu)(z) > 0$. The distributions of R_n and R are both invariant under relabelling uniformly at random. This implies that conditioned on having a fixed isomorphism type $\mathscr{T}$ of an $\mathcal{F}' \circ \mathcal{G}$-object with positive μ-weight, each possible labelling of $\mathscr{T}$ is equally likely. In particular, it follows that

$$(R_n \mid \tilde{R} = \mathscr{T}) = (R \mid \tilde{R} = \mathscr{T}).$$

Hence it suffices to establish total variational convergence of the isomorphism type of R_n to the isomorphism type of R, that is

$$\lim_{n\to\infty} d_{TV}(\tilde{R}_n, \tilde{R}) = 0. \tag{6.1}$$

Let F denote a random $\mathcal{F}$-object following a $\mathbb{P}_{\mathcal{F}^\omega, \mathcal{G}^\nu(\rho)}$-distribution, and for each $1 \leqslant i \leqslant |F|$ let G_i be an independent $\mathbb{P}_{\mathcal{G}^\nu,\rho}$ distributed $\mathcal{G}$-object. Lemma 2.2 states that, up to relabelling uniformly at random, the tupel

$$S := (F, G_1, \ldots, G_{|F|})$$

follows a $\mathbb{P}_{\mathcal{F}^\omega \circ \mathcal{G}^\nu, \rho}$ distribution. Consequently, if we condition S on having size n, then it corresponds to an element from $(\mathcal{F} \circ \mathcal{G})[n]$ that is sampled with probability proportional to its weight. That is, setting $f = |F|$ and $g_i = |G_i|$ for all i, it follows that as unlabelled objects

$$S_n \stackrel{d}{=} (S \mid g_1 + \ldots + g_f = n).$$

The random $\mathcal{F}' \circ \mathcal{G}$-object R follows a $\mathbb{P}_{(\mathcal{F})'^\omega \circ \mathcal{G}^\nu, \rho}$ distribution. Hence we may apply Lemma 2.2 again to identify it up to relabelling with a tuple

$$R = (F', G^1, \ldots, G^{|F'|}),$$

where F' follows a $\mathbb{P}_{(\mathcal{F}')^\omega, \mathcal{G}^\nu(\rho)}$ distribution, and the G^i are independent and $\mathbb{P}_{\mathcal{G}^\nu,\rho}$ distributed. To simplify notation, we set $f' := |F'|$ and $g^i = |G^i|$ for all i.

Let $\hat{S}_n$ denote the composite $\mathcal{F} \circ \mathcal{G}$-structure obtained by sampling a random $\mathcal{G}$-structure G^* from $\mathcal{G}[n - |R|]$ with probability proportional to its ν-weight, and assigning it to the $*$-vertex of F'. This is only well-defined if

$$n - |R| > 0 \qquad \text{and} \qquad [z^{n-|R|}]\mathcal{G}^\nu(z) > 0, \tag{6.2}$$

otherwise we set $\hat{S}_n$ to some place-holder value. The probability for the event (6.2) tends to 1 as n becomes large, since $|R|$ is almost surely finite and a multiple of d, and $[z^{kd}]\mathcal{G}^\nu(z) > 0$ for all sufficiently large k by assumption.

We are going to show that as unlabelled $\mathcal{F} \circ \mathcal{G}$-objects

$$\lim_{n\to\infty} d_{TV}(S_n, \hat{S}_n) = 0. \tag{6.3}$$

This implies Equation (6.1), since the probability, that R is the largest component of $\hat{S}_n$, tends to 1 as n becomes large.

Let g denote a random variable that is distributed like the size of a random $\mathcal{G}$-object with a $\mathbb{P}_{\mathcal{G}^\nu,\rho}$ distribution. Since $\mathcal{G}^\nu(z)$ belongs to $\mathscr{S}_d$, it holds that

$$\mathbb{P}(g = n + d) \sim \mathbb{P}(g = n), \qquad n \to \infty.$$

Consequently, there is a sequence t_n of non-negative integers such that $t_n \to \infty$ and

$$\lim_{n\to\infty} \sup_{\substack{0\leqslant y\leqslant t_n \\ y\equiv 0 \mod d}} |\mathbb{P}(g = n + y)/\mathbb{P}(g = n) - 1| = 0. \tag{6.4}$$

Indeed, for each $\epsilon > 0$ and $t > 0$ there is a constant $N_{\epsilon,t} \geqslant 1$ such that for all $n \geqslant N$

$$\sup_{\substack{0\leqslant y\leqslant t \\ y\equiv 0 \mod d}} |\mathbb{P}(g = n + y)/\mathbb{P}(g = n) - 1| \leqslant \epsilon.$$

Setting $t_n = 1$ for $n < N_{2,1/2}$, $t_n = 2$ for $N_{2,1/2} \leqslant x < N_{2,1/2} + N_{3,1/3}$, and in general $t_n = k$ for

$$N_{2,1/2} + \ldots + N_{k,1/k} \leqslant n \leqslant N_{2,1/2} + \ldots + N_{k+1,1/(k+1)},$$

yields a sequence with the desired properties.

Let $k, x_1, \ldots, x_k \geqslant 1$ be integers with $x_1 + \ldots + x_k = n$. If we condition on $f = k$ and $g_i = x_i$ for all $1 \leqslant i \leqslant k$, then F gets drawn from $\mathcal{F}[k]$ with probability proportional to its ω-weight, and likewise G_i gets drawn from $\mathcal{G}[x_i]$ with probability proportional to its ν-weight for all i. Conditioned on having size $k-1$, F' gets drawn from $\mathcal{F}[[k-1] \cup \{*\}]$ with probability proportional to its ω-weight. Thus, up to relabeling uniformly at random,

$$(F' \mid f' = k - 1) \overset{d}{=} (F \mid f = k).$$

Since $x_1 + \ldots + x_k = n$, it follows that as unlabelled $\mathcal{F} \circ \mathcal{G}$-objects

$$(S \mid f = k, g_i = x_i, 1 \leqslant i \leqslant k) \overset{d}{=} (\hat{S}_n \mid f' = k - 1, g^i = x_i, 1 \leqslant i \leqslant k - 1). \tag{6.5}$$

For any sequence $\mathbf{y} = (y_1, \ldots, y_{k-1})$ of positive integers with $D(\mathbf{y}) := y_1 + \ldots, y_{k-1} < n$ set

$$\sigma_n(\mathbf{y}) = \{(y_1, \ldots, y_{j-1}, n - D(\mathbf{y}), y_j, \ldots, y_k) \mid 1 \leqslant j \leqslant k\}.$$

In order for Equation (6.4) to hold, we may replace the sequence t_n by any other sequence that tends to infinity more slowly. So without loss of generality, we may assume that $t_n < n/2$ for all n, and set

$$M_n := \{(k, \mathbf{y}) \mid k \geqslant 1, \mathbf{y} \in \mathbb{N}^{k-1}, D(\mathbf{y}) \leqslant t_n\}.$$

We are going to verify that

$$\mathbb{P}(f = k, (g_1, \ldots, g_k) \in \sigma_n(\mathbf{y}) \mid g_1 + \ldots + g_f = n) \sim \mathbb{P}(f' = k - 1, (g^1, \ldots, g^{k-1}) = \mathbf{y}) \tag{6.6}$$

uniformly for all $(k, \mathbf{y}) \in M_n$. Since $t_n < n/2$, it holds that given $g_1 + \ldots g_f = n$ and $f = k$, the event $(g_1, \ldots, g_k) \in \sigma_n(\mathbf{y})$ corresponds to k distinct outcomes, depending on the unique location for the maximum of the g_i. Each outcome is equally likely, so the left-hand side in (6.6) divided by the right-hand side equals

$$\frac{k\mathbb{P}(f = k)\mathbb{P}(g = n - D(\mathbf{y}))}{\mathbb{P}(f' = k - 1)\mathbb{P}(g_1 + \ldots + g_f = n)},$$

with

$$\frac{k\mathbb{P}(f = k)}{\mathbb{P}(f' = k - 1)} = \frac{(\mathcal{F}')^{\omega}(\mathcal{G}^{\nu}(\rho))\mathcal{G}^{\nu}(\rho)}{\mathcal{F}^{\omega}(\mathcal{G}^{\nu}(\rho))} = \mathbb{E}[f].$$

As $D(\mathbf{y}) \leqslant t_n$, it follows by Equation (6.4) that uniformly for $(k, \mathbf{y}) \in M_n$

$$\mathbb{P}(g = n - D(\mathbf{y})) \sim \mathbb{P}(g = n).$$

By Theorem 2.4 it holds that

$$\mathbb{P}(g_1 + \ldots + g_f = n) \sim \mathbb{E}[f]\mathbb{P}(g = n),$$

and (6.6) follows.

To complete the proof, we first note that

$$\lim_{n \to \infty} \mathbb{P}((f' + 1, (g^1, \ldots, g^{f'})) \in M_n) = 1. \tag{6.7}$$

Hence (6.6) yields that with probability tending to 1 as n becomes large

$$((f, (g_1, \ldots, g_f)) \mid g_1 + \ldots g_f = n) \in \{k\} \times \sigma_n(\mathbf{y}) \quad \text{for some} \quad (k, \mathbf{y}) \in M_n.$$

Using Equation (6.5), it follows that uniformly for all sets $\mathcal{E}$ of n-sized unlabelled $\mathcal{F} \circ \mathcal{G}$-objects

$$\begin{aligned}\mathbb{P}(S_n \in \mathcal{E}) &= \mathbb{P}(S \in \mathcal{E} \mid g_1 + \ldots g_f = n) \\ &= o(1) + \sum_{(k,\mathbf{y}) \in M_n} \mathbb{P}(S \in \mathcal{E}, f = k, (g_1, \ldots, g_f) \in \sigma_n(\mathbf{y}) \mid g_1 + \ldots g_f = n).\end{aligned}$$

For each $(k, \mathbf{y}) \in M_n$, the corresponding summand may be simplified to

$$\mathbb{P}(S \in \mathcal{E}, f = k, (g_1, \ldots, g_f) \in \sigma_n(\mathbf{y}))/\mathbb{P}(g_1 + \ldots g_f = n),$$

and then expressed as the product

$$\mathbb{P}(S \in \mathcal{E} \mid f = k, (g_1, \ldots, g_k) \in \sigma_n(\mathbf{y}))\mathbb{P}(f = k, (g_1, \ldots, g_k) \in \sigma_n(\mathbf{y}) \mid g_1 + \ldots g_f = n).$$

We treat the two factors separately. For the first, Equation (6.5) yields

$$\mathbb{P}(S \in \mathcal{E} \mid f = k, (g_1, \ldots, g_k) \in \sigma_n(\mathbf{y})) = \mathbb{P}(\hat{S}_n \in \mathcal{E} \mid f' = k - 1, g^i = y_i, 1 \leqslant i \leqslant k - 1).$$

By (6.6) it holds that

$$\mathbb{P}(f = k, (g_1, \ldots, g_k) \in \sigma_n(\mathbf{y}) \mid g_1 + \ldots g_f = n) \sim \mathbb{P}(f' = k-1, (g^1, \ldots, g^{k-1}) = \mathbf{y})$$

uniformly for all $(k, \mathbf{y}) \in M_n$. Using Equation (6.7) it follows that

$$\begin{aligned} \mathbb{P}(S_n \in \mathcal{E}) &= o(1) + \sum_{(k,\mathbf{y}) \in M_n} (1 + o(1)) \mathbb{P}(\hat{S}_n \in \mathcal{E}, f' = k-1, (g^1, \ldots, g^{k-1}) = \mathbf{y}) \\ &= o(1) + \mathbb{P}(\hat{S}_n \in \mathcal{E}). \end{aligned}$$

This completes the proof. □

Proof of Proposition 3.2. By Theorem 3.1 we need only check the convergence of the moments. With $\mathcal{F}^\omega(z) = \sum_{i=0}^\infty f_i z^i$, set $f(z) = \sum_{i=1}^\infty i^k f_i z^i$. Theorem 2.4 implies that

$$\mathbb{E}[c(S_n)^k] = \frac{[z^n] f(\mathcal{G}^\nu(z))}{[z^n] \mathcal{F}^\omega(\mathcal{G}^\nu(z))} \sim \frac{f'(\mathcal{G}^\nu(\rho))}{(\mathcal{F}')^\omega(\mathcal{G}^\nu(\rho))} = \mathbb{E}[(c(R) + 1)^k].$$

□

Proof of Lemma 3.3. We will tacitly assume that $n \equiv 1 \mod d$. The number Z_n is known as the partition function of simply generated trees. That is, the random plane tree $\mathcal{T}_n$ with distribution given by

$$\mathbb{P}(\mathcal{T}_n = T) = Z_n^{-1} \prod_{v \in T} \omega_{d^+(v)}$$

for any plane tree T, with $d^+(v)$ denoting outdegree of a vertex v, that is, its number of sons. There is a well-known connection between simply generated trees and branching processes [18, 8]: Precisely when $\rho_{\mathcal{Z}} > 0$, there is a critical or subcritical Galton–Watson tree $\mathcal{T}$, with

$$\mathbb{P}(|\mathcal{T}| = n) = Z_n \rho_{\mathcal{Z}}^n / \mathcal{Z}(\rho_{\mathcal{Z}}),$$

such the simply generated tree $\mathcal{T}_n$ is distributed like the $\mathcal{T}$ conditioned on having $|\mathcal{T}| = n$ vertices:

$$\mathcal{T}_n \stackrel{d}{=} (\mathcal{T} \mid |\mathcal{T}| = n).$$

Hence in order to verify

$$Z_n^{-1} \sum_{i+j=n+1} Z_i Z_j \sim 2\mathcal{Z}(\rho_{\mathcal{Z}})/\rho_{\mathcal{Z}},$$

we need to show that

$$\mathbb{P}(|\mathcal{T}| + |\mathcal{T}'| = n + 1) \sim 2\mathbb{P}(|\mathcal{T}| = n), \tag{6.8}$$

with $\mathcal{T}'$ denoting an independent copy of $\mathcal{T}$. Let ξ with $\mathbb{E}[\xi] \leqslant 1$ denote the offspring distribution of the Galton–Watson tree $\mathcal{T}$. Let $(\xi_i)_{i \geqslant 1}$ a family of independent copies of ξ, and

$$S_n = \xi_1 + \ldots + \xi_n$$

the associated random-walk. Any list $d_1, \ldots, d_n$ in $\mathbb{N}_0$ corresponds to the out-degrees of a depth-first-search ordered list of vertices of a plane tree with size n, if and only if

$$\sum_{i=1}^{n} d_i = n-1 \qquad \text{and} \qquad \sum_{i=1}^{k} d_i \geqslant k \quad \text{for all } k < n.$$

A classical combinatorial observation, also called the cycle lemma, states that for any sequence $x_1, \ldots, x_s \geqslant -1$ of integers satisfying

$$\sum_{i=1}^{s} x_i = -r$$

for some $r \geqslant 1$, there are precisely r integers $1 \leqslant u \leqslant s$ such that the cyclically shifted sequence

$$x_i^{(u)} = x_{1+(i+u) \mod s}$$

satisfies

$$\sum_{i=1}^{\ell} x_i^{(u)} > r$$

for all $1 \leqslant \ell \leqslant s-1$; see for example [18, Lem. 15.3]. Hence

$$\begin{aligned}\mathbb{P}(|\mathcal{T}| = n) &= \mathbb{P}(\xi_1 + \ldots + \xi_n = n-1, \xi_1 + \ldots + \xi_k \geqslant k \text{ for } k < n) \\ &= \frac{1}{n}\mathbb{P}(S_n = n-1).\end{aligned}$$

Likewise, any list $d_1, \ldots, d_{n+1}$ in $\mathbb{N}_0$ corresponds to the concatenation of the depth-first-search ordered lists of outdegrees of two plane trees with total size $n+1$, if and only if

$$\sum_{i=1}^{n+1} d_i = n-1 \qquad \text{and} \qquad \sum_{i=1}^{k} d_i \geqslant k-1 \quad \text{for all } k \leqslant n.$$

This yields

$$\begin{aligned}\mathbb{P}(|\mathcal{T}| + |\mathcal{T}'| = n+1) &= \mathbb{P}(\xi_1 + \ldots + \xi_{n+1} = n-1, \xi_1 + \ldots + \xi_k \geqslant k-1 \text{ for } k \leqslant n) \\ &= \frac{2}{n+1}\mathbb{P}(S_{n+1} = n-1).\end{aligned}$$

Hence

$$\frac{\mathbb{P}(|\mathcal{T}| + |\mathcal{T}'| = n+1)}{\mathbb{P}(|\mathcal{T}| = n)} = 2\frac{n+1}{n}\frac{\mathbb{P}(S_{n+1} = n-1)}{\mathbb{P}(S_n = n-1)}.$$

By the strong ratio property [20], it holds that

$$\mathbb{P}(S_{n+1} = n-1) \sim \mathbb{P}(S_n = n-1).$$

This verifies (6.8) and completes the proof. $\square$

Proof of Theorem 3.4. Let F denote a random $\mathcal{F}$-object following a $\mathbb{P}_{\mathcal{F}^\omega,\mathcal{G}^\nu(\rho)}$-distribution, and for each $1 \leqslant i \leqslant |\mathsf{F}|$ let G_i be an independent $\mathbb{P}_{\mathcal{G}^\nu,\rho}$ distributed $\mathcal{G}$-object. Then Lemma 2.2 yields that the tupel

$$\mathsf{S} := (\mathsf{F}, \mathsf{G}_1, \dots, \mathsf{G}_{|\mathsf{F}|})$$

follows up to relabelling a $\mathbb{P}_{\mathcal{F}^\omega \circ \mathcal{G}^\nu,\rho}$ distribution. We set $f = |\mathsf{F}|$ and $g_i = |\mathsf{G}_i|$ for all i. The Gibbs partition S_n is a random structure sampled from $(\mathcal{F} \circ \mathcal{G})[n]$ with probability proportional to its weight. Hence it is distributed like the Boltzmann structure S conditioned on having size n:

$$\mathsf{S}_n \stackrel{d}{=} (\mathsf{S} \mid g_1 + \dots + g_f = n). \tag{6.9}$$

By assumption, there is an integer $0 \leqslant m < d$ such that $\mathcal{G}^\nu(z)/z^m$ lies in the class $\mathscr{S}_d$. In particular, we have

$$g_i \equiv m \mod d.$$

for all $1 \leqslant i \leqslant f$. Recall that $D = d/\gcd(m, d)$. If $g_1 + \dots + g_f = n$, then for all $0 \leqslant a < D$ it holds that

$$f \equiv a \mod D \qquad \text{if and only if} \qquad n \equiv am \mod d. \tag{6.10}$$

For $n \geqslant 1$, the event $f \equiv a \mod D$ has positive probability if and only if the restriction $\mathcal{F}_a^\omega$ to objects with size in $a + D\mathbb{Z}$ has a non-constant generating function $\mathcal{F}_a^\omega(z)$. Let us fix an integer $0 \leqslant a < D$ with this property, and suppose that $n \equiv am \mod d$. Then Equations (6.9) and (6.10) imply that

$$\mathsf{S}_n \stackrel{d}{=} (\mathsf{S} \mid g_1 + \dots g_f = n, f \equiv a \mod D).$$

Conditioned on having size in $a + D\mathbb{Z}$, the random $\mathcal{F}$-object F follows a $\mathbb{P}_{\mathcal{F}_a^\omega \circ \mathcal{G}^\nu,\rho}$ distribution. Let F_a denote a $\mathbb{P}_{\mathcal{F}_a^\omega,\mathcal{G}^\nu(\rho)}$ distributed $\mathcal{F}_a$-object that is independent from all previously considered random variables. It follows that

$$\mathsf{S}_n \stackrel{d}{=} ((\mathsf{F}_a, \mathsf{G}_1, \dots, \mathsf{G}_{|\mathsf{F}_a|}) \mid g_1 + \dots + g_{|\mathsf{F}_a|} = n).$$

It follows by Lemma 2.2, that the vector

$$(\mathsf{F}_a, \mathsf{G}_1, \dots, \mathsf{G}_{|\mathsf{F}_a|})$$

has a $\mathbb{P}_{\mathcal{F}_a^\omega \circ \mathcal{G}^\nu,\rho}$ Boltzmann distribution, and consequently S_n is distributed like the n-sized Gibbs partition for $\mathcal{F}_a^\omega \circ \mathcal{G}^\nu$. If we can verify that

$$\mathbb{P}(g_1 + \dots + g_{f_a} = n) \sim \mathbb{E}[f_a]\mathbb{P}(g = n - (a-1)m), \qquad n \to \infty, \qquad n \equiv am \mod d, \tag{6.11}$$

then the convergence in total variation of R_n toward $\mathsf{R}(a)$ follows in an entirely analogous manner as in the proof of Theorem 3.1.

Thus it remains to check (6.11). Let g be distributed according to the size of a random $\mathcal{G}$-object following a $\mathbb{P}_{\mathcal{G}^\nu,\rho}$ Boltzmann distribution. We may write

$$g = m + d\bar{g}, \qquad f_a = a + \bar{f}_a D, \qquad n = am + \bar{n}d,$$

with $\bar{n} \in \mathbb{N}_0$, and $\bar{g}, \bar{f}_a$ random non-negative integers. We let $(\bar{g}_i^{(j)})_{i,j \geqslant 0}$, denote independent copies of $\bar{g}$, and set

$$S_i^{(j)} = g_1^{(j)} + \dots + g_i^{(j)}.$$

Thus

$$\mathbb{P}(g_1 + \ldots + g_{f_a} = n) = \mathbb{P}(S_a^{(0)} + (S_D^{(1)} + Dm/d) + \ldots + (S_D^{(\bar{f}_a)} + Dm/d) = \bar{n}).$$

Since $\mathcal{G}^\nu(z)/z^m$ lies in the class $\mathscr{S}_d$ by assumption, it follows that the probability weight sequence of $\bar{g}$ lies in $\mathscr{S}_1$. By Theorem 2.4 it follows that the densities of $S_a^{(0)}$ and $S_D^{(j)} + Dm/d$ belong to $\mathscr{S}_1$. Applying Theorem 2.4 again yields that the same holds for the randomly stopped sum $\sum_{j=1}^{\bar{f}_a}(S_D^{(j)} + Dm/d)$, with

$$\mathbb{P}(S_a^{(0)} = \bar{n}) \sim a\mathbb{P}(\bar{g} = \bar{n}) \qquad \text{and} \qquad \mathbb{P}(\sum_{j=1}^{\bar{f}_a}(S_D^{(j)} + Dm/d) = x) \sim D\mathbb{E}[\bar{f}_a]\mathbb{P}(\bar{g} = \bar{n})$$

as $\bar{n} \to \infty$. Hence Lemma 2.6 yields

$$\mathbb{P}(S_a^{(0)} + (S_D^{(1)} + Dm/d) + \ldots + (S_D^{(\bar{f}_a)} + Dm/d) = \bar{n}) \sim (a + D\mathbb{E}[\bar{f}_a])\mathbb{P}(\bar{g} = \bar{n}).$$

This verifies (6.11) and thus completes the proof. □

6.2 Proofs from Section 4

Proof of Theorem 4.1. Equation (2.2) yields that

$$z\mathcal{C}'(z) = z\phi(z\mathcal{C}'(z))$$

for the power series $\phi(z) = \exp(\mathcal{B}'(z))$. Hence we may apply Lemma 3.3 and obtain that the series $\mathcal{C}'(z)$ belongs to the class $\mathscr{S}_d$. That is, the coefficients $x_n = [z^n]\mathcal{C}'(z) = (n+1)c_{n+1}$ satisfy

$$\mathcal{C}'(\rho) < \infty, \qquad \frac{x_n}{x_{n+d}} \sim \rho^d, \qquad \frac{1}{x_n}\sum_{i+j=n} x_i x_j \sim 2\mathcal{C}'(\rho) < \infty,$$

as $n \equiv 0 \mod d$ becomes large. It is clear, that this also implies

$$\mathcal{C}(\rho) < \infty \qquad \text{and} \qquad \frac{c_n}{c_{n+d}} \sim \rho^d, \qquad n \to \infty, \qquad n \equiv 1 \mod d.$$

A characterization of subexponential series given for example in Foss, Korshunov, Zachary [13, Thm. 4.21] states that for any sequence $k_n \to \infty$ with $k_n < n/2$ it holds that

$$\frac{1}{x_n}\sum_{\substack{i+j=n \\ i,j \geqslant k_n}} x_i x_j \to 0, \qquad n \to \infty, \qquad n \equiv 0 \mod d.$$

As $x_n/x_{n+d} \sim \rho^d$, we may choose a sequence k_n that tends to infinity slowly enough such that

$$\lim_{n\to\infty} \sup_{\substack{0 \leqslant y \leqslant k_n \\ y \equiv 0 \mod d}} \left| \frac{x_n}{x_{n+y}} - \rho^y \right| = 0, \qquad n \equiv 0 \mod d.$$

Without loss of generality we may additionally assume that $k_n = o(n)$. Hence

$$\begin{aligned}\frac{1}{c_n}\sum_{i+j=n+1} c_i c_j &= \frac{1}{x_{n-1}}\sum_{i+j=n+1} \frac{n}{ij} x_{i-1}x_{j-1} \\ &= o(1) + 2\sum_{1\leqslant i<k_n} c_i \frac{n}{n-i}\frac{x_{n-i}}{x_{n-1}}, \\ &\to 2\mathcal{C}(\rho)/\rho.\end{aligned}$$

as $n \equiv 1 \mod d$ becomes large. Thus, the shifted series $\mathcal{C}(z)/z$ belongs to the class $\mathscr{S}_d$. □

Proof of Theorem 4.2. The convergence of the small fragments follows directly by Theorem 3.4. The asymptotic expression of $[z^n]\mathcal{A}(z)$ follows from the observation that $n \equiv a \mod d$ implies

$$[z^n]\mathcal{A}(z) = [z^n](\mathrm{SET}_a \circ \mathcal{C})(z) \sim C_{a-1}[z^{n-(a-1)}]\mathcal{C}(z),$$

similar as in Equation (6.11). □

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