

# Gradient estimate in terms of a Hilbert-like distance, for minimal surfaces and Chaplygin gas

Denis Serre

► **To cite this version:**

Denis Serre. Gradient estimate in terms of a Hilbert-like distance, for minimal surfaces and Chaplygin gas. Communications in Partial Differential Equations, Taylor

Francis, 2016, 41, pp.774 - 784. <10.1080/03605302.2015.1127969>. <ensl-01402393>

**HAL Id: ensl-01402393**

**<https://hal-ens-lyon.archives-ouvertes.fr/ensl-01402393>**

Submitted on 24 Nov 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Gradient estimate in terms of a Hilbert-like distance, for minimal surfaces and Chaplygin gas

Denis Serre

UMPA, UMR CNRS–ENS Lyon # 5669  
École Normale Supérieure de Lyon  
46, allée d’Italie, F–69364 Lyon, cedex 07

November 30, 2015

## Abstract

We consider a quasilinear elliptic boundary value problem with homogenous Dirichlet condition. The data is a convex planar domain. The gradient estimate is needed to ensure the uniform ellipticity, before applying regularity theory. We establish this estimate in terms of a distance which is equivalent to the Hilbert metric.

This fills the proof of existence and uniqueness of a solution to this BVP, when the domain is only convex but not strictly, for instance if it is a polygon.

**Keywords:** Chaplygin gas, Riemann problem, Keldysh-type degeneracy, Hilbert distance.

**MSC classification:** 76N10, 35J93, 52A10.

## 1 Introduction

Let  $\Omega$  be a connected planar open domain. We are interested in the following elliptic boundary-value problem:

$$\begin{aligned} (1) \quad & \operatorname{div} \frac{\nabla w}{\sqrt{1 + |\nabla w|^2}} + \frac{2}{w \sqrt{1 + |\nabla w|^2}} = 0 \quad \text{in } \Omega, \\ (2) \quad & w > 0 \quad \text{in } \Omega, \\ (3) \quad & w = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Because of ellipticity we anticipate that the solution is classical, hence (1) amounts to

$$(4) \quad (1 + |\nabla w|^2)\Delta w - D^2w : \nabla w \otimes \nabla w + \frac{2}{w}(1 + |\nabla w|^2) = 0.$$

The principal part in (1) is the operator of minimal surfaces. There are at least three interpretations of the PDE (1) :

- Equation (1) is the Euler–Lagrange equation for the functional

$$A[w] = \int \frac{\sqrt{1 + |\nabla w|^2}}{w^2} dx.$$

The graph  $x_3 = w(x_1, x_2)$  is therefore a complete minimal surface in the half-space  $x_3 > 0$  endowed with the Poincaré Riemannian metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}.$$

The curve  $\partial\Omega$  is the asymptote of the surface at the “infinity”  $x_3 = 0$ . We notice that the total area of the graph is infinite; see [3], especially a remark after Theorem 5.1.

General minimal surfaces associated with the metric  $ds^2$ , with prescribed asymptote at infinity have been studied by Anderson [1] in terms of currents. When  $\partial\Omega$  is not a convex curve, a minimal surface is not a graph over  $\Omega$ .

- The hypersurface of revolution in  $\mathbb{R}^4$  defined by

$$\sqrt{x_3^2 + x_4^2} = w(x_1, x_2)$$

satisfies the equation

$$\kappa_3 = \frac{1}{2}(\kappa_1 + \kappa_2),$$

in terms of its principal curvatures; see [3], especially a remark after Theorem 5.1.

- Our original motivation in [3] was the resolution of a 2-dimensional Riemann problem for a gas that obeys the equation of state of Chaplygin.

The function  $\phi := \frac{1}{2}w^2$  is the potential of a self-similar irrotational isentropic flow. The independent variable is a self-similar coordinate  $x = \frac{y}{t}$ , where  $y, t$  are the space and time variables, respectively. The cone defined by  $y \in t\Omega$  is the domain where the flow is pseudo-subsonic. The velocity and the density are given by

$$u(y, t) = \frac{y}{t} + (\nabla_x \phi) \left( \frac{y}{t} \right), \quad \rho(y, t) = \frac{a}{\sqrt{2\phi + |\nabla \phi|^2}}$$

for some constant  $a > 0$ . The domain  $\Omega$  can be determined from the Riemann data, by solving explicitly the flow in its pseudo-supersonic regime.

The data of the BVP is nothing but the domain  $\Omega$ . The function  $\phi$  defined in the third item above helps us to guess for which domains the BVP is likely to be well-posed. The equation satisfied by  $\phi$  is

$$(5) \quad (2\phi + |\nabla \phi|^2)\Delta\phi - D^2\phi : \nabla\phi \otimes \nabla\phi + 4\phi + |\nabla\phi|^2 = 0.$$

The positivity of  $\phi$  is the condition that this second-order equation is elliptic. Of course the ellipticity degenerates at the boundary because of (3). However this degeneracy is of a rather

strong form, that called Keldysh type; formally, if  $\phi$  could be extended such that  $\phi < 0$  away from  $\overline{\Omega}$  (hyperbolic type), then the characteristic curves would be tangent to  $\partial\Omega$ . This is the symptom that if  $\phi \in \mathcal{C}^2(\overline{\Omega})$ , the normal derivative can be calculated in terms of the curvature of  $\partial\Omega$  (again, see [3], Paragraph 5.4) :

$$(6) \quad \kappa \frac{\partial\phi}{\partial\nu} = -1.$$

Because  $\phi$  is positive in  $\Omega$  and vanishes at the boundary, we must have

$$\frac{\partial\phi}{\partial\nu} \leq 0,$$

which yields the necessary condition that  $\kappa > 0$ . In other words,  $\Omega$  needs to be convex, in a strong sense.

We proved in [3] that this necessary condition is also sufficient:

**Theorem 1.1** *Let  $\Omega$  be a bounded convex planar domain, whose boundary  $\partial\Omega$  is a piecewise- $\mathcal{C}^2$  curve, with uniformly strictly positive curvature.*

*Then there exists one and only one strictly positive solution  $\phi \in Lip(\Omega) \cap \mathcal{C}^\infty(\Omega)$  of (5) satisfying the boundary condition  $\phi = 0$ .*

Of course,  $w = \sqrt{2\phi}$  is the unique solution of our BVP in this situation. We remark that, although  $w$  is smooth in  $\Omega$ , it is not globally Lipschitz, as it experiences a square root singularity at the boundary.

Theorem 1.1 raises the question whether a solution exists when  $\Omega$  is a convex domain but the curvature vanishes in some part of the boundary ; for instance, we are interested in the case where the domain is polygonal. Then (6) suggests that even  $\phi$  will not be globally Lipschitz over  $\Omega$ . This observation is meaningful because the Lipschitz estimate of  $\phi$  was a crucial step in the proof of Theorem 1.1. We recall now the procedure that we followed in [3] :

**Step #1.** Construct a sequence of approximate solutions  $w^m$ . There are several possible choices, but they need to be consistent with the estimates described below.

**Step #2.** Equation (1) satisfies the maximum principle. One may compare  $w$  to exact solutions  $w_{p,r} = \sqrt{2\phi_{p,r}}$  where

$$\phi_{p,r} = \frac{1}{2}(r^2 - |x - p|^2).$$

We obtain upper/lower bounds  $w_\pm$ , given by

$$\phi_- = \inf\{\phi_{p,r} \mid D(p;r) \subset \Omega\}, \quad \phi_+ = \sup\{\phi_{p,r} \mid \Omega \subset D(p;r)\},$$

where  $D(p;r)$  is the disk of radius  $r$ , with center at  $p$ .

The solution is therefore expected to satisfy

$$(7) \quad \phi_- \leq \phi \leq \phi_+.$$

The lower bound ensures that  $w > 0$  in  $\Omega$ , while the upper bound provides an  $L^\infty$  estimate. Of course, the approximate solutions must be constructed so as to fill the same inequality, or an approximate form of it.

**Step #3.** Because the curvature is everywhere strictly positive (this is the assumption that we intend to drop below), we have  $\phi_{\pm} \equiv 0$  on the boundary. Therefore (7) ensures that  $w$  is continuous at the boundary, with  $w = 0$  along  $\partial\Omega$ .

Actually, (7) implies the bounds

$$0 \geq \frac{\partial\phi}{\partial\nu} \geq \frac{\partial\phi_+}{\partial\nu} \geq -\frac{1}{\bar{\kappa}},$$

where  $\bar{\kappa}$  is a lower bound of the  $\kappa$  along  $\partial\Omega$ . This can be recast as

$$(8) \quad |\nabla\phi| \leq \frac{1}{\bar{\kappa}} \quad \text{along } \partial\Omega.$$

**Step #4.** It turns out that if  $\alpha \in (2, 3)$ , then  $\alpha\phi + |\nabla\phi|^2$  must reach its maximum on the boundary. This fact and (8) provide the needed *a priori* Lipschitz estimate.

**Step #5.** The Lipschitz estimate provides the relative compactness of  $w^m$  in  $\mathcal{C}(\bar{\Omega})$ . It also tells us that (1) is a *uniformly* elliptic equation. Therefore the regularity theory provides interior estimates for  $D^2w^m$  and higher derivatives (see [2]). With Ascoli–Arzela Theorem, this ensures the relative compactness of  $\nabla w^m$  in  $\mathcal{C}(K)$  for every compact  $K \subset \Omega$ . We may therefore pass to the limit in a subsequence : the limit  $w$  satisfies the PDE. We obtain in the limit  $\phi_- \leq \phi \leq \phi_+$ , which guaranties (2,3).

When the curvature vanishes somewhere, two arguments in the strategy above fall down. On the one hand, it may happen that  $\phi_+$  does not vanish on the boundary. When  $\Omega$  is a triangle, one finds that  $\phi_+$  coincide with  $\phi_{P;R}$  where  $D(P;R)$  is the circumscribed disk ; then  $\phi_+$  vanishes only at the vertices. On the other hand, even if  $\kappa$  vanishes only at isolated points and therefore  $\phi_+ \equiv 0$  on the boundary, its normal derivative must be infinite at these exceptional points. Therefore one cannot conclude in the Step #4 above.

The purpose of this article is to provide new *a priori* estimates for both  $w$  and  $\nabla w$ , which are valid for arbitrary bounded convex domains. The  $L^\infty$  estimates involve one-dimensional barrier functions ; this part is far from original. The technique employed to estimate the gradient is more interesting and, up to our knowledge, rather new. We use the scale invariance of (1) and compare  $w$  to *itself*, after translation and dilation. We obtain a Lipschitz estimate of the form

$$|\log w(y) - \log w(x)| \leq d(x, y),$$

where  $d$  is a distance on  $\Omega$ , which is locally equivalent to the Euclidian distance. For the metric  $d$ ,  $\partial\Omega$  is a horizon at infinity ; this ressembles the situation when  $\Omega$  is equipped with the Hilbert distance, although our metric does not coincide with the latter.

With these estimates in hands, we may proceed as in Step #5 above. Passing to the limit from uniformly convex domains to arbitrary ones, we obtain the more general conclusion :

**Theorem 1.2** *Let  $\Omega$  be a bounded planar convex domain. Then the boundary-value problem (1,2,3) admits one and only one solution  $w \in \mathcal{C}(\bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$ .*

**Plan of the paper:** We construct barrier functions in Section 2. Even if they are new with respect to [3], they are hardly surprising. We introduce our (new ?) distance on convex domains in Section 3, and compare it with the Euclidian distance and the Hilbert metric. Our most original idea is the Lipschitz estimate, presented in Section 4. We use both estimates to prove Theorem 1.2 in Section 5.

## 2 Barrier functions, $L^\infty$ -estimate

Let  $y := x \cdot \xi$  be a one-dimensional variable, where  $\xi$  is a unit vector. The function  $x \mapsto W(y)$  solves (1) if and only if

$$(9) \quad \frac{1}{2} WW'' + W'^2 + 1 = 0.$$

**Lemma 2.1** *Let  $I = (a, b)$  be a non-void interval. There exists a positive solution  $W^I$  of (9) over  $I$ , with the property that  $W^I(a) = W^I(b) = 0$ .*

*Proof*

Let  $v(t)$  be the solution of the Cauchy problem

$$v' = \sqrt{1 - 9v^{4/3}}, \quad v(0) = 0.$$

The function  $v$  is increasing from 0 to the maximum  $\frac{1}{3\sqrt{3}}$  reached at

$$y^* = \int_0^{\frac{1}{3\sqrt{3}}} \frac{dv}{\sqrt{1 - 9v^{4/3}}} = \frac{1}{3\sqrt{3}} \int_0^1 \frac{du}{\sqrt{1 - u^{4/3}}}.$$

The function  $v$  can be extended over  $(0, 2y^*)$  by  $v(2y^* - y) = v(y)$ , as a solution of

$$v'^2 + 9v^{4/3} = 1.$$

Then  $z := v^{1/3}$  satisfies  $z'^2 + 1 = \frac{1}{9} z^{-4}$ , or

$$z^4 z'^2 + z^4 = \frac{1}{9}.$$

Differentiating once, we obtain

$$\frac{1}{2} z z'' + z'^2 + 1 = 0.$$

There remains to define

$$W^I(y) = \frac{b-a}{2y^*} z \left( 2y^* \frac{y-a}{b-a} \right).$$

■

A more careful analysis provides a positive solution such that  $W(a) = W(b) = 0$ , and  $W(a+b-y) = W(y)$ . We don't need it to establish the following upper bound:

**Corollary 2.1** *Let  $\Omega$  be a bounded planar convex domain. Then there exists a super-solution  $w^+ \in \mathcal{C}(\overline{\Omega})$  of (1), positive in  $\Omega$  and vanishing along  $\partial\Omega$ . It satisfies*

$$w^+ \leq \frac{\text{diam } \Omega}{2y^* \sqrt{3}}.$$

*Proof*

Given a direction  $\xi \in \mathbf{S}^1$ , let us define

$$a(\xi) = \inf_{x \in \Omega} x \cdot \xi, \quad b(\xi) = \sup_{x \in \Omega} x \cdot \xi.$$

Let us denote  $I(\xi) = (a(\xi), b(\xi))$ . Lemma 2.1 provides a solution of (1) of the form  $w_\xi(x) = W^{I(\xi)}(x \cdot \xi)$ . Then

$$w^+(x) = \inf_{\xi} w_\xi(x)$$

is a continuous positive super-solution of (1).

The bound

$$w_+(x) \leq \frac{\text{diam} \Omega}{2y^*} z \left( \min_{\xi} \frac{x \cdot \xi - a(\xi)}{b(\xi) - a(\xi)} \right)$$

shows that  $w_+(x) \rightarrow 0$  as  $x \rightarrow \bar{x} \in \partial\Omega$  (consider the inward unit normal at  $\bar{x}$ ). ■

Of course, if  $\Omega$  is a polygon, a super-solution vanishing at the boundary can be defined as the minimum of finitely many one-dimensional solutions of (1).

### 3 A distance over a bounded convex domain

As above,  $\Omega$  is a non-void, bounded convex open domain. The fact that  $\Omega$  is 2-dimensional is not essential here.

Given two points  $p, q \in \Omega$ ,  $\Omega - p$  contains a ball centered at the origin and is therefore absorbing. Thus there exists some  $\lambda > 0$  such that  $\Omega - q \subset \lambda \cdot (\Omega - p)$ . If  $\mu > \lambda$ , then also  $\Omega - q \subset \mu \cdot (\Omega - p)$ , by convexity. Likewise, the infimum  $m(p, q)$  of all such numbers satisfies the same inclusion, by continuity. Hence the set of these numbers is of the form  $[m(p, q), +\infty)$ . Considering the volumes, we have

$$|\Omega| = |\Omega - q| \leq m(p, q)^2 |\Omega - p| = m(p, q)^2 |\Omega - q|,$$

which implies

$$(10) \quad m(p, q) \geq 1.$$

The equality in (10) stands only if

$$\Omega - q = \Omega - p,$$

that is if  $p = q$ .

If  $r \in \Omega$  is a third point, then

$$\Omega - r \subset m(q, r) \cdot (\Omega - q) \subset m(q, r)m(p, q) \cdot (\Omega - p)$$

and therefore

$$m(p, r) \leq m(q, r)m(p, q).$$

All this shows that the logarithm of  $m$  is a non-negative function over  $\Omega \times \Omega$ , which vanishes only along the diagonal and satisfies the triangle inequality. In other words, the function

$$d_\Omega(p, q) = \log m(p, q) + \log m(q, p)$$

is a *distance* over  $\Omega$ .

Our distance is finer than the Euclidian one:

**Proposition 3.1** *Denote  $M$  the diameter of  $\Omega$ . Then*

$$|q - p| \leq \frac{M}{2} d_\Omega(p, q).$$

*Proof*

Let  $J = (\alpha, \beta)$  be the segment in  $\Omega$  passing through  $p$  and  $q$ , with  $\alpha, \beta \in \partial\Omega$ . We may assume that  $\alpha, p, q, \beta$  are ordered in that way along the line, and we use the same order along  $J$  below. If

$$\Omega - q \subset \lambda \cdot (\Omega - p),$$

then  $J - q \subset \lambda \cdot (J - p)$ , which means  $\alpha - q \geq \lambda(\alpha - p)$ , that is

$$q - p \leq (q - \alpha) \left(1 - \frac{1}{\lambda}\right).$$

We deduce

$$|q - p| \leq M \left(1 - \frac{1}{m(p, q)}\right) \leq M \log m(p, q).$$

■

The opposite inequality is non-uniform at the boundary:

**Proposition 3.2** *The following inequality holds true:*

$$\begin{aligned} d_\Omega(p, q) &\leq \log \left(1 + \frac{|q - p|}{\text{dist}(p, \partial\Omega)}\right) + \log \left(1 + \frac{|q - p|}{\text{dist}(q, \partial\Omega)}\right) \\ &\leq |q - p| \left( \frac{1}{\text{dist}(p, \partial\Omega)} + \frac{1}{\text{dist}(q, \partial\Omega)} \right), \end{aligned}$$

where  $\text{dist}(\cdot, \partial\Omega)$  denotes the Euclidian distance to the boundary.



*Proof*  
One has

$$\Omega - q = p - q + \Omega - p \subset D(0; |q - p|) + \Omega - p \subset \left( \frac{|q - p|}{\text{dist}(p, \partial\Omega)} + 1 \right) \cdot (\Omega - p),$$

whence

$$m(p, q) \leq \frac{|q - p|}{\text{dist}(p, \partial\Omega)} + 1.$$

We conclude with the inequality  $\log(1 + t) \leq t$ . ■

Using the fact that the inclusion between two disks  $D(P; r) \subset D(Q; s)$  is equivalent to  $s - r \geq |P - Q|$ , we calculate easily the distance associated with the unit disk  $D$  :

$$d_D(p, q) = 2 \log \left( 1 - p \cdot q + \sqrt{|q - p|^2 - |p|^2 |q|^2 \sin^2 \theta} \right) - \log(1 - |p|^2) - \log(1 - |q|^2),$$

where  $\theta$  denotes the angle between the vectors  $p$  and  $q$ . When  $|q| \rightarrow 1$ , the first term above is equivalent to  $\log(2 - 2|p| \cos \theta)$ . If  $p$  is kept fixed, we thus have

$$d_D(p, q) \sim \log \frac{1}{1 - |q|} \rightarrow +\infty.$$

Therefore the boundary is the infinite horizon of  $D$ . In other words,  $(D, d_D)$  is a complete metric space.

For a general domain, we have

**Proposition 3.3** *The following inequality holds true:*

$$d_\Omega(p, q) \geq |\log \text{dist}(q, \partial\Omega) - \log \text{dist}(p, \partial\Omega)|.$$

*In particular,  $(\Omega, d_\Omega)$  is a complete metric space.*

*Proof*

Denote  $r = \text{dist}(q, \partial\Omega)$  and  $s = \text{dist}(p, \partial\Omega)$ . We have

$$D(0; r) \subset \Omega - q \leq m(p, q) \cdot (\Omega - p),$$

whence

$$D\left(0; \frac{r}{m(p, q)}\right) \subset \Omega - p.$$

We deduce

$$m(p, q) \geq \frac{r}{s}, \quad \text{or} \quad \log m(p, q) \geq (\log r - \log s)^+.$$

■

Propositions 3.2 and 3.3 show that when  $p$  is kept fixed and  $q$  tends to  $\partial\Omega$ , then  $d_\Omega(p, q)$  blows up like

$$\log \frac{1}{\text{dist}(q, \partial\Omega)}.$$

When  $\Omega = (a, b)$  is one-dimensional, our distance coincides with that defined by Hilbert:

$$d_{(a,b)}(p, q) = \left| \log \frac{b-q}{b-p} \frac{p-a}{q-a} \right|.$$

For a convex domain  $\Omega$  of arbitrary dimension, the Hilbert distance  $d_H(p, q)$  between two points  $p, q \in \Omega$  is defined as  $d_J(p, q)$  where  $J = \Omega \cap \text{Aff}(pq)$  is the segment obtained by intersecting  $\Omega$  with the line passing through  $p$  and  $q$ . The coincidence of  $d_\Omega$  with  $d_H$  holds true only in one space dimension. In higher dimension, it fails, as shown by our formula above for a disk. Instead, we have

**Proposition 3.4** *For general bounded convex domains, we have  $d_H \leq d_\Omega$ , where  $d_H$  denotes the Hilbert metric.*

*Proof*

Let  $L$  be an affine subspace, and define  $\omega := L \cap \Omega$ . If  $p, q \in \omega$  and if  $\Omega - q \subset \lambda \cdot (\Omega - p)$ , then also  $\omega - q \subset \lambda \cdot (\omega - p)$ , because  $L - q = L - p$  is a vector space. We deduce  $m_\omega(p, q) \leq m_\Omega(p, q)$ , hence  $d_\omega(p, q) \leq d_\Omega(p, q)$

Choose  $L$  the line passing through  $p$  and  $q$  and remark that  $d_L(p, q)$  is nothing but  $d_H(p, q)$ . ■

## 4 Lipschitz estimate

Suppose  $w$  is a solution of (1,2,3) in a bounded convex planar domain  $\Omega$ .

Observe that the equation (1) is translation invariant as well as dilation invariant: if  $r \in \mathbb{R}^2$  and  $\mu > 0$ , then

$$x \mapsto \frac{1}{\mu} w(\mu x + r)$$

is also a solution.

Given  $p, q \in \Omega$ , we thus define

$$\bar{w}(x) := \lambda w \left( p + \frac{x - q}{\lambda} \right), \quad \text{with } \lambda = m(p, q),$$

which is a positive solution of (1) in the domain

$$q + \lambda(\Omega - p).$$

Because the latter contains  $\Omega$ ,  $\bar{w}$  is a super-solution of (1,2,3) and we infer

$$w \leq \bar{w}.$$

Evaluating this inequality at  $q$ , we obtain

$$w(q) \leq \bar{w}(q) = m(p, q)w(p),$$

that is

$$(\log w(q) - \log w(p))^+ \leq \log m(p, q).$$

This yields our Lipschitz estimate:

**Proposition 4.1** *Let  $\Omega$  be a bounded convex planar domain, and  $w$  be a solution of the BVP (1,2,3) in  $\Omega$ . Then we have*

$$|\log w(q) - \log w(p)| \leq d_\Omega(p, q).$$

Using Proposition 3.2, we deduce

**Corollary 4.1** *We have the gradient estimate*

$$|\nabla w(p)| \leq \frac{2w(p)}{\text{dist}(p; \partial\Omega)}.$$

When the boundary has a positive curvature, we know that  $w^2$  is Lipschitz up to the boundary, with non-zero normal derivative. The estimate above is thus improved into

$$|\nabla w(p)| = O\left(\frac{1}{\sqrt{\text{dist}(p; \partial\Omega)}}\right).$$

For general convex domain, the accuracy of Corollary 4.1 is unclear. The barrier function  $w^+$  constructed in Section 2 satisfies a slightly better bound

$$|\nabla w_+(p)| = O\left(\frac{1}{(\text{dist}(p; \partial\Omega))^{2/3}}\right).$$

## 5 Proof of Theorem 1.2

### 5.1 Outer approximation of $\Omega$ by strictly convex domains

We begin by constructing a one-parameter family of bounded convex domains  $\Omega_\epsilon$  ( $\epsilon > 0$ ), with three properties:

1. The boundary  $\partial\Omega_\epsilon$  is a  $\mathcal{C}^1$ -curve, piecewise  $\mathcal{C}^2$ . Its curvature is bounded below by  $\epsilon$ .
2. The family  $\epsilon \mapsto \Omega_\epsilon$  is decreasing for the inclusion.
3. More precisely

$$\Omega_\epsilon \subset \Omega + D(0; h(\epsilon)), \quad \lim_{\epsilon \rightarrow 0^+} h(\epsilon) = 0.$$

To begin with, we define  $U^\epsilon$  as the intersection of all disks  $D(p; \epsilon + \frac{1}{\epsilon})$  such that

$$\Omega \subset D(p; \frac{1}{\epsilon}).$$

Each of this disk is convex, with a curvature less than  $\epsilon$ , and so is their intersection. The boundary has at most denumerably many vertices. Each vertex, being at a distance  $\geq \epsilon$  of  $\Omega$ , can be smoothed out by placing a bitangent arc of circle of radius  $\frac{\epsilon}{2}$ . This is done in a unique way and yields a domain  $\Omega_\epsilon \subset U^\epsilon$  which satisfies the requirements above.

## 5.2 A converging approximation

According to [3], Theorem 5.1, the BVP (1,2,3) admits a unique solution  $w_\epsilon \in \mathcal{C}(\bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$ . If  $\epsilon < \eta$  then  $w_\eta$  is a super-solution to the BVP in  $\Omega_\epsilon$  and thus  $w_\epsilon \leq w_\eta$  by the maximum principle. The map  $\epsilon \mapsto w_\epsilon$  is therefore non-decreasing.

A positive lower bound is the one established in [3]. It uses the solutions  $w_{p,r}$  of (1) defined in Step #2 above. If  $D(p; r) \subset \Omega$ ,  $w_{p,r}$  is a sub-solution for all these BVPs and we infer  $w_\epsilon \geq w_{p,r}$  in  $D(p; r)$ . We deduce

$$w_\epsilon \geq w_- := \sup\{w_{p,r} \mid D(p; r) \subset \Omega\}.$$

If  $K$  is a compact subset of  $\Omega$ , this lower bound is uniformly positive over  $K$ .

By monotonicity, the limit

$$w(x) = \lim_{\epsilon \rightarrow 0^+} w_\epsilon(x)$$

exists and satisfies  $w_- \leq w$ . Therefore  $w > 0$  in  $\Omega$ . On the other hand, passing to the limit in the bound  $w_\epsilon \leq w_\epsilon^+$  obtained in Section 2, yields the upper bound  $w \leq w_+$ , implying

$$(11) \quad \lim_{d(x; \partial\Omega) \rightarrow 0} w(x) = 0.$$

To each  $w_\epsilon$ , we apply the  $L^\infty$  estimate (Section 2) and the Lipschitz estimate (Corollary 4.1) above. This shows that the operator

$$L_\epsilon = (1 + |\nabla w_\epsilon|^2)\Delta - \nabla w_\epsilon \otimes \nabla w_\epsilon : \mathbb{D}^2$$

is uniformly (in  $\epsilon$  and  $x$ ) elliptic in  $\Omega'$ , provided  $\Omega' \Subset \Omega$ .

**$\mathcal{C}^{2,\alpha}$ -Estimates.** Let us fix a sub-domain  $\Omega' \Subset \Omega$ . Our Lipschitz estimate tells us that there exists a finite number  $\varpi$  such that  $|\nabla_\epsilon(x)| \leq \varpi$  for every  $\epsilon < \eta$  and  $x \in \Omega'$ . In this region,  $w_\epsilon$  is therefore a solution of the quasilinear uniformly elliptic equation

$$(1 + |f(\nabla w_\epsilon)|^2)\Delta w_\epsilon - \mathbb{D}^2 w_\epsilon : f(\nabla w_\epsilon) \otimes f(\nabla w_\epsilon) + \frac{2}{w_\epsilon} (1 + |\nabla w_\epsilon|^2) = 0,$$

where

$$f(p) = \begin{cases} p, & \text{if } |p| \leq \varpi, \\ \frac{\varpi}{|p|} \left(2 - \frac{\varpi}{|p|}\right) p, & \text{if } |p| \geq \varpi. \end{cases}$$

We may therefore apply Theorem 13.6 in [2] to derive uniform estimates of  $\mathbb{D}^2 w_\epsilon$  in  $\mathcal{C}^\alpha(\Omega'')$  whenever  $\Omega'' \Subset \Omega'$ .

**Concluding.** Applying repeatedly Theorem 6.17, we actually obtain uniform estimates of higher derivatives in every sub-domain  $\omega \Subset \Omega$ . This ensures that  $w \in \mathcal{C}^\infty(\Omega)$ . With (11), we have in particular  $w \in \mathcal{C}(\overline{\Omega})$ . Finally,  $w_\epsilon \rightarrow w$  in  $\mathcal{C}^\infty(\Omega)$  and we may pass to the limit in

$$(1 + |\nabla w_\epsilon|^2)\Delta w_\epsilon - D^2 w_\epsilon : \nabla w_\epsilon \otimes \nabla w_\epsilon + \frac{2}{w_\epsilon} (1 + |\nabla w_\epsilon|^2) = 0$$

and infer that  $w$  solves (1). This proves the existence part of Theorem 1.2. The uniqueness is just a consequence of the maximum principle, as in [3].

**Acknowledgement.** I am indebted to the referee for his/her careful reading and pointing out the correct reference for the regularity estimates.

## References

- [1] M. Anderson. Complete minimal varieties in hyperbolic space. *Inventiones Math.*, **69** (1982), pp 477–494.
- [2] D. Gilbarg, N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer-Verlag (2001), Heidelberg.
- [3] D. Serre. Multi-dimensional shock interaction for a Chaplygin gas. *Arch. Rational Mech. Anal.*, **191** (2009), pp 539–577.