The relative entropy method for the stability of intermediate shock waves ; the rich case

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Dedicated to Peter Lax, with admiration and gratitude

Abstract

M.-J. Kang and one of us [2] developed a new version of the relative entropy method, which is efficient in the study of the long-time stability of extreme shocks. When a system of conservation laws is rich, we show that this can be adapted to the case of intermediate shocks.

1   Introduction

We consider a strictly hyperbolic system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \qquad u(x,t) \in \mathcal{U},$$

where $\mathcal{U}$ is a convex open domain in $\mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}^n$ is a smooth vector field. By assumption, the Jacobian $\mathrm{d}f(u)$ is diagonalizable with real eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$ and eigenvectors $r_k(u)$. The fields $u \mapsto (\lambda_k, r_k)$ are smooth.

An entropy-flux pair is made of functions $\eta(u), q(u)$ for which every classical solution satisfies in addition

$$\partial_t \eta(u) + \partial_x q(u) = 0.$$

We are interested only in entropies such that $\mathrm{D}^2\eta$ is positive definite, which we call *strongly convex* entropies. Then we say that a bounded measurable solutions is admissible if it satisfies the *entropy inequality*

$$\partial_t \eta(u) + \partial_x q(u) \leq 0.$$

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for every convex entropy.

If a is a fixed state, the *relative* entropy $\eta(u|a)$ and its relative flux $q(u;a)$ are defined as usual by

$$\eta(u|a) = \eta(u) - \eta(a) - \mathrm{d}\eta_a \cdot (u - a), \qquad q(u;a) = q(u) - q(a) - \mathrm{d}\eta_a \cdot (f(u) - f(a)).$$

We point out that $\eta(u|a) > 0$ unless $u = a$. The relative entropy-flux pair still satisfies

$$\partial_t \eta(u|a) + \partial_x q(u;a) \leq 0$$

for every admissible solution.

Let U denote a pure shock wave:

$$U(x,t) = \begin{cases} u_\ell, & \text{if } x < \sigma t, \\ u_r, & \text{if } x > \sigma t. \end{cases}$$

Its data $u_{\ell,r}$ satisfy the Rankine–Hugoniot jump relation

$$(1) \qquad\qquad f(u_r) - f(u_\ell) = \sigma(u_r - u_\ell).$$

We treat here the case of a Lax shock : there is an index $1 \leq k \leq n$ (we speak of a k-shock) such that the following inequalities hold true :

$$(2) \qquad\qquad \lambda_{k-1}(u_\ell) < \sigma < \lambda_{k+1}(u_r), \qquad \lambda_k(u_r) < \sigma < \lambda_k(u_\ell).$$

P. Lax [5] proved that such shocks exist with $u_{\ell,r} \sim u^*$ if the k-th characteristic field is genuinely nonlinear at u^* : $(\mathrm{d}\lambda_k \cdot r_k)(u^*) \neq 0$. The jump $[u] := u_r - u_\ell$ is then approximately colinear with $r_k(u^*)$.

The assumption (2), together with the Implicit Function Theorem, ensures that u_r can be locally defined as a smooth function of the other parameters: $(u_\ell, \sigma) \mapsto u_r$.

Leger & Vasseur showed in [3, 4] that the relative entropy method applied to a scalar (case $n = 1$) conservation law, with a convex (or concave) flux f, yield the strong property that

$$t \mapsto \inf_{h \in \mathbb{R}} \left(\int_{-\infty}^{h} \eta(u|u_\ell) \, dx + \int_{h}^{+\infty} \eta(u|u_r) \, dx \right)$$

is non-increasing for every shock U and every entropy solution u, whenever the initial data $u(\cdot, 0)$ belongs to $U + L^2(\mathbb{R})$. We interpret this property as saying that scalar shock waves are *attractors up to a shift*.

We considered in [13] the relative entropy method applied to systems ($n \geq 2$). We were interested in those shock waves that are attractors up to a shift. We found that, although the so-called Keyfitz–Kranzer system does admit such stable shocks, most systems resist to the

method. A new tool was therefore needed, which was elaborated recently by Vasseur & M.-J. Kang [2]. Their idea was, instead of integrating the relative entropy associated with the same η on both sides of the shock, to involve η on the left side of the shock, and $a\eta$ on the right side, for some appropriate constant $a > 0$. Depending on whether a is small or large, they obtained that either 1-shocks or n-shocks (the so-called *extreme shocks*) are local attractors up to a shift. The method however, even with this improvement, does not work for intermediate shocks, as those encountered in elasticity or MHD. Our goal is therefore to adapt it to the case of a rich system (see [10] for this notion). We shall use two entropies $\eta_\pm$ on both sides of the shock to measure the distance from u to v. We thus consider the quantity

$$E[u; h] := \int_{-\infty}^{h} \eta_-(u|u_\ell) \, dx + \int_{h}^{+\infty} \eta_+(u|u_r) \, dx,$$

which incorporates a shift h. For intermediate shocks, η_+ and η_- will not be linearly dependent *modulo* the affine functions. Our main result (Theorem 3.1) is that given a Lax shock of a rich system, strongly convex entropies $\eta_\pm$ can be chosen so that the shock be a local attractor up to a shift.

The necessity of a shift tolerance

Let U be a Lax shock associated with a genuinely nonlinear field. Without loss of generality, we may assume that it is steady. If $u^0 = U + \phi$ is a compactly supported perturbation where ϕ is small, then the asymptotic behaviour as $t \to +\infty$ has been described by T.-P. Liu [6]. It consists of a superposition of so-called N-waves (which decay in L^2-norm like $t^{-1/4}$), of linear waves that just propagate at constant velocity, and of a shift of the shock from $x = 0$ to $x = h$. To determine h, we split the mass $m = \int_{\mathbb{R}} \phi \, dx$ into three parts $X_- + h[u] + X_+$, where X_- (resp. X_+) belongs to the stable (resp. unstable) subspace of $\mathrm{d}f(u_\ell)$ (resp. $\mathrm{d}f(u_r)$) ; thus $h[u] = \pi m$ for some projection operator in $\mathbb{R}^n$. All these waves asymptotically separate from each other and therefore the limit of $E[u(t); 0]$ equals either $|h|\eta_-(u_r|u_\ell)$ or $|h|\eta_+(u_\ell|u_r)$, depending on the sign of h.

If $E[\cdot; 0]$ was a Lyapunov function for the system, then this limit would not be greater than

$$E[U + \phi; 0] = \int_{-\infty}^{0} \eta_-(u_\ell + \phi|u_\ell) \, dx + \int_{0}^{+\infty} \eta_+(u_r + \phi|u_r) \, dx \sim \int_{\mathbb{R}} \phi^2 \, dx.$$

This would imply an inequality

$$|u_r - u_l|^2 \left| \int_{\mathbb{R}} \pi\phi \, dx \right| \leq C \cdot \int_{\mathbb{R}} \phi^2 dx,$$

which is obviously false because $L^2(\mathbb{R})$ does not embed into $L^1(\mathbb{R})$. Therefore the uniform stability of shock waves cannot be obtained without involving a shift, as it was done by Leger & Vasseur in the scalar case. This explains why we consider

$$E_{\min}(t) := \inf_{h} E[u(t); h] = \inf_{h} \left(\int_{-\infty}^{h} \eta_-(u(x, t)|u_\ell) \, dx + \int_{h}^{+\infty} \eta_+(u(x, t)|u_r) \, dx \right).$$

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2 Local analysis

If a shock U is stable, we expect, following [6], that for every admissible solution u, close enough to U, there is a shock curve $x = X(t)$ with $X'(t) \to \sigma$ and $u_\pm := u(X(t) \pm 0, t) \to u_{\ell,r}$ as $t \to +\infty$. For such a solution, we expect that $E_{\min}(t)$ is achieved by the shift $h = X(t)$, for large enough t.

The function

$$\psi(h) := \int_{-\infty}^{h} \eta_-(u|u_\ell) \, dx + \int_{h}^{+\infty} \eta_+(u|u_r) \, dx$$

has left and right derivatives at $h = X(t)$, with

$$\psi'(X(t) - 0) = \eta_-(u_-|u_\ell) - \eta_+(u_-|u_r), \qquad \psi'(X(t) + 0) = \eta_-(u_+|u_\ell) - \eta_+(u_+|u_r).$$

When t is large enough, we obtain $\psi'(X(t) - 0) \sim -\eta_+(u_\ell|u_r) < 0$ and $\psi'(X(t) + 0) \sim \eta_-(u_r|u_\ell) > 0$, which ensures that ψ has a local minimum at $h = X(t)$.

If instead $u(\cdot, t)$ is continuous at $\bar{h}$ where $\psi(\bar{h})$ is minimum, then we must have

(3) $$\eta_-(\bar{u}|u_\ell) = \eta_+(\bar{u}|u_r), \qquad \bar{u} := u(\bar{h}).$$

Let us assume that the minimum $E_{\min}(t)$ is achieved at some $h(t)$, a piecewise smooth function of time. Then $t \mapsto E_{\min}(t)$ is a continuous function, piecewise $\mathcal{C}^1$, and its decay is equivalent to the fact that the derivative be non-positive. We compute

$$
\begin{aligned}
\frac{d}{dt} E_{\min} &= \frac{d}{dt} \left(\int_{-\infty}^{h(t)} \eta_-(u|u_\ell) \, dx + \int_{h(t)}^{+\infty} \eta_+(u|u_r) \, dx \right) \\
&= \dot{h}(\eta_-(u_-|u_\ell) - \eta_+(u_+|u_r)) + \int_{-\infty}^{h(t)} \partial_t \eta_-(u|u_\ell) \, dx + \int_{h(t)}^{+\infty} \partial_t \eta_+(u|u_r) \, dx \\
&\leq \dot{h}(\eta_-(u_-|u_\ell) - \eta_+(u_+|u_r)) - \int_{-\infty}^{h(t)} \partial_x q_-(u; u_\ell) \, dx - \int_{h(t)}^{+\infty} \partial_x q_+(u; u_r) \, dx \\
&= q_+(u_+; u_r) - \dot{h} \eta_+(u_+|u_r) - q_-(u_-; u_\ell) + \dot{h} \eta_-(u_-|u_\ell),
\end{aligned}
$$

where we have used the entropy inequalities for $\eta_\pm$.

If u is continuous at $h(t)$, we have

$$\frac{d}{dt} E_{\min} \leq D_c(\bar{u}; u_{\ell,r}) := q_+(\bar{u}; u_r) - q_-(\bar{u}; u_\ell),$$

where $\bar{u} = u(h, t)$ is constrained by (3).

If instead u is discontinuous at $h(t)$, we have

$$\frac{d}{dt} E_{\min} \leq D_s(u_\pm; u_{\ell,r}) := q_+(u_+; u_r) - \dot{h} \eta_+(u_+|u_r) - q_-(u_-; u_\ell) + \dot{h} \eta_-(u_-|u_\ell),$$

where $(u_\pm; \dot{h})$ are constrained by the Rankine–Hugoniot relation and the Lax entropy inequality

$$(4) \qquad f(u_+) - f(u_-) = \dot{h}(u_+ - u_-), \qquad q_\pm(u_+) - q_\pm(u_-) \le \dot{h}(\eta_\pm(u_+) - \eta_\pm(u_-)).$$

We ask therefore whether, given a shock U, there is a pair of strongly convex entropies $\eta_\pm$ such that $D_c(\bar{u}; u_{\ell,r}) \le 0$ for all $\bar{u}$ satisfying (3), and $D_s(u_\pm; u_{\ell,r}) \le 0$ for every (u_-, u_+) satisfying (4). If this holds true, we may say that the shock is a *global attractor*. If we only have $D_s(u_\pm; u_{\ell,r}) \le 0$ for every (u_-, u_+) close to (u_ℓ, u_r) and satisfying (4), then we say that the shock is a *local attractor*. Notice that we exclude the "continuous" case in the latter definition, because a continuous $u(x)$ cannot be uniformly close to the shock U, even up to a shift.

2.1 Local attractors

Because

$$\eta(b|a) \sim \frac{1}{2} \mathrm{D}^2 \eta_a(b - a, b - a), \qquad q(b; a) \sim \frac{1}{2} \mathrm{D}^2 \eta_a(\mathrm{d}f_a(b - a), b - a),$$

we see on the one hand that D_s behaves approximately quadratically around the pair (u_ℓ, u_r) :

$$2D_s(u_\pm; u_{\ell,r}) \sim \mathrm{D}^2 \eta_{+r}((\mathrm{d}f_r - \sigma)\delta u_+, \delta u_+) - \mathrm{D}^2 \eta_{-\ell}((\mathrm{d}f_\ell - \sigma)\delta u_-, \delta u_-),$$

where $\delta u_+ := u_+ - u_r$ and $\delta u_- := u_- - u_\ell$. We have written $\mathrm{D}^2 \eta_{+r}$ for $\mathrm{D}^2(\eta_+)_r = \mathrm{D}^2 \eta_+|_{u=u_r}$.

On the other hand, differentiating the Rankine–Hugoniot relation (remember that u_+ is a smooth function of u_- and σ), we have

$$(\mathrm{d}f_r - \sigma)\mathrm{d}u_r - (\mathrm{d}f_\ell - \sigma)\mathrm{d}u_\ell = (\mathrm{d}\sigma)(u_r - u_\ell),$$

from which we infer

$$(\mathrm{d}f_r - \sigma)\delta u_+ - (\mathrm{d}f_\ell - \sigma)\delta u_- \sim \epsilon(u_r - u_\ell)$$

for $\epsilon = \dot{h} - \sigma$. In order that U be a local attractor, it is therefore necessary that the quadratic form

$$(v_-, v_+) \overset{Q}{\longmapsto} \mathrm{D}^2 \eta_{+r}((\mathrm{d}f_r - \sigma)v_+, v_+) - \mathrm{D}^2 \eta_{-\ell}((\mathrm{d}f_\ell - \sigma)v_-, v_-)$$

be negative semi-definite over the linear space defined by

$$(\mathrm{d}f_r - \sigma)v_+ - (\mathrm{d}f_\ell - \sigma)v_- \in \mathbb{R}(u_r - u_\ell).$$

Conversely, if this quadratic form is negative definite, then there is a neighbourhood of (u_ℓ, u_r) in which D_s has a constant negative sign, and then the shock is a local attractor.

Let us recall that $\mathrm{d}f_a$ is $\mathrm{D}^2 \eta_a$-symmetric: the product

$$S := \mathrm{D}^2 \eta_a \mathrm{d}f_a$$

is a symmetric matrix. In other words, the eigen-basis of $\mathrm{d}f_a$ is orthogonal with respect to $\mathrm{D}^2 \eta_a$, that is

$$(i \ne j) \implies (\mathrm{D}^2 \eta(r_i, r_j) = 0).$$

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We deduce that, if the matrix $\mathrm{d}f_a - \sigma$ is non-singular, then

$$M_a := (\mathrm{d}f_a - \sigma)^{-T}\mathrm{D}^2\eta_a \in \mathbf{Sym}_n(\mathbb{R}).$$

We point out that, because of $\mathrm{D}^2\eta_a$ is positive definite, the signature (n_-, n_+) of M_a is made of the number $n_\pm$ of positive/negative eigenvalues of $\mathrm{d}f_a - \sigma$.

Our quadratic form can be re-written in the form

$$Q(v, s) := (v + s[u])^T M_{+r}(v + s[u]) - v^T M_{-\ell} v,$$

where $(\mathrm{d}f_\ell - \sigma)v_- =: v$ and $(\mathrm{d}f_r - \sigma)v_+ =: v + s[u]$. The negative definiteness of Q is equivalent to that of the matrix

$$\mathcal{Q} := \begin{pmatrix} M_{+r} - M_{-\ell} & M_{+r}[u] \\ [u]^T M_{+r} & [u]^T M_{+r}[u] \end{pmatrix}.$$

2.2 The choice of entropies $\eta_\pm$

A general system of conservation laws does not always admit a non-trivial entropy (that is, a non-affine one). If $n = 1$, every function is an entropy, thus every convex function is a convex entropy. If $n = 2$, the system admits a lot of entropy-flux pairs, among which there is an infinite dimensional cone of convex entropies (Serre [10], Chapter 12). More generally, we may declare that physically relevant systems do admit at least one strongly convex entropy. However, if $n \geq 3$, this convex entropy η is unique up to the addition of an affine function and the multiplication by a positive constant. The former modification does not change the relative entropy.

In the general case, we have therefore only the choice $\eta_\pm = a_\pm \eta$ where $a_\pm$ are positive constants and only the ratio a_+/a_- matters. This is the approach made by Kang & Vasseur [2]. In the rich case (see [10]), we have a more general choice, which we exploit in Section 3.

2.2.1 One entropy only doesn't work

It turns out that in the scalar case with a convex flux f, we may choose $a_+/a_- = 1$, that is $\eta_+ = \eta_-$. It is therefore of some importance to recognize whether this is also possible for systems. The following example shows that the answer is negative.

If U is a local attractor when we choose only one entropy (denoted η), then the matrix

$$\begin{pmatrix} M_r - M_\ell & M_r[u] \\ [u]^T M_r & [u]^T M_r[u] \end{pmatrix}$$

is non-positive. We show below that this does not happen in the simple example of the p-system

$$v_t + w_x = 0, \qquad w_t + p(v)_x = 0, \qquad u = \begin{pmatrix} v \\ w \end{pmatrix},$$

where $p' > 0$ and p'' does not vanish. The natural entropy is $\eta(u) = \frac{1}{2}w^2 + e(v)$ with e a primitive of p. We have

$$\mathrm{D}^2\eta = \begin{pmatrix} c^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad c := \sqrt{p'}$$

and

$$\mathrm{d}f(u) = \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix}.$$

This yields

$$M = \frac{1}{c^2 - \sigma^2} \begin{pmatrix} \sigma & c^2 \\ 1 & \sigma \end{pmatrix} \begin{pmatrix} c^2 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{c^2 - \sigma^2} \begin{pmatrix} \sigma c^2 & c^2 \\ c^2 & \sigma \end{pmatrix}.$$

We obtain therefore

$$M_r - M_\ell = \frac{\sigma(c_\ell^2 - c_r^2)}{(c_r^2 - \sigma^2)(c_\ell^2 - \sigma^2)} \begin{pmatrix} \sigma^2 & \sigma \\ \sigma & 1 \end{pmatrix} = \kappa YY^T, \qquad Y := \begin{pmatrix} \sigma \\ 1 \end{pmatrix}.$$

Because $M_r - M_\ell$ is only rank-one, we see that Q cannot be negative definite. We show below that it is not even negative semi-definite.

On another hand, the Rankine–Hugoniot conditions are

$$[w] = \sigma[v], \quad [p(v)] = \sigma[w],$$

hence

$$\sigma^2 = [p(v)]/[v], \qquad [u] = [v]\begin{pmatrix} 1 \\ \sigma \end{pmatrix}.$$

When U is a 2-shock, that is $\sigma > 0$, the Lax shock inequalities give $c_r < \sigma < c_\ell$; we thus have $\kappa < 0$. If U is a 1-shock, then $-c_r < \sigma < -c_\ell$ and we obtain again $\kappa < 0$. In both cases, the block $M_r - M_\ell$ is negative semi-definite. We also have

$$[u]^T M_r[u] = \frac{\sigma[v]^2}{c_r^2 - \sigma^2}(3c_r^2 + \sigma^2) < 0.$$

The Sherman–Morrison criterion therefore says that Q is negative semi-definite if and only if

$$M_r - M_\ell - \frac{1}{[u]^T M_r[u]} M_r[u][u]^T M_r \leq 0_2,$$

or equivalently

$$[u]^T M_r[u](M_r - M_\ell) \geq M_r[u][u]^T M_r.$$

Both sides are rank-one non-negative matrices ; this inequality has the form $ZZ^T \leq \beta YY^T$ with $\beta > 0$ and $Z = M_r[u]$. It implies that Z is colinear to Y ; in other words, we must have

$$0 = \begin{pmatrix} 1 & -\sigma \end{pmatrix} Z = [v]\begin{pmatrix} 1 & -\sigma \end{pmatrix} M_r \begin{pmatrix} 1 \\ \sigma \end{pmatrix} = \sigma[v].$$

This is absurd because $\sigma \neq 0$ and $[v] \neq 0$.

In conclusion, a shock U is not a local attractor for the p-system if we only choose $\eta_+ = \eta_-$ equal to the mechanical energy.

2.2.2 Extreme shocks (after Kang & Vasseur)

We now allow that $\eta_- = a\eta_+$ for some positive constant a. We denote $\eta = \eta_+$, a convex entropy usually provided by some physical principle.

An "extreme" shock is either a 1-shock or an n-shock. Up to a switch $x \mapsto -x$, we may suppose that U is a 1-shock:

$$\lambda_1(u_r) < \sigma < \lambda_1(u_\ell), \quad \sigma < \lambda_2(u_{r,\ell}).$$

In the construction above, $\mathrm{d}f(u_\ell) - \sigma$ has positive eigenvalues, while $\mathrm{d}f(u_r) - \sigma$ has $n-1$ positive and one negative eigenvalues. Therefore $M_{-\ell}$ is positive definite while M_{+r} is of signature $(n-1, 1)$.

Because $M_{-\ell}$ is positive definite, $\mathcal{Q}$ is negative definite provided a is large enough, and $[u]^T M_r [u] < 0$. Therefore, the shock is a local attractor if

$$(5) \qquad\qquad \mathrm{D}^2 \eta_{u_r}((\mathrm{d}f(u_r) - \sigma)^{-1}[u], [u]) < 0.$$

Remarking that the restriction of M_r to the hyperplane spanned by $r_2(u_r), \ldots, r_n(u_r)$ is positive definite, we see that (5) is slightly stronger than the Liu–Majda's criterion

$$\det([u], r_2(u_r), \ldots, r_n(u_r)) \neq 0.$$

We recall that the latter is the condition under which the local-in-time stability of the shock wave U holds true: the free boundary value problem associated with the evolution of the shock under initial disturbances is locally well-posed in spaces of smooth functions. : See [7, 8].

If the shock strength is small, the direction $[u]$ is approximately that of $r_1(u_r)$. Then

$$\mathrm{D}^2 \eta_{u_r}((\mathrm{d}f(u_r) - \sigma)^{-1}[u], [u]) \sim \frac{(\ell_1 \cdot [u])^2}{\lambda_1(u_r) - \sigma} \, \mathrm{D}^2 \eta(r_1, r_1) < 0,$$

because of the Lax shock inequality. Then the criterion (5) is satisfied trivially.

Kang & Vasseur actually proved that when $1 < k < n$ (non extreme case), a small shock cannot be an attractor if we deal with a pair $\eta_\pm = (a\eta, \eta)$. This can be checked by following the arguments developed in the next paragraph.

3 Intermediate shocks in rich systems

The class of rich systems generalizes that of 2×2 systems ; it was introduced in [9]. A rich system has a lot of entropies. Specifically, if $\bar{u} \in \mathcal{U}$ and $R_j(\bar{u})$ is the integral curve of the j-th characteristic field passing through $\bar{u}$, then for every set of prescribed functions $\phi_j : R_j(\bar{u}) \to \mathbb{R}$ vanishing at $\bar{u}$, there exists one and only one entropy coinciding with ϕ_j over R_j for all $j = 1, \ldots, n$.

In particular, we may specify the Hessian matrix of an entropy at one given point $\bar{u}$, provided that it is diagonal in the basis $\{r_1(\bar{u}), \ldots, r_n(\bar{u})\}$. Choosing a positive definite Hessian, the entropy is strongly convex in some neighbourhood of $\bar{u}$.

Let U be a k-shock. The stable subspace S_r of $\mathrm{d}f(u_r) - \sigma$ is spanned by $r_1(u_r), \ldots, r_k(u_r)$. Likewise, the unstable subspace U_r is spanned by $r_{k+1}(u_r), \ldots, r_n(u_r)$. They have dimensions k and $n - k$, respectively. The matrix $\mathrm{d}f(u_\ell) - \sigma$ has also a stable and an unstable subspaces S_ℓ and U_ℓ, but then $\dim S_\ell = k - 1$ and $\dim U_\ell = n - k + 1$.

Because we may choose arbitrarily the Hessian of η_+ at u_r, among the positive diagonal matrices in the eigenbasis $\mathcal{B}_r = \{r_1(u_r), \ldots, r_n(u_r)\}$, we see that the quadratic form q_+ associated with M_{+r} can be any difference $q_{Sr} \ominus q_{Ur}$, where q_{Ur} and q_{Sr} are diagonal in $\mathcal{B}_r$ as well, and are positive semi-definite, with $\ker q_{Ur} = U_r$ and $\ker q_{Sr} = S_r$ respectively. Likewise the quadratic form q_- associated with $M_{-\ell}$ has the form $q_{S\ell} \ominus q_{U\ell}$ where each form is diagonal in the eigenbasis $\mathcal{B}_\ell$, *et cætera*.

By choosing $q_{S\ell}$ large enough and then $q_{U\ell}$ small enough we see that the form Q can be made negative definite, provided that the restriction of

$$Q_r := \begin{pmatrix} M_{+r} & M_{+r}[u] \\ [u]^T M_{+r} & [u]^T M_{+r}[u] \end{pmatrix}$$

to $S_\ell \times \mathbb{R}$ is negative definite. Because

$$Q_r = X^T M_r X, \qquad X = \begin{pmatrix} I_n & [u] \end{pmatrix},$$

this amounts to requiring that M_r is negative definite over $S_\ell + \mathbb{R}[u]$, *and* that the last sum is a direct sum: $[u] \notin S_\ell$. Finally, choosing q_{Ur} large enough and then q_{Sr} small enough, we see that this M_r can be made negative definite over $S_\ell \oplus [u]$ if and only if $(S_\ell \oplus [u]) \cap U_r = \{0\}$. Because the dimensions sum up to n, we obtain

Theorem 3.1 *For a rich system, the following properties are equivalent to each other.*

- *There exist two entropies $\eta_\pm$, strongly convex about $u_{\ell,r}$ respectively, such that the matrix $\mathcal{Q}$ is negative definite (and then the shock U is a local attractor up to a shift for the corresponding E).*

- *One has*
$$\mathbb{R}^n = S_\ell \oplus [u] \oplus U_r.$$

The latter property is nothing but the Liu–Majda's condition

$$\det(r_1(u_\ell), \ldots, r_{k-1}(u_\ell), [u], r_{k+1}(u_r), \ldots, r_n(u_r)) \neq 0,$$

under which the evolution of the shock wave (a free boundary-value problem) is locally well-posed (again, see [7, 8]). It can be viewed as a Lopatinskii condition for the shock stability.

Comments

- The result stated in the theorem suggests that every Lax shock satisfying Liu–Majda's condition is actually asymptotically stable ; a similar result is known for viscous profiles, but only for extreme shocks, see [1].

- When we consider an extreme shock, say a 1-shock, Theorem 3.1 actually gives a slightly better result than that of Kang & Vasseur : if we had to choose $\eta_- = a\eta_+$ with a large enough, the attractivity would require the stronger condition

  $$(6) \qquad \mathrm{D}^2\eta_r\left((\mathrm{d}f_r - \sigma)^{-1}[u], [u]\right) < 0,$$

  which is not needed here. Notice that our result is a genuine improvement only for large amplitude shocks: if $[u]$ is small, then its direction is approximately that of r_1, and (6) is automatically satisfied.

- Kang & Vasseur's restricted choice is nevertheless relevant if the system is the limit of some dissipation process, because then we cannot prove the entropy inequality for every convex entropy, but only for one of them ; see [11].

- When the shock strength tends to zero, the matrices $\mathrm{d}f(u_{r,\ell}) - \sigma$ become singular. A good question is whether we can choose $\eta_\pm$ independently of the shock triplet $(u_\ell, u_r; \sigma)$ in the neighbourhood of $(\bar{u}, \bar{u}; \lambda_k(\bar{u}))$ when the k-th field is genuine nonlinear. The answer is positive in the scalar case (convex flux), where we may just choose $\eta_+ = \eta_-$, a strongly convex function.

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