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Beyond Universality in Random Matrix Theory

A. Edelman, A. Guionnet and S. Péché

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Abstract

In order to have a better understanding of finite random matrices with non-Gaussian entries, we study the $1/N$ expansion of local eigenvalue statistics in both the bulk and at the hard edge of the spectrum of random matrices. This gives valuable information about the smallest singular value not seen in universality laws. In particular, we show the dependence on the fourth moment (or the kurtosis) of the entries. This work makes use of the so-called complex Gaussian divisible ensembles for both Wigner and sample covariance matrices.

1 Beyond Universality

The desire to assess the applicability of universality results in random matrix theory has pressed the need to go beyond universality, in particular the need to understand the influence of finite n and what happens if the matrix deviates from Gaussian normality. In this note, we provide exact asymptotic correction formulas for the smallest singular value of complex matrices and bulk statistics for complex Wigner matrices.

"Universality," a term encountered in statistical mechanics, is widely found in the field of random matrix theory. The universality principle loosely states that eigenvalue statistics of interest will behave asymptotically as if the matrix elements were Gaussian. The spirit of the term is that the eigenvalue statistics will not care about the details of the matrix elements.

It is important to extend our knowledge of random matrices beyond universality. In particular, we should understand the role played by

- finite n and
- non Gaussian random variables.

From an application viewpoint, it is very valuable to have an estimate for the departure from universality. Real problems require that n be finite, not infinite, and it has long been observed computationally that ∞ comes very fast in random matrix theory. The applications beg to know how fast. From a theoretical viewpoint, there is much to be gained in searching for proofs that closely follow

the underlying mechanisms of the mathematics. We might distinguish "mechanism oblivious" proofs whose bounds require n to be well outside imaginably useful ranges, with "mechanism aware" proofs that hold close to the underlying workings of random matrices. We encourage such "mechanism aware" proofs.

In this article, we study the influence of the fourth cumulant on the local statistics of the eigenvalues of random matrices of Wigner and Wishart type.

On one hand, we study the asymptotic expansion of the smallest eigenvalue density of large random sample covariance matrices. The behavior of smallest eigenvalues of sample covariance matrices when p/n is close to one (and more generally) is somewhat well understood now. We refer the reader to [15], [33], [18], [6], [7]. The impact of the fourth cumulant of the entries is of interest here; we show its contribution to the distribution function of the smallest eigenvalue density of large random sample covariance matrices as an additional error term of order of the inverse of the dimension (see Theorem 3.1).

On the other hand, we consider the influence of the fourth moment in the local fluctuations in the bulk. Here, we consider Wigner matrices and discuss a conjecture of Tao and Vu [31] that the fourth moment brings a correction to the fluctuation of the expectation of the eigenvalues in the bulk of order of the inverse of the dimension. We prove (cf. Theorem 3.3) that the quantiles of the one point correlation function fluctuate according to the formula predicted by Tao and Vu for the fluctuations of the expectation of the eigenvalues.

In both cases, we consider the simplest random matrix ensembles that are called Gaussian divisible, that is whose entries can be described as the convolution of a distribution by the Gaussian law. To be more precise, we consider the so-called Gaussian-divisible ensembles, also known as Johansson-Laguerre and Johansson-Wigner ensembles. These ensembles, defined hereafter, have been first considered in [24] and have the remarkable property that the induced joint eigenvalue density can be computed. It is given in terms of the Itzykson-Zuber-Harich-Chandra integral. From such a formula, saddle point analysis allows to study the local statistics of the eigenvalues. It turns out that in both cases under study, the contribution of the fourth moment to the local statistics can be inferred from the fluctuations of the one-point correlation function, that is of the mean linear statistics of Wigner and Wishart random matrices. The covariance of the latter is well known, since [25], to depend on the fourth moments, from which our results follow.

2 Discussion and Simulations

2.1 Preliminaries: Real Kurtosis

We will only consider distributions whose real and imaginary parts are independent and are identically distributed.

Definition 1. *The* **kurtosis** *of a distribution is*

$$\gamma = \frac{\kappa_4^{\Re}}{\sigma_{\Re}^4} = \frac{\mu_4}{\sigma_{\Re}^4} - 3,$$

where $\kappa_4^{\Re}$ is the fourth cumulant of the real part, $\sigma_{\Re}^2$ is the variance of the real part, and μ_4 is the fourth moment about the mean. The fourth cumulant of a centered complex distribution P with i.i.d real and complex part with variance $\sigma_{\Re}^2$, is given by

$$\kappa_4 = \int |zz^*|^2 dP(z) - 8\sigma_{\Re}^4 = 2\kappa_4^{\Re} = 2\gamma\sigma_{\Re}^4 .$$

Note: From a software viewpoint, commands such as `randn` make it natural to take the real and the imaginary parts to separately have mean 0, variance 1, and also to consider the real kurtosis.

Example of Kurtoses γ for distributions with mean 0, and $\sigma^2 = 1$:

Distribution	γ	Univariate Code
normal	0	`randn`
Uniform $[-\sqrt{3}, \sqrt{3}]$	-1.2	`(rand -.5)*sqrt(12)`
Bernoulli	-2	`sign(randn)`
Gamma	6	`rand(Gamma()) - 1`

For the matrices themselves, we compute the smallest eigenvalues of the Gram matrix constructed from $(n+\nu) \times n$ complex random matrices with Julia [8] code provided for the reader's convenience:

RM	Complex Matrix Code
normal	`randn(n+`ν`,n)+im*randn(n+`ν`,n)`
Uniform	`((rand(n+`ν`,n)-.5)+im*rand(n+`ν`,n) -.5))*sqrt(12)`
Bernoulli	`sign(randn(n+`ν`, n))+im*sign(randn(n+`ν`,n))`
Gamma	`(rand(Gamma(),n+`ν`,n)-1) + im*(rand(Gamma(),n+`ν`,n)-1)`

2.2 Smallest Singular Value Experiments

Let A be a random $n+\nu$ by n complex matrix with iid real and complex entries all with mean 0, variance 1 and kurtosis γ. In the next several subsections we display special cases of our results, with experiment vs. theory curves for $\nu = 0, 1$, and 2.

We consider the cumulative distribution function

$$F(x) = \mathbb{P}\left(\lambda_{\min}(AA^*) \leq \frac{x}{n}\right) = \mathbb{P}\left((\sigma_{\min}(A))^2 \leq \frac{x}{n}\right),$$

where $\sigma_{\min}(A)$ is the smallest singular value of A. We also consider the density

$$f(x) = \frac{d}{dx}F(x).$$

In the plots to follow we took a number of cases when $n = 20, 40$ and sometimes $n = 80$. We computed $2,000,000$ random samples on each of 60 processors using Julia [8], for a total of $120,000,000$ samples of each experiment.

The runs used 75% of the processors on a machine equipped with 8 Intel Xeon E7-8850-2.0 GHz-24M-10 Core Xeon MP Processors. This scale experiment, which is made easy by the Julia system, allows us to obtain visibility on the higher order terms that would be hard to see otherwise. Typical runs took about an hour for $n = 20$, three hours for $n = 40$, and twelve hours for $n = 80$.

We remark that we are only aware of two or three instances where parallel computing has been used in random matrix experiments. Working with Julia is pioneering in showing just how easy this can be, giving the random matrix experimenter a new tool for honing in on phenomena that would have been nearly impossible to detect using conventional methods.

2.3 Example: Square Complex Matrices ($\nu = 0$)

Consider taking, a 20 by 20 random matrix with independent real and imaginary entries that are uniformly distributed on $[-\sqrt{3}, \sqrt{3}]$.

```
((rand(20,20)-.5) + im*(randn(20,20)-.5))*sqrt(12).
```

This matrix has real and complex entries that have mean 0, variance 1, and kurtosis $\gamma = -1.2$.

An experimenter wants to understand how the smallest singular value compares with that of the complex Gaussian matrix

```
randn(20,20) + im*randn(20,20).
```

The law for complex matrices [14, 15] in this case valid for all finite sized matrices, is that $n\lambda_{\min} = n\sigma_{\min}^2$ is exactly exponentially distributed: $f(x) = \frac{1}{2}e^{-x/2}$. Universality theorems say that the uniform curve will match the Gaussian in the limit as matrix sizes go to ∞. The experimenter obtains the curves in Figure 1 (taking both $n = 20$ and $n = 40$).

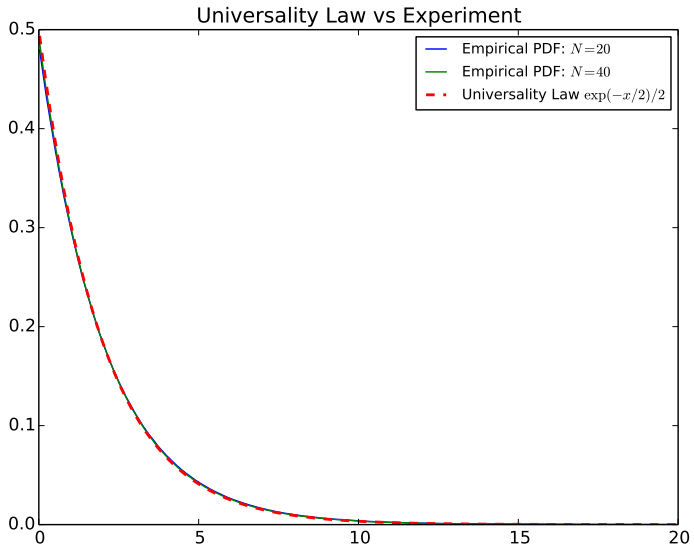


Figure 1: Universality Law vs Experiment: $n = 20$ and $n = 40$ already resemble $n = \infty$

Impressed that $n = 20$ and $n = 40$ are so close, he or she might look at the proof of the universality theorem only to find that no useful bounds are available at $n = 20, 40$.

The results in this paper gives the following correction in terms of the kurtosis (when $\nu = 0$):

$$f(x) = e^{-x/2}\left(\frac{1}{2} + \frac{\gamma}{n}(\frac{1}{4} - \frac{x}{8})\right) + O\left(\frac{1}{n^2}\right).$$

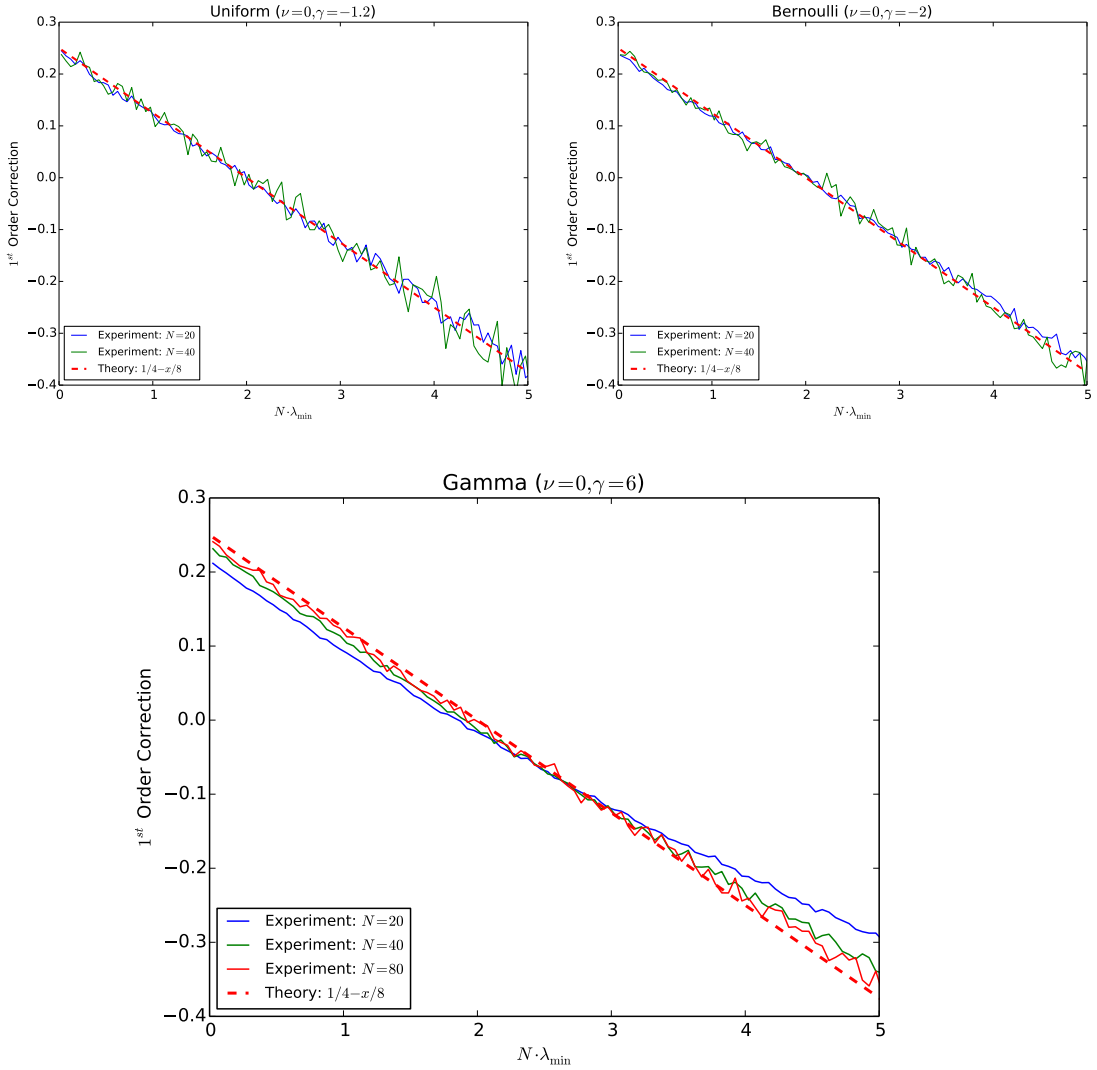


Figure 2: Correction for square matrices Uniform, Bernoulli, ($\nu = 0$). Monte carlo simulations are histogrammed, 0th order term subtracted, and result multiplied by $ne^{x/2}/\gamma$. Bottom curve shows convergence for $n = 20, 40, 80$ for a distribution with positive kurtosis.

On the bottom of Figure 1, with the benefit of 60 computational processors, we can magnify the departure from universality with Monte Carlo experiments, showing that the departure truly fits $\frac{\gamma}{n}(\frac{1}{4} - \frac{x}{8})e^{-x/2}$. This experiment can be run and rerun many times, with many distributions, kurtoses that are positive and negative, small values of n, and the correction term works very well.

2.4 Example: $n+1$ by n complex matrices ($\nu = 1$)

The correction to the density can be written as

$$f(x) = e^{-x/2}\left(\frac{1}{2}I_2(s) + \frac{1+\gamma}{8n}\left(sI_1(s) - xI_2(s)\right)\right) + O(\frac{1}{n^2}),$$

where $I_1(x)$ and $I_2(x)$ are Bessel functions and $s = \sqrt{2x}$.

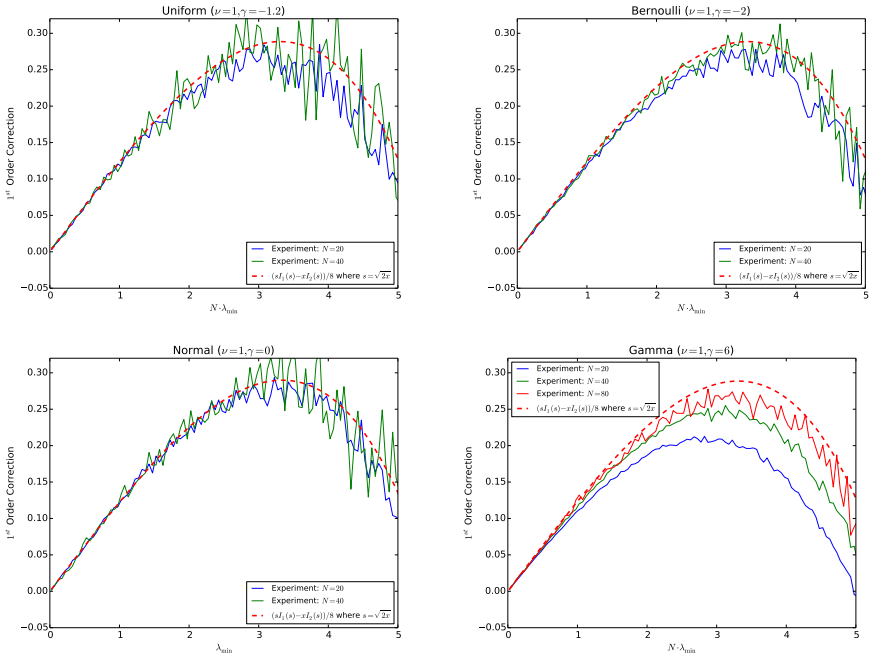


Figure 3: Correction for $\nu = 1$. Uniform, Bernoulli, normal, and Gamma; Monte carlo simulations are histogrammed, 0th order term subtracted, and result multiplied by $ne^{x/2}/(1+\gamma)$. Bottom right curve shows convergence for $n = 20, 40, 80$ for a distribution with positive kurtosis.

2.5 Example: $n+2$ by n complex matrices ($\nu = 2$)

The correction to the density for $\nu = 2$ can be written

$$f(x) = \frac{1}{2}e^{-x/2}\left([I_2^2(s) - I_1(s)I_3(s)] + \frac{2+\gamma}{2n}\left[(x+4)I_1^2(s) - 2sI_0(s)\,I_1(s) - (x-2)I_2^2(s)\right]\right),$$

where I_0, I_1, I_2, and I_3 are Bessel functions, and $s = \sqrt{2x}$.

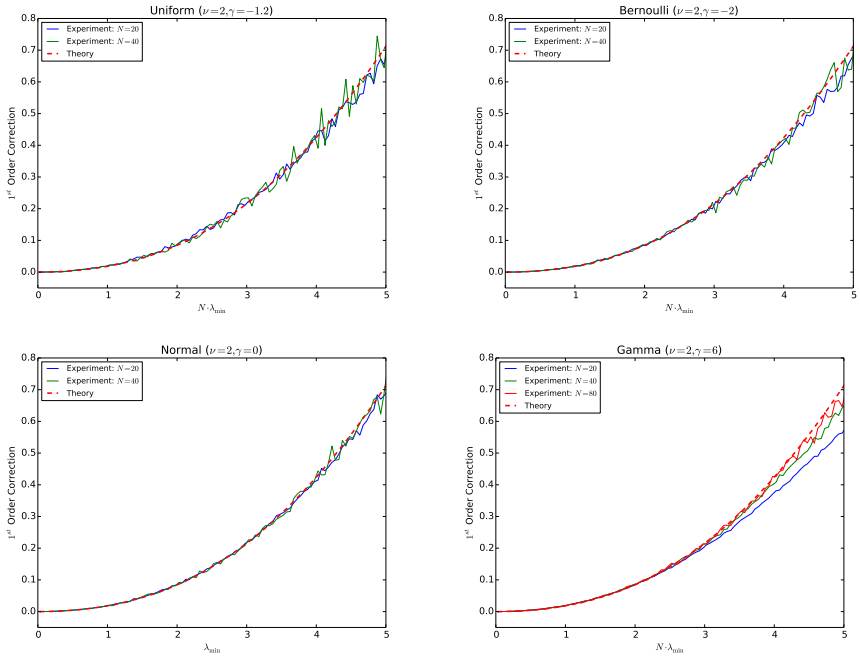


Figure 4: Correction for $\nu = 2$. Uniform, Bernoulli, normal, and Gamma; Monte carlo simulations are histogrammed, 0th order term subtracted, and result multiplied by $ne^{x/2}/(2+\gamma)$. Bottom right curve shows convergence for $n = 20, 40, 80$ for a distribution with positive kurtosis.

3 Models and Results

In this section, we define the models we will study and state the results. Let some real parameter $a > 0$ be given. Consider a matrix M of size $p \times n$:

$$M = W + aV$$

where

- $V = (V_{ij})_{1 \le i \le p; 1 \le j \le n}$ has i.i.d. entries with complex $\mathcal{N}_{\mathbb{C}}(0,1)$ distribution, which means that both $\Re V_{ij}$ and $\Im V_{ij}$ are real i.i.d. $\mathcal{N}(0, 1/2)$ random variables,
- $W = (W_{ij})_{1 \le i \le p; 1 \le j \le n}$ is a random matrix with entries being mutually independent random variables with distribution $P_{ij}, 1 \le j \le n$ independent of n and p, with uniformly bounded fourth moment,
- W is independent of V,
- $\nu := p - n \ge 0$ is a fixed integer independent of n.

We then form the Gaussian divisible ensemble (also known as the Johansson-Laguerre matrix):

$$\frac{1}{n}M^*M = \left(\frac{1}{\sqrt{n}}(W+aV)\right)^* \left(\frac{1}{\sqrt{n}}(W+aV)\right). \tag{1}$$

When W is fixed, the above ensemble is known as the *Deformed Laguerre Ensemble.*

We assume that the probability distributions $P_{j,k}$ satisfy

$$\int z dP_{j,k}(z) = 0, \ \int |zz^*| dP_{j,k}(z) = \sigma_{\mathcal{C}}^2 = \frac{1}{4}. \tag{2}$$

Here the complex $\sigma_{\mathcal{C}}^2 = 2\sigma_{\Re}^2$ represents the complex variance. Hypothesis (2) ensures the convergence of the spectral measure of $\frac{1}{n}W^*W$ to the Marchenko-Pastur distribution with density

$$\rho_{\mathrm{PM}}(x) = \frac{2}{\pi}\frac{\sqrt{1-x}}{\sqrt{x}}, \quad 0 \le x \le 1. \tag{3}$$

Condition (2) implies also that the limiting spectral measure of $\frac{1}{n}M^*M$ is then given by Marchenko-Pastur's law with parameter $1/4 + a^2$; we denote ρ_a the density of this probability measure, i.e, $\rho_a(x) = (1+4a^2)^{-1/2}\rho_{\mathrm{PM}}(x/\sqrt{1+4a^2})$.

For technical reasons, we assume that the entries of W have sub-exponential tails: There exist $C, c, \theta > 0$ so that for all $j, \in \mathbb{N}$, all $t \ge 0$

$$P_{j,k}(|z| \ge t) \le Ce^{-ct^\theta}. \tag{4}$$

This hypothesis could be weakened to requiring enough finite moments.

Finally we assume that the fourth moments do not depend on j, k and let κ_4 be the difference between the fourth moment of $P_{j,k}$ and the complex Gaussian case,

$$\kappa_4 = \int |zz^*|^2 dP_{j,k} - 8^{-1}$$

(Thus, with the notation of Definition 1, $\kappa_4 = 2\gamma\sigma_{\Re}^4 = 2\kappa_4^{\Re}$.)

Then our main result is the following. Let $\sigma := \sqrt{4^{-1} + a^2}$ and for an Hermitian matrix A denote $\lambda_{min}(A) = \lambda_1(A) \le \lambda_2(A) \cdots \le \lambda_n(A)$ the eigenvalues of A.

Theorem 3.1. *Let F_n be the cumulative density function of the hard edge in the Gaussian case with entries with complex variance σ^2:*

$$F_n(s) = \mathbb{P}\left(\sigma^2\lambda_{min}(VV^*) \le \frac{s}{n}\right)$$

Then, for all $s > 0$, if our distribution has complex fourth cumulant $\kappa_4 = 2\kappa_4^{\Re}$,

$$\mathbb{P}\left(\lambda_{min}(MM^*) \le \frac{s}{n}\right) = F_n(s) + \frac{sF_n'(s)}{\sigma^4 n}\kappa_4 + o(\frac{1}{n}).$$

We note that this formula is scale invariant. It is equivalent to

$$\mathbb{P}\left(\lambda_{min}(MM^*) \geq \frac{s}{n}\right) = 1 - F_n(s) + \frac{s(1-F_n)'(s)}{\sigma^4 n}\kappa_4 + o(\frac{1}{n}).$$

Let $F_\infty(s) = \lim_{n\to\infty} F_n(s)$ be the limiting cumulative distributive function in the Gaussian case.

Corollary 3.2. *For the ν for which Conjecture 2 is true (see section 4.2),*

$$\mathbb{P}\left(\lambda_{min}(MM^*) \leq \frac{s}{n}\right) = F_\infty(s) + (\nu + \frac{\kappa_4}{\sigma^4})\frac{sF'_\infty(s)}{n} + o(\frac{1}{n}).$$

Note: Since the posting of this report on ArXiv, Corollary 3.2 has been proved correct first by Schehr [30], and then with a different proof by Bornemann [10]. Thus at this time, Conjecture 2 has not been explicitly verified, but the more important Corollary 3.2 is now proved correct for all ν.

Note: Stated simply Conjecture 2 was confirmed correct for $\nu = 0, \ldots, 25$ symbolically in Mathematica and Maple, and numerically for larger values at this time. Numerically means in the presence of roundoff error.

Corollary 3.2 is the convenient formulation we used for $\nu = 0, 1, 2$ in our showcase examples in Sections 2.3, 2.4, and 2.5 respectively.

Note: Corollary 3.2 is remarkable because it states that the correction term for n being a finite Gaussian as opposed to being infinite, and the correction term for n being non Gaussian as opposed to Gaussian "line up," in that either way the corrections are multiples of $sF'_\infty(s)$. This could not be predicted by Theorem 3.1 alone.

Note: It is worth taking more of a close look between the formulation in Theorem 3.1 and Corollary 3.2. The first term in Theorem 3.1 is n dependent, while in Corollary 3.2 the first term has reached its $n \to \infty$ limit. Also of note is that the F_n formulation in Theorem 3.1 involves Laguerre polynomials (and exponentials.) The F_∞ formulation in Corollary 3.2 involves Bessel functions (and exponentials).

Logical Note: It might be difficult at first glance to follow the sequence of logic of this set of corollaries, lemmas, and conjectures. We therefore outline the basic steps.

- Bessel Function Identity Conjecture 2 in Section 4.2 is a Bessel Function algebraic identity involving three ν by ν determinants of Bessel functions. It is the one conjecture that needs to be verified from which the other conjectures would logically follow. One can read and try to solve the statement in Conjecture 2 without any knowledge of random matrix theory and independently of other material in this paper.

- Laguerre Polynomial Asymptotic Lemma 4.1 expands scaled Laguerre polynomials in a two term asymptotic series involving Bessel functions. Lemma 4.1 is also independent of the remainder of this paper and involves special function manipulations.

- Gaussian entry Conjecture 1 is a statement about the distribution of the smallest singular value of $n+\nu$ by n random matrices with complex Gaussian entries of equal variance. For any ν for which the identity in Conjecture 2 is confirmed, one immediately obtains Conjecture 1 in light of Lemma 4.1. The knowledge required to follow Conjecture 1 may be found in Section 4.1.1

- General entry Corollary 3.2 is a statement about the distribution of the smallest singular value of $n + \nu$ by n random matrices based on the cumulant κ_4. For any ν for which the identity in Conjecture 2 is confirmed, one immediately obtains Corollary 3.2 in light of Theorem 3.1. In light of [10, 30], we known the truth of Corollary 3.2 anyway and there is no longer any direct need to verify Conjecture 2.

For the Wigner ensemble we consider the matrix

$$M_n = \frac{1}{\sqrt{n}}(W + aV)$$

where W a Wigner matrix with complex (resp. real) independent entries above (resp. along) the diagonal $W_{ij}, 1 \leq i \leq j \leq N$ with law P_{ij}. We assume that the distributions P_{ij} have sub exponential moments: there exists $C, c > 0$, and $\alpha > 0$ such that for all $t \geq 0$ and all $1 \leq i \leq j \leq N$

$$P_{ij}(|x| \geq t) \leq C \exp\{-ct^\alpha\}, \tag{5}$$

and satisfies

$$\int x dP_{ij}(x) = 0, \int |x|^2 dP_{ij}(x) = 1/4, \int x^3 dP_{ij}(x) = 0. \tag{6}$$

Again we assume that the fourth moments do not depend on i, j and let κ_4 be the difference between the fourth moment of P_{ij} $(j \neq i)$ and the Gaussian case,

$$\kappa_4 = \int |zz^*|^2 dP_{ij} - 1/8.$$

The other matrix V is a GUE random matrix with i.i.d. $\mathcal{N}_{\mathbb{C}}(0, 1)$ entries. We denote by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the ordered eigenvalues of M_n. By Wigner's theorem, it is known that the spectral measure of M_n

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}$$

converges weakly to the semi-circle distribution with density

$$\sigma_{sc}^{2\sigma}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\mathbb{1}_{|x|\leq 2\sigma}; \sigma^2 = 1/4 + a^2. \tag{7}$$

This is the Gaussian-divisible ensemble studied by Johansson [24]. We study the dependency of the one point correlation function ρ_n of this ensemble, given as the probability measure on $\mathbb{R}$ so that for any bounded measurable function f

$$\mathbb{E}[\frac{1}{n}\sum_{i=1}^{n} f(\lambda_i)] = \int f(x)\rho_n(x)dx$$

as well as the localization of the quantiles of ρ_n with respect to the quantiles of the limiting semi-circle distribution. In particular, we study the $1/n$ expansion of this localization, showing that it depends on the fourth moment of μ. Define $N_n(x) := \frac{1}{n}\sharp\{i, \lambda_i \leq x\}$, and $N_{sc}(x) = \int_{-\infty}^{x} d\sigma_{sc}(u)$, with σ_{sc} defined in (34). Let us define the quantiles $\hat{\gamma}_i$ (resp. γ_i) by

$$\hat{\gamma}_i := \inf\left\{y, \mathbb{E}N_n(y) = \frac{i}{n}\right\} \text{ resp. } N_{sc}((-\infty, \gamma_i]) = \frac{i}{n}.$$

We shall prove that

Theorem 3.3. *Let $\varepsilon > 0$. There exists functions C, D on $[-2+\varepsilon, 2-\varepsilon]$, independent of the distributions μ, μ', such that for all $x \in [-2+\varepsilon, 2-\varepsilon]$*

$$\rho_n(x) = \sigma_{sc}(x) + \frac{1}{n}C(x) + \frac{1}{n}\kappa_4 D(x) + o\left(\frac{1}{n}\right).$$

For all $i \in [n\varepsilon, n(1-\varepsilon)]$ for some $\varepsilon > 0$, there exists a constant C_i independent of κ_4 so that

$$\hat{\gamma}_i - \gamma_i = \frac{C_i}{n} + \frac{\kappa_4}{2n}(2\gamma_i^3 - \gamma_i) + o(\frac{1}{n}). \tag{8}$$

This is a version of the rescaled Tao-Vu conjecture 1.7 in [31] where $\mathbb{E}[\lambda_i]$ is replaced by $\hat{\gamma}_i$. A similar result could be derived for Johansson-Laguerre ensembles. We do not present the details of the computation here, which would resemble the Wigner case. The functions C and D are computed explicitly in Proposition 6.1.

4 Smallest Singular Values of $n+\nu$ by n complex Gaussian matrices

Theorem 3.1 depends on the partition function for Gaussian matrices, which itself depends on ν and n. In this section, we investigate these dependencies.

4.1 Known exact results

It is worthwhile to review what exact representations are known for the smallest singular values of complex Gaussians.

We consider the finite n density $f_n^\nu(x)$, the finite n cumulative distribution $F_n^\nu(x)$ (we stress the ν-dependency in this section), and their asymptotic values $f_\infty^\nu(x)$ and $F_\infty^\nu(x)$. We have found the first form in the list below useful for symbolic and numerical computation. In the formulas to follow, we assume $\sigma_\Re^2 = 1$ so that a command such as `randn()` can be used without modification for the real and imaginary parts. All formulas concern $n\lambda_{\min} = n\sigma_{\min}^2$ and its asymptotics. We present in the array below eight different formulations of the exact distribution F_n^ν.

1. Determinant: ν by ν	[19, 20]
2. Painléve III	[19, Eq. (8.93)]
3. Determinant: n by n	[12]
4. Fredholm Determinant	[9, 32]
5. Multivariate Integral Recurrence	[15, 20]
6. Finite sum of Schur Polynomials (evaluated at $\mathbb{I}$)	[13]
7. Hypergeometric Function of Matrix Argument	[13]
8. Confluent Hypergeometric Function of Matrix Argument	[29]

Table 1: Exact Results for smallest singular values of complex Gaussians (smallest eigenvalues of complex Wishart or Laguerre Ensembles)

Some of these formulations allow one or both of ν or n to extend beyond integers to real positive values. Assuming ν and n are integers [15, Theorem 5.4], the probability density $f_n^\nu(x)$ takes the form $x^\nu e^{-x/2}$ times a polynomial of degree $(n-1)\nu$ and $1 - F_n^\nu(x)$ is $e^{-x/2}$ times a polynomial of degree $n\nu$.

Remark: A helpful trick to compare normalizations used by different authors is to inspect the exponential term. The 2 in $e^{-x/2}$ denotes total complex variance 2 (twice the real variance of 1). In general the total complex variance $\sigma^2 = 2\sigma_\Re^2$ will appear in the denominator.

In the next paragraphs, we discuss the eight formulations introduced above.

4.1.1 Determinant: ν by ν determinant

The quantities of primary use are the beautiful ν by ν determinant formulas for the distributions by Forrester and Hughes [20] in terms of Bessel functions and Laguerre polynomials. The infinite formulas also appear in [19, Equation (8.98)].

$$F_\infty^\nu(x) = 1 - e^{-x/2} \det[I_{i-j}(\sqrt{2x})]_{i,j=1,\ldots,\nu}.$$
$$f_\infty^\nu(x) = \tfrac{1}{2} e^{-x/2} \det[I_{2+i-j}(\sqrt{2x})]_{i,j=1,\ldots,\nu}.$$
$$F_n^\nu(x) = 1 - e^{-x/2} \det\left[L_{n+i-j}^{(j-i)}(-x/2n)\right]_{i,j=1,\ldots,\nu}.$$
$$f_n^\nu(x) = \left(\tfrac{x}{2n}\right)^\nu \tfrac{(n-1)!}{2(n+\nu-1)!} e^{-x/2} \det\left[L_{n-1+i-j}^{(j-i+2)}(-x/2n)\right]_{i,j=1,\ldots,\nu}.$$

Recall that $I_j(x) = I_{-j}(x)$. To facilitate reading of the relevant ν by ν determinants we provide expanded views:

$$\det[I_{i-j}(\sqrt{2x})]_{i,j=1,\ldots,\nu} = \left| \begin{array}{ccccc} I_0 & I_1 & I_2 & \cdots & I_{\nu-1} \\ I_1 & I_0 & I_1 & \cdots & I_{\nu-2} \\ I_2 & I_1 & I_0 & \cdots & I_{\nu-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{\nu-1} & I_{\nu-2} & I_{\nu-3} & \cdots & I_0 \end{array} \right|_{\text{Bessel functions evaluated at } \sqrt{2x}}$$

$$\det[I_{2+i-j}(\sqrt{2x})]_{i,j=1,\ldots,\nu} = \left| \begin{array}{ccccc} I_2 & I_1 & I_0 & \cdots & I_{\nu-3} \\ I_3 & I_2 & I_1 & \cdots & I_{\nu-4} \\ I_4 & I_1 & I_2 & \cdots & I_{\nu-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{\nu+1} & I_\nu & I_{\nu-1} & \cdots & I_2 \end{array} \right|_{\text{Bessel functions evaluated at } \sqrt{2x}}$$

$$\det\left[L_{n+i-j}^{(j-i)}(-\frac{x}{2n})\right]_{i,j=1,\ldots,\nu} = \left| \begin{array}{ccccc} L_n & L_{n-1}^{(1)} & L_{n-2}^{(2)} & \cdots & L_{n-\nu+1}^{(\nu-1)} \\ L_{n+1}^{(-1)} & L_n & L_{n-1}^{(1)} & \cdots & L_{n-\nu+2}^{(\nu-2)} \\ L_{n+2}^{(-2)} & L_{n+1}^{(-1)} & L_n & \cdots & L_{n-\nu+3}^{(\nu-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n+\nu-1}^{(1-\nu)} & L_{n+\nu-2}^{(2-\nu)} & L_{n+\nu-3}^{(3-\nu)} & \cdots & L_n \end{array} \right|_{\text{evaluated at } -x/2n}$$

$$\det\left[L_{n-1+i-j}^{(j-i+2)}(-\frac{x}{2n})\right]_{i,j=1,\ldots,\nu} = \left| \begin{array}{ccccc} L_{n-1}^{(2)} & L_{n-2}^{(3)} & L_{n-3}^{(4)} & \cdots & L_{n-\nu}^{(\nu+1)} \\ L_n^{(1)} & L_{n-1}^{(2)} & L_{n-2}^{(3)} & \cdots & L_{n-\nu+1}^{(\nu)} \\ L_{n+1} & L_n^{(1)} & L_{n-1}^{(2)} & \cdots & L_{n-\nu+2}^{(\nu-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n+\nu-2}^{(3-\nu)} & L_{n+\nu-3}^{(4-\nu)} & L_{n+\nu-4}^{(5-\nu)} & \cdots & L_{n-1}^{(2)} \end{array} \right|_{\text{evaluated at } -x/2n}$$

The following Mathematica code symbolically computes these distributions

```
M[x_, v_]     := Table[ BesselI[Abs[i - j], x],     {i,v}, {j,v}];
m[x_, v_]     := Table[ BesselI[Abs[2 + i - j], x], {i,v}, {j,v}]
M[x_, n_, v_] := Table[ LaguerreL[n+i-j, j - i,  -x/(2*n)], {i,v}, {j,v}];
```

```
m[x_, n_, v_] := Table[ LaguerreL[n-1+i-j,j-i+2, -x/(2*n)], {i,v}, {j,v}];
F[x_, v_  ]   :=  1 - Exp[-x/2]*Det [M[Sqrt[2 x], v]]
f[x_, v_]     := (1/2)*Exp[-x/2]*Det[m[Sqrt[2 x], v]]
F[x_, n_, v_] :=  1 - Exp[-x/2]*Det [M[x,n,v]]
f[x_, n_, v_] := (x/(2 n))^v*((n - 1)!/(2 (n+v-1)!))*Exp[-x/2]*Det[m[x,n,v]]
```

4.1.2 Painléve III

According to [19, Eq. (8.93)], [9, p. 814-815], [32,33] we have the formula valid for all $\nu > 0$

$$F_\infty^\nu(x) = \exp\left(-\int_0^{2t} \sigma(s)\frac{ds}{s}\right),$$

where $\sigma(s)$ is the solution to a Painléve III differential equation. Please consult the references taking care to match the normalization.

4.1.3 n by n determinant:

Following standard techniques to set up the multivariate integral and applying a continuous version of the Cauchy-Binet theorem (Gram's Formula) [27, e.g., Appendix A.12] or [34, e.g. Eqs. (1.3) and (5.2)] one can work out an $n \times n$ determinant valid for any ν, so long as n is an integer [12].

$$F_n^\nu(x) = \frac{\det(M(m,\nu,x/2))}{\det(M(m,\nu,0))}.$$

where

$$M(m,\nu,x) = \begin{pmatrix} \Gamma(\nu+1,x) & \Gamma(\nu+2,x) & \Gamma(\nu+3,x) & \cdots & \Gamma(\nu+m,x) \\ \Gamma(\nu+2,x) & \Gamma(\nu+3,x) & \Gamma(\nu+4,x) & \cdots & \Gamma(\nu+m+1,x) \\ \Gamma(\nu+3,x) & \Gamma(\nu+4,x) & \Gamma(\nu+5,x) & \cdots & \Gamma(\nu+m+2,x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Gamma(\nu+m,x) & \Gamma(\nu+m+1,x) & \Gamma(\nu+m+2,x) & \cdots & \Gamma(\nu+2m-1,x) \end{pmatrix}.$$

4.1.4 Remaining Formulas in Table 1

The Fredholm determinant is a standard procedure. The multivariate integral recurrence was computed in the real case in [15] and in the complex case in [20]. Various hypergeometric representations may be found in [13], but to date we are not aware of the complex representation of the confluent representation in [29] which probably is worth pursuing.

4.2 Asymptotics of Smallest Singular Value Densities of Complex Gaussians

A very useful expansion extends a result from [20, (3.29)]

Lemma 4.1. *As $n \to \infty$, we have the first two terms in the asymptotic expansion of scaled Laguerre polynomials whose degree and constant parameter sum to n:*

$$L_{n-k}^{(k)}(-x/n) \sim n^k \left\{ \frac{I_k(2\sqrt{x})}{x^{k/2}} - \frac{1}{2n} \left(\frac{I_{k-2}(2\sqrt{x})}{x^{(k-2)/2}} \right) + O\left(\frac{1}{n^2} \right) \right\}$$

Proof. We omit the tedious details but this (and indeed generalizations of this result) may be computed either through direct expansion of the Laguerre polynomial or through the differential equation it satisfies.

We can use the lemma above to obtain asymptotics of the distribution $F_n^{(\nu)}(x)$. As a result, we have ample evidence to believe the following conjecture:

Conjecture 1. *(Verified symbolically correct for $\nu = 0, 1, 2, \ldots, 25$) Let $F_n^{(\nu)}(x)$ be the distribution of $n\sigma_{\min}^2$ of an $n + \nu$ by n complex Gaussian. We propose that*

$$F_n^{(\nu)}(x) = F_\infty^{(\nu)}(x) + \frac{\nu}{2n} x f_\infty^{(\nu)}(x) + O(\frac{1}{n^2})$$

Note: The above is readily checked to be scale invariant, so it is not necessary to state the particular variances in the matrix as long as they are equal.
Note: We are delighted to report that two very different proofs of Conjecture 1 the first by Anthony Perret and Grégory Schehr [30] and another by Folkmar Bornemann [10], have appeared fairly quickly after posting this report as a preprint on ArXiv. The first uses properties of the Jacobi matrix associated with modified Laguerre polynomials and its implication to Painlevé while the second takes a close look at the Fredholm determinant and the asymptotics of Laguerre polynomials.

In light of Lemma 4.1, Conjecture 1 could have been deduced from

Conjecture 2. *Consider the Bessel function (evaluated at x) determinant*

$$\begin{vmatrix} I_0 & I_1 & I_2 & \cdots & I_{\nu-1} \\ I_1 & I_0 & I_1 & \cdots & I_{\nu-2} \\ I_2 & I_1 & I_0 & \cdots & I_{\nu-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{\nu-1} & I_{\nu-2} & I_{\nu-3} & \cdots & I_0 \end{vmatrix}.$$

We propose that the following determinant equation is an equality for $\nu \geq 2$, where the first/second determinant below on the left side of the equal sign is

identical to the above except for the first/second column respectively.

$$\begin{vmatrix} I_2 & I_1 & I_2 \cdots I_{\nu-1} \\ I_3 & I_0 & I_1 \cdots I_{\nu-2} \\ I_4 & I_1 & I_0 \cdots I_{\nu-3} \\ \vdots & \vdots & \vdots \ddots \vdots \\ I_{\nu+1} & I_{\nu-2} & I_{\nu-3} \cdots I_0 \end{vmatrix} + \begin{vmatrix} I_0 & I_1 & I_2 \cdots I_{\nu-1} \\ I_1 & I_2 & I_1 \cdots I_{\nu-2} \\ I_2 & I_3 & I_0 \cdots I_{\nu-3} \\ \vdots & \vdots & \vdots \ddots \vdots \\ I_{\nu-1} & I_\nu & I_{\nu-3} \cdots I_0 \end{vmatrix} = \nu \begin{vmatrix} I_2 & I_1 & I_0 \cdots I_{\nu-3} \\ I_3 & I_2 & I_1 \cdots I_{\nu-4} \\ I_4 & I_1 & I_2 \cdots I_{\nu-5} \\ \vdots & \vdots & \vdots \ddots \vdots \\ I_{\nu+1} & I_\nu & I_{\nu-1} \cdots I_2 \end{vmatrix}.$$

Proof. This may be obtained by comparing the asymptotics of $F_n^\nu(x)$ using Lemma 4.1, and taking the derivative of the determinant for $F_\infty^\nu(x)$, using the derivative of $\frac{d}{dx} I_j(x) = \frac{1}{2}(I_{j+1}(x) + I_{j-1}(x))$ and the usual multilinear properties of determinants.

Remark: This conjecture has been verified for $\nu = 2, \ldots, 25$ symbolically in Mathematica and Maple, and numerically for larger values.

Our main interest in this conjecture is that once granted it would also prove the Corollary 3.2 to Theorem 3.1.

5 The hard edge of complex Gaussian divisible ensembles

The hard edge denotes the location of the smallest eigenvalues of sample covariance matrices when $\nu = p - n$ is a fixed integer.

5.1 Reminder on Johansson-Laguerre ensemble

We here recall some important facts about the Johansson-Laguerre ensemble, that we use in the following.

Notations: We call $\mu_{n,p}$ the law of the sample covariance matrix $\frac{1}{n} M^* M$ defined in (1). We denote by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the ordered eigenvalues of the random sample covariance matrix $\frac{1}{n} M^* M$. We also set

$$H = \frac{W}{\sqrt{n}},$$

and denote the distribution of the random matrix H by P_n. The ordered eigenvalues of HH^* are denoted by $y_1(H) \leq y_2(H) \leq \cdots \leq y_n(H)$.

We can now state the known results about the joint eigenvalue density (j.e.d.) induced by the Johansson-Laguerre ensemble. By construction, this is obtained as the integral w.r.t. P_n of the j.e.d. of the Deformed Laguerre Ensemble. We recall that the Deformed Laguerre Ensemble denotes the distribution of the covariance matrix $n^{-1} MM^*$ when H is given. The latter has been first computed by [21] and [23].

We now set

$$s = \frac{a^2}{n}.$$

Proposition 5.1. *The symmetrized eigenvalue measure on $\mathbb{R}_+^n$ induced by $\mu_{n,p}$ has a density w.r.t. Lebesgue measure given by*

$$g(x_1,\dots,x_n) = \int dP_n(H) \frac{\Delta(x)}{\Delta(y(H))} det\Big(\frac{e^{-\frac{y_i(H)+x_j}{2t}}}{2t} I_\nu\left(\frac{\sqrt{y_i(H)x_j}}{t}\right) \Big(\frac{x_j}{y_i(H)}\Big)^{\frac{\nu}{2}}\Big)_{i,j=1}^n, \tag{9}$$

where $t = \frac{a^2}{2n} = \frac{s}{2}$, and $\Delta(x) = \prod_{i<j}(x_i - x_j)$.

From the above computation, all eigenvalue statistics can in principle be computed. In particular, the m-point correlation functions of $\mu_{n,p}$ defined by $R_m(u_1,\dots,u_m) = \frac{n!}{(n-m)!}\int_{\mathbb{R}_+^{n-m}} g(u_1,\dots,u_n)\prod_{i=m+1}^n du_i$ are given by the integral w.r.t. to $dP_n(H)$ of those of the Deformed Laguerre Ensemble. Let $R_m(u,v;y(H))$ be the m-point correlation function of the Deformed Laguerre Ensemble (defined by the fixed matrix H). Then

Proposition 5.2.

$$R_m(u_1,\dots,u_m) = \int_{M_{p,n}(\mathbb{C})} dP_n(H) R_m(u_1,\dots,u_m;y(H)).$$

The second remarkable fact is that the Deformed Laguerre Ensemble induces a determinantal random point field, that is all the m-point correlation functions are given by the determinant of a $m\times m$ matrix involving the same *correlation kernel*.

Proposition 5.3. *Let m be a given integer. Then one has that*

$$R_m(u_1,\dots,u_m;y(H)) = \det\left(K_n(u_i,u_j;y(H))\right)_{i,j=1}^m,$$

where the correlation kernel K_n is defined in Theorem 5.4 below.

There are two important facts about this determinantal structure. The fundamental characteristic of the correlation kernel is that it depends only on the spectrum of HH^* and more precisely on its spectral measure. Since we are interested in the determinant of matrices with entries $K_n(x_i,x_j;y)$, we can consider the correlation kernel up to a conjugation: $K_n(x_i,x_j)\frac{f(x_i)}{f(x_j)}$. This has no impact on correlation functions and we may use this fact later.

Theorem 5.4. *The correlation kernel of the Deformed Laguerre Ensemble (H is fixed) is also given by*

$$K_n(u,v;y(H)) = \frac{e^{\nu i\pi}}{i\pi s^3}\int_\Gamma\int_\gamma dwdzwzK_B\Big(\frac{2z\sqrt{u}}{s},\frac{2w\sqrt{v}}{s}\Big)\Big(\frac{w}{z}\Big)^\nu \exp\{\frac{w^2-z^2}{s}\}$$
$$\times\prod_{i=1}^n\frac{w^2-y_i(H)}{z^2-y_i(H)}\Big(1-s\sum_{i=1}^n\frac{y_i(H)}{(w^2-y_i(H))(z^2-y_i(H))}\Big).$$

where the contour Γ *is symmetric around 0 and encircles the* $\pm\sqrt{y_i(H)}$, γ *is the imaginary axis oriented positively* $0 \longrightarrow +\infty$, $0 \longrightarrow -\infty$, *and* K_B *is the kernel defined by*

$$K_B(x,y) = \frac{xI_\nu'(x)I_\nu(y) - yI_\nu'(y)I_\nu(x)}{x^2 - y^2}. \tag{10}$$

For ease of exposition, we drop from now on the dependency of the correlation kernel K_n on the spectrum of H and write $K_n(u,v)$ for $K_n(u,v;y(H))$. The goal of this section is to deduce Theorem 3.1 by a careful asymptotic analysis of the above formulas. Set

$$\alpha = \sigma^2/4, \tag{11}$$

with $\sigma = \sqrt{1/4 + a^2}$ and hereafter denote $\lambda_{min} = \lambda_{min}(\frac{MM^*}{n})$.

We recall that it was proved in [7] that

$$\lim_{n\to\infty} \mathbb{P}\left(\lambda_{min} \geq \frac{\alpha s}{n^2}\right) = \det(I - \widetilde{K_B})_{L^2(0,s)} \tag{12}$$

where $\widetilde{K_B}$ is the usual Bessel kernel

$$\widetilde{K_B}(u,v) := e^{\nu i\pi} K_B(i\sqrt{u}, i\sqrt{v}) \tag{13}$$

with K_B defined in (10). This is a universality result as the limiting distribution function is the same as that of the smallest eigenvalue of the Laguerre Ensemble (see e.g. [18]).

5.2 Asymptotic expansion of the partition function at the hard edge

The main result of this section is to prove the following expansion for the partition function at the hard edge: recall that α is given by (11).

Theorem 5.5. *There exists a non-negative function* g_n^0, *depending on* n, *so that*

$$\mathbb{P}\left(\lambda_{min} \geq \frac{\alpha s}{n^2}\right) = g_n^0(s) + \frac{1}{n}\partial_\beta g_n^0(\beta s)|_{\beta=1} \int dP_n(H)[\Delta_n(H)] + o\left(\frac{1}{n}\right)$$

where

$$\Delta_n(H) = \frac{-1}{v_c^{\pm} m'_{\mathrm{MP}}(v_c^+)} X_n(v_c^+)$$

with $X_n(z) = \sum_{i=1}^n \frac{1}{y_i(H)-z} - n m_{\mathrm{MP}}(z)$, $m_{\mathrm{MP}}(z)$ *is the Stieltjes transform of the Marchenko-Pastur distribution* ρ_{MP}, $(y_i(H))_{1\leq i\leq n}$ *are the eigenvalues of* H, *and* $v_c^+ = (w_c^\pm)^2$ *where*

$$w_c^\pm = \pm i(R - 1/R)/2, \quad R := \sqrt{1+4a^2}. \tag{14}$$

We will estimate the term $\int dP_n(H)[\Delta_n(H)]$ in terms of the kurtosis in the next section.

Remark 5.6. The function g_n^0 is universal, in the sense that it does not depend on the detail of the distributions P_{jk} (provided the assumptions (4) and (2) are satisfied).

5.2.1 Expansion of the correlation kernel

Let $z_c^{\pm}$ be the critical points of

$$F_n(w) := w^2/a^2 + \frac{1}{n}\sum_{i=1}^{n} \ln(w^2 - y_i(H)), \tag{15}$$

where the $y_i(H)$ are the eigenvalues of H^*H. Then we have the following Lemma. Let K_n be the kernel defined in Theorem 5.4.

Lemma 5.7. *There exists a smooth function A such that for all x, y*

$$\frac{\alpha}{n^2} K_n(u\alpha n^{-2}, v\alpha n^{-2}; y(H)) =$$
$$\widetilde{K_B}(u,v) + \frac{A(u,v)}{n} + ((\frac{z_c^+}{w_c^+})^2 - 1)\frac{\partial}{\partial\beta}\Big|_{\beta=1} \beta\widetilde{K_B}(\beta u, \beta v) + o\left(\frac{1}{n}\right).$$

where $\widetilde{K_B}$ has been defined in (13). Note that $\frac{z_c^+}{w_c^+} = \frac{z_c^-}{w_c^-}$.

Proof

To focus on local eigenvalue statistics at the hard edge, we consider

$$u = \left(\frac{a^2}{2nr_0}\right)^2 x; \quad v = \left(\frac{a^2}{2nr_0}\right)^2 y, \text{ where } r_0 \text{ will be fixed later.}$$

As $\nu = p - n$ is a fixed integer independent of n, this readily implies that the Bessel kernel shall not play a role in the large exponential term of the correlation kernel. In other words, the large exponential term to be considered is F_n defined in (15). The correlation kernel can then be re-written as

$$\begin{aligned} K_n(u,v) = \quad & \frac{1}{i\pi s^3} e^{\nu i\pi} \int_\Gamma \int_\gamma dwdz\, w\, z\, K_B\Big(\frac{zx^{1/2}}{r_0}, \frac{wy^{1/2}}{r_0}\Big)\Big(\frac{w}{z}\Big)^\nu \\ & \times \exp\{nF_n(w) - nF_n(z)\}\tilde{g}(w,z), \end{aligned} \tag{16}$$

where

$$\tilde{g}(w,z) := a^2 g(w,z) = 1 - s\sum_{i=1}^{n} \frac{y_i(H)}{(w^2 - y_i(H))(z^2 - y_i(H))} = \frac{a^2}{2}\frac{wF_n'(w) - zF_n'(z)}{w^2 - z^2}.$$

We note that $F_n(w) = H_n(w^2)$ where $H_n(w) = w/a^2 + \frac{1}{n}\sum_{i=1}^{n} \ln(w - y_i(H))$.

We may compare the exponential term F_n to its "limit", using the convergence of the spectral measure of $H^*H = \frac{1}{n}W^*W$ to the Marchenko-Pastur distribution ρ_{MP}. Set

$$F(w) := w^2/a^2 + \int \ln(w^2 - y) d\rho_{\mathrm{MP}}(y).$$

It was proved in [7] that this term has two conjugated critical points satisfying $F'(w) = 0$ and are given by $w_c^{\pm}$ defined in (14). Let us also denote by $z_c^{\pm}$ the

true non real critical points (which can be seen to exist and be conjugate [7]) associated to F_n. These critical points do depend on n but for ease of notation we do not stress this dependence. These critical points satisfy

$$F_n'(z_c^{\pm}) = 0, \quad z_c^+ = -z_c^-$$

and it is not difficult to see that they are also on the imaginary axis.

We now refer to the results established in [7] to claim the following facts:

- there exist constants C and $\xi > 0$ such that

$$|z_c^{\pm} - w_c^{\pm}| \leq Cn^{-\xi}. \tag{17}$$

  This comes from concentration results for the spectral measure of H established in [22] and [2].

- Fix $\theta > 0$. By the saddle point analysis performed in [7], the contribution of the parts of the contours γ and Γ within $\{|w - z_c^{\pm}| \geq n^{\theta} n^{-1/2}\}$ is $O(e^{-cn^{\theta}})$ for some $c > 0$. This contribution "far from the critical points " is thus exponentially negligible. In the sequel we will choose $\theta = 1/11$. The choice of $1/11$ is arbitrary.

- We can thus restrict both the w and z integrals to neighborhoods of width $n^{1/11}n^{-1/2}$ of the critical points $z_c^{\pm}$.

Also , we can assume that the parts of the contours Γ and γ that will contribute to the asymptotics are symmetric w.r.t. $z_c^{\pm}$. This comes from the fact that the initial contours exhibit this symmetry and from the location of the critical points. A plot of the oriented contours close to critical points is given in Figure 5.2.1.

Let us now make the change of variables

$$w = z_c^1 + sn^{-1/2}; \quad z = z_c^2 + tn^{-1/2};$$

where z_c^1, z_c^2 are equal to z_c^+ or z_c^- depending on the part of the contours γ and Γ under consideration and s, t satisfy $|s|, |t| \leq n^{1/11}$. Then we perform the Taylor expansion of each of the terms arising in both z and w integrands. Then one has that

$$\begin{aligned}
&e^{nF_n(z_c^{\pm}+sn^{-1/2})-nF_n(z_c^{\pm})} \\
&= e^{F_n''(z_c^{\pm})\frac{s^2}{2}+\sum_{i=3}^{5} F_n^{(i)}(z_c^{\pm})\frac{s^i}{i!n^{i/2-1}}}(1+O(n^{-23/22})) \\
&= e^{F_n''(z_c^{\pm})\frac{s^2}{2}} + \frac{1}{n^{1/2}} \underbrace{e^{F_n''(z_c^{\pm})\frac{s^2}{2}} \frac{F_n^{(3)}(z_c^{\pm})}{6} s}_{e_1(s)} \\
&+ \frac{1}{n} \underbrace{e^{F_n''(z_c^{\pm})\frac{s^2}{2}} \left(\frac{F_n^{(4)}(z_c^{\pm})s^4}{4!} + \left(\frac{F_n^{(3)}(z_c^{\pm})}{6} \right)^2 \frac{s^6}{2} \right)}_{e_2(s)} + o\left(\frac{1}{n}\right) e^{F_n''(z_c^{\pm})\frac{s^2}{2}}.
\end{aligned}$$

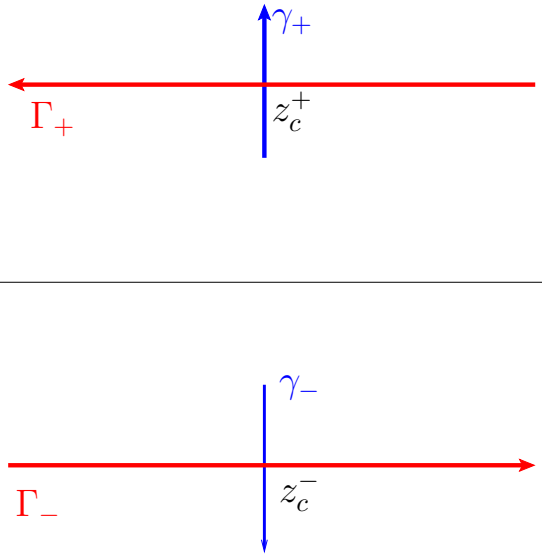


Figure 5: Contours close to the critical points

(18)

as $|s| \leq n^{1/11}$. For each term in the integrand, one has to consider the contribution of equal or opposite critical points. In the following, we denote by z_c, z_c^1, z_c^2 any of the two critical points (allowing z_c to take different values with a slight abuse of notation). We then perform the Taylor expansion of each of the functions arising in the integrands. This yields the following four expansions:

$$wz = z_c^1 z_c^2 + n^{-1/2} \underbrace{(sz_c^2 + tz_c^1)}_{v_1(s,t)} + \frac{1}{n} \underbrace{st}_{v_2(s,t)}, \tag{19}$$

and

$$g\left(z_c^1 + \frac{s}{n^{1/2}}, z_c^2 + \frac{t}{n^{1/2}}\right) = \frac{F_n''(z_c)}{2} \mathbb{1}_{z_c^1 = z_c^2} + \frac{1}{\sqrt{n}} \underbrace{\left(s\frac{\partial}{\partial x_1} + t\frac{\partial}{\partial x_2}\right) g(x_1, x_2)\Big|_{z_c^1, z_c^2}}_{g_1(s,t)}$$
$$+ \frac{1}{n} \underbrace{\left((\frac{s^2}{2} \frac{\partial^2}{\partial x_1^2} g(x_1, x_2) + \frac{t^2}{2} \frac{\partial^2}{\partial x_2^2} + st \frac{\partial^2}{\partial x_2 \partial x_1}) g(x_1, x_2)\Big|_{z_c^1, z_c^2} \right)}_{g_2(s,t)} + o\left(\frac{1}{n}\right). \tag{20}$$

One also has

$$\left(\frac{w}{z}\right)^\nu = (z_c^1/z_c^2)^\nu + n^{-1/2} \underbrace{(z_c^1/z_c^2)^\nu (\frac{\nu s}{z_c^1} - \frac{\nu t}{z_c^2})}_{r_1(s,t)}$$

$$+\frac{1}{n}\underbrace{(z_c^1/z_c^2)^\nu\left(\frac{\nu(\nu-1)s^2}{(z_c^1)^2}+\frac{\nu(\nu+1)t^2}{(z_c^2)^2}-\frac{\nu^2 st}{z_c^1 z_c^2}\right)}_{r_2(s,t)}+o\left(\frac{1}{n}\right). \tag{21}$$

Last, one has that

$$\begin{aligned}
&K_B\left(\frac{zx^{1/2}}{r_0},\frac{wy^{1/2}}{r_0}\right)=K_B\left(\frac{z_c x^{1/2}}{r_0},\frac{z_c y^{1/2}}{r_0}\right)\\
&+\frac{1}{\sqrt{n}}\underbrace{\left(s\frac{\partial}{\partial x_1}+t\frac{\partial}{\partial x_2}\right)\Big|_{z_c,z_c}K_B\left(\frac{x_1x^{1/2}}{r_0},\frac{x_2y^{1/2}}{r_0}\right)}_{h_1(s,t)}\\
&+\frac{1}{n}\underbrace{\left(\frac{s^2}{2}\frac{\partial^2}{\partial x_1^2}+\frac{t^2}{2}\frac{\partial^2}{\partial x_2^2}+st\frac{\partial^2}{\partial x_1\partial x_2}\right)\Big|_{z_c,z_c}K_B\left(\frac{x_1x^{1/2}}{r_0},\frac{x_2y^{1/2}}{r_0}\right)}_{h_2(s,t)}+o\left(\frac{1}{n}\right).
\end{aligned} \tag{22}$$

In all the lines above, $z_c^1/z_c^2=\pm 1$ as critical points are either equal or opposite. Also one can note that the o are uniform as long as $|s|,|t|<n^{1/11}$.

We now choose

$$r_0=|w_c^+|.$$

Combining the whole contribution of neighborhoods of a pair of equal critical points e.g., denoted by $K_n(u,v)_{equal}$, we find that it has an expansion of the form

$$\begin{aligned}
&\frac{a^4}{4n^2r_0^2}K_n(u,v)_{equal}=\sum_{z_c=z_c^\pm}\\
&\frac{\pm}{4i\pi}e^{\nu i\pi}\int_{\mathbb{R}}\int_{i\mathbb{R}}dsdt\frac{|z_c|^2}{r_0^2}\left(K_B(\frac{z_cx^{1/2}}{|w_c^+|},\frac{z_cy^{1/2}}{|w_c^+|})+\sum_{i=1}^{2}\frac{h_i(s,t)}{n^{i/2}}+o\left(\frac{1}{n}\right)\right)\\
&\times\left(\frac{F_n''(z_c)}{2}+\sum_{i=1}^{2}\frac{g_i(s,t)}{n^{i/2}}+o\left(\frac{1}{n}\right)\right)\left(1+\sum_{i=1}^{2}\frac{r_i(s,t)}{n^{i/2}}+o\left(\frac{1}{n}\right)\right)\\
&\times(1+\sum_{i=1}^{2}v_i(s,t)n^{-i/2}z_c^{-2})\Big(\exp\{F_n''(z_c)(s^2-t^2)/2\}\Big(1+o\left(\frac{1}{n}\right)\\
&+n^{-1/2}(e_1(s)-e_1(t))+\frac{1}{n}(-e_1(s)e_1(t)+e_2(s)-e_2(t))\Big)\Big),
\end{aligned} \tag{23}$$

where h_i,e_i,r_i,v_i and g_i defined above have no singularity.

It is not difficult also to see that h_1,g_1,r_1,e_1 are odd functions in s as well as in t: because of the symmetry of the contour, their contribution will thus vanish. The first non zero lower order term in the asymptotic expansion will thus come from the combined contributions $h_1g_1,g_1r_1,r_1h_1,h_1e_1,g_1e_1,r_1e_1,r_1v_1...$

and those from h_2, g_2, r_2, e_2, v_2. Therefore one can check that one gets the expansion

$$\frac{\alpha}{n^2}K_n\left(\frac{\alpha x}{n^2},\frac{\alpha y}{n^2}\right)_{equal}=\frac{e^{i\nu\pi}}{2}\left(\frac{|z_c^{\pm}|}{|w_c^{\pm}|}\right)^2 K_B\left(\frac{z_c^{\pm}x^{1/2}}{|w_c^{+}|},\frac{z_c^{\pm}y^{1/2}}{|w_c^{+}|}\right)+\frac{a_1(z_c^{\pm};x,y)}{n}+o\left(\frac{1}{n}\right),$$

where a_1 is a function of $z_c^{\pm}, x, y$ only. a_1 is a smooth and non vanishing function a priori.

We can write the first term above as $\left(\frac{z_c^{\pm}}{w_c^{\pm}}\right)^2\widetilde{K_B}\left(\left(\frac{z_c^{\pm}}{w_c^{\pm}}\right)^2 x,\left(\frac{z_c^{\pm}}{w_c^{\pm}}\right)^2 y\right)$ so that we deduce that

$$\begin{aligned}&e^{i\nu\pi}\left(\frac{z_c^{\pm}}{w_c^{\pm}}\right)^2 K_B\left(\frac{z_c^{\pm}x^{1/2}}{|w_c^{+}|},\frac{z_c^{\pm}y^{1/2}}{|w_c^{+}|}\right)\\&=\widetilde{K_B}(x,y)+\left(\left(\frac{z_c^{\pm}}{w_c^{\pm}}\right)^2-1\right)\partial_\beta(\beta\widetilde{K_B}(\beta x,\beta y))|_{\beta=1}+o(z_c^{\pm}-w_c^{\pm}).\end{aligned}$$

One can do the same thing for the combined contribution of opposite critical points and get a similar result. We refer to [7] for more detail about this fact.

5.2.2 Asymptotic expansion of the density

The distribution of the smallest eigenvalue of M_n is defined by

$$\mathbb{P}\left(\lambda_{min}\geq\frac{\alpha s}{n^2}\right)=\int dP_n(H)\det(I-\tilde{K}_n)_{L^2(0,s)},$$

where $\tilde{K}_n$ is the rescaled correlation kernel $\frac{\alpha}{n^2}K_n(x\alpha n^{-2}, y\alpha n^{-2})$. In the above we choose $\alpha=(a^2/2r_0)^2$. The limiting correlation kernel is then, at the first order, the Bessel kernel:

$$\widetilde{K_B}(x,y):=e^{\nu i\pi}K_B(i\sqrt{x},i\sqrt{y}).$$

The error terms are ordered according to their order of magnitude: the first order error term, in the order of $O(n^{-1})$, can thus come from two terms, namely
-the deterministic part that is $a_1(z_c^{\pm};x,y)$. These terms yield a contribution of order $\frac{1}{n}$. However it is clear that, as a_1 is smooth and using (17),

$$a_1(z_c^{\pm};x,y)=a_1(w_c^{\pm};x,y)+o(1). \tag{24}$$

Note that $w_c^{\pm}$ does not depend on the exact distributions P_{jk}, but only on the limiting Marchenko-Pastur distribution ρ_{MP}. As a consequence, there is no fourth moment contribution in these $\frac{1}{n}$ terms. We denote the contribution of the deterministic error from all the combined (equal or not) critical points by $A(x,y)/n$.
-the kernel (arising 4 times due to the combination of critical points)

$$e^{\nu i\pi}(\frac{z_c^{+}}{|w_c^{+}|})^2 K_B\left(\frac{z_c^{+}}{|w_c^{+}|}(\sqrt{x},\sqrt{y})\right)=\widetilde{K_B}(x,y)+\int_1^{(z_c^{+}/w_c^{+})^2}\frac{\partial}{\partial\beta}\beta\widetilde{K_B}(\beta x,\beta y)d\beta. \tag{25}$$

Lemma 5.7 and the arguments above (24), (25) gives the following:

$$\begin{aligned}&\frac{\alpha}{n^2}K_n(x\alpha n^{-2}, y\alpha n^{-2})\\&= \widetilde{K_B}(x,y) + \frac{A(x,y)}{n} + ((z_c^+/w_c^+)^2 - 1)\frac{\partial}{\partial\beta}\Big|_{\beta=1}\beta\widetilde{K_B}(\beta x, \beta y) + o\left(\frac{1}{n}\right).\end{aligned}$$

We insist that the kernel A is universal in the sense that it does not depend on the detail of the distributions P_{jk}.

The Fredholm determinant can be developed to obtain that

$$\begin{aligned}&\det(I-\tilde{K}_n)_{L^2(0,s)}\\&= \sum_k \frac{(-1)^k}{k!}\int_{[0,s]^k}\det\left(\tilde{K}_n(x_i,x_j)\right)_{i,j=1}^k \prod dx_i\\&= \sum_k \frac{(-1)^k}{k!}\int_{[0,s]^k}\det\left(\widetilde{K_B}(x_i,x_j)\right)_{i,j=1}^k \det\left(I+G(x_i,x_j)\right)_{i,j=1}^k \prod dx_i + o(\frac{1}{n}),\end{aligned} \tag{26}$$

where we have set

$$G(x_i,x_j) = \left(\widetilde{K_B}(x_i,x_j)\right)^{-1}_{1\le i,j\le k}(B(x_i,x_j))_{i,j=1}^k$$

with

$$B(x_i,x_j) = \frac{A(x_i,x_j)}{n} + \left(\left(\frac{z_c^+}{w_c^+}\right)^2 - 1\right)\frac{\partial}{\partial\beta}\Big|_{\beta=1}\beta\widetilde{K_B}(\beta x_i, \beta x_j) + o\left(\frac{1}{n}\right).$$

The matrix $\left(\widetilde{K_B}(x_i,x_j)\right)_{i,j=1}^k$ is indeed invertible for any k.

Therefore, up to an error term in the order $o\left(\frac{1}{n}\right)$ at most,

$$\begin{aligned}&\det(I-\tilde{K}_n)_{L^2(0,s)} = \det(I-\widetilde{K_B})\\&+\sum_k \frac{(-1)^k}{k!}\int_{[0,s]^k}\det\left(\widetilde{K_B}(x_i,x_j)\right)_{i,j=1}^k \mathrm{Tr}\left(G(x_i,x_j)\right)_{i,j=1}^k dx + o(\frac{1}{n}).\end{aligned}$$

Now if we just consider the term which is linear in $((z_c^+/w_c^+)^2-1)$ which will bring the contribution depending on the fourth cumulant, we have that the correction is

$$L := \sum_k \frac{(-1)^k}{k!}\int_{[0,s]^k}\det\left(\widetilde{K_B}(x_i,x_j)\right)_{i,j=1}^k \mathrm{Tr}(\widetilde{K_B}^{-1}\partial_\beta\beta\widetilde{K_B}(\beta x_i,\beta x_j))_{i,j=1}^k dx|_{\beta=1}$$

$$= \partial_\beta \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det\left(\widetilde{K_B}(x_i,x_j)\right)_{i,j=1}^k \mathrm{Tr}(\log \beta\widetilde{K_B}(\beta x_i,\beta x_j))_{i,j=1}^k dx|_{\beta=1}.$$

As $\widetilde{K_B}$ is trace class, we can write

$$\mathrm{Tr}(\log \beta\widetilde{K_B}(\beta x_i,\beta x_j))_{i,j=1}^k = \log\det\left(\beta\widetilde{K_B}(\beta x_i,\beta x_j)\right)_{i,j=1}^k.$$

Therefore, we have

$$\begin{aligned} L &= \partial_\beta \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det\left(\beta\widetilde{K_B}(\beta x_i,\beta x_j)\right)_{i,j=1}^k \log\det\left(\beta\widetilde{K_B}(\beta x_i,\beta x_j)\right)_{i,j=1}^k dx|_{\beta=1} \\ &= \partial_\beta \sum_k \frac{(-1)^k}{k!} \int_{[0,s]^k} \det\left(\beta\widetilde{K_B}(\beta x_i,\beta x_j)\right)_{i,j=1}^k dx|_{\beta=1} \\ &= \partial_\beta \sum_k \frac{(-1)^k}{k!} \int_{[0,s\beta]^k} \det\left(\widetilde{K_B}(y_i,y_j)\right)_{i,j=1}^k dy_i|_{\beta=1} \\ &= \partial_\beta \det(I-\widetilde{K_B})_{L^2(0,s\beta)}\Big|_{\beta=1}. \end{aligned} \tag{27}$$

Hence, since $\det(I-\widetilde{K_B})_{L^2(0,s\beta)}$ is the leading order in the expansion of $\mathbb{P}\left(\lambda_{min} \geq \frac{\alpha s}{n^2}\right)$ plugging (**??**) into (27) shows that there exists a function g_n^0 (whose leading order is $\det(I-\widetilde{K_B})_{L^2(0,s\beta)}$) so that

$$\mathbb{P}\left(\lambda_{min} \geq \frac{\alpha s}{n^2}\right) = g_n^0(s) + \partial_\beta g_n^0(\beta s)|_{\beta=1} \int dP_n(H)\Big[\left(\frac{z_c^+}{w_c^+}\right)^2 - 1\Big] + o\left(\frac{1}{n}\right). \tag{28}$$

5.2.3 An estimate for $\left(\frac{z_c^+}{w_c^+}\right)^2 - 1$

Let

$$X_n(z) = \sum_{i=1}^n \frac{1}{y_i(H)-z} - n m_{\mathrm{MP}}(z)$$

where $z \in \mathbb{C}\backslash\mathbb{R}_+$. Let us express $(z_c^+)^2 - (w_c^+)^2$ in terms of X_n. The critical point z_c^+ of F_n lies in a neighborhood of the critical point w_c^+ of F. So $u_c^+ = (z_c^+)^2$ is in a neighborhood of $v_c^+ = (w_c^+)^2$. These points are the solutions with non negative imaginary part of

$$\frac{1}{a^2} + \frac{1}{n}\sum_{i=1}^n \frac{1}{u_c^+ - y_i(H)} = 0, \quad \frac{1}{a^2} + \int \frac{1}{v_c^+ - y} d\rho_{\mathrm{MP}}(y) = 0.$$

Therefore it is easy to check that

$$-\int \frac{u_c^+ - v_c^+}{(v_c^+ - y)^2} d\rho_{\mathrm{MP}}(y) + \frac{1}{n} X_n(v_c^+) = o\left(\frac{1}{n}, (z_c^+ - w_c^+)\right)$$

which gives

$$\left(\frac{z_c^+}{w_c^+}\right)^2 - 1 = -\frac{1}{v_c^+ m'_{\mathrm{MP}}(v_c^+)} \frac{1}{n} X_n(v_c^+) + o\left(\frac{1}{n}\right). \tag{29}$$

The proof of Theorem 5.5 is therefore complete. In the next section we estimate the expectation of $X_n(v_c^+)$ to get the correction in (28).

5.3 The role of the fourth moment

In this section we compute $\mathbb{E}[X_n(v_c^+)]$, which with Theorem 5.5, will allow to prove Theorem 3.1.

5.3.1 The expected value $\mathbb{E}[X_n(v_c^+)]$

In this section we give the asymptotics of the mean of $X_n(z)$. Such type of estimates is now well known, and can for instance be found in Bai and Silverstein book [3] for either Wigner matrices or Wishart matrices with $\kappa_4 = 0$. We refer to [3, Theorem 9.10] for a precise statement. In the more complicated setting of F-matrices, we refer the reader to [35]. In the case where $\kappa_4 \neq 0$, the asymptotics of the mean have been computed in [28] and [5].

Proposition 5.8. *Let $z \in \mathbb{C} \setminus \mathbb{R}_+ \cap \{Z \in \mathbb{C}, \Im Z \geq 0\}$. Under the hypothesis (4), we have*

$$\lim_{n\to\infty} \mathbb{E}[X_n(z)] = A(z) - \kappa_4 B(z)$$

with A independent of κ_4, and if $m_{\mathrm{MP}}(z) = \int (x-z)^{-1} d\rho_{\mathrm{MP}}(x)$,

$$B(z) = \frac{m_{\mathrm{MP}}(z)^2}{(1+\frac{m_{\mathrm{MP}}(z)}{4})^2 (z + \frac{z m_{\mathrm{MP}}(z)}{2})}. \tag{30}$$

Note that the above result follows from a simple expansion (up to the $1/N$ order) of the normalized trace of the resolvent $\frac{1}{n}\mathrm{Tr}\left(\frac{WW^*}{n} - zI\right)^{-1}$ for a complex number z with non zero imaginary part. We recall [26] that

$$m_{\mathrm{MP}}(z) := \lim_{n\to\infty} \frac{1}{n}\mathrm{Tr}\left(\frac{WW^*}{n} - zI\right)^{-1}$$

is uniquely defined as the solution with non negative imaginary part of the equation

$$\frac{1}{1+\frac{1}{4}m_{\mathrm{MP}}(z)} = -z m_{\mathrm{MP}}(z). \tag{31}$$

5.3.2 Estimate at the critical point

Since $m_{\mathrm{MP}}(v_c^+) = a^{-2}$, we deduce from (30) that there exists a constant $c(v_c^+)$ independent of κ_4 such that

$$\mathbb{E}[X_n(v_c^+)] = \quad c(v_c^+) - \kappa_4 \frac{a^{-4}}{(1+\frac{1}{4}a^{-2})^2} \frac{1}{v_c^+(1+\frac{1}{2}a^{-2})} + o(1).$$

Moreover, we have

$$v_c^+ = -\frac{a^4}{\frac{1}{4}+a^2} = -\frac{4a^4}{1+4a^2}. \tag{32}$$

and by (31), after taking the derivative, we find

$$m'_{\mathrm{MP}}(z) = -\frac{m_{\mathrm{MP}}(z)(1+m_{\mathrm{MP}}(z)/4)}{z(1+m_{\mathrm{MP}}(z)/2)},$$

so that at the critical point we get

$$\begin{aligned} m'_{\mathrm{MP}}(v_c^+) &= \frac{(4a^2+1)^2}{16a^6(a^2+\frac{1}{2})}, \\ v_c^+ m'_{\mathrm{MP}}(v_c^+) &= -\frac{(1+4a^2)}{4(\frac{1}{2}+a^2)} = -\frac{a^{-2}(1+\frac{1}{4a^2})}{1+\frac{1}{2a^2}}. \end{aligned} \tag{33}$$

Therefore, with the notations of Theorem 5.5, and using (32), (32) and (33), we find constants C independent of κ_4 (and which may change from line to line) so that

$$\begin{aligned} \int dP_n(H)[\Delta_n(H)] &= -\frac{1}{v_c^+ m'_{\mathrm{MP}}(v_c^+)} \mathbb{E}[n(m_n(v_c^+) - m_{\mathrm{MP}}(v_c^+))] + o(1) \\ &= -\frac{1+\frac{1}{2a^2}}{a^{-2}(1+\frac{1}{4a^2})} \kappa_4 \frac{a^{-4}}{(1+\frac{1}{4a^2})^2} \frac{1}{1+\frac{1}{2a^2}} \frac{1+4a^2}{4a^4} + C + o(1) \\ &= -\frac{16\kappa_4}{(1+4a^2)^2} + C + o(1) = -\frac{\kappa_4}{(\frac{1}{4}+a^2)^2} + C + o(1). \end{aligned}$$

Rescale the matrix M by dividing it by σ so as to standardize the entries. Combining Theorem 5.5 and the above, we have therefore found that the deviation of the smallest eigenvalue are such that

$$\mathbb{P}\left(\lambda_{\min}(\frac{MM^*}{n\sigma^2}) \geq \frac{s}{n^2}\right) = \mathbf{g}_n(s) + \frac{\gamma}{2n} s\mathbf{g}'_n(s) + o\left(\frac{1}{n}\right),$$

where γ is the kurtosis defined in Definition (1). At this point $\mathbf{g}_n$ is identified to be the distribution function at the Hard Edge of the Laguerre ensemble with variance 1, as it corresponds to the case where $\gamma = 0$. Theorem 3.1 follows.

6 The bulk of Gaussian divisible ensembles

We here choose to consider the Deformed GUE instead of the Deformed Laguerre ensemble. Indeed, while the arguments are completely similar, the technicalities in the Deformed Laguerre ensemble are more involved. To ease the reading, we here present the simplest ensemble.

6.1 Deformed GUE in the bulk

Let $W = (W_{ij})_{i,j=1}^n$ be a Hermitian Wigner matrix of size n. The entries W_{ij} $1 \leq i < j \leq n$ are i.i.d. with distribution P_{ij}. The entries along the diagonal are i.i.d. real random variables with law P_{ii} independent of the off diagonal entries. We assume that $P_{ij}, P_{ii}, 1 \leq i \leq j \leq N$ have sub exponential tails and satisfy (5) and (6). The fourth moment of the P_{ij}'s is also assumed not to depend on i, j. Let also V be a GUE random matrix with i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$ entries and consider the rescaled matrix

$$M_n = \frac{1}{\sqrt{n}}(W + aV).$$

We denote by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the ordered eigenvalues of M_n. By Wigner's theorem, it is well known that the spectral measure of M_n

$$\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}$$

converges weakly to the semi-circle distribution with density

$$\sigma_{sc}^{2\sigma}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}1_{|x|\leq 2\sigma}; \quad \sigma^2 = 1/4 + a^2. \tag{34}$$

This is the Deformed GUE ensemble studied by Johansson [24]. In this section, we study the localization of the eigenvalues λ_i with respect to the quantiles of the limiting semi-circle distribution. We study the $\frac{1}{n}$ expansion of this localization, showing that it depends on the fourth moment of P_{ij}, and prove Theorem 3.3.

The route we follow is similar to that we took in the previous section for Wishart matrices: we first obtain a $\frac{1}{n}$ expansion of the correlation functions of the Deformed GUE. The dependency of this expansion in the fourth moment of P_{ij} is then derived.

6.2 Asymptotic analysis of the correlation functions

Let ρ_n be the one point correlation function of the Deformed GUE. We prove in this subsection the following result, with $z_c^{\pm}, w_c^{\pm}$ critical points similar to those of the last section, which we will define precisely in the proof.

Proposition 6.1. *For all $\varepsilon > 0$, uniformly on $u \in [-2\sigma + \varepsilon, 2\sigma - \varepsilon]$, we have*

$$\rho_n(u) = \sigma_{sc}^{2\sigma}(u) + \mathbb{E}[\left(\frac{\Im z_c^+(u)}{\Im w_c^+(u)} - 1\right)]\sigma_{sc}^{2\sigma}(u) + \frac{C'(u)}{n} + o\left(\frac{1}{n}\right),$$

where the function $u \mapsto C'(u)$ does not depend on the distribution of the entries of W whereas z_c^+ depends on the eigenvalues of W.

Proof of Proposition 6.1: Denote by $y_1 \leq y_2 \leq \cdots \leq y_n$ the ordered eigenvalues of $W/\sqrt{n}$. [24] (see also [11]) proves that, for a fixed $W/\sqrt{n}$, the eigenvalue density of M_n induces a determinantal process with correlation kernel given by

$$K_n\Big(u, v; y\left(\frac{W}{\sqrt{n}}\right)\Big) = \frac{n}{(2i\pi)^2}\int_\Gamma dz \int_\gamma dw e^{n(F_v(w) - F_v(z))}\frac{1 - e^{\frac{(u-v)zn}{a^2}}}{z(u-v)} g_n(z, w),$$

where

$$F_v(z) = \frac{(z-v)^2}{2a^2} + \frac{1}{n}\sum \ln(z - y_i(H)),$$

and

$$g_n(z, w) = F_u'(z) + z\frac{F_v'(z) - F_v'(w)}{z - w}.$$

The contour Γ has to encircle all the $y_i(H)$'s and γ is parallel to the imaginary axis.

We now consider the asymptotics of the correlation kernel in the bulk, that is close to some point $u_0 \in (-2\sigma + \delta, 2\sigma - \delta)$ for some $\delta > 0$ (small). We recall that we can consider the correlation kernel up to conjugation: this follows from the fact that $\det\left(K_n(x_i, x_j; y)\right) = \det\left(K_n(x_i, x_j; y)\frac{h(x_i)}{h(x_j)}\right)$, for any non vanishing function h. We omit some details in the next asymptotic analysis as it closely follows the arguments of [24] and those of Subsection 5.2.

Let then u, v be points in the bulk with

$$u = u_0 + \frac{\alpha x}{n}, v = u_0 + \frac{\alpha \tilde{x}}{n}; u_0 = \sqrt{1 + 4a^2}\cos(\theta_0), \theta_0 \in (2\epsilon, \pi - 2\epsilon).$$

The constant α will be fixed afterwards. Then the approximate large exponential term to lead the asymptotic analysis is given by

$$\tilde{F}_v(z) = \frac{(z-v)^2}{2a^2} + \int \ln(z - y) d\sigma_{sc}^1(y)\,.$$

In the following we note $R_0 = \sqrt{1 + 4a^2} = 2\sigma$.

We recall the following facts from [24], Section 3. Let $u_0 = \sqrt{1 + 4a^2}\cos(\theta_0)$ be a given point in the bulk.

- The approximate critical points, i.e. the solutions of $\tilde{F}_{u_0}'(z) = 0$ are given by

$$w_c^\pm(u_0) = (R_0 e^{i\theta_c} \pm \frac{1}{R_0 e^{i\theta_c}})/2.$$

The true critical points satisfy $F'_{u_0}(z) = 0$. Among the solutions, we disregard the $n-1$ real solutions which are interlaced with the eigenvalues $y_1, \dots, y_n$. The two remaining solutions are complex conjugate with non zero imaginary part and we denote them by $z_c^{\pm}(u_0)$. Furthermore [24] proves that

$$|z_c(u_0)^+ - w_c(u_0)^+| \leq n^{-\xi}$$

for any point u_0 in the bulk of the spectrum.

- We now fix the contours for the saddle point analysis. The steep descent/ascent contours can be chosen as :

$$\begin{aligned} \gamma = & \quad z_c^+(v) + it, t \in \mathbb{R}, \\ \Gamma = & \quad \{z_c^{\pm}(r), r = R_0 \cos(\theta), \theta \in (\epsilon, \pi - \epsilon)\} \bigcup \{z_c^{\pm}(R_0 \cos(\epsilon)) + x, x > 0\} \\ & \bigcup \{z_c^{\pm}(-R_0 \cos(\epsilon)) - x, x > 0\}. \end{aligned}$$

It is an easy computation (using that $\Re F''_{u_0}(w) > 0$ along γ) to check that the contribution of the contour $\gamma \cap \{|w - z_c^{\pm}(v)| \geq n^{1/12-1/2}\}$ is exponentially negligible. Indeed there exists a constant $c > 0$ such that

$$\Big| \int_{\gamma \cap \{|w - z_c^{\pm}(v)| \geq n^{1/12-1/2}\}} e^{n\Re(F_{u_0}(w) - F_{u_0}(z_c^+(v)))} dw \Big| \leq e^{-cn^{1/6}}.$$

Similarly the contribution of the contour $\Gamma \cap \{|w - z_c^{\pm}(v)| \geq n^{1/12-1/2}\}$ is of order $e^{-cn^{1/6}}$ that of a neighborhood of $z_c^{\pm}(v)$.

For ease of notation, we now denote $z_c(v) := z_c^+(v)$. We now modify slightly the contours so as to make the contours symmetric around $z_c^{\pm}(v)$. To this aim we slightly modify the Γ contour: in a neighborhood of width $n^{1/12-1/2}$ we replace Γ by a straight line through $z_c^{\pm}(v)$ with slope $z'_c(v)$. This slope is well defined as

$$z'_c(v) = \frac{1}{F''_v(z_c(v))} \neq 0,$$

using that $|z_c^{\pm}(v) - w_c^{\pm}(u_0)| \leq n^{-\xi}$. We refer to Figure 6.2, to define the new contour Γ' which is more explanatory.

Denote by E the leftmost point of $\Gamma \cap \{w, |w - z_c(v)| = n^{1/12-1/2}\}$. Then there exists v_1 such that $E = z_c(v_1)$. We then define e by $e = z_c(v) + z'_c(v)(v_1 - v)$. We then draw the segment $[e, z_c(v)]$ and draw also its symmetric to the right of $z_c(v)$. Then it is an easy fact that

$$|E - e| \leq Cn^{2(1/12-1/2)}, \text{ for some constant C.}$$

Here we have used that e, E both lie within a distance $n^{1/12-1/2}$ from $z_c(v)$. It follows that

$$\forall z \in [e, E], \quad \Big| \Re\Big(nF_v(z) - nF_v(E) \Big) \Big| \leq Cnn^{3(1/12-1/2)} << n^{1/6}.$$

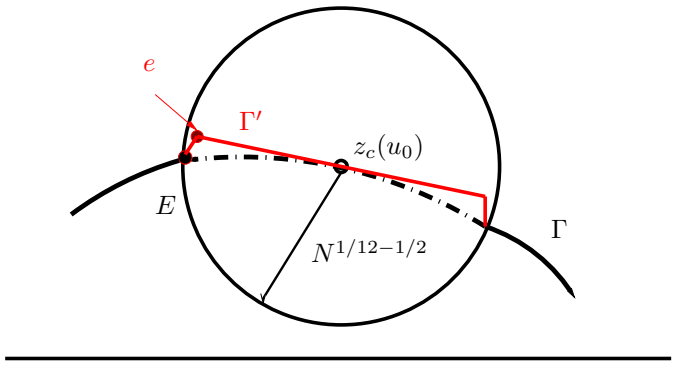


Figure 6: Modification of the Γ contour

This follows from the fact that $|F_v'(z)| = O(n^{1/12-1/2})$ along the segment $[e, E]$. This is now enough as $\Re nF_v(E) > \Re nF_v(z_c) + cn^{1/6}$ to ensure that the deformation has no impact on the asymptotic analysis.

We now make the change of variables $z = z_c^{\pm}(v) + \frac{t}{\sqrt{n}}, w = z_c^{\pm}(v) + \frac{s}{\sqrt{n}}$ where $|s|, |t| \leq n^{1/12}$. We examine the contributions of the different terms in the integrand. We first consider g_n. We start with the combined contribution of equal critical points, e.g. z and w close to the same critical point. In this case we have that

$$\frac{g_n(w,z)}{z} = F_v''(z_c(v)) + \frac{\alpha(x-\tilde{x})}{n} + \frac{F_v^{(3)}(z_c(v))}{2\sqrt{n}}(s+t) + \frac{F_v^{(4)}(z_c(v))}{3!n}(s^2+t^2+st) + o\left(\frac{1}{n}\right).$$

On the other hand when w and z lie in the neighborhood of different critical points, one gets that

$$\frac{g_n(w,z)}{z} = \frac{\alpha(x-\tilde{x})}{nz_c^{\pm}}(1 - \frac{t}{z_c^{\pm}\sqrt{N}}) + \frac{F_v^{(2)}(z_c^{\pm}(v))t - F_v^{(2)}(z_c^{\mp}(v))s}{2(z_c^{\pm} - z_c^{\mp})\sqrt{n}} + o\left(\frac{1}{n}\right),$$

where the $o\left(\frac{1}{n}\right)$ depends on the third derivative of F_v only.
We next turn to the second term, which depends on z only. One has that

$$1 - e^{(x-\tilde{x})\alpha a^{-2} z_c^{\pm}} = 1 - e^{(x-\tilde{x})\alpha a^{-2}\Re z_c^{+}} e^{i\pm(x-\tilde{x})\alpha a^{-2}\Im z_c}.$$

We then perform the same Taylor expansion as in Subsection 5.2 of all the terms in the integrands. As the contours are symmetric around $z_c(u_0)$, the first non zero term in the expansion is in the scale of $\frac{1}{n}$. Furthermore apart from constants, one has that

$$\frac{\alpha}{n} K_n\left(u, v; y(\frac{W}{\sqrt{n}})\right) = \frac{e^{(x-\tilde{x})\frac{\alpha}{a^2}\Re z_c^{+}}}{2i\pi(x-\tilde{x})}\left(e^{i(x-\tilde{x})\frac{\alpha}{a^2}\Im z_c^{+}} - e^{-i(x-\tilde{x})\frac{\alpha}{a^2}\Im z_c^{+}}\right) + \frac{C(x,\tilde{x})}{n} + o\left(\frac{1}{n}\right).$$

The function $C(x, \tilde{x})$ does not depend on the detail of the distributions of the entries of W. We now choose $\alpha = \sigma_{sc}^{2\sigma}(u_0)^{-1}$ where $\sigma_{sc}^{2\sigma}$ is the density of

the semi-circle distribution defined in (34). It has been proved in [24] that $\Im w_c^+(u_0) = \pi a^2 \sigma_{sc}^{2\sigma}(u_0)$. Setting then

$$\beta := \Im z_c^+(u_0)/\Im(w_c^+(u_0))$$

we then obtain that

$$\frac{\alpha}{n} K_n\left(u, v; y(\frac{W}{\sqrt{n}})\right) e^{-(x-\tilde{x})\frac{\alpha}{a^2}\Re z_c^+} = \frac{\sin \pi\beta(x-\tilde{x})}{\pi(x-\tilde{x})} + \frac{C'(x,\tilde{x})}{n} + o\left(\frac{1}{n}\right).$$

The constant $C'(x,\tilde{x})$ does not depend on the distribution of the entries of W. This proves Proposition 6.1 since

$$\begin{aligned}
\rho_n(x) &= \mathbb{E}[\frac{1}{n} K_n\left(u, u; y(\frac{W}{\sqrt{n}})\right)] \\
&= \frac{1}{\alpha}\mathbb{E}[\beta] + \frac{C'(x,x)}{n} + o\left(\frac{1}{n}\right) \\
&= \sigma_{sc}^{2\sigma}(u_0) + \sigma_{sc}^{2\sigma}(u_0)\mathbb{E}[(\beta-1)] + \frac{C'(x,x)}{n} + o\left(\frac{1}{n}\right).
\end{aligned}$$

It can easily be checked e.g. in the case where W is Gaussian that $C'(x,x) = 0$.

6.3 An estimate for $z_c - w_c$ and the role of the fourth moment

We follow the route developed for Wishart matrices, showing first that the fluctuations of $z_c^{\pm}(u_0)$ around $w_c^{\pm}(u_0)$ depend on the fourth moment of the entries of W.

We fix a point u_0 in the bulk of the spectrum. Set now

$$\beta_4 = \mathbb{E}(4|W_{ij}|^2 - 1)^2 = 4^2(\kappa_4 + 1/16).$$

Proposition 6.2. *There exists a constant $C_n = C_n(u_0)$ independent of the distributions $(P_{ij})_{i\leq j}$ and $l = l(u_0) \in \mathbb{R}$ such that $nC_n \to l$ such that*

$$\mathbb{E}[z_c^+(u_0) - w_c^+(u_0)] = \frac{C_n + \beta_4 m_{\rm sc}(w_c^+(u_0))^4/(16n)}{(a^{-2} - m'_{\rm sc}(w_c^+(u_0)))(w_c + m_{\rm sc}(w_c^+(u_0))/2)} + o\left(\frac{1}{n}\right).$$

As a consequence, for any $\varepsilon > 0$ uniformly on $u \in [-2\sigma+\varepsilon, 2\sigma-\varepsilon]$,

$$\rho_n(u_0) = \sigma_{sc}^{2\sigma}(u_0) + \frac{C'(u_0)}{n} + \kappa_4 \frac{D(u_0)}{n} + o\left(\frac{1}{n}\right), \tag{35}$$

where $D(u_0) = d(w_c^+(u_0)), C'(u_0) = c(w_c^+(u_0))$ are given by

$$d(z) = \frac{1}{16\pi a^2}\Im\left(\frac{m_{\rm sc}(z)^4}{(a^{-2} - m'_{\rm sc}(z))(z + m_{\rm sc}(z)/2)}\right), \tag{36}$$

$$c(z) = d(z) + \frac{1}{\pi a^2}\Im\left(\left(\frac{m_{\rm sc}^2(z)}{4(z+\frac{m_{\rm sc}(z)}{2})} - \frac{m_{\rm sc}^2(z)}{16}\frac{|m_{\rm sc}(z)|^4}{4(1-|m_{\rm sc}(z)|^2/4)}\right)\frac{1}{(\frac{1}{a^2} - m'_{\rm sc}(z))(z+\frac{m_{\rm sc}(z)}{2})}\right).$$

Proof of Proposition 6.2: We first relate the critical points z_c^+ and w_c^+ to the difference of the Stieltjes transforms $m_n - m_{\rm sc}$. The true and approximate critical points satisfy the following equations:

$$\frac{z_c^+ - u_0}{a^2} - m_n(z_c^+) = 0; \quad \frac{w_c^+ - u_0}{a^2} - m_{\rm sc}(w_c^+) = 0.$$

Hence,

$$(\frac{1}{a^2} - m'_{\rm sc}(w_c^+))(z_c^+ - w_c^+) = m_n(w_c^+) - m_{\rm sc}(w_c^+) + o\left(\frac{1}{n}\right) \tag{37}$$

where we have used that $m_n - m_{\rm sc}$ is of order $\frac{1}{n}$. Indeed, the estimate will again rely on the estimate of the mean of the central limit theorem for Wigner matrices, see [3, Theorem 9.2]. For the sake of completeness we recall the main steps. Using Schur complement formulae (see [1] Section 2.4 e.g.) one has that

$$m_n(z) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{-z + W_{ii}n^{-1/2} - h_i^* R^{(i)}(z) h_i},$$

where h_i is the ith column of $W/\sqrt{n}$ with ith entry removed and $R^{(i)}$ is the resolvent of the $(n-1)\times(n-1)$ matrix formed from $W/\sqrt{n}$ by removing column and row i. We write

$$m_n(z) + \frac{1}{z + \frac{1}{4}m_n(z)} = \frac{1}{n}\sum_{i=1}^{n} \frac{\delta_n}{(z + \frac{1}{4}m_n(z))(z + \frac{1}{4}m_n(z) + \delta_n)} =: E_n,$$

where $\delta_n = W_{ii}n^{-1/2} + \frac{1}{4}m_n - h_i^* R^{(i)(z)} h_i$. We refer the reader to [4] where similar computations have been done to estimate $\mathbb{E}[E_n]$: for $z \in \mathbb{C}\setminus\mathbb{R}$, one finds that

$$\mathbb{E}[E_n] = c_n + \frac{\beta_4 m_{\rm sc}(z)^2}{16n(z + \frac{1}{4}m_{\rm sc}(z))^3} + o\left(\frac{1}{n}\right), \tag{38}$$

where the sequence $c_n = c_n(z)$ is given by

$$c_n = \frac{1}{4n(z + \frac{m_{\rm sc}(z)}{4})^2} m'_{\rm sc}(z) - \frac{1}{16n(z + \frac{m_{\rm sc}(z)}{4})^3}\left(\frac{\Im m_{\rm sc}(z)}{\Im z} - |m_{\rm sc}(z)|^2\right) + o\left(\frac{1}{n}\right).$$

We recall that the limiting Stieltjes transform satisfies

$$m_{\rm sc}(z) + \frac{1}{z + \frac{1}{4}m_{\rm sc}(z)} = 0.$$

As a consequence , we get

$$(m_n(z) - m_{\rm sc}(z))(z + \frac{1}{4}(m_n(z) + m_{\rm sc}(z)) = E_n(z + \frac{1}{4}m_n(z)), \tag{39}$$

from which we deduce (using that $m_n(z) - m_{\rm sc}(z) \to 0$ as $n \to \infty$) that

$$\begin{aligned}\mathbb{E}[(m_n(z)-m_{\rm sc}(z))](z+m_{\rm sc}(z)/2) &\sim \mathbb{E}[E_n](z+\frac{1}{4}m_{\rm sc}(z))\\ &= [c_n(z+\frac{1}{4}m_{\rm sc}(z))+\frac{\beta_4 m_{\rm sc}(z)^4}{16n}]+o\left(\frac{1}{n}\right).\end{aligned} \tag{40}$$

Combining (38), (39) and (40) and using the fact that $|z_c(v)^+ - w_c(v)^+| \leq n^{-\xi}$ for any point v in the bulk of the spectrum, we deduce the first part of Proposition 6.2. In particular one has that $\mathbb{E}[z_c^+(u) - w_c^+(u)] = C_n^1(w_c^+) + \beta_4 D_n^1(w_c^+)$ where

$$\begin{aligned}C_n^1(z) &= -\left(\frac{m_{\rm sc}(z)m'_{\rm sc}(z)}{4n}+\frac{m_{\rm sc}(z)^2}{16n}\left(\frac{\Im m_{\rm sc}(z)}{\Im z}-|m_{\rm sc}(z)|^2\right)\right)\frac{1}{(\frac{1}{a^2}-m'_{\rm sc}(z))(z+\frac{m_{\rm sc}(z)}{2})}\\ &= \left(\frac{m_{\rm sc}^2(z)}{4n(z+\frac{m_{\rm sc}(z)}{2})}-\frac{m_{\rm sc}^2(z)}{16n}\frac{|m_{\rm sc}(z)|^4}{4(1-|m_{\rm sc}(z)|^2/4)}\right)\frac{1}{(\frac{1}{a^2}-m'_{\rm sc}(z))(z+\frac{m_{\rm sc}(z)}{2})},\end{aligned} \tag{41}$$

and

$$D_n^1(z) = \left(\frac{m_{\rm sc}(z)^4}{16n(a^{-2}-m'_{\rm sc}(z))(z+m_{\rm sc}(z)/2)}\right), \tag{42}$$

where in both formulas $z = z_c(u)$. To derive the above formulas, we have used that $m'_{\rm sc}(z)(z+m_{\rm sc}(z)/2) = -m_{\rm sc}(z)$ and that $|m_{\rm sc}(z)^2|(1+1/4\frac{\Im m_{\rm sc}(z)}{\Im z}) = \frac{\Im m_{\rm sc}(z)}{\Im z}$. Using Proposition 6.1, the expansion for the one point correlation function follows.

6.4 The localization of eigenvalues

We now use (35) to obtain a precise localization of eigenvalues in the bulk of the spectrum. A conjecture of Tao and Vu (more precisely Conjecture 1.7 in [31]) states that (when the variance of the entries of W is $\frac{1}{4}$), there exists a constant $c > 0$ and a function $x \mapsto C'(x)$ independent of κ_4 such that

$$\mathbb{E}\left(\lambda_i - \gamma_i\right) = \frac{1}{n\sigma_{sc}^{2\sigma}(\gamma_i)}\int_0^{\gamma_i} C'(x)dx + \frac{\kappa_4}{2n}(2\gamma_i^3-\gamma_i) + O\left(\frac{1}{n^{1+c}}\right) \tag{43}$$

where γ_i is given by $N_{sc}(\gamma_i) = i/n$ if $N_{sc}(x) = \int_{-\infty}^x d\sigma_{sc}^{2\sigma}(u)$. We do not prove the conjecture but another version instead. More precisely we obtain the following estimate. Fix $\delta > 0$ and an integer i such that $\delta < i/n < 1-\delta$. Define also

$$N_n(x) := \frac{1}{n}\sharp\{i, \lambda_i \leq x\}, \text{ with } \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\,. \tag{44}$$

Let us define the quantile $\hat{\gamma}_i$ by

$$\hat{\gamma}_i := \inf\left\{y, \int_{-\infty}^y \rho_n(x)dx = \frac{i}{n}\right\}.$$

By definition $\mathbb{E}N_n(\hat{\gamma}_i) = i/n$. We prove the following result.

Proposition 6.3. *There exists a constant $c > 0$ and a function $x \mapsto C'(x)$ independent of κ_4 such that*

$$\hat{\gamma}_i - \gamma_i = \frac{1}{n\sigma_{sc}^{2\sigma}(\gamma_i)} \int_0^{\gamma_i} C'(x)dx + \frac{\kappa_4}{2n}(2\gamma_i^3 - \gamma_i) + O\left(\frac{1}{n^{1+c}}\right) \quad (45)$$

The main step to prove this proposition is the following.

Proposition 6.4. *Assume that $i \geq n/2$ without loss of generality. There exists a constant $c > 0$ such that*

$$\hat{\gamma}_i - \gamma_i - \hat{\gamma}_{[n/2]} + \gamma_{[n/2]} = \frac{1}{\sigma_{sc}^{2\sigma}(\gamma_i)} \int_{\gamma_{[n/2]}}^{\gamma_i} [\rho_n(x) - \sigma_{sc}^{2\sigma}(x)]dx + O\left(\frac{1}{n^{1+c}}\right). \quad (46)$$

Note here that $\gamma_{[n/2]} = 0$ when n is even.

Proof of Proposition 6.4: The proof is divided into Lemma 1 and Lemma 2 below.

Lemma 1. *For any $\varepsilon > 0$, there exists $c > 0$ such that uniformly on $i \in [\varepsilon N, (1-\varepsilon)N]$*

$$\gamma_i - \hat{\gamma}_i = \mathbb{E}\left(N_n(\hat{\gamma}_i) - N_{sc}(\hat{\gamma}_i)\right) \frac{1}{\sigma_{sc}^{2\sigma}(\gamma_i)} + O\left(\frac{1}{n^{1+c}}\right). \quad (47)$$

Proof of Lemma 1: Under assumptions of sub exponential tails, it is proved in [16] and [17] (see also Remark 2.4 of [31]) that given $\eta > 0$ for n large enough

$$\mathbb{P}\left(\max_{\varepsilon N \leq i \leq (1-\varepsilon)n} |\gamma_i - \lambda_i| \geq n^{\eta-1}\right) \leq n^{-\log n}. \quad (48)$$

This implies that

$$\mathbb{E}[N_n(\gamma_i + n^{\eta-1})] \geq \frac{i}{n} + n^{-\log n}, \quad \mathbb{E}[N_n(\gamma_i - n^{\eta-1})] \leq \frac{i}{n} - n^{-\log n},$$

from which it follows that for n large enough

$$\max_{\varepsilon N \leq i \leq (1-\varepsilon)n} |\gamma_i - \hat{\gamma}_i| \leq 2n^{\eta-1}. \quad (49)$$

From the fact that $\mathbb{E}N_n(\hat{\gamma}_i) = N_{sc}(\gamma_i)$, we deduce that

$$\begin{aligned} &\mathbb{E}N_n(\hat{\gamma}_i) - N_{sc}(\hat{\gamma}_i) = N_{sc}(\gamma_i) - N_{sc}(\hat{\gamma}_i) \\ &= N'_{sc}(\gamma_i)(\gamma_i - \hat{\gamma}_i) - \int_{\gamma_i}^{\hat{\gamma}_i} \int_{\gamma_i}^{u} N''_{sc}(s)dsdu. \end{aligned} \quad (50)$$

Using that $N'_{sc}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}1_{|x|\leq 2\sigma}$ and that both γ_i and $\hat{\gamma}_i$ lie within $(-2\sigma + \epsilon, 2\sigma - \epsilon)$ for some $0 < \epsilon < 2\sigma$, we deduce that

$$\mathbb{E}N_n(\hat{\gamma}_i) - N_{sc}(\hat{\gamma}_i) = \sigma_{sc}^{2\sigma}(\gamma_i)(\gamma_i - \hat{\gamma}_i) + O(\gamma_i - \hat{\gamma}_i)^2.$$

We now make the following replacement.

Lemma 2. *Let $\varepsilon > 0$. There exist a constant $c > 0$ such that uniformly on $i \in [\varepsilon n, (1-\varepsilon)n]$,*

$$\mathbb{E}\left(N_n(\hat{\gamma}_i) - N_{sc}(\hat{\gamma}_i)\right) \quad = \mathbb{E}\left(N_n(\gamma_i) - N_{sc}(\gamma_i)\right) + O\left(\frac{1}{n^{1+c}}\right). \tag{51}$$

Proof of Lemma 2: We write that

$$\begin{aligned}&\mathbb{E}\left(N_n(\hat{\gamma}_i) - N_{sc}(\hat{\gamma}_i)\right)\\ &= \mathbb{E}\left(N_n(\gamma_i) - N_{sc}(\gamma_i)\right) + \mathbb{E}\left(N_n(\hat{\gamma}_i) - N_n(\gamma_i) - N_{sc}(\hat{\gamma}_i) + N_{sc}(\gamma_i)\right).\end{aligned} \tag{52}$$

We show that the second term in (52) is negligible with respect to n^{-1}. In fact, by (35) and (49), for $\varepsilon > 0$, there exists $\delta > 0$ such that for any $i \in [\varepsilon n, (1-\varepsilon)n]$,

$$\begin{aligned}\left|\mathbb{E}\left(N_n(\hat{\gamma}_i) - N_n(\gamma_i) - N_{sc}(\hat{\gamma}_i) + N_{sc}(\gamma_i)\right)\right| &\leq \left|\int_{\gamma_i}^{\hat{\gamma}_i} (\rho_n(x) - \sigma_{sc}^{2\sigma}(x))dx\right| \\ &\leq n^{\eta-1}\frac{1}{n} \leq \frac{1}{n^{2-\eta}}.\end{aligned} \tag{53}$$

In the last line, we have used (35). This finishes the proof of Lemma 2.

Combining Lemma 1 and Lemma 2 yields Proposition 6.4:

$$\begin{aligned}\delta_n &:= \sigma_{sc}^{2\sigma}(\gamma_i)(\gamma_i - \hat{\gamma}_i) - \sigma_{sc}^{2\sigma}(\gamma_{[n/2]})(\gamma_{[n/2]} - \hat{\gamma}_{[n/2]}) \\ &= \int_{\gamma_{[n/2]}}^{\gamma_i} [\rho_n(x) - \sigma_{sc}^{2\sigma}(x)]dx + O\left(\frac{1}{n^{1+c}}\right) \\ &= \frac{1}{n}\int_{\gamma_{[n/2]}}^{\gamma_i} (C'(x) + \kappa_4 D(x))dx + O\left(\frac{1}{n^{1+c}}\right) \\ &= \frac{1}{n}\int_{0}^{\gamma_i} (C'(x) + \kappa_4 D(x))dx + O\left(\frac{1}{n^{1+c}}\right)\end{aligned} \tag{54}$$

where we used that $\gamma_{[n/2]}$ vanishes or is at most of order $1/n$. This formula will be the basis for identifying the role κ_4 in the $\frac{1}{n}$ expansion of $\hat{\gamma}_i$. We now write for a point x in the bulk $(-R(1-\delta), R(1-\delta))$ that

$$x = \sqrt{1+4a^2}\cos\theta.$$

We also write that $\gamma_i = \sqrt{1+4a^2}\cos\theta_0$. We then have that

$$w_c(x) = \frac{\cos\theta}{R} + \frac{2a^2}{R}e^{\pm i\theta};\ m_{\mathrm{sc}}(w_c(x)) = \pm i\pi\sigma_{sc}^{2\sigma}(x) - \frac{2}{1+4a^2}x.$$

By combining Proposition 6.2 and (36), we have that

$$D(x) = \Im\left(\frac{m_{\mathrm{sc}}(w_c(x))^4}{16(w_c(x) + m_{\mathrm{sc}}(w_c(x)))\pi}(1 + o(1))\right). \tag{55}$$

When $a \to 0$, we then have the following estimates

$$x \sim \cos\theta;\ m_{\mathrm{sc}}(w_c(x)) \sim -2e^{-i\theta};\ \sigma(x) \sim \frac{2}{\pi}\sin\theta; w_c + m_{\mathrm{sc}}(w_c)/2 \sim i\sin\theta.$$

Using (54) and identifying the term depending on κ_4 in the limit $a \to 0$, we then find that

$$\begin{aligned}\delta_n &= \frac{1}{n}\int_0^{\gamma_i}(C'(x)+\kappa_4 D(x))dx + O\left(\frac{1}{n^{1+c}}\right)\\ &= \frac{1}{n}\int_0^{\gamma_i} C'(x)dx + \frac{\kappa_4}{n}\int_{\theta_0}^{\pi/2}\frac{\cos(4\theta)}{\pi}d\theta + O\left(\frac{1}{n^{1+c}}\right)\\ &= \frac{1}{n}\int_0^{\gamma_i} C'(x)dx - \frac{\kappa_4}{2n}\sigma_{sc}^{2\sigma}(\gamma_i)\cos\theta_0(2\cos^2\theta_0-1) + O\left(\frac{1}{n^{1+c}}\right), \quad (56)\end{aligned}$$

where in the last line we used that $\frac{1}{4}\sin(4\theta) = \sin\theta\cos\theta\cos(2\theta)$. Thus we have that

$$\delta_n = \frac{1}{n}\int_0^{\gamma_i} C'(x)dx - \frac{\kappa_4}{2n}\sigma_{sc}^{2\sigma}(\gamma_i)(2\gamma_i^3-\gamma_i) + O\left(\frac{1}{n^{1+c}}\right). \quad (57)$$

We finally show that

$$\lim_{n\to\infty} n\left(-\gamma_{[n/2]} + \hat{\gamma}_{[n/2]}\right) = 0$$

which completes the proof of Proposition 6.3.

To that end, let us first notice that for any C^8 function f whose support is strictly included in that of $\sigma_{sc}^{2\sigma}$, we have

$$\lim_{n\to\infty}\mathbb{E}[\sum_{i=1}^n f(\lambda_i)] = m(f) + \kappa_4\int_{-1}^1 f(t)T_4(t)\frac{dt}{\sqrt{1-t^2}} := m_{\kappa_4}(f), \quad (58)$$

with T_4 the fourth Tchebychev polynomials and $m(f)$ a linear form independent of κ_4. This is an extension of the formulas found in [3, Theorem 9.2, formula (9.2.4)] up to the normalization (the variance is $\frac{1}{4}$ here) to C^8 functions. We can extend the convergence (58) to functions which are only C^8 by noticing that the error in (40) still goes to zero uniformly on $\Im z \geq n^{-1/7}$ and then using that for f C^8 compactly supported, we can find by [1, (5.5.11)] a function Ψ so that $\Psi(t,0) = f(t)$ compactly supported and bounded by $|y|^8$ so that for any probability measure μ

$$\Re\int_0^\infty dy\int dx\Psi(x,y)\int\frac{1}{t-x-iy}d\mu(t) = \int\Psi(t,0)d\mu(t)$$

Hence,

$$\mathbb{E}[\sum f(\lambda_i)] - n\sigma_{sc}^{2\sigma}(f) = \Re\int_0^\infty dy\int dx\Psi(x,y)n(m_n(x+iy)-m_{\rm sc}(x+iy)).$$

Applying the previous estimate for $y \geq n^{-1/7}$ and on $y \in [0, n^{-1/7}]$ simply bounding $|n(m_n(x+iy)-m_{\rm sc}(x+iy))| \leq 2ny^{-2}$ s well as $|\Psi|(x,y) \leq 1_{x\in[-M,M]}y^8$ provide the announced convergence (58).

Next we can rewrite (58) in terms of the quantiles $\hat{\gamma}_i$ as

$$\begin{aligned} m_{\kappa_4}(f) &= n\int f(x)\rho_n(x)dx + o(1) \\ &= \sum_i f(\hat{\gamma}_i) + \sum_i f'(\hat{\gamma}_i)(\hat{\gamma}_{i+1} - \hat{\gamma}_i) + o(1) \end{aligned}$$

where we used that $\hat{\gamma}_{i+1} - \hat{\gamma}_i$ is of order $n^{-1+\eta}$ by (49) and the fact that $\gamma_{i+1} - \gamma_i$ is at most of order $1/n$. Now, again by (49), we have

$$\sum_i f(\hat{\gamma}_i) = \sum_i f(\gamma_i) + \sum_i f'(\gamma_i)(\hat{\gamma}_i - \gamma_i) + O(\frac{1}{n^{-1+2\eta}})\,.$$

Moreover

$$\sum_i f(\gamma_i) = n\int f(x)\sigma_{sc}^{2\sigma}(x)dx - \sum_i f'(\gamma_i)(\gamma_{i+1} - \gamma_i) + o(1)$$

Noting that the first term in the right hand side vanishes we deduce that

$$m_{\kappa_4}(f) = \sum_i f'(\gamma_i)[\hat{\gamma}_{i+1} - \hat{\gamma}_i - \gamma_{i+1} + \gamma_i + \hat{\gamma}_i - \gamma_i] + o(1)$$

where $\hat{\gamma}_{i+1} - \hat{\gamma}_i - \gamma_{i+1} + \gamma_i$ is at most of order n^{-1-c} by (57). Hence, we find by (57) that

$$\begin{aligned} -m_{\kappa_4}(f) &= \sum f'(\gamma_i)(\gamma_i - \hat{\gamma}_i) + o(1) \\ &= \frac{1}{n}\sum_i f'(\gamma_i)\frac{\sigma_{sc}^{2\sigma}(\gamma_{[n/2]})}{\sigma_{sc}^{2\sigma}(\gamma_i)}[n(\gamma_{[n/2]} - \hat{\gamma}_{[n/2]})] \\ &\quad + \frac{1}{n}\sum_i \frac{f'(\gamma_i)}{\sigma_{sc}^{2\sigma}(\gamma_i)}\int_0^{\gamma_i} C'(x)dx + \frac{\kappa_4}{2n}\sum_i f'(\gamma_i)(2\gamma_i^3 - \gamma_i) + o(1) \\ &= \sigma_{sc}^{2\sigma}(\gamma_{[n/2]})[n(\gamma_{[n/2]} - \hat{\gamma}_{[n/2]})]\int_{-2\sigma}^{2\sigma} f'(x)dx + \int f'(x)\int_0^x C'(y)dydx \\ &\quad + \frac{\kappa_4}{2}\int f'(x)(2x^3 - x)\sigma_{sc}^{2\sigma}(x)dx + o(1)\,. \end{aligned}$$

We finally take f' even, that is f odd in which case the last term in κ_4 vanishes, as well as the term depending on κ_4 in $m_{\kappa_4}(f)$ as T_4 is even and f odd. Moreover, $\sigma_{sc}^{2\sigma}(\gamma_{[n/2]})$ goes to $1/2$. Hence, we deduce that there exists a constant independent of κ_4 such that

$$\lim_{n\to\infty} n(\gamma_{[n/2]} - \hat{\gamma}_{[n/2]}) = C\,.$$

In fact, this constant must vanish as in the case where the distribution is symmetric, and n even, both $\gamma_{[n/2]}$ and $\hat{\gamma}_{[n/2]}$ vanish by symmetry.

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