Regulators for Rankin-Selberg products of modular forms

François Brunault, Masataka Chida

REGULATORS FOR RANKIN-SELBERG PRODUCTS OF MODULAR FORMS

FRANÇOIS BRUNAULT AND MASATAKA CHIDA

Dedicated to Professor Glenn Stevens on the occasion of his 60th birthday

Abstract. We prove a weak version of Beilinson's conjecture for non-critical values of L-functions for the Rankin-Selberg product of two modular forms.

Introduction

In his fundamental paper [2, §6], Beilinson introduced the so-called *Beilinson-Flach elements* in the higher Chow group of a product of two modular curves and related their image under the regulator map to special values of Rankin L-series of the form $L(f \otimes g, 2)$, where f and g are newforms of weight 2, as predicted by his conjectures on special values of L-functions. These elements were later exploited by Flach [17] to prove the finiteness of the Selmer group associated to the symmetric square of an elliptic curve. More recently, Bertolini, Darmon and Rotger [4] established a p-adic analogue of Beilinson's result, while Lei, Loeffler and Zerbes [26] constructed a cyclotomic Euler system whose bottom layer are the Beilinson-Flach elements. These results have many important arithmetic applications ([5], [26]).

Our aim in this paper is to define an analogue of the Beilinson-Flach elements in the motivic cohomology of a product of two Kuga-Sato varieties and to prove an analogue of Beilinson's formula for special values of Rankin L-series associated to newforms f and g of any weight ≥ 2. More precisely, we prove the following theorem.

Theorem 0.1. *Let $f \in S_{k+2}(\Gamma_1(N_f), \chi_f)$ and $g \in S_{\ell+2}(\Gamma_1(N_g), \chi_g)$ be newforms with $k, \ell \geq 0$. Let $N = \operatorname{lcm}(N_f, N_g)$, and let j be an integer satisfying $0 \leq j \leq \min\{k, \ell\}$. In the case $j = k = \ell$, assume that $g \neq f^*$ and $N > 1$. Assume that the automorphic factor $R_{f,g,N}(j+1)$ defined in Section 5 is non-zero (this holds for example if $\gcd(N_f, N_g) = 1$ or if $k + \ell - 2j \notin \{0, 1, 2\}$). Then the weak version of Beilinson's conjecture for $L(f \otimes g, k + \ell + 2 - j)$ holds.*

The range of critical values (in the sense of Deligne) for the Rankin-Selberg L-function $L(f \otimes g, s)$ is given by $\min\{k, \ell\} + 2 \leq s \leq \max\{k, \ell\} + 1$, so that our L-value $L(f \otimes g, k + \ell + 2 - j)$ is *non-critical.* In fact, the integers $0 \leq j \leq \min\{k, \ell\}$ are precisely those at which the dual L-function $L(f^* \otimes g^*, s+1)$ vanishes *at order 1.*

We refer to Theorem 7.7 for the explicit formula giving the regulator of our generalized Beilinson-Flach elements. In the weight 2 case, an explicit version of Beilinson's formula for $L(f \otimes g, 2)$, similar to Theorem 7.7, was proved by Baba and Sreekantan [1] and by Bertolini, Darmon and Rotger [4]. In the higher weight case, a similar formula for the regulator of generalized Beilinson-Flach elements was proved by Scholl (unpublished) and recently by Kings, Loeffler and Zerbes [25]. While the article [25] uses motivic cohomology with coefficients, we choose to work directly with motivic cohomology of the Kuga-Sato varieties. Despite this different language, the motivic elements and their construction are the same. However, as a difference with [25], we prove that our generalized Beilinson-Flach elements extend to the boundary of the Kuga-Sato varieties, in accordance with Beilinson's conjectures which are formulated for motives of smooth proper varieties (see Sections 8 and 9).

Another interesting problem is the integrality of the generalized Beilinson-Flach elements. In the case f and g have weight 2, Scholl proved that if g is not a twist of f, then the Beilinson-Flach elements belong to the integral subspace of motivic cohomology [32, Theorem 2.3.4]. We do not investigate integrality in this article, but it would be interesting to do so using Scholl's techniques.

Date: October 31, 2015.

Key words and phrases. Rankin-Selberg product, Regulator, Beilinson conjecture.

The plan of this article is as follows. In Section 1, we state a version of Beilinson's conjecture for Grothendieck motives. In Section 2, we recall some basic results about modular curves, Kuga-Sato varieties and motives of modular forms, and we describe explicitly the Deligne-Beilinson cohomology associated to the Rankin product of two modular forms. After recalling Beilinson's theory of the Eisenstein symbol in Section 3, we construct in Section 4 special elements $\Xi^{k,\ell,j}(\beta)$ in the motivic cohomology of the product of two Kuga-Sato varieties. We recall standard properties of the Rankin-Selberg L-function in Section 5 and give the description of Deligne-Beilinson cohomology of smooth open varieties in terms of currents on a smooth compactification in Section 6. We carry the computation of the regulator of our elements $\Xi^{k,\ell,j}(\beta)$ in Section 7. We then show in Sections 8 and 9, using motivic techniques, that a suitable modification of the elements $\Xi^{k,\ell,j}(\beta)$ extends to the boundary of the Kuga-Sato varieties. Finally, we give in Section 10 the application of our results to Beilinson's conjecture.

We would like to thank Frédéric Déglise and Jörg Wildeshaus for very stimulating discussions on the topic of motives, José Ignacio Burgos Gil for helpful advice on Deligne-Beilinson cohomology, and Francesco Lemma and Jan Nekovář for helpful discussions and comments on our work. We also thank the anonymous referee for improvement suggestions. The first author is partly supported by the research grant ANR Régulateurs (Grant Number ANR-12-BS01-0002). The second author is partly supported by JSPS KAKENHI (Grant Number 23740015) and the research grant of Hakubi project of Kyoto University.

1. Beilinson's conjecture

Let X be a smooth projective variety over $\mathbb{Q}$. For a non-negative integer i and an integer j, let $H^i_{\mathcal{M}}(X,\mathbb{Q}(j))$ be the motivic cohomology and $H^i_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(j))$ be the Deligne-Beilinson cohomology. Then one can define natural $\mathbb{Q}$-structures $\mathcal{B}_{i,j}$ and $\mathcal{D}_{i,j}$ in $\det_{\mathbb{R}}(H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(j)))$ (see Beilinson [2, 3.2], Deninger-Scholl [15, 2.3.2] or Nekovář [27, (2.2)]). Denote the integral part of motivic cohomology by $H^i_{\mathcal{M}}(X,\mathbb{Q}(j))_{\mathbb{Z}}$ (see Beilinson [2, 2.4.2] if X has a regular model over $\mathbb{Z}$ and Scholl [32, 1.1.6. Theorem] for the general case). Then Beilinson defined the regulator map

$$r_{\mathcal{D}} : H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(j)) \to H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(j))$$

and formulated a conjecture for the special values of the L-function $L(h^i(X),s)$ as follows.

Conjecture 1.1 (Beilinson [2]). *Assume $j > (i+2)/2$.*

(1) *The map $r_{\mathcal{D}} \otimes \mathbb{R} : H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(j))_{\mathbb{Z}} \otimes \mathbb{R} \to H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(j))$ is an isomorphism.*

(2) *We have $r_{\mathcal{D}}(\det H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(j))_{\mathbb{Z}}) = L(h^i(X),j)\cdot \mathcal{D}_{i,j} = L^*(h^i(X),i+1-j)\cdot \mathcal{B}_{i,j}$, where $L^*(h^i(X),m)$ is the leading term of the Taylor expansion of $L(h^i(X),s)$ at $s=m$.*

In the case of the near-central point $j=(i+2)/2$, we have the following modified conjecture. Let $N^{j-1}(X) = \mathrm{CH}^{j-1}(X)_{\mathrm{hom}} \otimes \mathbb{Q}$ be the group of $(j-1)$-codimensional cycles modulo homological equivalence. The cycle class map into de Rham cohomology defines an extended regulator map

$$\hat{r}_{\mathcal{D}} : H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(j)) \oplus N^{j-1}(X) \to H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(j)).$$

Conjecture 1.2 (Beilinson [2]). *Assume $j=(i+2)/2$.*

(1) *(Tate's conjecture) We have $\mathrm{ord}_{s=j} L(h^i(X),s) = -\dim_{\mathbb{Q}} N^{j-1}(X)$.*

(2) *The map $\hat{r}_{\mathcal{D}} \otimes \mathbb{R} : (H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(j))_{\mathbb{Z}} \oplus N^{j-1}(X)) \otimes \mathbb{R} \to H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(j))$ is an isomorphism.*

(3) *We have $\hat{r}_{\mathcal{D}}(\det(H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(j))_{\mathbb{Z}} \oplus N^{j-1}(X))) = L^*(h^i(X),j)\cdot \mathcal{D}_{i,j} = L^*(h^i(X),j-1)\cdot \mathcal{B}_{i,j}$.*

Although Beilinson's conjectures are most naturally formulated for Chow motives, the motives associated to modular forms of general weight are Grothendieck motives, but are not known to be Chow motives. We therefore need to formulate a (weak) version of Beilinson's conjectures for Grothendieck motives.

Let $M=(X,p)$ be a Grothendieck motive over $\mathbb{Q}$ with coefficients in L, where X is a smooth projective variety over $\mathbb{Q}$ and p is a projector in $\mathrm{CH}^{\dim X}(X\times X)_{\mathrm{hom}} \otimes_{\mathbb{Q}} L$. We define the Deligne-Beilinson cohomology of M by

$$H^{\cdot}_{\mathcal{D}}(M,j) = p_*(H^{\cdot}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(j)) \otimes L).$$

We have an L-function $L(h^i(M),s)$ taking values in $L\otimes\mathbb{C}$. Moreover, there are natural L-structures $\mathcal{B}_{i,j}(M)$ and $\mathcal{D}_{i,j}(M)$ in $\det_{L\otimes\mathbb{R}} H^{i+1}_{\mathcal{D}}(M,j)$. We define Beilinson's regulator as the composition

$$r_{\mathcal{D}} : H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(j)) \otimes L \to H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(j)) \otimes L \to H^{i+1}_{\mathcal{D}}(M,j),$$

where the last map is the projection induced by p_*. Similarly, we define an extended regulator map $\hat{r}_{\mathcal{D}}$ in the case $j = (i+2)/2$. Then the weak version of Beilinson's conjecture for $L(h^i(M), s)$ can be formulated as follows.

Conjecture 1.3. *Let $M^\vee = (X, {}^t p)$ be the dual motive of M.*

(1) *If $j > (i+2)/2$, then there exists a subspace V of $H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(j))$ such that $p_*(r_{\mathcal{D}}(V \otimes L))$ gives an L-structure of $H^{i+1}_{\mathcal{D}}(M, j)$ and*
$$\det p_*(r_{\mathcal{D}}(V \otimes L)) = L(h^i(M), j) \cdot \mathcal{D}_{i,j}(M) = L^*(h^i(M^\vee), i+1-j) \cdot \mathcal{B}_{i,j}(M).$$
(2) *If $j = (i+2)/2$, then there exists a subspace V of $H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(j)) \oplus N^{j-1}(X)$ such that $p_*(\hat{r}_{\mathcal{D}}(V \otimes L))$ gives an L-structure of $H^{i+1}_{\mathcal{D}}(M, j)$ and*
$$\det p_*(\hat{r}_{\mathcal{D}}(V \otimes L)) = L^*(h^i(M), j) \cdot \mathcal{D}_{i,j}(M) = L^*(h^i(M^\vee), i+1-j) \cdot \mathcal{B}_{i,j}(M).$$

Remark 1.4. We could have required a stronger property in Conjecture 1.3, namely that V is a subspace of $H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(j))_{\mathbb{Z}}$. But since we don't consider the problem of integrality of elements of motivic cohomology in this paper, we leave Conjecture 1.3 as it is.

2. Motives associated to Rankin products

In this section we recall some basic properties of motives associated to Rankin products.

2.1. **Modular curves and Kuga-Sato varieties.** Let $N \geq 3$ be an integer. Let $Y(N)$ be the open modular curve with full level N structure defined over $\mathbb{Q}$. It represents the functor sending a $\mathbb{Q}$-scheme S to the set of isomorphism classes of pairs (E, α), where E is an elliptic curve over S, and $\alpha : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\cong} E[N]$. The complex points of $Y(N)$ are described as follows [14, (3.4)]. Let $\mathcal{H}$ be the upper half-plane. Then we have an isomorphism

$$Y(N)(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{Z}) \backslash (\mathcal{H} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})) \tag{2.1}$$

where $\mathrm{SL}_2(\mathbb{Z})$ acts by Möbius transformations on $\mathcal{H}$ and by left multiplication on $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. The isomorphism (2.1) sends the class of $(\tau, g) \in \mathcal{H} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ to the $\mathbb{C}$-valued point $(E_\tau, \alpha_{\tau,g})$ with
$$E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \qquad \alpha_{\tau,g}(v) = \left(-\frac{1}{N}, \frac{\tau}{N}\right) gv.$$
The group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts from the left on $Y(N)$ by the rule $\gamma \cdot (E, \alpha) = (E, \alpha \circ {}^t\gamma)$. On the complex points, this action is given by $\gamma \cdot [(\tau, g)] = [(\tau, g\,{}^t\gamma)]$.

The modular curve $Y(N)$ is not geometrically connected. Indeed, we have an isomorphism of Riemann surfaces

$$\begin{aligned} \mu : (\mathbb{Z}/N\mathbb{Z})^\times \times \Gamma(N) \backslash \mathcal{H} &\xrightarrow{\cong} Y(N)(\mathbb{C}) \\ (a, [\tau]) &\mapsto \left[\left(\tau, \begin{pmatrix} 0 & -1 \\ a & 0 \end{pmatrix}\right)\right], \end{aligned} \tag{2.2}$$

where $\Gamma(N)$ is the kernel of the reduction map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Note that $\mu(a, \tau)$ corresponds to the $\mathbb{C}$-valued point (E_τ, α) with $\alpha(e_1) = [a\tau/N]$ and $\alpha(e_2) = [1/N]$, where (e_1, e_2) is the canonical basis of $(\mathbb{Z}/N\mathbb{Z})^2$.

For an integer $N \geq 4$, define the modular curve $Y_1(N) = G_1 \backslash Y(N)$ where G_1 is the subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ given by
$$G_1 = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$
The modular curve $Y_1(N)$ represents the functor sending a $\mathbb{Q}$-scheme S to the set of isomorphism classes of pairs (E, P), where E is an elliptic curve over S, and P is a section of E/S of exact order N: for every geometric point $s : \operatorname{Spec} k \to S$, the point $P \circ s$ has order N in $E(k)$ (see [24, 2.1] and [16, 8.2]). The canonical map $Y(N) \to Y_1(N)$ sends a pair (E, α) to the pair $(E, \alpha(e_2))$. We have an isomorphism $Y_1(N)(\mathbb{C}) \cong \Gamma_1(N) \backslash \mathcal{H}$, where
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0, d \equiv 1 \mod N \right\}.$$

This isomorphism sends the class of $\tau \in \mathcal{H}$ to the $\mathbb{C}$-valued point $(E_\tau, [\frac{1}{N}])$. Note that we have a commutative diagram

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\nu"] \arrow[d] & Y(N)(\mathbb{C}) \arrow[d] \\ \Gamma_1(N)\backslash\mathcal{H} \arrow[r, "\cong"] & Y_1(N)(\mathbb{C}) \end{tikzcd}$$

where ν is the holomorphic map defined by

$$\nu(\tau) := \left[\left(\tau, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\right].$$

Let $X(N)$ (resp. $X_1(N)$) be the smooth compactification of $Y(N)$ (resp. $Y_1(N)$), obtained by adding a finite number of cusps. By (2.2), the set of cusps of $X(N)(\mathbb{C})$ is in bijection with $(\mathbb{Z}/N\mathbb{Z})^\times \times \Gamma(N)\backslash\mathbb{P}^1(\mathbb{Q})$.

The set of cusps of $X_1(N)(\mathbb{C})$ is in bijection with $\Gamma_1(N)\backslash\mathbb{P}^1(\mathbb{Q})$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$, and the stabilizer of ∞ is given by

$$\Gamma_\infty = \left\{\pm\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \,\middle|\, k \in \mathbb{Z}\right\}.$$

Moreover, let $\mathcal{E}_N$ be the set of pairs $(c,d) \in (\mathbb{Z}/N\mathbb{Z})^2$ such that $(c,d,N) = 1$. Then we have a bijection

$$\Gamma_1(N)\backslash\mathrm{SL}_2(\mathbb{Z}) \cong \mathcal{E}_N$$

sending the class of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to the pair $(c,d) \mod N$. It follows that the set of cusps of $X_1(N)(\mathbb{C})$ is in bijection with $\mathcal{E}_N/\Gamma_\infty$.

We now recall the interpretation of the cusps of $X_1(N)$ in terms of generalized elliptic curves as in [12]. Recall that for $N \geq 5$, the modular curve $X_1(N)$ represents the functor sending a $\mathbb{Q}$-scheme S to the set of isomorphism classes of pairs (E,P), where E is a generalized elliptic curve over S, and P is a section of E^{reg}/S of exact order N such that for every geometric point $s : \operatorname{Spec} k \to S$, the image of the immersion $(\mathbb{Z}/N\mathbb{Z})_k \to E_k^{\mathrm{reg}}$ meets every component [12, IV.4.14].

Let $x \in X_1(N)(\mathbb{C})$ be a cusp corresponding to a pair $(c,d) \in \mathcal{E}_N$. By [16, 9.3], the cusp x corresponds to the pair (E,P), where E is the Néron $N/(c,N)$-gon defined in [16, 9.2], and P is the point $e^{2\pi i d/N}$ in the $c/(c,N)$-th component of E. We may check this for the cusp ∞ as follows: when $\tau = iy \in \mathcal{H}$ approches the cusp $i\infty$, the elliptic curve $(\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}), [\frac{1}{N}])$ degenerates to $(\mathbb{C}/\mathbb{Z}, [\frac{1}{N}])$. By the exponential map, this is the same as $(\mathbb{C}^\times, e^{2\pi i/N})$, which is precisely the pointed Néron 1-gon associated to the pair $(0,1) \in \mathcal{E}_N$.

The Galois action on the cusps of $X_1(N)(\mathbb{C})$ is given as follows. Let $x \in X_1(N)(\mathbb{C})$ be a cusp corresponding to a pair $(c,d) \in \mathcal{E}_N$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, the cusp $\sigma(x)$ corresponds to the pair $(c, t_N(\sigma)d)$, where $t_N(\sigma) \in (\mathbb{Z}/N\mathbb{Z})^\times$ denotes the cyclotomic character defined by $\sigma(e^{2\pi i/N}) = e^{2\pi i t_N(\sigma)/N}$. In particular, the cusp $\infty \in X_1(N)(\mathbb{C})$, corresponding to $(0,1) \in \mathcal{E}_N$, is not defined over $\mathbb{Q}$, but rather over the cyclotomic field $\mathbb{Q}(e^{2\pi i/N})^+$.

Let E be the universal elliptic curve over $Y(N)$. For an integer $k \geq 0$, let E^k be the k-fold fiber product of E over $Y(N)$. The complex points of E^k are given by [14, (3.4), (3.6)]

$$E^k(\mathbb{C}) \cong (\mathbb{Z}^{2k} \rtimes \mathrm{SL}_2(\mathbb{Z}))\backslash\left(\mathcal{H} \times \mathbb{C}^k \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\right) \tag{2.3}$$

where the action of $\mathrm{SL}_2(\mathbb{Z})$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau; z_1, \ldots, z_k; g) = \left(\frac{a\tau+b}{c\tau+d}; \frac{z_1}{c\tau+d}, \ldots, \frac{z_k}{c\tau+d}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} g\right)$$

and the action of $\mathbb{Z}^{2k}$ is given by

$$(u_1, v_1, \ldots, u_k, v_k) \cdot (\tau; z_1, \ldots, z_k; g) = (\tau; z_1 + u_1 - v_1\tau, \ldots, z_k + u_k - v_k\tau; g).$$

The map $(\tau; z_1, \ldots, z_k; g) \mapsto \det(g)$ induces a bijection between the set of connected components of $E^k(\mathbb{C})$ and $(\mathbb{Z}/N\mathbb{Z})^\times$. The group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts from the left on E^k. Note that the subgroup of matrices of the form $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$ acts simply transitively on the set of connected components of $E^k(\mathbb{C})$.

If $f \in M_{k+2}(\Gamma_1(N))$ is a modular form of weight $k+2$ on the group $\Gamma_1(N)$, then there is a unique G_1-invariant holomorphic $(k+1)$-form ω_f on $E^k(\mathbb{C})$ such that

$$\nu^* \omega_f = (2\pi i)^{k+1} f(\tau) d\tau \wedge dz_1 \wedge \ldots \wedge dz_k,$$

where ν is the holomorphic map

$$\nu : \mathcal{H} \times \mathbb{C}^k \to E^k(\mathbb{C}) \tag{2.4}$$

$$(\tau; z_1, \ldots, z_k) \mapsto \left[\left(\tau; z_1, \ldots, z_k; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \right].$$

Since the image of ν is a connected component of $E^k(\mathbb{C})$, the G_1-invariance determines ω_f uniquely.

Let $\overline{E}$ be the minimal regular proper model of E over $X(N)$. Then $\overline{E} \to X(N)$ is the universal generalized elliptic curve in the sense of [12]. The scheme $\overline{E}$ is smooth and proper over $\mathbb{Q}$. The fiber of $\overline{E}$ over a cusp is a Néron N-gon, that is a chain of N copies of $\mathbb{P}^1$ meeting transversally at the points $0, \infty$.

For an integer $k \geq 0$, let $\overline{E}^k$ be the k-fold fiber product of $\overline{E}$ over $X(N)$. Note that $\overline{E}^k$ is singular if $k \geq 2$. Deligne [9] has constructed a canonical desingularization $\overline{\overline{E}}^k \to \overline{E}^k$. The scheme $\overline{\overline{E}}^k$ is a smooth compactification of E^k, and the complement $\overline{\overline{E}}^k - E^k$ is a simple normal crossings divisor. If $f \in S_{k+2}(\Gamma_1(N))$ is a cusp form, then ω_f extends to an holomorphic form on $\overline{\overline{E}}^k(\mathbb{C})$ [31, 1.2.0].

2.2. **Motives associated to modular forms.** Let $f = \sum_{n \geq 1} a_n q^n \in S_{k+2}(\Gamma_1(N), \chi)^{\mathrm{new}}$ be a normalized eigenform of weight $k+2 \geq 2$, level N and character χ. Let $K_f \subset \mathbb{C}$ be the number field generated by the Fourier coefficients of f. Then we may consider ω_f as a $K_f \otimes \mathbb{C}$-valued holomorphic form on $\overline{\overline{E}}^k(\mathbb{C})$.

Scholl has constructed in [31] a Grothendieck motive $M(f)$ associated to f. The motive $M(f)$ is a direct factor of $h^{k+1}(\overline{\overline{E}}^k) \otimes K_f$. It has rank 2, is defined over $\mathbb{Q}$ and has coefficients in K_f. By Grothendieck's theorem, we have an isomorphism of $K_f \otimes \mathbb{C}$-modules

$$H_B^{k+1}(M(f)) \otimes \mathbb{C} \cong H_{\mathrm{dR}}^{k+1}(M(f)) \otimes \mathbb{C}$$

between Betti and de Rham cohomology. The $K_f \otimes \mathbb{C}$-module $H_{\mathrm{dR}}^{k+1}(M(f)) \otimes \mathbb{C}$ is free of rank 2, with basis $(\omega_f, \overline{\omega_{f^*}})$, where $f^*(\tau) = \sum_{n \geq 1} \overline{a_n} q^n$ is the newform with complex conjugate Fourier coefficients.

Since the cusp ∞ is not defined over $\mathbb{Q}$, the differential form ω_f is not de Rham rational. We therefore need to introduce the differential form $\omega_f' := G(\chi)^{-1} \omega_f$, where

$$G(\chi) = \sum_{u=1}^{N_\chi} \chi(u) e^{2\pi i u / N_\chi}$$

is the Gauss sum of the Dirichlet character χ and N_χ is the conductor of χ. By [25, Lemma 6.1.1], we have $\omega_f' \in H_{\mathrm{dR}}^{k+1}(M(f))$ and the Hodge filtration of $H_{\mathrm{dR}}^{k+1}(M(f))$ is given by

$$\mathrm{Fil}^i\, H_{\mathrm{dR}}^{k+1}(M(f)) = \begin{cases} H_{\mathrm{dR}}^{k+1}(M(f)) & \text{if } i \leq 0, \\ K_f \cdot \omega_f' & \text{if } 1 \leq i \leq k+1, \\ 0 & \text{if } i \geq k+2. \end{cases}$$

By Poincaré duality, we have a perfect pairing of K_f-vector spaces

$$H_B^{k+1}(M(f)) \times H_B^{k+1}(M(f^*)(k+1)) \to K_f.$$

2.3. **Motives associated to Rankin products.** Let $f \in S_{k+2}(\Gamma_1(N_f), \chi_f)^{\mathrm{new}}$, $g \in S_{\ell+2}(\Gamma_1(N_g), \chi_g)^{\mathrm{new}}$ be normalized eigenforms, with $0 \leq k \leq \ell$. We consider the Grothendieck motive

$$M(f \otimes g) := M(f) \otimes M(g).$$

This motive has coefficients in $K_{f,g} := K_f K_g$. Let N be any integer divisible by N_f and N_g. Then $M(f \otimes g)$ is a direct factor of

$$h^{k+1}(\overline{\overline{E}}^k) \otimes h^{\ell+1}(\overline{\overline{E}}^\ell) \otimes K_{f,g} \subset h^{k+\ell+2}(\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell) \otimes K_{f,g},$$

where E is the universal elliptic curve of level N. We emphasize that $\overline{\overline{E}}^{k} \times \overline{\overline{E}}^{\ell}$ denotes the absolute product, and is thus distinct from $\overline{\overline{E}}^{k+\ell}$.

Let j be an integer such that $0 \leq j \leq k$ and put $n = k + \ell + 2 - j$. The Deligne-Beilinson cohomology of $M(f \otimes g)(n)$ can be expressed as follows. The de Rham realization

$$H^{k+\ell+2}_{\mathrm{dR}}(M(f \otimes g)) = H^{k+1}_{\mathrm{dR}}(M(f)) \otimes H^{\ell+1}_{\mathrm{dR}}(M(g))$$

has dimension 4 over $K_{f,g}$. Moreover $\mathrm{Fil}^n H^{k+\ell+2}_{\mathrm{dR}}(M(f \otimes g))$ is the $K_{f,g}$-line generated by $\omega'_f \otimes \omega'_g$. Then we have an exact sequence (see [30, §2, page 9 (∗)] and [22, 4.9])

$$0 \to \mathrm{Fil}^n H^{k+\ell+2}_{\mathrm{dR}}(M(f \otimes g)) \otimes \mathbb{R} \xrightarrow{\pi} H^{k+\ell+2}_{B}(M(f \otimes g)(n-1))^{+} \otimes \mathbb{R} \to H^{k+\ell+3}_{\mathcal{D}}(M(f \otimes g)(n)) \to 0. \tag{2.5}$$

In particular $H^{k+\ell+3}_{\mathcal{D}}(M(f \otimes g)(n))$ is a free $K_{f,g} \otimes \mathbb{R}$-module of rank 1.

The exact sequence (2.5) induces a $K_{f,g}$-rational structure on $H^{k+\ell+3}_{\mathcal{D}}(M(f \otimes g)(n))$. Let us make explicit a generator of this rational structure. Let $e_f^{\pm}$ be a K_f-basis of $H^{k+1}_B(M(f))^{\pm}$, and let $e_g^{\pm}$ be a K_g-basis of $H^{\ell+1}_B(M(g))^{\pm}$. Under the comparison isomorphism $H^{k+1}_B(M(f)) \otimes \mathbb{C} \cong H^{k+1}_{\mathrm{dR}}(M(f)) \otimes \mathbb{C}$, we have $\omega_f = \alpha_f^+ e_f^+ + \alpha_f^- e_f^-$ for some $\alpha_f^+, \alpha_f^- \in \mathbb{C}$. Note that $\alpha_f^+ \in \mathbb{R}$ and $\alpha_f^- \in i\mathbb{R}$. Similarly, let $\omega_g = \alpha_g^+ e_g^+ + \alpha_g^- e_g^-$. The $K_{f,g}$-vector space $H^{k+\ell+2}_B(M(f \otimes g)(n-1))^+$ admits as a $K_{f,g}$-basis (e_1, e_2) where

$$\begin{aligned} e_1 &= e_f^+ \otimes e_g^{(-1)^{n+1}} \otimes (2\pi i)^{n-1}, \\ e_2 &= e_f^- \otimes e_g^{(-1)^{n}} \otimes (2\pi i)^{n-1}. \end{aligned}$$

The image of $\omega'_f \otimes \omega'_g$ in $H^{k+\ell+2}_B(M(f \otimes g)(n-1))^+ \otimes \mathbb{R}$ under (2.5) is given by

$$\begin{aligned} \pi(\omega'_f \otimes \omega'_g) &= G(\chi_f)^{-1} G(\chi_g)^{-1} \alpha_f^- \alpha_g^{(-1)^n} e_f^- \otimes e_g^{(-1)^n} + G(\chi_f)^{-1} G(\chi_g)^{-1} \alpha_f^+ \alpha_g^{(-1)^{n+1}} e_f^+ \otimes e_g^{(-1)^{n+1}} \\ &= G(\chi_f)^{-1} G(\chi_g)^{-1} (2\pi i)^{1-n} (\alpha_f^+ \alpha_g^{(-1)^{n+1}} e_1 + \alpha_f^- \alpha_g^{(-1)^n} e_2). \end{aligned}$$

Thus a rational structure of $H^{k+\ell+3}_{\mathcal{D}}(M(f \otimes g)(n))$ is given by

$$t := G(\chi_f) G(\chi_g) (2\pi i)^{n-1} (\alpha_f^- \alpha_g^{(-1)^n})^{-1} e_1. \tag{2.6}$$

Since $M(f \otimes g)(n-1)^\vee \cong M(f^* \otimes g^*)(j+1)$, we have a perfect pairing

$$H^{k+\ell+2}_B(M(f \otimes g)(n-1)) \times H^{k+\ell+2}_B(M(f^* \otimes g^*)(j+1)) \to K_{f,g}.$$

Now, let us define a canonical element $\Omega \in H^{k+\ell+2}_B(M(f^* \otimes g^*)(j+1)) \otimes \mathbb{C}$, which we will use to pair with the regulator of our generalized Beilinson-Flach element. Under the canonical isomorphism

$$\phi_{\mathrm{dR}} : H^{k+\ell+2}_{\mathrm{dR}}(M(f^*) \otimes M(g^*)(j+1)) \xrightarrow{\cong} H^{k+\ell+2}_{\mathrm{dR}}(M(f) \otimes M(\chi_f) \otimes M(g) \otimes M(\chi_g)(j+1)),$$

the element $G(\overline{\chi_f})^{-1} G(\overline{\chi_g})^{-1} \omega_{f^*} \otimes \omega_{g^*}$ corresponds to a $K_{f,g}^\times$-rational multiple of $\omega'_f \otimes \omega(\chi_f) \otimes \omega'_g \otimes \omega(\chi_g)$, where $\omega(\chi_f)$ is a basis of $H^0_{\mathrm{dR}}(M(\chi_f))$.

We recall the periods for motives associated to Dirichlet characters with coefficients in E. Let $M(\chi)$ be the motive associated to χ with coefficients in a number field E. Then the period of the comparison isomorphism $H^0_B(M(\chi)) = E(\chi) \to H^0_{\mathrm{dR}}(M(\chi)) = G(\chi) \cdot E$ is given by $G(\chi)^{-1}$, where $E(\chi)$ is the rank one E-vector space on which the Galois group $\mathrm{Gal}(\mathbb{Q}(e^{2\pi i/N_\chi})/\mathbb{Q})$ acts via χ and $G(\chi) \cdot E$ is the E-vector space generated by $G(\chi)$ (for details, see [11, Section 6]).

Under the comparison isomorphism

$$\phi : H^{k+\ell+2}_B(M(f) \otimes M(\chi_f) \otimes M(g) \otimes M(\chi_g)(j+1)) \otimes \mathbb{C} \xrightarrow{\cong} H^{k+\ell+2}_{\mathrm{dR}}(M(f) \otimes M(\chi_f) \otimes M(g) \otimes M(\chi_g)(j+1)) \otimes \mathbb{C},$$

we have

$$\phi^{-1}(\omega'_f \otimes \omega(\chi_f) \otimes \omega'_g \otimes \omega(\chi_g)) = (\alpha_f^+ e_f^+ + \alpha_f^- e_f^-) \otimes (\alpha_g^+ e_g^+ + \alpha_g^- e_g^-) \otimes e(\chi_f) \otimes e(\chi_g).$$

Let $e_f^{\pm,\vee}$ be a K_f-basis of $H^{k+1}_B(M(f)^\vee)^{\pm}$ with $\langle e_f^{\pm}, e_f^{\pm,\vee} \rangle = 1$, and let $e_g^{\pm,\vee}$ be a K_g-basis of $H^{\ell+1}_B(M(g)^\vee)^{\pm}$ with $\langle e_g^{\pm}, e_g^{\pm,\vee} \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the Poincaré duality pairing. We have an isomorphism $\phi_B(f)$:

$H_B^{k+1}(M(f)\otimes M(\chi_f)) \xrightarrow{\cong} H_B^{k+1}(M(f)^\vee(-k-1))$ sending $e_f^\pm \otimes e(\chi_f)$ to a $K_{f,g}^\times$-rational multiple of $(2\pi i)^{-k-1}e_f^{\mp,\vee}$. Note that $(2\pi i)^{k+1}e(\chi_f) \in H_B^0(M(\chi_f)(k+1))^-$, since $\chi_f(-1) = (-1)^k$. Therefore we have an isomorphism

$$\phi_B : H_B^{k+\ell+2}(M(f)\otimes M(g)\otimes M(\chi_f)\otimes M(\chi_g)(j+1)) \xrightarrow{\cong} H_B^{k+\ell+2}(M(f)^\vee \otimes M(g)^\vee(1-n))$$

sending $(2\pi i)^{j+1}e_f^\pm \otimes e_g^\pm \otimes e(\chi_f)\otimes e(\chi_g)$ to a rational multiple of $(2\pi i)^{1-n}e_f^{\mp,\vee}\otimes e_g^{\mp,\vee}$. Let us define

$$\nu_f := \phi_B \circ \phi^{-1}(\omega_f' \otimes \omega(\chi_f)) = (2\pi i)^{-k-1}(\alpha_f^+ e_f^{-,\vee} + \alpha_f^- e_f^{+,\vee}) = (2\pi i)^{-k-1}(\alpha_f^+ e_f^{-,\vee} + \alpha_f^- e_f^{+,\vee})$$

and

$$\nu_g := (2\pi i)^{-\ell-1}(\alpha_g^+ e_g^{-,\vee} + \alpha_g^- e_g^{+,\vee}).$$

Also we define

$$\overline{\nu_{g^*}} = \overline{F}_\infty^*(\nu_g) = (2\pi i)^{-\ell-1}(-\alpha_g^+ e_g^{-,\vee} + \alpha_g^- e_g^{+,\vee}),$$

where $\overline{F}_\infty^*$ is the involution defined in [11, 1.4]. We define

$$\Omega := G(\overline{\chi_f})G(\overline{\chi_g})\nu_f \otimes \overline{\nu_{g^*}} \in H_B^{k+\ell+2}(M(f\otimes g)^\vee(1-n))\otimes\mathbb{C} = H_B^{k+\ell+2}(M(f^*\otimes g^*)(j+1))\otimes\mathbb{C}.$$

Since $\phi_B\circ\phi^{-1}\circ\phi_{\mathrm{dR}}(\omega_{f^*}'\otimes\overline{\omega_g'}) = \phi_B\circ\phi^{-1}\circ\phi_{\mathrm{dR}}(\omega_{f^*}'\otimes\overline{F}_\infty^*(\omega_{g^*}'))$ is a $K_{f,g}^\times$-rational multiple of $\nu_f\otimes\overline{F}_\infty^*(\nu_g) = \nu_f\otimes\overline{\nu_{g^*}}$, it follows that $\phi_{\mathrm{dR}}^{-1}\circ\phi\circ\phi_B^{-1}(\Omega)$ is a $K_{f,g}^\times$-rational multiple of

$$G(\overline{\chi_f})G(\overline{\chi_g})\omega_{f^*}'\otimes\overline{\omega_g'} = \omega_{f^*}\otimes\overline{\omega_g} \in H_{\mathrm{dR}}^{k+\ell+2}(M(f^*\otimes g^*)(j+1))\otimes\mathbb{C}.$$

Lemma 2.1. *The map*

$$\langle\,\cdot\,,\Omega\rangle : H_B^{k+\ell+2}(M(f\otimes g)(n-1))^+\otimes\mathbb{R}\to K_{f,g}\otimes\mathbb{C}$$

factors through $H_{\mathcal{D}}^{k+\ell+3}(M(f\otimes g)(n))$.

Proof. It suffices to check that $\langle\pi(\omega_f'\otimes\omega_g'),\Omega\rangle = 0$. We have

$$\begin{aligned}
\langle\pi(\omega_f'\otimes\omega_g'),\Omega\rangle &= \langle G(\chi_f)^{-1}G(\chi_g)^{-1}(\alpha_f^+\alpha_g^{(-1)^{n+1}}e_f^+\otimes e_g^{(-1)^{n+1}} + \alpha_f^-\alpha_g^{(-1)^n}e_f^-\otimes e_g^{(-1)^n}),\\
&\qquad G(\overline{\chi_f})G(\overline{\chi_g})(\alpha_f^+e_f^{-,\vee}+\alpha_f^-e_f^{+,\vee})\otimes(-\alpha_g^+e_g^{-,\vee}+\alpha_g^-e_g^{+,\vee})\cdot(2\pi i)^{-k-\ell-2}\rangle\\
&= \left(\alpha_f^+\alpha_g^{(-1)^{n+1}}\alpha_f^-(-1)^{n+1}\alpha_g^{(-1)^n} + \alpha_f^-\alpha_g^{(-1)^n}\alpha_f^+(-1)^n\alpha_g^{(-1)^{n+1}}\right)\cdot\frac{G(\overline{\chi_f})G(\overline{\chi_g})}{G(\chi_f)G(\chi_g)}(2\pi i)^{-k-\ell-2}\\
&= 0.
\end{aligned}$$

∎

Lemma 2.2. *Let t be the rational structure of $H_{\mathcal{D}}^{k+\ell+3}(M(f\otimes g)(n))$ given by (2.6). We have*

$$\langle t,\Omega\rangle = (-1)^{n+1}\chi_f(-1)\chi_g(-1)N_{\chi_f}N_{\chi_g}(2\pi i)^{k+\ell-2j}.$$

Proof. We have

$$\begin{aligned}
\langle t,\Omega\rangle &= \langle G(\chi_f)G(\chi_g)(2\pi i)^{2n-2}(\alpha_f^-\alpha_g^{(-1)^n})^{-1}e_f^+\otimes e_g^{(-1)^{n+1}},\\
&\qquad G(\overline{\chi_f})G(\overline{\chi_g})(\alpha_f^+e_f^{-,\vee}+\alpha_f^-e_f^{+,\vee})\otimes(-\alpha_g^+e_g^{-,\vee}+\alpha_g^-e_g^{+,\vee})\cdot(2\pi i)^{k-\ell-2}\rangle\\
&= G(\chi_f)G(\overline{\chi_f})G(\chi_g)G(\overline{\chi_g})(2\pi i)^{k+\ell-2j}(\alpha_f^-\alpha_g^{(-1)^n})^{-1}\alpha_f^-(-1)^{n+1}\alpha_g^{(-1)^n}\\
&= (-1)^{n+1}\chi_f(-1)\chi_g(-1)N_{\chi_f}N_{\chi_g}(2\pi i)^{k+\ell-2j},
\end{aligned}$$

since for any Dirichlet character χ, we have $G(\chi)G(\overline{\chi}) = \chi(-1)N_\chi$. ∎

3. Eisenstein symbols

Here we recall Beilinson's theory of the Eisenstein symbol [3].

3.1. **The Eisenstein symbol map.** Let $N \geq 3$ be an integer. Let $X^\infty = X(N) - Y(N)$ be the set of cusps of $X(N)$. We have a bijection $X^\infty(\mathbb{C}) \cong \left\{\pm\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}\right\}\backslash \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, and the group $\mathrm{Aut}(\mathbb{C})$ acts on $X^\infty(\mathbb{C})$ by $\sigma([g]) = \left[\begin{pmatrix}t_N(\sigma) & 0 \\ 0 & 1\end{pmatrix} g\right]$, where $t_N(\sigma) \in (\mathbb{Z}/N\mathbb{Z})^\times$ denotes the cyclotomic character. In particular X^∞, seen as a set of closed points of $X(N)$, can be identified with $\left\{\pm\begin{pmatrix}* & * \\ 0 & 1\end{pmatrix}\right\}\backslash \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. For any integer $k \geq 0$, define

$$\mathcal{F}_N^k = \left\{ f : \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{Q} \,\middle|\, f\left(\begin{pmatrix}a & b \\ 0 & 1\end{pmatrix} g\right) = f(g) = (-1)^k f(-g) \text{ for all } a \in (\mathbb{Z}/N\mathbb{Z})^\times \text{ and } b \in \mathbb{Z}/N\mathbb{Z} \right\}.$$

Note that $\mathcal{F}_N^k$ is non-canonically isomorphic to the space of divisors on X^∞.

Recall that for any integer $k \geq 0$, E^k denotes the k-fold fiber product of the universal elliptic curve E over $Y(N)$. Beilinson constructed a residue map [3, 2.1.2]

$$\mathrm{Res}^k : H^{k+1}_{\mathcal{M}}(E^k, \mathbb{Q}(k+1)) \to \mathcal{F}_N^k.$$

In the case $k = 0$, this is just the map sending a modular unit to its divisor. Moreover, Beilinson [3, §3] constructed a canonical right inverse of Res^k. In the case $k \geq 1$, it is a map

$$\text{Б}^k : \mathcal{F}_N^k \to H^{k+1}_{\mathcal{M}}(E^k, \mathbb{Q}(k+1))$$

such that $\mathrm{Res}^k \circ \text{Б}^k = \mathrm{id}_{\mathcal{F}_N^k}$. In the case $k = 0$, it is a map

$$\text{Б}^0 : \mathcal{F}_N^{0,0} \to H^1_{\mathcal{M}}(Y(N), \mathbb{Q}(1))$$

such that $\mathrm{Res}^0 \circ \text{Б}^0 = \mathrm{id}_{\mathcal{F}_N^{0,0}}$, where $\mathcal{F}_N^{0,0}$ denotes the subspace of $\mathcal{F}_N^0$ of divisors of degree 0.

Define the horospherical map $\omega^k : \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2] \to \mathcal{F}_N^k$ by

$$\omega^k(\beta)(g) = \sum_{x=(x_1,x_2)\in(\mathbb{Z}/N\mathbb{Z})^2} \beta(g^{-1}x) B_{k+2}\left(\left\{\frac{x_2}{N}\right\}\right),$$

where B_{k+2} denotes the Bernoulli polynomial and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. In the case $k = 0$, ω^0 induces a map $\mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2 - \{0\}] \to \mathcal{F}_N^{0,0}$.

Definition 3.1. The Eisenstein symbol map is defined by $\mathrm{Eis}^k = \text{Б}^k \circ \omega^k$. In the case $k \geq 1$, it is a map $\mathrm{Eis}^k : \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2] \to H^{k+1}_{\mathcal{M}}(E^k, \mathbb{Q}(k+1))$. In the case $k = 0$, it is a map $\mathrm{Eis}^0 : \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2 - \{0\}] \to H^1_{\mathcal{M}}(Y(N), \mathbb{Q}(1))$.

3.2. **The $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-action.** The group $G = \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts from the left on E^k and thus from the right on $H^{k+1}_{\mathcal{M}}(E^k, \mathbb{Q}(k+1))$. The maps Res^k and Б^k are G-equivariant, where the right action of G on $\mathcal{F}_N^k$ is given by

$$(f|g)(h) = f(h^t g) \qquad (f \in \mathcal{F}_N^k, g \in G, h \in G).$$

Moreover, the horospherical map ω^k is G-equivariant, where the right action of G on $\mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]$ is given by

$$(\beta|g)(v) = \beta({}^t g^{-1} v) \qquad (\beta \in \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2], g \in G, v \in (\mathbb{Z}/N\mathbb{Z})^2).$$

We thus get the following result.

Lemma 3.2. *Let $k \geq 0$ and $\beta \in \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]$. In the case $k = 0$, assume that β is supported on $(\mathbb{Z}/N\mathbb{Z})^2 - \{0\}$. Then for any $g \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, we have*

$$g^* \mathrm{Eis}^k(\beta) = \mathrm{Eis}^k(\beta|g).$$

3.3. **Realization of the Eisenstein symbol.** We now recall an explicit formula for the realization of the Eisenstein symbol. Let

$$r_{\mathcal{D}} : H^{k+1}_{\mathcal{M}}(E^k, \mathbb{Q}(k+1))^{e_k} \to H^{k+1}_{\mathcal{D}}(E^k_{\mathbb{R}}, \mathbb{R}(k+1))^{e_k}$$

be the Beilinson regulator map. By [27, (7.3)], the Deligne-Beilinson cohomology group is given by

$$H^{k+1}_{\mathcal{D}}(E^k_{\mathbb{R}}, \mathbb{R}(k+1)) \simeq \frac{\{\varphi \in H^0(E^k_{\mathbb{R},\mathrm{an}}, \mathcal{A}^k \otimes \mathbb{R}(k)) \,|\, d\varphi = \frac{1}{2}(\omega + (-1)^k \overline{\omega}), \omega \in \Omega^{k+1}(\overline{\overline{E}}^k)\langle D \rangle)\}}{dH^0(E^k_{\mathbb{R},\mathrm{an}}, \mathcal{A}^{k-1} \otimes \mathbb{R}(k))},$$

where $\mathcal{A}^{\cdot}$ is the de Rham complex of real valued $\mathcal{C}^\infty$-forms and $\Omega^{k+1}(\overline{\overline{E}}^k)\langle D \rangle$ denotes the space of holomorphic $(k+1)$-forms on $E^k(\mathbb{C})$ with logarithmic singularities along $D = \overline{\overline{E}}^k(\mathbb{C}) \setminus E^k(\mathbb{C})$.

We denote by $(\tau; z_1, \dots, z_k; h)$ the coordinates on $E^k(\mathbb{C})$ using the isomorphism (2.3). For any integer $0 \leq j \leq k$, define

$$\psi_{k,j} = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) d\overline{z}_{\sigma(1)} \wedge \cdots \wedge d\overline{z}_{\sigma(j)} \wedge dz_{\sigma(j+1)} \wedge \cdots \wedge dz_{\sigma(k)}.$$

Let $\beta \in \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]$. In the case $k = 0$, assume that β is supported on $(\mathbb{Z}/N\mathbb{Z})^2 - \{0\}$. Then by [14, (3.12), (3.28)] and [20, Remark after Lemma 7.1], $r_{\mathcal{D}}(\mathrm{Eis}^k(\beta))$ is represented by

$$\Phi^k(\beta) := -\frac{k!(k+2)}{N(2\pi i)} \cdot \frac{\tau - \overline{\tau}}{2} \sum_{a=0}^{k} \psi_{k,a} \cdot \left(\sideset{}{'}\sum_{(c,d) \in \mathbb{Z}^2} \sum_{v \in (\mathbb{Z}/N\mathbb{Z})^2} \frac{\beta(h^{-1}v) \cdot e^{\frac{2\pi i(cv_1 + dv_2)}{N}}}{(c\tau + d)^{k+1-a}(c\overline{\tau} + d)^{a+1}} \right) \mod d\tau, d\overline{\tau},$$

where $\sum'$ denotes that we omit the term $(c, d) = (0, 0)$. For brevity, for any $a, b \geq 1$ we put

$$\mathcal{E}^{a,b}_{\beta}(\tau, h) := \sideset{}{'}\sum_{(c,d) \in \mathbb{Z}^2} \sum_{v \in (\mathbb{Z}/N\mathbb{Z})^2} \frac{\beta(h^{-1}v) \cdot e^{\frac{2\pi i(cv_1 + dv_2)}{N}}}{(c\tau + d)^a (c\overline{\tau} + d)^b}.$$

4. Construction of elements in the motivic cohomology

Let k, ℓ be non-negative integers with $k \leq \ell$ and choose an integer j such that $0 \leq j \leq k$. Write $k' = k - j \geq 0$ and $\ell' = \ell - j \geq 0$. We construct the following diagram

$$\begin{array}{ccccc} E^{k'+j+\ell'} & \xrightarrow{\Delta} & E^{k+\ell} & \xrightarrow{i} & E^k \times E^\ell \\ \Big\downarrow{\scriptstyle p} & & & & \\ E^{k'+\ell'}, & & & & \end{array}$$

where the morphisms p, Δ and i are defined as follows:

(1) $p : E^{k'+j+\ell'} \to E^{k'+\ell'}$ is given by forgetting the middle j coordinates:

$$(\tau; u_1, \dots, u_{k'}, t_1, \dots, t_j, v_1, \dots, v_{\ell'}; h) \mapsto (\tau; u_1, \dots, u_{k'}, v_1, \dots, v_{\ell'}; h).$$

(2) $\Delta : E^{k'+j+\ell'} \to E^{k'+2j+\ell'} = E^{k+\ell}$ is given by duplicating the middle j coordinates:

$$(\tau; u_1, \dots, u_{k'}, t_1, \dots, t_j, v_1, \dots, v_{\ell'}; h) \mapsto (\tau; u_1, \dots, u_{k'}, t_1, \dots, t_j, t_1, \dots, t_j, v_1, \dots, v_{\ell'}; h).$$

(3) $i : E^{k+\ell} \to E^k \times E^\ell$ is given by

$$(\tau; u_1, \dots, u_k, v_1, \dots, v_\ell; h) \mapsto ((\tau; u_1, \dots, u_k; h), (\tau; v_1, \dots, v_\ell; h)).$$

Note that $E^k \times E^\ell$ denotes the absolute product and is distinct from $E^{k+\ell}$. In particular, i is a closed embedding of codimension 1. Note also that $i \circ \Delta$ is a closed embedding and is given by

$$(i \circ \Delta)(\tau; u, t, v; h) = ((\tau; u, t; h), (\tau; t, v; h)).$$

Definition 4.1. Let $\beta \in \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]$. In the case $j = k = \ell$, assume that β is supported on $(\mathbb{Z}/N\mathbb{Z})^2 - \{0\}$. Denote by $\Xi^{k,\ell,j}(\beta)$ the image of $\mathrm{Eis}^{k'+\ell'}(\beta)$ under the composition of morphisms:

$$\begin{aligned} H_{\mathcal{M}}^{k'+\ell'+1}(E^{k'+\ell'}, \mathbb{Q}(k'+\ell'+1)) &\xrightarrow{p^*} H_{\mathcal{M}}^{k'+\ell'+1}(E^{k'+j+\ell'}, \mathbb{Q}(k'+\ell'+1)) \\ &\xrightarrow{\Delta_*} H_{\mathcal{M}}^{k+\ell+1}(E^{k+\ell}, \mathbb{Q}(k+\ell-j+1)) \\ &\xrightarrow{i_*} H_{\mathcal{M}}^{k+\ell+3}(E^k \times E^\ell, \mathbb{Q}(k+\ell-j+2)). \end{aligned}$$

5. The Rankin-Selberg method

Let V_f denote the 2-dimensional $K_f \otimes \mathbb{Q}_\ell$-representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ associated to f [9]. Let $L(f \otimes g, s)$ denote the L-function associated to the 4-dimensional Galois representation $V(f \otimes g) = V_f \otimes V_g$. We have

$$L(f \otimes g, s) = \prod_{p \text{ prime}} P_p(f \otimes g, s)^{-1},$$

where the p-Euler factor is defined by $P_p(f \otimes g, s) = \det(1 - \mathrm{Frob}_p \cdot p^{-s} | (V(f \otimes g))^{I_p})$. The polynomial $P_p(f \otimes g, s)$ coincides up to the shift $s \mapsto s - \frac{k+\ell+2}{2}$ with the automorphic L-factor defined by Jacquet in [21], and $L(f \otimes g, s)$ converges for $\mathrm{Re}(s) > \frac{k+\ell}{2} + 2$.

Assume that $k \leq \ell$. Denote $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$ and define the completed L-function $\Lambda(f \otimes g, s)$ by

$$\Lambda(f \otimes g, s) = L_\infty(f \otimes g, s) L(f \otimes g, s),$$

where $L_\infty(f \otimes g, s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k-1)$. Then we have the functional equation [21, Theorem 19.14]

$$\Lambda(f \otimes g, s) = \varepsilon(f \otimes g, s)\Lambda(f^* \otimes g^*, k+\ell+3-s), \tag{5.7}$$

where $\varepsilon(f \otimes g, s)$ is the (automorphic) global ε-factor.

Remark 5.1. The global epsilon factor $\varepsilon(f \otimes g, s)$ is a product of local epsilon factors:

$$\varepsilon(f \otimes g, s) = \prod_v \varepsilon_v(f \otimes g, s, \psi_v),$$

where v runs through the places of $\mathbb{Q}$, $\psi : \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \to \mathbb{C}^\times$ is a non-trivial character, and $\psi_v : \mathbb{Q}_v \to \mathbb{C}^\times$ is the restriction of ψ to $\mathbb{Q}_v$. By a characterization of local Langlands correspondence for $\mathrm{GL}(n)$ [19] (proved by Harris-Taylor [18]), it is known that the automorphic local epsilon factor $\varepsilon_v(f \otimes g, s, \psi_v)$ coincides with the local epsilon factor $\varepsilon(\rho_v(f) \otimes \rho_v(g), s, \psi_v)$ defined by Deligne [10], where $\rho_v(f)$ and $\rho_v(g)$ are the Weil-Deligne representations associated to f and g at v. Therefore the global epsilon factor $\varepsilon(f \otimes g, s)$ coincides with the epsilon constant $\varepsilon(M(f \otimes g), s)$ defined in [11, Section 5].

Let N be an integer divisible by N_f and N_g. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be the Dirichlet character induced by $\chi_f\chi_g$. Put $D(f, g, s) := \sum_{n=1}^\infty a_n(f)a_n(g)n^{-s}$. By [34, Lemma 1], we have

$$L(\chi, 2s-k-\ell-2)D(f, g, s) = R_{f,g,N}(s)L(f \otimes g, s),$$

where

$$R_{f,g,N}(s) := \left(\prod_{p|N} P_p(f \otimes g, s)\right) \sum_{n \in S(N)} \frac{a_n(f)a_n(g)}{n^s}$$

is a polynomial in the variables p^{-s} for $p|N$ by [21, Theorem 15.1]. Here $S(N)$ denotes the set of integers all of whose prime factors divide N.

Remark 5.2. Since $P_p(f \otimes g, s)$ is the p-Euler factor associated to a Grothendieck motive $M(f \otimes g)$ with coefficients in $K_{f,g}$, $P_p(f \otimes g, s)$ is a polynomial of p^{-s} with coefficients in $K_{f,g}$. Also by [34, Lemma 1], $\sum_{n \in S(N)} \frac{a_n(f)a_n(g)}{n^s}$ is a polynomial in the variables p^{-s} for $p|N$ with coefficients in $K_{f,g}$. Therefore $R_{f,g,N}(s)$ is a polynomial with coefficients in $K_{f,g}$.

For any Dirichlet character $\omega : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, define the Eisenstein series

$$E_{\ell-k,N}(\tau, s, \omega) = \sideset{}{'}\sum_{m,n \in \mathbb{Z}} \frac{\omega(n)}{(Nm\tau + n)^{\ell-k}|Nm\tau + n|^{2s}}. \tag{5.8}$$

Theorem 5.3 (Shimura [34, (2.4)]). *We have*

$$\int_{\Gamma_0(N)\backslash\mathcal{H}} f(\tau)g(-\overline{\tau})E_{\ell-k,N}(\tau,s-1-\ell,\chi)y^{s-1}dxdy = 2(4\pi)^{-s}\Gamma(s)L(\chi,2s-k-\ell-2)D(f,g,s).$$

Remark 5.4. Let us assume $k=\ell$. Then by [34, (2.5)] and [35, page 220, Correction], $D(f,g,s)$ has a pole at $s=k+2$ if and only if $\langle f^*,g\rangle \neq 0$. This is equivalent to $g=f^*$. In this case, we have a decomposition of the L-function

$$L(f\otimes f^*,s) = L(\mathrm{Sym}^2(f)\otimes\chi_f^{-1},s)\zeta(s-k-1).$$

By [29, Lemma 2.1], the L-function $L(\mathrm{Sym}^2(f)\otimes\chi_f^{-1},s)$ is critical and nonzero at $s=k+1$. Therefore $L(f\otimes f^*,s)$ is always nonzero at $s=k+1$. Hence, by [2, Conjecture 3.7 (a)] it is expected that the integral subspace of motivic cohomology $H^{2k+3}_{\mathcal{M}}(M(f\otimes f^*),\mathbb{Q}(k+2))_{\mathbb{Z}}$ is zero (note that the definition of this group is conjectural, since $M(f\otimes f^*)$ is not known to be a Chow motive).

6. Deligne-Beilinson cohomology

In this section, we recall the explicit description of Deligne-Beilinson cohomology in terms of currents [7, §5].

Let X be a smooth quasi-projective complex variety of pure dimension d. Let $\overline{X}$ be a smooth compactification of X such that $D:=\overline{X}-X$ is a simple normal crossings divisor. Let $i: D\to\overline{X}$ be the canonical embedding.

Let $S^{\cdot}_{\overline{X}}$ be the complex of smooth differential forms on $\overline{X}$. Define the subcomplex

$$\Sigma_D S^{\cdot}_{\overline{X}} = \left\{\omega\in S^{\cdot}_{\overline{X}}\,\middle|\, i^*\omega=0\right\}.$$

Let $T^{\cdot}_{\overline{X}}$ denote the complex of currents on $\overline{X}$; that is, for any integer $n\in\mathbb{Z}$, $T^{n}_{\overline{X}}$ is the topological dual of $S^{-n}_{\overline{X}}$. Define the subcomplex

$$T^{\cdot}_D = \left\{T\in T^{\cdot}_{\overline{X}}\,\middle|\,\forall\omega\in\Sigma_D S^{\cdot}_{\overline{X}}, T(\omega)=0\right\}.$$

For any integer $p\in\mathbb{Z}$, let $\mathcal{D}^{\cdot}(T^{\cdot}_{\overline{X}}/T^{\cdot}_D[-2d](-d),p)$ denote the Deligne complex associated to $T^{\cdot}_{\overline{X}}/T^{\cdot}_D[-2d](-d)$ (see [7, Definition 5.10] for the definition). By [7, Theorem 5.22 and Theorem 5.44], we have the following description of the Deligne-Beilinson cohomology of X in terms of currents on $\overline{X}$.

Theorem 6.1. *For any integer $p\in\mathbb{Z}$, we have an isomorphism*

$$H^*_{\mathcal{D}}(X,\mathbb{R}(p)) \cong H^*(\mathcal{D}^{\cdot}(T^{\cdot}_{\overline{X}}/T^{\cdot}_D[-2d](-d),p)).$$

Moreover, if an element $x\in H^n_{\mathcal{D}}(X,\mathbb{R}(p))$ is represented by a differential form $\eta(x)$ on X with logarithmic singularities at infinity, then the image of x under this isomorphism is represented by a current $T_{\eta(x)}$ on $\overline{X}$ satisfying

$$T_{\eta(x)}(\omega) = \frac{1}{(2\pi i)^d}\int_{\overline{X}}\omega\wedge\eta(x) = \frac{1}{(2\pi i)^d}\int_X \omega\wedge\eta(x) \tag{6.9}$$

for every $\omega\in\Sigma_D S^{\cdot}_{\overline{X}}$.

Remark 6.2. The fact that the integral $\int_{\overline{X}}\omega\wedge\eta(x)$ is (absolutely) convergent if $\omega\in\Sigma_D S^{\cdot}_{\overline{X}}$ is proved in [6, Proposition 3.3]. Since D has measure 0, the convergence of the integral implies the last equality in (6.9). We caution the reader that for an arbitrary differential form $\omega\in S^{\cdot}_{\overline{X}}$, the integral $\int_{\overline{X}}\omega\wedge\eta(x)$ needs not converge. In fact, the form with logarithmic singularities $\eta(x)$ might not be locally integrable around D. For example, if D is given locally by the equation $z=0$, the form $\eta(x)$ might well be of the form $\log|z|\frac{dz}{z}\wedge\frac{d\overline{z}}{\overline{z}}$, which is not integrable. But in Theorem 6.1 the test form ω vanishes along D (that is to say $i^*\omega=0$), and this makes $\omega\wedge\eta(x)$ integrable.

We will use Theorem 6.1 in the case $X=E^k(\mathbb{C})\times E^\ell(\mathbb{C})$ is the product of Kuga-Sato varieties and $\omega=\Omega_{f,g}=\omega_{f^*}\otimes\overline{\omega_g}$. In our case ω extends to a smooth form on the compactification $\overline{X}$, and this smooth form vanishes along the boundary $\overline{X}-X$, so that the formula (6.9) applies.

We also need to recall the functoriality of Deligne-Beilinson cohomology with respect to proper morphisms [7, Theorem 5.46]. Let Y be a smooth quasi-projective complex variety. Let $f : X \to Y$ be a proper morphism of relative dimension e. Using the Poincaré duality isomorphism [23, 6.1.1) j)] and the covariance of Deligne-Beilinson homology with respect to proper morphisms [23, 6.1.1) b)], we get an induced map

$$f_* : H^n_{\mathcal{D}}(X, \mathbb{R}(p)) \to H^{n-2e}_{\mathcal{D}}(Y, \mathbb{R}(p-e)).$$

Let $\overline{Y}$ be a smooth compactification of Y such that $E := \overline{Y} - Y$ is a simple normal crossings divisor. Assume that f extends to a morphism $\overline{f} : \overline{X} \to \overline{Y}$ such that $\overline{f}^{-1}(E) = D$. By [7, Theorem 5.46], we have the following result.

Theorem 6.3. *If an element $x \in H^n_{\mathcal{D}}(X, \mathbb{R}(p))$ is represented by a current T on $\overline{X}$, then the element $f_*(x) \in H^{n-2e}_{\mathcal{D}}(Y, \mathbb{R}(p-e))$ is represented by the current $\overline{f}_*(T)$ on $\overline{Y}$.*

7. Computation of the regulator integral

Let j be an integer satisfying $0 \leq j \leq k \leq \ell$. Recall that we have a $K_{f,g} \otimes \mathbb{C}$-valued differential form $\Omega_{f,g} := \omega_{f^*} \otimes \overline{\omega_g}$ on $E^k(\mathbb{C}) \times E^\ell(\mathbb{C})$. It extends to a smooth differential form on Deligne's compactification $\overline{\overline{E}}^k(\mathbb{C}) \times \overline{\overline{E}}^\ell(\mathbb{C})$. We first define a pairing

$$\langle \cdot, \Omega_{f,g} \rangle : H^{k+\ell+3}_{\mathcal{D}}(E^k_{\mathbb{R}} \times E^\ell_{\mathbb{R}}, \mathbb{R}(k+\ell+2-j)) \to K_{f,g} \otimes \mathbb{C}.$$

Note that $D := (\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell) - (E^k \times E^\ell)$ is a simple normal crossings divisor and we have

$$D = (\overline{\overline{E}}^{k,\infty} \times \overline{\overline{E}}^\ell) \cup (\overline{\overline{E}}^k \times \overline{\overline{E}}^{\ell,\infty}),$$

where $\overline{\overline{E}}^{k,\infty} := \overline{\overline{E}}^k - E^k$. Let $i_\infty : D \to \overline{\overline{E}}^k \times \overline{\overline{E}}^\ell$ be the canonical embedding. Since ω_f (resp. ω_g) is a holomorphic form of top degree on $\overline{\overline{E}}^k(\mathbb{C})$ (resp. $\overline{\overline{E}}^\ell(\mathbb{C})$), we have $i_\infty^* \Omega_{f,g} = 0$. Therefore $\Omega_{f,g} \in \Sigma_D S^{k+\ell+2}_{\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell}$ and it makes sense to evaluate a current in $T^{-k-\ell-2}_{\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell} / T^{-k-\ell-2}_D$ on $\Omega_{f,g}$. Since $\Omega_{f,g}$ is the wedge product of two closed forms, it is also a closed form. Hence if η is an exact form on $\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell$, then we have $T_\eta(\Omega_{f,g}) = 0$ by [6, Proposition 3.3] (also see [6, page 561]). Using Theorem 6.1, we get a well-defined pairing

$$\langle \cdot, \Omega_{f,g} \rangle : H^{k+\ell+3}_{\mathcal{D}}(E^k_{\mathbb{R}} \times E^\ell_{\mathbb{R}}, \mathbb{R}(k+\ell+2-j)) \to K_{f,g} \otimes \mathbb{C}.$$

Note that we have a commutative diagram

$$\begin{array}{ccc} H^{k+\ell+3}_{\mathcal{D}}(\overline{\overline{E}}^k_{\mathbb{R}} \times \overline{\overline{E}}^\ell_{\mathbb{R}}, \mathbb{R}(k+\ell+2-j)) & \xrightarrow{\langle \cdot, \Omega_{f,g} \rangle} & K_{f,g} \otimes \mathbb{C} \\ \downarrow & & \| \\ H^{k+\ell+3}_{\mathcal{D}}(E^k_{\mathbb{R}} \times E^\ell_{\mathbb{R}}, \mathbb{R}(k+\ell+2-j)) & \xrightarrow{\langle \cdot, \Omega_{f,g} \rangle} & K_{f,g} \otimes \mathbb{C}. \end{array} \tag{7.10}$$

Furthermore, the upper map in this diagram induces a pairing

$$\langle \cdot, \Omega_{f,g} \rangle : H^{k+\ell+3}_{\mathcal{D}}(M(f \otimes g), k+\ell+2-j) \to K_{f,g} \otimes \mathbb{C}.$$

This pairing coincides, up to a $K^\times_{f,g}$-rational multiple, with the pairing defined in Lemma 2.1 (since Ω is a $K^\times_{f,g}$-rational multiple of $\Omega_{f,g}$).

Let $\beta \in \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]$. In the case $j = k = \ell$, we assume that β is supported on $(\mathbb{Z}/N\mathbb{Z})^2 - \{0\}$. In this section, we compute $\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta)), \Omega_{f,g} \rangle$ in terms of the Rankin-Selberg L-function of f and g. At the beginning β is arbitrary, but from Definition 7.4 on, we will use a particular choice of β.

Lemma 7.1. *Let $m, n \geq 0$ be two integers and let $p : E^{m+n} \to E^m$ be the canonical projection. Then there exists a morphism $\overline{p} : \overline{\overline{E}}^{m+n} \to \overline{\overline{E}}^m$ fitting into the commutative diagram*

$$\begin{array}{ccc} \overline{\overline{E}}^{m+n} & \xrightarrow{\overline{p}} & \overline{\overline{E}}^m \\ \uparrow & & \uparrow \\ E^{m+n} & \xrightarrow{p} & E^m. \end{array}$$

Proof. We refer to [9] and [33, Chapter 7] for the construction of $\overline{\overline{E}}^m$. Let κ be a field and $C = \mathbf{A}^2_\kappa$, seen as a scheme over $\mathbf{A}^1_\kappa$ by the map $\pi : (x, y) \mapsto xy$. Recall that around a singular point $\overline{E}^m$ is given étale-locally by the m-fold fiber product $C^m = C \times_{\mathbf{A}^1_\kappa} \cdots \times_{\mathbf{A}^1_\kappa} C$. We thus have $C^m = \operatorname{Spec} \kappa[x_1, y_1, \ldots, x_m, y_m]/I_m$ where I_m is the ideal generated by $x_i y_i - x_j y_j$ with $i \neq j$. The semi-direct product $\Gamma^m = \{\pm 1\}^m \rtimes \mathfrak{S}_m$ acts on C^m. Then Deligne's desingularization is given étale-locally by blowing up C^m along the ideal

$$J_m = \left\langle \gamma^*(x_1^{m-1} x_2^{m-2} \cdots x_{m-1}) \,\middle|\, \gamma \in \Gamma^m \right\rangle.$$

The projection $p : C^{m+n} \to C^m$ is induced by the obvious map

$$\phi : \kappa[x_1, y_1, \ldots, x_m, y_m] \to \kappa[x_1, y_1, \ldots, x_{m+n}, y_{m+n}].$$

It is easy to see that J_{m+n} is contained in the ideal generated by $\phi(J_m)$. Hence p extends to a morphism $B_{J_{m+n}} C^{m+n} \to B_{J_m} C^m$ between the blow-ups. □

As in Section 4, write $k' = k - j$ and $\ell' = \ell - j$.

Lemma 7.2. *There is a commutative diagram*

$$\begin{array}{ccc}
\overline{\overline{E}}^{k'+j+\ell'} & \xrightarrow{\ \bar\iota\ } & \overline{\overline{E}}^{k} \times \overline{\overline{E}}^{\ell} \\
\uparrow & & \uparrow \\
E^{k'+j+\ell'} & \xrightarrow{\ i\circ\Delta\ } & E^k \times E^\ell
\end{array}$$

such that $\bar\iota^{-1}(D) = \overline{\overline{E}}^{k'+j+\ell',\infty}$.

Proof. Let $\mathrm{pr}_1 : E^k \times E^\ell \to E^k$ and $\mathrm{pr}_2 : E^k \times E^\ell \to E^\ell$ be the two projections. Then the map $\mathrm{pr}_1 \circ i \circ \Delta : E^{k'+j+\ell'} \to E^k$ is the projection on the first k components. By Lemma 7.1, it extends to a morphism

$$\bar\iota_1 : \overline{\overline{E}}^{k'+j+\ell'} \to \overline{\overline{E}}^{k}.$$

Similarly $\mathrm{pr}_2 \circ i \circ \Delta$ extends to a morphism

$$\bar\iota_2 : \overline{\overline{E}}^{k'+j+\ell'} \to \overline{\overline{E}}^{\ell}.$$

We then define $\bar\iota := (\bar\iota_1, \bar\iota_2)$. The property $\bar\iota^{-1}(D) = \overline{\overline{E}}^{k'+j+\ell',\infty}$ follows from the commutative diagram

$$\begin{array}{ccc}
\overline{\overline{E}}^{k'+j+\ell'} & \xrightarrow{\ \bar\iota\ } & \overline{\overline{E}}^{k} \times \overline{\overline{E}}^{\ell} \\
\downarrow & & \downarrow \\
X(N) & \xrightarrow{\ \Delta\ } & X(N) \times X(N).
\end{array}$$

□

Lemma 7.3. *We have*

$$\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta)), \Omega_{f,g} \rangle = \frac{1}{(2\pi i)^{k'+j+\ell'+1}} \int_{E^{k'+j+\ell'}(\mathbb{C})} \Delta^* i^* \Omega_{f,g} \wedge p^* \Phi^{k'+\ell'}(\beta). \tag{7.11}$$

Proof. Recall that the realization of the Eisenstein symbol $r_{\mathcal{D}}(\mathrm{Eis}^{k'+\ell'}(\beta))$ is represented by the differential form with logarithmic singularities $\Phi^{k'+\ell'}(\beta)$ defined in Section 3. Therefore $r_{\mathcal{D}}(p^*\mathrm{Eis}^{k'+\ell'}(\beta))$ is represented by the differential form $p^*\Phi^{k'+\ell'}(\beta)$. Using Theorem 6.1 with $X = E^{k'+j+\ell'}$, $\overline{X} = \overline{\overline{E}}^{k'+j+\ell'}$ and $D = \overline{\overline{E}}^{k'+j+\ell',\infty}$, it follows that $r_{\mathcal{D}}(p^*\mathrm{Eis}^{k'+\ell'}(\beta))$ is represented by a current $T = T_{p^*\Phi^{k'+\ell'}(\beta)}$ on $\overline{\overline{E}}^{k'+j+\ell'}(\mathbb{C})$ satisfying

$$T(\omega) = \frac{1}{(2\pi i)^{k'+j+\ell'+1}} \int_{E^{k'+j+\ell'}(\mathbb{C})} \omega \wedge p^* \Phi^{k'+\ell'}(\beta) \qquad (\omega \in \Sigma_D S^{k+\ell+2}_{\overline{X}}).$$

Using Theorem 6.3 with $f = i \circ \Delta$ and using Lemma 7.2, we see that $r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta)) = (i \circ \Delta)_* r_{\mathcal{D}}(p^* \mathrm{Eis}^{k'+\ell'}(\beta))$ is represented by the current $\overline{\iota}_* T$ on $\overline{\overline{E}}^k(\mathbb{C}) \times \overline{\overline{E}}^\ell(\mathbb{C})$. Therefore

$$\begin{aligned}\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta)), \Omega_{f,g} \rangle &= (\overline{\iota}_* T)(\Omega_{f,g}) \\ &= T(\overline{\iota}^* \Omega_{f,g}) \\ &= \frac{1}{(2\pi i)^{k'+j+\ell'+1}} \int_{E^{k'+j+\ell'}(\mathbb{C})} \Delta^* i^* \Omega_{f,g} \wedge p^* \Phi^{k'+\ell'}(\beta).\end{aligned}$$

□

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be the Dirichlet character induced by $\chi_f \chi_g$.

Definition 7.4. Let $\beta_\chi \in \mathbb{Q}(\chi)[(\mathbb{Z}/N\mathbb{Z})^2] \subset K_{f,g}[(\mathbb{Z}/N\mathbb{Z})^2]$ be the divisor defined by

$$\beta_\chi(v_1, v_2) = \begin{cases} \overline{\chi}(-v_2) & \text{if } v_1 = 0, \\ 0 & \text{if } v_1 \neq 0. \end{cases}$$

In the case $j = k = \ell$, we assume that $N > 1$, so that β_χ is supported on $(\mathbb{Z}/N\mathbb{Z})^2 - \{0\}$. In particular, we may consider $\Xi^{k,l,j}(\beta_\chi)$.

Recall that G_1 denotes the subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ of matrices of the form $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$.

Lemma 7.5. *The differential form $\Delta^* i^* \Omega_{f,g} \wedge p^* \Phi^{k'+\ell'}(\beta_\chi)$ is G_1-invariant.*

Proof. Since ω_{f^*} and ω_g are G_1-invariant, it follows that $\Delta^* i^* \Omega_{f,g}$ is G_1-invariant. Moreover β_χ is G_1-invariant, and Lemma 3.2 implies that $\mathrm{Eis}^{k'+\ell'}(\beta_\chi)$ is G_1-invariant. □

By Lemma 7.5, it is enough to compute the regulator integral (7.11) on the connected component given by the image of the map $\nu : \mathcal{H} \times \mathbb{C}^{k+\ell-j} \to E^{k+\ell-j}(\mathbb{C})$ defined in (2.4).

Let $\tau, z_1, \ldots, z_{k+\ell-j}$ denote the coordinates on $E^{k+\ell-j}(\mathbb{C})$. Note that we have

$$\nu^*(\Delta^* i^* \Omega_{f,g}) = (-1)^{k+\ell+1} (2\pi i)^{k+\ell+2} f^*(\tau) \overline{g(\tau)} d\tau \wedge d\overline{\tau} \wedge dz_1 \wedge \cdots \wedge dz_k \wedge d\overline{z}_{k-j+1} \wedge \cdots \wedge d\overline{z}_{k+\ell-j}$$

Since this differential form already contains $d\tau \wedge d\overline{\tau}$, we may neglect the terms of $\Phi^{k'+\ell'}(\beta_\chi)$ involving $d\tau, d\overline{\tau}$. Moreover, we have

$$\nu^*(\Delta^* i^* \Omega_{f,g} \wedge p^* \psi_{k'+\ell',a}) = \begin{cases} C_1 f^*(\tau) \overline{g(\tau)} d\tau \wedge d\overline{\tau} \wedge \bigwedge_{i=1}^{k+\ell-j} dz_i \wedge d\overline{z}_i & \text{if } a = k', \\ 0 & \text{if } a \neq k', \end{cases}$$

with

$$C_1 = (-1)^{1+k'^2+j(k'+\ell')+(k'+j+\ell')(k'+j+\ell'-1)/2} \frac{k'! \cdot \ell'!}{(k'+\ell')!} (2\pi i)^{k+\ell+2}.$$

It follows that

$$\begin{aligned}&\nu^* \left(\Delta^* i^* \Omega_{f,g} \wedge p^* \Phi^{k'+\ell'}(\beta_\chi) \right) \\ &= -\frac{(k'+\ell')! \cdot (k'+\ell'+2)}{N(2\pi i)} \cdot \frac{\tau - \overline{\tau}}{2} \cdot \mathcal{E}_{\beta_\chi}^{\ell'+1,k'+1}\left(\tau, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \cdot \nu^* \left(\Delta^* i^* \Omega_{f,g} \wedge p^* \psi_{k'+\ell',k'} \right) \\ &= -\frac{C_1 \cdot (k'+\ell')! \cdot (k'+\ell'+2)}{N(2\pi i)} \cdot \frac{\tau - \overline{\tau}}{2} \cdot \mathcal{E}_{\beta_\chi}^{\ell'+1,k'+1}\left(\tau, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \cdot f^*(\tau) \overline{g(\tau)} d\tau \wedge d\overline{\tau} \wedge \bigwedge_{i=1}^{k+\ell-j} dz_i \wedge d\overline{z}_i.\end{aligned}$$

Integrating over the fibers of the projection $E^{k'+j+\ell'} \to Y(N)$ and using that $\int_{\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})} dz \wedge d\overline{z} = -2i\mathrm{Im}(\tau)$, we get

$$\begin{aligned}
\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta_\chi)), \Omega_{f,g}\rangle &= \frac{1}{(2\pi i)^{k'+j+\ell'+1}} \int_{E^{k'+j+\ell'}(\mathbb{C})} \Delta^* i^* \Omega_{f,g} \wedge p^* \Phi^{k'+\ell'}(\beta_\chi) \\
&= \frac{\phi(N)}{(2\pi i)^{k'+j+\ell'+1}} \int_{\nu(\mathcal{H}\times\mathbb{C}^{k'+j+\ell'})} \Delta^* i^* \Omega_{f,g} \wedge p^* \Phi^{k'+\ell'}(\beta_\chi) \\
&= -\frac{(-2i)^{k+\ell-j} \cdot i \cdot C_1 \cdot \phi(N) \cdot (k'+\ell'+2)!}{(2\pi i)^{k+\ell-j+2} \cdot N \cdot (k'+\ell'+1)} \\
&\quad \cdot \int_{\Gamma(N)\backslash\mathcal{H}} f^*(\tau)\overline{g(\tau)} \mathcal{E}_{\beta_\chi}^{\ell'+1,k'+1}\left(\tau, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \mathrm{Im}(\tau)^{k+\ell-j+1} d\tau \wedge d\overline{\tau}.
\end{aligned}$$

For an integer $w \geq 0$, $\alpha \in \mathbb{Q}/\mathbb{Z}$, $\tau \in \mathcal{H}$ and $s \in \mathbb{C}$, define the following standard real-analytic Eisenstein series as in [26, Definition 4.2.1]:

$$E_\alpha^{(w)}(\tau, s) = (-2\pi i)^{-w} \pi^{-s} \Gamma(s+w) {\sum_{m,n\in\mathbb{Z}}}' \frac{\mathrm{Im}(\tau)^s}{(m\tau+n+\alpha)^w |m\tau+n+\alpha|^{2s}},$$

where $\sum'$ denotes that the term $(m,n) = (0,0)$ is omitted if $\alpha = 0$, and

$$F_\alpha^{(w)}(\tau, s) = (-2\pi i)^{-w} \pi^{-s} \Gamma(s+w) {\sum_{m,n\in\mathbb{Z}}}' \frac{e^{2\pi i \alpha m}\mathrm{Im}(\tau)^s}{(m\tau+n)^w |m\tau+n|^{2s}},$$

where $\sum'$ denotes that the term $(m,n) = (0,0)$ is omitted. For fixed w, α and τ, these functions have meromorphic continuations to the whole s-plane, and are holomorphic everywhere if $w \neq 0$. Note that

$$\sum_{\alpha\in(\mathbb{Z}/N\mathbb{Z})^\times} \omega(\alpha) E_{\alpha/N}^{(\ell-k)}(\tau, s) = (-2\pi i)^{-\ell+k} \pi^{-s} \Gamma(s+\ell-k) \mathrm{Im}(\tau)^s N^{\ell-k+2s} E_{\ell-k,N}(\tau, s, \omega),$$

where $E_{\ell-k,N}(\tau, s, \omega)$ is the Eisenstein series defined by (5.8).

Lemma 7.6. *We have*

$$\mathcal{E}_{\beta_\chi}^{\ell'+1,k'+1}\left(\tau, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \frac{\pi^{k'+\ell'+1}}{\ell'! N^{k'+\ell'} \cdot \mathrm{Im}(\tau)^{k'+\ell'+1}} \lim_{s\to-\ell'} \Gamma(s+\ell-k) E_{\ell-k,N}(\tau, s, \overline{\chi}). \tag{7.12}$$

Proof. We have

$$\begin{aligned}
\mathcal{E}_{\beta_\chi}^{\ell'+1,k'+1}\left(\tau, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) &= {\sum_{(c,d)\in\mathbb{Z}^2}}' \sum_{v_1,v_2\in\mathbb{Z}/N\mathbb{Z}} \frac{\beta_\chi(v_2, -v_1) \cdot e^{\frac{2\pi i(cv_1+dv_2)}{N}}}{(c\tau+d)^{\ell'+1}(c\overline{\tau}+d)^{k'+1}} \\
&= {\sum_{(c,d)\in\mathbb{Z}^2}}' \sum_{v_1\in(\mathbb{Z}/N\mathbb{Z})^\times} \frac{\overline{\chi}(v_1) \cdot e^{\frac{2\pi i c v_1}{N}}}{(c\tau+d)^{\ell'+1}(c\overline{\tau}+d)^{k'+1}} \\
&= \frac{(-2\pi i)^{\ell-k}\pi^{k'+1}}{\ell'! \cdot \mathrm{Im}(\tau)^{k'+1}} \sum_{v_1\in(\mathbb{Z}/N\mathbb{Z})^\times} \overline{\chi}(v_1) F_{v_1/N}^{(\ell-k)}(\tau, k'+1) \\
&= \frac{(-2\pi i)^{\ell-k}\pi^{k'+1}}{\ell'! \cdot \mathrm{Im}(\tau)^{k'+1}} \sum_{v_1\in(\mathbb{Z}/N\mathbb{Z})^\times} \overline{\chi}(v_1) E_{v_1/N}^{(\ell-k)}(\tau, -\ell') \qquad \text{by [26, 4.2.2(iv)]} \\
&= \frac{\pi^{k'+\ell'+1}}{\ell'! N^{k'+\ell'} \cdot \mathrm{Im}(\tau)^{k'+\ell'+1}} \lim_{s\to-\ell'} \Gamma(s+\ell-k) E_{\ell-k,N}(\tau, s, \overline{\chi}).
\end{aligned}$$

□

Using Lemma 7.6, we get

$$\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta_\chi)), \Omega_{f,g}\rangle = \frac{C_2}{(2\pi i)^{k+\ell-j+1}} \int_{\Gamma(N)\backslash\mathcal{H}} f^*(\tau) g^*(-\overline{\tau}) \mathrm{Im}(\tau)^j \lim_{s\to-\ell'} \Gamma(s+\ell-k) E_{\ell-k,N}(\tau, s, \overline{\chi}) dxdy$$

with
$$C_2 = \frac{(-2i)^{k+\ell-j} \cdot i \cdot \pi^{k'+\ell'} \cdot (k'+\ell'+2)! \cdot \phi(N)}{N^{k'+\ell'+1} \cdot \ell'! \cdot (k'+\ell'+1)} \cdot C_1.$$
Since the integrand is invariant under the group $\Gamma_0(N)$, this can be rewritten as
$$\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta_\chi)), \Omega_{f,g}\rangle = \frac{C_2 \cdot N \cdot \phi(N)}{2(2\pi i)^{k+\ell-j+1}} \lim_{s\to -\ell'} \Gamma(s+\ell-k) \int_{\Gamma_0(N)\backslash\mathcal{H}} f^*(\tau) g^*(-\overline{\tau}) E_{\ell-k,N}(\tau, s, \overline{\chi}) y^{s+\ell} dx dy.$$
Using Theorem 5.3 with f^* and g^*, we get
$$\begin{aligned}\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta_\chi)), \Omega_{f,g}\rangle &= \frac{C_2 \cdot N \cdot \phi(N)}{2(2\pi i)^{k+\ell-j+1}} \lim_{s\to -\ell'} \Gamma(s+\ell-k) \cdot 2 \cdot (4\pi)^{-s-1-\ell}\Gamma(s+1+\ell) \cdot \\ &\qquad \cdot R_{f^*,g^*,N}(s+1+\ell) L(f^* \otimes g^*, s+1+\ell) \\ &= \frac{C_2 \cdot N \cdot \phi(N)}{(2\pi i)^{k+\ell-j+1}} (4\pi)^{-j-1} \cdot j! \cdot R_{f^*,g^*,N}(j+1) \lim_{s\to -\ell'} \Gamma(s+\ell-k) L(f^*\otimes g^*, s+1+\ell) \\ &= \frac{C_2 \cdot N \cdot \phi(N)}{(2\pi i)^{k+\ell-j+1}} (4\pi)^{-j-1} \cdot j! \cdot R_{f^*,g^*,N}(j+1) \frac{(-1)^{k-j}}{(k-j)!} L'(f^*\otimes g^*, j+1).\end{aligned}$$
Putting everything together, we have the following theorem.

Theorem 7.7. *Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be the Dirichlet character induced by $\chi_f \chi_g$. Let $\Omega_{f,g} = \omega_{f^*} \otimes \overline{\omega_g}$. Then we have the following identity in $K_{f,g} \otimes \mathbb{C}$:*
$$\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta_\chi)), \Omega_{f,g}\rangle = \pm (2\pi i)^{k+\ell-2j} \cdot \frac{(k+\ell-2j+2) \cdot j! \cdot \phi(N)^2}{2 \cdot N^{k+\ell-2j}} \cdot R_{f^*,g^*,N}(j+1) \cdot L'(f^* \otimes g^*, j+1).$$

Note that since $R_{f^*,g^*,N}(s)$ is a polynomial in the variables p^{-s} with coefficients in $K_{f,g}$ by Remark 5.2, the number $R_{f^*,g^*,N}(j+1)$ belongs to $K_{f,g}$.

8. Computation of residues

Since Beilinson's conjecture is formulated for direct factors of motives associated to smooth projective varieties and $E^k \times E^\ell$ is not projective, we want to extend our motivic elements $\Xi^{k,\ell,j}(\beta)$ to the motivic cohomology of the smooth projective variety $\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell$. We will do this in two steps. In this section, we extend the motivic element $\Xi^{k,\ell,j}(\beta)$ to the product of Néron models $\hat{E}^k \times \hat{E}^\ell$. In the next section, we extend the motivic element to $\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell$. In the case $k = \ell = 0$, Beilinson [2, Section 6] already extended the motivic element to the boundary of the product of two modular curves. Therefore we may assume that $k > 0$ or $\ell > 0$. By symmetry, we may assume that $\ell \geq k \geq 0$ and $\ell \geq 1$.

8.1. **Voevodsky's category of motives and motivic cohomology.** For a field k, let $DM^{\mathrm{eff}}_{gm}(k)$ be the category of effective geometrical motives over k. For a scheme X over k, we have the motive $M_{gm}(X)$ and the motive with compact support $M^c_{gm}(X)$. We consider the $\mathbb{Q}$-linear analogue of $DM^{\mathrm{eff}}_{gm}(k)$ denoted by $DM^{\mathrm{eff}}_{gm}(k)_{\mathbb{Q}}$. For any object M of $DM^{\mathrm{eff}}_{gm}(k)_{\mathbb{Q}}$, we define the motivic cohomology by
$$H^i_{\mathcal{M}}(M, \mathbb{Q}(j)) = \mathrm{Hom}_{DM^{\mathrm{eff}}_{gm}(k)_{\mathbb{Q}}}(M, \mathbb{Q}(j)[i]).$$
Then it is known that
$$H^i_{\mathcal{M}}(M_{gm}(X), \mathbb{Q}(j)) \simeq H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \simeq CH^j(X, 2j-i)$$
for a smooth separated scheme X over k, where $CH^n(X,m)$ is Bloch's higher Chow group.

8.2. **Motives for Kuga-Sato varieties.** Fix an integer $N \geq 3$ and an integer $k \geq 0$. Let $Y = Y(N)$ and $X = X(N)$. Denote $X^\infty = X \setminus Y$. Recall that E is the universal elliptic curve over Y and E^k the k-fold fiber product of E over Y. The symmetric group $\mathfrak{S}_k$ acts on E^k by permutation, $(\mathbb{Z}/N\mathbb{Z})^{2k}$ by translations, and μ_2^k by inversion in the fiber. Therefore we have the action of $G = ((\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mu_2)^k \rtimes \mathfrak{S}_k$. Let $\varepsilon_k : G \to \{\pm 1\}$ be the character which is trivial on $(\mathbb{Z}/N\mathbb{Z})^{2k}$, is the product on μ_2^k, and is the sign character on $\mathfrak{S}_k$. Then define the idempotent
$$e_k := \frac{1}{(2N^2)^k \cdot k!} \sum_{g\in G} \varepsilon_k(g)^{-1} \cdot g \in \mathbb{Z}[\frac{1}{2N \cdot k!}][G].$$

Let $\hat{E}^k$ be the Néron model of E^k over X and let $\hat{E}^{k,*}$ be the connected component of the identity. We denote $Z^k = \hat{E}^{k,*} \setminus E^k = \hat{E}^{k,*} \times_X X^\infty \simeq \mathbb{G}_m^k \times_{\mathbb{Q}} X^\infty$ (non-canonically), $E^{k,\ell} = E^k \times E^\ell$, $\hat{E}^{k,\ell,*} = \hat{E}^{k,*} \times \hat{E}^{\ell,*}$, $Z^{k,\ell} = Z^k \times Z^\ell$ and $U^{k,\ell} = \hat{E}^{k,\ell,*} \setminus Z^{k,\ell}$. Let $i' : E^{k+\ell} \to U^{k,\ell}$ be the canonical closed immersion. Then i' induces the morphism

$$i'_* : H^{k+\ell+1}_{\mathcal{M}}(E^{k+\ell}, \mathbb{Q}(k+\ell-j+1)) \to H^{k+\ell+3}_{\mathcal{M}}(U^{k,\ell}, \mathbb{Q}(k+\ell-j+2)).$$

Recall that we defined the morphisms:

$$\begin{tikzcd} E^{k+\ell-j} \arrow[r, "\Delta"] \arrow[d, "p"] & E^{k+\ell} \arrow[r, "i"] & E^k \times E^\ell \\ E^{k+\ell-2j}. \end{tikzcd}$$

Similarly we define the morphisms:

$$\begin{tikzcd} \hat{E}^{k+\ell-j,*} \arrow[r, "\hat{\Delta}"] \arrow[d, "\hat{p}"] & \hat{E}^{k+\ell,*} \arrow[r, "\hat{i}"] & \hat{E}^{k,*} \times \hat{E}^{\ell,*} \\ \hat{E}^{k+\ell-2j,*} \end{tikzcd}$$

and

$$\begin{tikzcd} Z^{k+\ell-j} \arrow[r, "\Delta_\infty"] \arrow[d, "p_\infty"] & Z^{k+\ell} \arrow[r, "i_\infty"] & Z^k \times Z^\ell \\ Z^{k+\ell-2j}. \end{tikzcd}$$

By [36, Proposition 3.5.4], for a smooth scheme X and a smooth closed subscheme Z of codimension c we have the following Gysin distinguished triangle

$$M_{gm}(X \setminus Z) \to M_{gm}(X) \to M_{gm}(Z)(c)[2c] \to M_{gm}(X \setminus Z)[1].$$

Put $m = k + \ell - 2j$. Then the diagram

$$\begin{tikzcd} M_{gm}(E^m) \arrow[r] & M_{gm}(\hat{E}^{m,*}) \arrow[r] & M_{gm}(Z^m)(1)[2] \arrow[r, "+1"] & {} \\ M_{gm}(E^{m+j}) \arrow[r] \arrow[u, "p_*"] & M_{gm}(\hat{E}^{m+j,*}) \arrow[r] \arrow[u, "\hat{p}_*"] & M_{gm}(Z^{m+j})(1)[2] \arrow[r, "+1"] \arrow[u, "p_{\infty,*}"] & {} \\ M_{gm}(E^{m+2j})(-j)[-2j] \arrow[r] \arrow[u, "\Delta^*"] & M_{gm}(\hat{E}^{m+2j,*})(-j)[-2j] \arrow[r] \arrow[u, "\hat{\Delta}^*"] & M_{gm}(Z^{m+2j})(-j+1)[-2j+2] \arrow[r, "+1"] \arrow[u, "\Delta_\infty^*"] & {} \\ M_{gm}(U^{k,\ell})(-j-1)[-2j-2] \arrow[r] \arrow[u, "i'^*"] & M_{gm}(\hat{E}^{k,\ell,*})(-j-1)[-2j-2] \arrow[r] \arrow[u, "\hat{i}^*"] & M_{gm}(Z^{k,\ell})(-j+1)[-2j+2] \arrow[r, "+1"] \arrow[u, "{i_\infty}^*"] & {} \end{tikzcd}$$

is commutative by [8, Proposition 4.10, Theorem 4.32]. Taking cohomology, we get the following commutative diagram with exact rows:

$$\begin{tikzcd} H^{m+1}_{\mathcal{M}}(\hat{E}^{m,*}, \mathbb{Q}(m+1)) \arrow[r] \arrow[d, "\hat{p}^*"] & H^{m+1}_{\mathcal{M}}(E^m, \mathbb{Q}(m+1)) \arrow[r] \arrow[d, "p^*"] & H^m_{\mathcal{M}}(Z^m, \mathbb{Q}(m)) \arrow[d, "p_\infty^*"] \\ H^{m+1}_{\mathcal{M}}(\hat{E}^{m+j,*}, \mathbb{Q}(m+1)) \arrow[r] \arrow[d, "\hat{\Delta}_*"] & H^{m+1}_{\mathcal{M}}(E^{m+j}, \mathbb{Q}(m+1)) \arrow[r] \arrow[d, "\Delta_*"] & H^m_{\mathcal{M}}(Z^{m+j}, \mathbb{Q}(m)) \arrow[d, "\Delta_{\infty,*}"] \\ H^{k+\ell+1}_{\mathcal{M}}(\hat{E}^{k+\ell,*}, \mathbb{Q}(m+j+1)) \arrow[r] \arrow[d, "\hat{i}_*"] & H^{k+\ell+1}_{\mathcal{M}}(E^{k+\ell}, \mathbb{Q}(m+j+1)) \arrow[r] \arrow[d, "i'_*"] & H^{k+\ell}_{\mathcal{M}}(Z^{k+\ell}, \mathbb{Q}(m+j)) \arrow[d, "i_{\infty,*}"] \\ H^{k+\ell+3}_{\mathcal{M}}(\hat{E}^{k,\ell,*}, \mathbb{Q}(m+j+2)) \arrow[r] & H^{k+\ell+3}_{\mathcal{M}}(U^{k,\ell}, \mathbb{Q}(m+j+2)) \arrow[r] & H^{k+\ell}_{\mathcal{M}}(Z^{k,\ell}, \mathbb{Q}(m+j)). \end{tikzcd}$$

Consider the subgroup $G' = \mu_2^k \rtimes \mathfrak{S}_k$ of G. Let ε'_k be the restriction of ε_k to G' and let e'_k be the idempotent corresponding to ε'_k.

Denote $\tilde{\Xi}^{k,\ell,j}(\beta) = i'_* \circ \Delta_* \circ p^*(\mathrm{Eis}^{k+\ell-2j}(\beta))$. Consider the image of $\tilde{\Xi}^{k,\ell,j}(\beta)$ under the residue map

$$\mathrm{Res}^{k,\ell,j} : H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(k+\ell-j+2))^{(e'_k,e'_\ell)} \to H_{\mathcal{M}}^{k+\ell}(Z^{k,\ell}, \mathbb{Q}(k+\ell-j))^{(e'_k,e'_\ell)}.$$

Note that $H_{\mathcal{M}}^{k+\ell}(Z^{k,\ell}, \mathbb{Q}(k+\ell))^{(e'_k,e'_\ell)}$ can be identified with $H_{\mathcal{M}}^{k}(Z^{k}, \mathbb{Q}(k))^{e'_k} \otimes_{\mathbb{Q}} H_{\mathcal{M}}^{\ell}(Z^{\ell}, \mathbb{Q}(\ell))^{e'_\ell} \simeq \mathcal{F}_N^k \otimes_{\mathbb{Q}} \mathcal{F}_N^\ell$.

Proposition 8.1. (1) *If $j > 0$, then we have* $\mathrm{Res}^{k,\ell,j} \circ \tilde{\Xi}^{k,\ell,j} = 0$.
(2) *If $j = 0$, then $\mathrm{Res}^{k,\ell,0} \circ \tilde{\Xi}^{k,\ell,0}(\beta)$ is a nonzero multiple of $\omega_N^{k+\ell}(\beta) \otimes \omega_N^{k+\ell}(\beta)$.*

Proof. (1) The image of Eisenstein symbol is contained in $H_{\mathcal{M}}^{k+\ell-2j}(Z^{k+\ell-j}, \mathbb{Q}(k+\ell-2j))^{e_{k+\ell-2j}}$. Let $\Delta : \mathbb{G}_m^{k+\ell-j} \to \mathbb{G}_m^{k+\ell}$ be the diagonal embedding. We have

$$\Delta_* : H_{\mathcal{M}}^{k+\ell-2j}(\mathbb{G}_m^{k+\ell-j}, \mathbb{Q}(k+\ell-2j))^{e'_{k+\ell-2j}} \to H_{\mathcal{M}}^{k+\ell}(\mathbb{G}_m^{k+\ell}, \mathbb{Q}(k+\ell-j))^{e'_{k+\ell-2j}}.$$

By [31, 1.3.1 Lemma], one has

$$H_{\mathcal{M}}^{k+\ell}(\mathbb{G}_m^{k+\ell}, \mathbb{Q}(k+\ell-j))^{e'_{k+\ell-2j}} \simeq H_{\mathcal{M}}^{2j}(\mathbb{G}_m^{2j}, \mathbb{Q}(j)) \simeq \mathrm{CH}^j(\mathbb{G}_m^{2j}).$$

Since $j > 0$, we have $\mathrm{CH}^j(\mathbb{G}_m^{2j}) = 0$. Therefore $\Delta_* = 0$.
(2) This follows from the commutativity of the diagram.

□

The closed embedding

$$i_{\mathrm{cusp}} : Z^k \times E^\ell \hookrightarrow U^{k,\ell}$$

induces

$$H_{\mathcal{M}}^{k+\ell+1}(Z^k \times E^\ell, \mathbb{Q}(k+\ell+1)) \xrightarrow{i_{\mathrm{cusp},*}} H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(k+\ell+2)).$$

Let us consider Gysin morphisms

$$\partial : M_{gm}(Z^\ell)(1)[2] \to M_{gm}(E^\ell)[1]$$

for the pair $(\hat{E}^{\ell,*}, Z^\ell)$ and

$$\partial' : M_{gm}(Z^k \times Z^\ell)(1)[2] \simeq M_{gm}(Z^k) \otimes M_{gm}(Z^\ell)(1)[2] \to M_{gm}(Z^k \times E^\ell)[1] = M_{gm}(Z^k) \otimes M_{gm}(E^\ell)[1]$$

for the pair $(Z^k \times \hat{E}^{\ell,*}, Z^k \times Z^\ell)$. By [8, Lemma 4.12], it follows that $\partial' = 1_{Z^k,*} \otimes \partial$. Therefore we have the following commutative diagram:

$$\begin{tikzcd}
H_{\mathcal{M}}^{k}(Z^k, \mathbb{Q}(k))^{e'_k} \otimes_{\mathbb{Q}} H_{\mathcal{M}}^{\ell+1}(E^\ell, \mathbb{Q}(\ell+1))^{e'_\ell} \arrow[r, "1_{Z^k,*}\otimes\partial"] \arrow[d, "\mu"] & H_{\mathcal{M}}^{k}(Z^k, \mathbb{Q}(k))^{e'_k} \otimes_{\mathbb{Q}} H_{\mathcal{M}}^{\ell}(Z^\ell, \mathbb{Q}(\ell))^{e'_\ell} \arrow[d, "\simeq"] \\
H_{\mathcal{M}}^{k+\ell+1}(Z^k \times E^\ell, \mathbb{Q}(k+\ell+1))^{(e'_k,e'_\ell)} \arrow[r, "\partial'"] \arrow[d, "i_{\mathrm{cusp},*}"] & H_{\mathcal{M}}^{k+\ell}(Z^{k,\ell}, \mathbb{Q}(k+\ell))^{(e'_k,e'_\ell)} \arrow[d, "="] \\
H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(k+\ell+2))^{(e'_k,e'_\ell)} \arrow[r, "\mathrm{Res}^{k,\ell,0}"] & H_{\mathcal{M}}^{k+\ell}(Z^{k,\ell}, \mathbb{Q}(k+\ell))^{(e'_k,e'_\ell)},
\end{tikzcd}$$

where $\mu : H_{\mathcal{M}}^{k}(Z^k, \mathbb{Q}(k))^{e'_k} \otimes_{\mathbb{Q}} H_{\mathcal{M}}^{\ell+1}(E^\ell, \mathbb{Q}(\ell+1))^{e'_\ell} \to H_{\mathcal{M}}^{k+\ell+1}(Z^k \times E^\ell, \mathbb{Q}(k+\ell+1))^{(e'_k,e'_\ell)}$ is the exterior product.

Since $\ell \geq 1$, the map ∂ is surjective by [3, Corollary 3.1.8], and $1_{Z^k,*} \otimes \partial$ is also surjective. Hence $\mathrm{Res}^{k,\ell,0} \circ i_{\mathrm{cusp},*}$ is surjective. It follows that there exists an element $\xi_\beta \in H_{\mathcal{M}}^{k+\ell+1}(Z^k \times E^\ell, \mathbb{Q}(k+\ell+1))^{(e'_k,e'_\ell)}$ such that $\mathrm{Res}^{k,\ell,0} \circ i_{\mathrm{cusp},*}(\xi_\beta) = \mathrm{Res}^{k,\ell,0}(\tilde{\Xi}^{k,\ell,0}(\beta))$.

Definition 8.2. We define the *generalized Beilinson-Flach element* $\widetilde{\mathrm{BF}}^{k,\ell,j}(\beta)$ by

$$\widetilde{\mathrm{BF}}^{k,\ell,j}(\beta) := \begin{cases} \tilde{\Xi}^{k,\ell,j}(\beta) & \text{if } j > 0, \\ \tilde{\Xi}^{k,\ell,0}(\beta) - i_{\mathrm{cusp},*}(\xi_\beta) & \text{if } j = 0. \end{cases}$$

From the definition, it is clear that $\widetilde{\mathrm{BF}}^{k,\ell,j}(\beta)$ belongs to the image of the canonical injection

$$H^{k+\ell+3}_{\mathcal{M}}(\hat{E}^{k,\ell,*},\mathbb{Q}(k+\ell-j+2))^{(e'_k,e'_\ell)} \to H^{k+\ell+3}_{\mathcal{M}}(U^{k,\ell},\mathbb{Q}(k+\ell-j+2))^{(e'_k,e'_\ell)}.$$

We may thus consider $\widetilde{\mathrm{BF}}^{k,\ell,j}(\beta)$ as an element of $H^{k+\ell+3}_{\mathcal{M}}(\hat{E}^{k,\ell,*},\mathbb{Q}(k+\ell-j+2))^{(e'_k,e'_\ell)}$.

9. Extension to the boundary

Recall that $\overline{E} \to X$ is the universal generalized elliptic curve over X. Consider the k-fold fiber product $\overline{E}^k = \overline{E}\times_X \cdots \times_X \overline{E}$ of $\overline{E}$ over X. Let $\overline{\overline{E}}^k \to \overline{E}^k$ be Deligne's desingularization. Then $\overline{\overline{E}}^k$ is a smooth projective variety over $\mathbb{Q}$. The action of $G = ((\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mu_2)^k \rtimes \mathfrak{S}_k$ can be extended to $\overline{\overline{E}}^k$ as explained in [31, 1.1.1].

To extend the motivic element $\Xi^{k,\ell,j}(\beta)$ to the boundary of the product of Kuga-Sato varieties $\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell$, we use the following proposition.

Proposition 9.1. *We have an isomorphism*

$$H^{k+\ell+3}_{\mathcal{M}}(\hat{E}^{k,*}\times\hat{E}^{\ell,*},\mathbb{Q}(k+\ell-j+2))^{(e'_k,e'_\ell)} \simeq H^{k+\ell+3}_{\mathcal{M}}(\overline{\overline{E}}^k\times\overline{\overline{E}}^\ell,\mathbb{Q}(k+\ell-j+2))^{(e_k,e_\ell)}.$$

To show the proposition, we prepare the following lemma.

Lemma 9.2. *Let $M_{gm}(\hat{E}^{k,*})^{e'_k}$, $M_{gm}(\overline{\overline{E}}^k)^{e_k} \in DM^{\mathrm{eff}}_{gm}(\mathbb{Q})_{\mathbb{Q}}$ be the images of the idempotents e'_k, e_k on $M_{gm}(\hat{E}^{k,*})$, $M_{gm}(\overline{\overline{E}}^k)$ respectively. Then we have $M_{gm}(\hat{E}^{k,*})^{e'_k} \simeq M_{gm}(\overline{\overline{E}}^k)^{e_k}$ in $DM^{\mathrm{eff}}_{gm}(\mathbb{Q})_{\mathbb{Q}}$.*

Proof of Lemma 9.2. Let $\overline{\overline{E}}^{k,\infty}$ be the complement of the smooth scheme E^k in the smooth proper scheme $\overline{\overline{E}}^k$. Let $\overline{\overline{E}}^{k,\infty,\mathrm{reg}}$ be the intersection of $\overline{\overline{E}}^{k,\infty}$ with the non-singular part $\overline{E}^{k,\mathrm{reg}}$ of $\overline{E}^k$ and $\overline{\overline{E}}^{k,\infty,0} \subset \overline{\overline{E}}^{k,\infty,\mathrm{reg}}$ the intersection of $\overline{\overline{E}}^{k,\infty,\mathrm{reg}}$ with $\hat{E}^{k,*}$. Note that the morphism $\overline{\overline{E}}^k \to \overline{E}^k$ is an isomorphism over $\overline{E}^{k,\mathrm{reg}}$ by [31, Theorem 3.1.0 (ii)], hence $\overline{E}^{k,\mathrm{reg}}$ can be identified with a subscheme of $\overline{\overline{E}}^k$ and the open immersion $\overline{E}^{k,\mathrm{reg}} \hookrightarrow \overline{\overline{E}}^k$ induces an isomorphism

$$M^c_{gm}(\overline{\overline{E}}^k)^{e_k} \xrightarrow{\sim} M^c_{gm}(\overline{E}^{k,\mathrm{reg}})^{e_k}$$

by [37, Remark 3.8 (a)]. Also the connected component $\hat{E}^{k,*}$ is identified with an open subscheme of $\overline{E}^{k,\mathrm{reg}}$ by [31, Theorem 3.1.0 (iii)]. From these facts and [37, Proof of Theorem 3.3], one has

$$M^c_{gm}(\overline{\overline{E}}^{k,\infty})^{e_k} \xrightarrow{\sim} M^c_{gm}(\overline{\overline{E}}^{k,\infty,\mathrm{reg}})^{e_k} \xrightarrow{\sim} M^c_{gm}(\overline{\overline{E}}^{k,\infty,0})^{e'_k}.$$

By [36, Proposition 4.1.5] we have the distinguished triangles:

$$\begin{tikzcd}
M^c_{gm}(\overline{\overline{E}}^k)^{e_k} \arrow[r] \arrow[d, "\simeq"] & M^c_{gm}(E^k)^{e_k} \arrow[r] \arrow[d, "="] & M^c_{gm}(\overline{\overline{E}}^{k,\infty})^{e_k}[1] \arrow[r, "+1"] \arrow[d, "\simeq"] & {} \\
M^c_{gm}(\overline{E}^{k,\mathrm{reg}})^{e_k} \arrow[r] \arrow[d] & M^c_{gm}(E^k)^{e_k} \arrow[r] \arrow[d] & M^c_{gm}(\overline{\overline{E}}^{k,\infty,\mathrm{reg}})^{e_k}[1] \arrow[r, "+1"] \arrow[d, "\simeq"] & {} \\
M^c_{gm}(\hat{E}^{k,*})^{e'_k} \arrow[r] & M^c_{gm}(E^k)^{e'_k} \arrow[r] & M^c_{gm}(\overline{\overline{E}}^{k,\infty,0})^{e'_k}[1] \arrow[r, "+1"] & {}.
\end{tikzcd}$$

Moreover one has

$$M^c_{gm}(E^k)^{e_k} \xrightarrow{\simeq} M^c_{gm}(E^k)^{e'_k},$$

since we have a decomposition $E^k = \coprod_{0\le q\le k} \mathring{Y}^k_q$ of E^k into locally closed subsets which are invariant under the action of $\mathfrak{S}_{k+1}\cdot(\mathbb{Z}/N\mathbb{Z})^{2k}$ as in [28, Proof of 4.2 Theorem], where

$$\mathring{Y}^k_q = \left\{(x_1,\ldots,x_k)\in E^k \,\middle|\, \text{exactly } q \text{ of the } x_i\text{'s are in } E[N]\right\}.$$

From this fact, it follows that the inclusion $\hat{E}^{k,*} \hookrightarrow \overline{E}^{k,\mathrm{reg}}$ induces

$$M^c_{gm}(\overline{E}^{k,\mathrm{reg}})^{e_k} \xrightarrow{\simeq} M^c_{gm}(\hat{E}^{k,*})^{e'_k}$$

and hence

$$M^c_{gm}(\overline{\overline{E}}^k)^{e_k} \xrightarrow{\simeq} M^c_{gm}(\hat{E}^{k,*})^{e'_k}.$$

By duality for smooth schemes [36, Theorem 4.3.7 (3)], we have

$$M_{gm}(\overline{\overline{E}}^k)^{e_k} \xleftarrow{\simeq} M_{gm}(\overline{E}^{k,\mathrm{reg}})^{e_k} \xleftarrow{\simeq} M_{gm}(\hat{E}^{k,*})^{e'_k}.$$

□

Proof of Proposition 9.1. Applying Künneth formula [36, Proposition 4.1.7], we have

$$M_{gm}(\hat{E}^{k,*} \times \hat{E}^{\ell,*})^{(e'_k,e'_\ell)} \simeq M_{gm}(\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell)^{(e_k,e_\ell)}.$$

By Voevodsky's definition of motivic cohomology, we have

$$H^i_{\mathcal{M}}(\hat{E}^{k,*} \times \hat{E}^{\ell,*}, \mathbb{Q}(j))^{(e'_k,e'_\ell)} \simeq H^i_{\mathcal{M}}(\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell, \mathbb{Q}(j))^{(e_k,e_\ell)}$$

for any i, j. This completes the proof. □

Definition 9.3. We define the *generalized Beilinson-Flach element*

$$\mathrm{BF}^{k,\ell,j}(\beta) \in H^{k+\ell+3}_{\mathcal{M}}(\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell, \mathbb{Q}(k+\ell-j+2))^{(e_k,e_\ell)}$$

as the image of the element $\widetilde{\mathrm{BF}}^{k,\ell,j}(\beta)$ under the isomorphism of Proposition 9.1.

Proposition 9.4. *We have* $\langle r_{\mathcal{D}}(\mathrm{BF}^{k,\ell,j}(\beta)), \Omega_{f,g}\rangle = \langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta)), \Omega_{f,g}\rangle$.

Proof. Let η be the open embedding $E^k \times E^\ell \to \overline{\overline{E}}^k \times \overline{\overline{E}}^\ell$. It factors as $\eta = \eta_2 \circ \eta_1$, where η_1 is the open embedding $E^k \times E^\ell \to \hat{E}^{k,*} \times \hat{E}^{\ell,*}$ and η_2 is the open embedding $\hat{E}^{k,*} \times \hat{E}^{\ell,*} \to \overline{\overline{E}}^k \times \overline{\overline{E}}^\ell$. By the commutative diagram (7.10), we have

$$\langle r_{\mathcal{D}}(\mathrm{BF}^{k,\ell,j}(\beta)), \Omega_{f,g}\rangle = \langle r_{\mathcal{D}}(\eta^*\mathrm{BF}^{k,\ell,j}(\beta)), \Omega_{f,g}\rangle.$$

Since the isomorphism of Proposition 9.1 is induced by η_2^*, we have $\eta^*\mathrm{BF}^{k,\ell,j}(\beta) = \eta_1^*\widetilde{\mathrm{BF}}^{k,\ell,j}(\beta)$. Now $\eta_1^*\tilde{\Xi}^{k,\ell,j}(\beta) = \Xi^{k,\ell,j}(\beta)$ and it remains to show that in the case $j=0$, we have $(\eta'_1)^* i_{\mathrm{cusp},*}(\xi_\beta) = 0$, where η'_1 is the open embedding $E^k \times E^\ell \to U^{k,\ell}$.

Let $i_1 : Z^k \times E^\ell \to \hat{E}^{k,*} \times E^\ell$ be the canonical closed embedding, and let $j_1 : E^k \times E^\ell \to \hat{E}^{k,*} \times E^\ell$ be the complementary open embedding. Then η'_1 factors as $j_2 \circ j_1$ in the following commutative diagram

$$\begin{tikzcd}
E^k \times E^\ell \arrow[r, "j_1"] & \hat{E}^{k,*} \times E^\ell \arrow[r, "j_2"] & U^{k,\ell} \\
& Z^k \times E^\ell \arrow[u, "i_1"] \arrow[r, equal] & Z^k \times E^\ell \arrow[u, "i_{\mathrm{cusp}}"']
\end{tikzcd}$$

where the horizontal (resp. vertical) arrows are open (resp. closed) embeddings. Since motivic cohomology satisfies axiom [23, 6.1.1) e)], we have $j_2^*(i_{\mathrm{cusp},*}(\xi_\beta)) = (i_1)_*(\xi_\beta)$. By the long exact localization sequence for motivic cohomology [23, 6.1.1) c)], we have $j_1^* \circ (i_1)_* = 0$. □

10. Application to Beilinson's conjecture

Consider the projection to the $f \otimes g$-component

$$\mathrm{pr}_{f,g} : H^{k+\ell+3}_{\mathcal{D}}(\overline{\overline{E}}^k_{\mathbb{R}} \times \overline{\overline{E}}^\ell_{\mathbb{R}}, \mathbb{Q}(k+\ell+2-j))^{(e_k,e_\ell)} \to H^{k+\ell+3}_{\mathcal{D}}(M(f \otimes g), \mathbb{R}(k+\ell+2-j)).$$

Our results admit the following consequence for Beilinson's conjecture for the motive $M(f \otimes g)(k+\ell+2-j)$.

Theorem 10.1. *Let $f \in S_{k+2}(\Gamma_1(N_f), \chi_f)$ and $g \in S_{\ell+2}(\Gamma_1(N_g), \chi_g)$ be newforms with $k, \ell \geq 0$. Let N be an integer divisible by N_f and N_g, and let j be an integer satisfying $0 \leq j \leq \min\{k, \ell\}$. In the case $j = k = \ell$, assume that $g \neq f^*$ and $N > 1$. Assume that $R_{f,g,N}(j+1) \neq 0$. Then there is an element $\alpha \in H^{k+\ell+3}_{\mathcal{M}}(\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell, \mathbb{Q}(k+\ell+2-j))^{(e_k, e_\ell)}$ such that*

$$\mathrm{pr}_{f,g} \circ r_{\mathcal{D}}(\alpha) = L^*(M(f \otimes g)(k+\ell+2-j)^\vee(1), 0) \cdot t \mod K^\times_{f,g},$$

where t is a generator of the $K_{f,g}$-rational structure in $H^{k+\ell+3}_{\mathcal{D}}(M(f \otimes g), \mathbb{R}(k+\ell+2-j))$.

Proof. Note that $R_{f,g,N}(j+1) \neq 0$ is equivalent to $R_{f^*,g^*,N}(j+1) \neq 0$. The theorem follows from Lemma 2.2, Proposition 7.7, Proposition 9.4 and the fact that $\Omega_{f,g}$ is a $K^\times_{f,g}$-rational multiple of Ω. □

Using an observation for the compatibility of Beilinson's conjecture with respect to the functional equation [27, (2.2)], we get the following corollary.

Corollary 10.2. *Under the assumptions of Theorem 10.1, the weak version of Beilinson's conjecture for $L(f \otimes g, k+\ell+2-j)$ holds.*

Proof. Here we follow the discussion in [27, Section 2] and [11, Section 5]. Let M_{dR} and M_B be the de Rham and Betti realization of $M = M(f \otimes g)(k+\ell+2-j)$ and let $I : M_B \otimes \mathbb{C} \simeq M_{dR} \otimes \mathbb{C}$ be the comparison isomorphism. Denote the determinant of I by $\delta(M)$ and put $d(M) = \dim_{K_{f,g}} M_B = 4$ and $d^-(M) = \dim_{K_{f,g}} M_B^- = 2$. Let $\varepsilon(M) = \varepsilon(M, 0)$ be the global epsilon constant for M as defined in [11, Section 5]. Then $\varepsilon(M) = \varepsilon(f \otimes g, k+\ell+2-j)$ by Remark 5.1. Since $D = \det(M)$ is of the form $[\chi_f \chi_g](n)$ with an integer n, $\varepsilon(D) \equiv (2\pi)^{w(D)/2} \cdot i^{d^-(D)} \cdot \delta(D) \mod K^\times_{f,g}$ by [11, (5.6.1) and Proposition 6.5], where $w(D)$ is the weight of D. By the definitions, it is easy to see $\delta(M) = \delta(D)$, $d^-(M) \equiv d^-(D) \mod 2$ and $w(M)d(M) = w(D)$. Hence one has

$$(2\pi i)^{-d^-(M)} \cdot \delta(M) \equiv (2\pi)^{-d^-(M)} \cdot (2\pi)^{-w(M)d(M)/2} \cdot \varepsilon(D) \mod K^\times_{f,g}.$$

Since $\varepsilon(M) \equiv \varepsilon(\det M) \mod K^\times_{f,g}$ by [11, Proposition 5.5], this implies

$$(2\pi i)^{-d^-(M)} \cdot \delta(M) \equiv (2\pi)^{-d^-(M)} \cdot (2\pi)^{-w(M)d(M)/2} \cdot \varepsilon(M) \mod K^\times_{f,g}.$$

Moreover we have

$$L_\infty(M, 0) L_\infty(M^\vee(1), 0)^{-1} \equiv (2\pi)^{w(M)d(M)/2 + d^-(M)} \mod K^\times_{f,g}$$

and

$$\mathcal{B}_{k+\ell+2,0}(M) = (2\pi i)^{-d^-(M)} \delta(M) \mathcal{D}_{k+\ell+2,0}(M)$$

by [27, (2.2)]. Therefore we have $L^*(M^\vee(1), 0)\mathcal{B}_{k+\ell+2,0}(M) = L(M, 0)\mathcal{D}_{k+\ell+2,0}(M)$ by the functional equation (5.7). □

Remark 10.3.
(1) The factor $R_{f,g,N}(j+1)$ is a product of local terms $R_{f,g,p}(j+1)$, where p runs through the prime factors of N. If p divides exactly one of the integers N_f and N_g, then $R_{f,g,p}(s) = 1$ by [21, Theorem 15.1]. If p divides N but doesn't divide $N_f N_g$, then $R_{f,g,p}(s) = 1 - \chi_f(p)\chi_g(p)p^{k+\ell+2-2s}$ by [34, Lemma 1] and it may happen that $R_{f,g,p}(j+1) = 0$, for example if $j = k = \ell$ and $p \equiv 1 \mod \mathrm{lcm}(N_f, N_g)$. Therefore, it is best to choose $N = \mathrm{lcm}(N_f, N_g)$ in Theorem 10.1. Moreover, it is easy to see that $R_{f,g,N}(j+1) \neq 0$ if $k+\ell-2j \notin \{0, 1, 2\}$ by [21, Theorem 15.1].
(2) The assumption $R_{f,g,N}(j+1) \neq 0$ is necessary. We give an example. There is a newform f of weight 8, level 39 with character $\left(\frac{13}{\cdot}\right)$ such that $a_3(f) = -27$ (it is called 39.8.5a in the modular forms database `http://www.lmfdb.org/`). Also there is a newform g of weight 8, level 3 with trivial character such that $a_3(g) = -27$ (it is called 3.8.a in the modular forms database). Let $\pi_f = \bigotimes'_v \pi_{f,v}$ and $\pi_g = \bigotimes'_v \pi_{g,v}$ be the automorphic representations generated by f and g. Then it is easy to see that $\pi_{f,3}$ and $\pi_{g,3}$ are special representations of the form $\mathrm{sp}(\sigma_{f,3}|\ |^{-\frac{1}{2}}, \sigma_{f,3}|\ |^{\frac{1}{2}})$ and $\mathrm{sp}(\sigma_{g,3}|\ |^{-\frac{1}{2}}, \sigma_{g,3}|\ |^{\frac{1}{2}})$, where $\sigma_{f,3}$ and $\sigma_{g,3}$ are unramified characters of $\mathbb{Q}_3^\times$ satisfying $\sigma_{f,3}(3) = \sigma_{g,3}(3) = -1$. By [21, Theorem 15.1], we have

$$L(\pi_{f,3} \otimes \pi_{g,3}, s) = L(\sigma_{f,3}\sigma_{g,3}, s) L(\sigma_{f,3}\sigma_{g,3}, s+1) = (1 - 3^{-s})(1 - 3^{-s-1}).$$

Hence the Euler factor of $L(f \otimes g, s) = L(\pi_f \otimes \pi_g, s-7)$ at 3 is given by $(1 - a_3(f)a_3(g)3^{-s+1})(1 - a_3(f)a_3(g)3^{-s}) = (1 - 3^6 \cdot 3^{-s+1})(1 - 3^6 \cdot 3^{-s})$. On the other hand, the Euler factor of $D(f,g,s)$ at 3 is $1 - a_3(f)a_3(g)3^{-s} = 1 - 3^6 \cdot 3^{-s}$. Therefore $R_{f,g,N}(s) = 1 - 3^6 \cdot 3^{-s+1} = 1 - 3^7 \cdot 3^{-s}$. If $j = 6$, then $R_{f,g,39}(j+1) = 0$.

References

[1] S. Baba and R. Sreekantan, *An analogue of circular units for products of elliptic curves*, Proc. Edinb. Math. Soc. (2) **47** (2004), no. 1, 35–51.

[2] A. A. Beilinson, *Higher regulators and values of L-functions*, J. Soviet Math. **30** (1985), 2036–2070.

[3] A. A. Beilinson, *Higher regulators of modular curves*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, Proceedings of Summer Research Conference held June 12–18, 1983, in Boulder, Colorado, Contemporary Mathematics 55, American Mathematical Society, Providence, Rhode Island, 1–34.

[4] M. Bertolini, H. Darmon and V. Rotger, *Beilinson-Flach elements and Euler systems I: syntomic regulators and p-adic Rankin L-series*. J. Algebraic Geom. **24** (2015), 355–378.

[5] M. Bertolini, H. Darmon and V. Rotger, *Beilinson-Flach elements and Euler systems II: the Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-series*, J. Algebraic Geom. **24** (2015), 569–604.

[6] J. I. Burgos Gil, *Green forms and their product*, Duke Math. J. **75** (1994), no. 3, 529–574.

[7] J. I. Burgos Gil, J. Kramer and U. Kühn, *Cohomological arithmetic Chow rings*, J. Inst. Math. Jussieu **6** (1) (2007), 1–172.

[8] F. Déglise, *Around the Gysin triangle II*, Doc. Math. **13** (2008), 613–675.

[9] P. Deligne, *Formes modulaires et représentations l-adiques*, Sém. Bourbaki vol. 1968/69 Exposés 347-363, Lect. Notes in Math., Vol. 179, Springer-Verlag, 1971.

[10] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L*, In Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 501–597. Lect. Notes in Math., Vol. 349, Springer, Berlin, 1973.

[11] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Sympos. Pure Math., vol. 33. Automorphic Forms. Representations and L-functions (Oregon State Univ. Corvallis. 1977), Part 2, Amer. Math. Soc. Providence. R.1., 1979, pp. 313–346.

[12] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 143–316. Lect. Notes in Math., Vol. 349, Springer, Berlin, 1973.

[13] C. Deninger, *Higher regulators and Hecke L-series of imaginary quadratic fields I*, Invent. Math. **96** (1989), no. 1, 1–69.

[14] C. Deninger, *Extensions of motives associated to symmetric powers of elliptic curves and to Hecke characters of imaginary quadratic fields*, in: F. Catanese (ed.): Proc. Arithmetic Geometry, Cortona, 1994, Cambridge University Press, 1997, pp. 99–137.

[15] C. Deninger and A. Scholl, *The Beilinson conjectures*, in L-functions and arithmetic (Durham, 1989), 173–209, London Math. Soc. Lecture Note Ser., 153, Cambridge Univ. Press, Cambridge, 1991.

[16] F. Diamond and J. Im, *Modular forms and modular curves*, in Seminar on Fermat's Last Theorem (Toronto, ON, 1993–1994), CMS Conf. Proc., vol. 17, Amer. Math. Soc., Providence, RI, 1995, pp. 39–133.

[17] M. Flach, *A finiteness theorem for the symmetric square of an elliptic curve*, Invent. Math. **109** (1992), no. 2, 307–327.

[18] M. Harris and R. Taylor, *On the geometry and cohomology of some simple Shimura varieties*, Annals of Math. Studies, Vol. **151**, Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[19] G. Henniart, *Caractérisation de la correspondence de Langlands locale par les facteurs ϵ de paires*, Invent. Math. **113** (1993), 339–350 .

[20] A. Huber and G. Kings, *Dirichlet motives via modular curves*, Ann. Sci. Éc. Norm. Sup. **32** (1999), 313–345.

[21] H. Jacquet, *Automorphic forms on* GL(2). *Part II*, Lect. Notes in Math., Vol. 278, Springer-Verlag, Berlin, 1972.

[22] U. Jannsen, *Deligne homology, Hodge-$\mathcal{D}$-conjecture, and motives*, in Beilinson's Conjectures on Special Values of L-functions, 305–372, Perspect. Math., 4, Academic Press, Boston, MA, 1988.

[23] U. Jannsen, *Mixed motives and algebraic K-theory*, Springer Lect. Notes in Math., Vol. 1400, xiii+246 pp, Springer-Verlag, Berlin, 1990.

[24] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, Cohomologies p-adiques et applications arithmétiques. III. Astérisque No. 295 (2004), ix, 117–290.

[25] G. Kings, D. Loeffler and S. L. Zerbes, *Rankin-Eisenstein classes for modular forms*, Preprint, `http://arxiv.org/abs/1501.03289`.

[26] A. Lei, D. Loeffler and S. L. Zerbes, *Euler systems for Rankin-Selberg convolutions of modular forms*, Ann. of Math. (2) **180** (2014), no. 2, 653–771.

[27] J. Nekovář, *Beilinson's conjectures*, In: Motives (Seattle, WA, 1991), Proc. Symp. Pure Math., **55** Part I, 537–570, Amer. Math. Soc., Providence, RI, 1994.

[28] N. Schappacher and A. J. Scholl, *The boundary of the Eisenstein symbol*, Math. Ann. **290** (1991), 303–321.

[29] C.-G. Schmidt, *p-adic measures attached to automorphic representations of* GL(3), Invent. Math. **92** (1988), 597–631.

[30] P. Schneider, *Introduction to the Beilinson conjectures*, in Beilinson's conjectures on special values of L-functions, 1–35, Perspect. Math., 4, Academic Press, Boston, MA, 1988.

[31] A. J. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), 419–430.

[32] A. J. Scholl, *Integral elements in K-theory and products of modular curves*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 467–489, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.

[33] A. J. Scholl, *L-functions of modular forms and higher regulators*, Unpublished book.

[34] G. Shimura, *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. **29** (1976), no. 6, 783–804.

[35] G. Shimura, *On the periods of modular forms*, Math. Ann. **229** (1977), no. 3, 211–221.

[36] V. Voevodsky, *Triangulated categories of motives*, in Cycles, transfers, and motivic homology theories, Annals of Mathematics Studies, vol. 143 (Princeton University Press, Princeton, NJ, 2000).

[37] J. Wildeshaus, *Chow motives without projectivity*, Compositio Math. **145** (2009), 1196–1226.

ÉNS Lyon, UMPA, 46 allée d'Italie, 69007 Lyon, France
E-mail address: francois.brunault@ens-lyon.fr

Mathematical Institute, Tohoku University, 6-3 Aramaki-Aza-Aoba, Aoba-ku, Sendai 980-8578, Japan
E-mail address: chida@math.tohoku.ac.jp