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► **To cite this version:**

Ignacio Garcia-Marco, Pascal Koiran. Lower Bounds by Birkhoff Interpolation. 2015. <ensl-01172296>

**HAL Id: ensl-01172296**

**<https://hal-ens-lyon.archives-ouvertes.fr/ensl-01172296>**

Submitted on 7 Jul 2015

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# Lower Bounds by Birkhoff Interpolation\*

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July 7, 2015

## Abstract

In this paper we give lower bounds for the representation of real univariate polynomials as sums of powers of degree 1 polynomials. We present two families of polynomials of degree  $d$  such that the number of powers that are required in such a representation must be at least of order  $d$ . This is clearly optimal up to a constant factor. Previous lower bounds for this problem were only of order  $\Omega(\sqrt{d})$ , and were obtained from arguments based on Wronskian determinants and "shifted derivatives." We obtain this improvement thanks to a new lower bound method based on Birkhoff interpolation (also known as "lacunary polynomial interpolation").

## 1 Introduction

In this paper we obtain lower bounds for the representation of a univariate polynomial  $f \in \mathbb{R}[X]$  of degree  $d$  under the form:

$$f(x) = \sum_{i=1}^l \beta_i (x + y_i)^{e_i} \tag{1}$$

where the  $y_i$  are real constants and the exponents  $e_i$  nonnegative integers.

We give two families of polynomials such that the number  $l$  of terms required in such a representation must be at least of order  $d$ . This is clearly optimal up to a constant factor. Previous lower bounds for this problem [11] were only of order  $\Omega(\sqrt{d})$ . The polynomials in our first family are of the form  $H_1(x) = \sum_{i=1}^k \alpha_i (x + x_i)^d$  with all  $\alpha_i$  nonzero and the  $x_i$ 's distinct. We show that that they require at least  $l \geq k$  terms whenever  $k \leq (d + 2)/4$ . In particular, for  $k = (d + 2)/4$  we obtain  $l = k = (d + 2)/4$  as a lower bound. The polynomials in our second family are of the form  $H_2(x) =$

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\*This work was supported by ANR project CompA (project number: ANR-13-BS02-0001-01). Authors' email addresses: [ignacio.garcia-marco, pascal.koiran]@ens-lyon.fr.

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$(x+1)^{d+1} - x^{d+1}$  and we show that they require more than  $(d-1)/2$  terms. This improves the lower bound for  $H_1$  by a factor of 2, but this second lower bound applies only when the exponents  $e_i$  are required to be bounded by  $d$  (obviously, if larger exponents are allowed we only need two terms to represent  $H_2$ ). It is easily shown that every polynomial of degree  $d$  can be represented with  $\lceil (d+1)/2 \rceil$  terms. This implies that of all polynomials of degree  $d$ ,  $H_2$  is essentially (up to a small additive constant) the hardest one.

Our lower bound results are specific to polynomials with real coefficients. It would be interesting to obtain similar lower bounds for other fields, e.g., finite fields or the field of complex numbers. As an intermediate step toward our lower bound theorems, we obtain a result on the linear independence of polynomials which may be of independent interest.

**Theorem 1.** *Let  $f_1, \dots, f_k \in \mathbb{R}[X]$  be  $k$  distinct polynomials of the form  $f_i(x) = (x + a_i)^{e_i}$ . Let us denote by  $n_j$  the number of polynomials of degree less than  $j$  in this family.*

*If  $n_1 \leq 1$  and  $n_j + n_{j-1} \leq j$  for all  $j$ , the family  $(f_i)$  is linearly independent.*

We will see later (in Section 4, Remark 17) that this theorem is optimal up to a small additive constant when  $d$  is even, and exactly optimal when  $d$  is odd.

## Motivation and connection to previous work

Lower bounds for the representation of univariate polynomials as sums of powers of *low degree* polynomials were recently obtained in [11]. We continue this line of work by focusing on powers of *degree one* polynomials. This problem is still challenging because the exponents  $e_i$  may be different from  $d = \deg(f)$ , and may be possibly larger than  $d$ . The lower bounds obtained in [11] are of order  $\Omega(\sqrt{d})$ . We obtain  $\Omega(d)$  lower bounds with a new method based on polynomial interpolation (more on this below).

The work in [11] and in the present paper is motivated by recent progress in arithmetic circuit complexity. It was shown that strong enough lower bounds for circuits of depth four [1, 14, 19] or even depth three [9, 19] would yield a separation of Valiant's [20] algebraic complexity classes VP and VNP. Moreover, lower bounds for such circuits were obtained thanks to the introduction by Neeraj Kayal of the method of *shifted partial derivatives*, see e.g. [10, 7, 8, 12, 13, 15, 16]. Some of these lower bounds seem to come close to separating VP from VNP, but there is evidence that the method of shifted derivatives by itself will not be sufficient to achieve this goal. It is therefore desirable to develop new lower bounds methods. We view the models studied in [11] and in the present paper as "test beds" for the development of such methods in a fairly simple setting. We note also that (as explained above) strong lower bounds in slightly more general models would

imply a separation of VP from VNP. Indeed, if the affine function  $x + y_i$  in (1) are replaced by a multivariate affine functions we obtain the model of "depth 3 powering arithmetic circuits." In general depth 3 arithmetic circuits, instead of powers of affine functions we have products of (possibly distinct) affine functions. We note that the depth reduction result of [9] yields circuits where the number of factors in such products can be much larger than the degree of the polynomial represented by the circuit. It is therefore quite natural to allow exponents  $e_i > d$  in (1). Likewise, the model studied in [11] is close to depth 4 arithmetic circuits, see [11] for details.

### **Birkhoff interpolation**

As mentioned above, our results are based on polynomial interpolation and more precisely on Birkhoff interpolation (also known as "lacunary interpolation"). The most basic form of polynomial interpolation is Lagrange interpolation. In a typical Lagrange interpolation problem, one may have to find a polynomial  $g$  of degree at most 2 satisfying the 3 constraints  $g(-1) = 1$ ,  $g(0) = 4$ ,  $g(1) = 3$ . At a slightly higher level of generality we find Hermite interpolation, where at each point we must interpolate not only values of  $g$  but also the values of its first few derivatives. As an example, we may have to find a polynomial  $g$  of degree 3 satisfying the 4 constraints  $g(0) = 1$ ,  $g(1) = 0$ ,  $g'(1) = -1$ ,  $g''(1) = 2$ . Birkhoff interpolation is even more general as there may be "holes" in the sequence of derivatives to be interpolated at each point. An example of such a problem is:  $g(0) = 0$ ,  $g'(1) = 0$ ,  $g(2) = g''(2) = 0$ . We have set the right hand side of all constraints to 0 because the interpolation problems that we need to handle in this paper all turn out to be of that form (in general, one may naturally allow nonzero values). Our interest is in the existence of a nonzero polynomial of degree at most  $d$  satisfying the constraints, and more generally in the dimension of the solution space. In fact, we need to know whether it has the dimension that one would expect by naively counting the number of constraints. Contrary to Lagrange or Hermite interpolation in one variable, where the existence of a nonzero solution can be easily decided by comparing the number of constraints to  $d + 1$  (the number of coefficients of  $g$ ), this is a nontrivial problem and a rich theory was developed to address it [18]. Results of the real (as opposed to complex) theory of Birkhoff interpolation turn out to be very well suited to our lower bound problems. This is the reason why we work with real polynomials in this paper.

### **The Waring problem**

Any homogenous (multivariate) polynomial  $f$  can be written as a sum of powers of linear forms. In the Waring problem for polynomials one attempts to determine the smallest possible number of powers in such a representa-

tion. This number is called the *Waring rank* of  $f$ . Obtaining lower bounds from results in polynomial interpolation seems to be a new method in complexity theory, but it may not come as a surprise to experts on the Waring problem. Indeed, a major result in this area, the Alexander-Hirschowitz theorem ([2], see [4] for a survey), is usually stated as a result on (multivariate, Hermite) polynomial interpolation. Classical work on the Waring problem was focused on the Waring rank of *generic polynomials*, and this question was completely answered by Alexander and Hirschowitz. The focus on generic polynomials is in sharp contrast with complexity theory, where a main goal is to prove lower bounds on the complexity of *explicit* polynomials (or of explicit Boolean functions in Boolean complexity). A few recent papers [17, 6] have begun to investigate the Waring rank of specific (or explicit, in computer science parlance) polynomials such as monomials, sums of coprime monomials, the permanent and the determinant. We expect that more connections between lower bounds in algebraic complexity, polynomial interpolation and the Waring problem will be uncovered in the future.

## Organization of the paper

In Section 2 we begin a study of the linear independence of polynomials of the form  $(x + y_i)^{e_i}$ . We show that this problem can be translated into a problem of Birkhoff interpolation, and in fact we show that Birkhoff interpolation and linear independence are dual problems. In Section 3 we present the notions and results on Birkhoff interpolation that are needed for this paper, and we use them to prove Theorem 1. We build on this result to prove our lower bound results in Section 4, and we discuss their optimality. The lower bound problem studied in this paper is over the field of real numbers. In Section 5 we briefly discuss the situation in other fields and in particular the field of complex numbers. Finally, we give an illustration of our methods in the appendix by completely working out a small example.

## 2 From linear independence to polynomial interpolation

There is a clear connection between lower bounds for representations of polynomials under form (1) and linear independence. Indeed, proving a lower bound for a polynomial  $f$  amounts to showing that  $f$  is linearly independent from  $(x + y_1)^{e_1}, \dots, (x + y_l)^{e_l}$  for some  $l$  and for any sequence of  $l$  pairs  $(y_1, e_1), \dots, (y_l, e_l)$ . Moreover, if the "hard polynomial"  $f$  is itself presented as a sum of powers of degree 1 polynomials (which is the case in this paper), we can obtain a lower bound for  $f$  from linear independence results for such powers. This motivates the following study.

Let us denote by  $\mathbb{R}_d[X]$  the linear subspace of  $\mathbb{R}[X]$  made of polynomials

of degree at most  $d$ , and by  $g^{(k)}$  the  $k$ -th order derivative of a polynomial  $g$ .

**Proposition 2.** *Let  $f_1, \dots, f_k \in \mathbb{R}_d[X]$  be  $k$  distinct polynomials of the form  $f_i(x) = (x + a_i)^{e_i}$ . The family  $(f_i)_{1 \leq i \leq k}$  is linearly independent if and only if*

$$\dim\{g \in \mathbb{R}_d[X]; g^{(d-e_i)}(a_i) = 0 \text{ for all } i\} = d + 1 - k.$$

Let  $V$  be the subspace of  $\mathbb{R}_d[X]$  spanned by the  $f_i$ . The orthogonal  $V^\perp$  of  $V$  is the space of linear forms  $\phi \in \mathbb{R}_d[X]^*$  such that  $\langle \phi, f \rangle = 0$  for all  $f \in V$ . We will use the fact that  $\dim V^\perp = d + 1 - \dim V$ . We will identify  $\mathbb{R}_d[X]$  with its dual  $\mathbb{R}_d[X]^*$  via the symmetric bilinear form

$$\langle g, f \rangle = \sum_{k=0}^d \frac{f_k g_{d-k}}{\binom{d}{k}}.$$

This is reminiscent of Weyl's unitarily invariant inner product (see e.g. chapter 16 of [5] for a recent exposition) but we provide here a self-contained treatment. Proposition 2 follows immediately from the next lemma:

**Lemma 3.** *The orthogonal  $f_i^\perp$  of  $f_i$  is equal to  $\{g \in \mathbb{R}_d[X]; g^{(d-e_i)}(a_i) = 0\}$ .*

*Proof.* We begin with the case  $e_i = d$ . We need to show that for a polynomial  $f(x) = (x + a)^d$ ,  $\langle g, f \rangle = 0$  iff  $g(a) = 0$ . This follows from the definition of  $\langle g, f \rangle$  since by expanding  $(x + a)^d$  in powers of  $x$  we have

$$\langle g, (x + a)^d \rangle = \sum_{k=0}^d g_{d-k} a^{d-k} = g(a). \quad (2)$$

Consider now the general case  $f(x) = (x + a)^{d-k}$  where  $k \geq 0$ . We will show that

$$g^{(k)}(a) = \frac{d!}{(d-k)!} \langle g, f \rangle, \quad (3)$$

thereby completing the proof of the lemma. In order to obtain (3) from (2) we introduce a new variable  $\epsilon$  and expand in two different ways  $\langle g, (x + a + \epsilon)^d \rangle$  in powers of  $\epsilon$ . From (2) we have

$$\langle g, (x + a + \epsilon)^d \rangle = g(a + \epsilon) = \sum_{k=0}^d \frac{g^{(k)}(a)}{k!} \epsilon^k. \quad (4)$$

On the other hand, since  $(x + a + \epsilon)^d = \sum_{k=0}^d \binom{d}{k} \epsilon^k (x + a)^{d-k}$  we have from bilinearity

$$\langle g, (x + a + \epsilon)^d \rangle = \sum_{k=0}^d \binom{d}{k} \langle g, (x + a)^{d-k} \rangle \epsilon^k. \quad (5)$$

Comparing (4) and (5) shows that  $\frac{g^{(k)}(a)}{k!} = \binom{d}{k} \langle g, (x + a)^{d-k} \rangle$ .  $\square$

Since  $\mathbb{R}_d[X]$  has dimension  $d + 1$  we must have  $k \leq d + 1$  for the  $f_i$  to be linearly independent. More generally, let  $n_j$  be the number of  $f_i$ 's which are of degree less than  $j$ . Again, for the  $f_i$  to be linearly independent we must have  $n_j \leq j$  for all  $j = 1, \dots, d + 1$ . The polynomial identity  $(x+1)^2 - (x-1)^2 - 4x = 0$  shows that the converse is not true, but Theorem 1 from the introduction shows that a weak converse holds true. We will use Proposition 2 to prove this theorem at the end of the next section.

### 3 Interpolation matrices

In Birkhoff interpolation we look for a polynomial  $g \in \mathbb{R}_d[X]$  satisfying a system of linear equations of the form

$$g^{(k)}(x_i) = c_{i,k}. \quad (6)$$

The system may be lacunary, i.e., we may not have an equation in the system for every value of  $i$  and  $k$ . We set  $e_{i,k} = 1$  if such an equation appears, and  $e_{i,k} = 0$  otherwise. We arrange this combinatorial data in an *interpolation matrix*  $E = (e_{i,k})_{1 \leq i \leq m, 0 \leq k \leq n}$ . We assume that the *knots*  $x_1, \dots, x_m$  are distinct. It is usually assumed [18] that  $|E| = \sum_{i,k} e_{i,k}$ , the number of 1's in  $E$ , is equal to  $d + 1$  (the number of coefficients of  $g$ ). Here we will only assume that  $|E| \leq d + 1$ . We can also assume that  $n \leq d$  since  $g^{(k)} = 0$  for  $k > d$ . In the sequel we will in fact assume that  $n = d$ : this condition can be enforced by adding empty columns to  $E$  if necessary.

Let  $X = \{x_1, \dots, x_m\}$  be the set of knots. When  $|E| = d + 1$ , the pair  $(E, X)$  is said to be *regular* if (6) has a unique solution for any choice of the  $c_{i,k}$ . Finding necessary or sufficient conditions for regularity has been a major topic in Birkhoff interpolation [18]. For  $|E| \leq d + 1$ , we may expect (6) to have a set of solutions of dimension  $d + 1 - |E|$ . We therefore extend the definition of regularity to this case as follows.

**Definition 4.** *The pair  $(E, X)$  is regular if for any choice of the  $c_{i,k}$  the set of solutions of (6) is an affine subspace of dimension  $d + 1 - |E|$ .*

Note in particular that the set of solutions is nonempty since  $|E| \leq d + 1$ .

Basic linear algebra provides a link between regularity for different values of  $|E|$ .

**Proposition 5.** *Let  $E$  be an  $m \times (d + 1)$  interpolation matrix. For an interpolation matrix  $F$  of the same format, we write  $F \subseteq E$  if  $e_{i,k} = 0$  implies  $f_{i,k} = 0$  (i.e., the set of 1's of  $F$  is included in the set of 1's of  $E$ ).*

*If the pair  $(E, X)$  is regular and  $F \subseteq E$  then  $(F, X)$  is regular as well.*

*Proof.* Consider the interpolation problem:

$$g^{(k)}(x_i) = c_{i,k} \text{ for } f_{i,k} = 1.$$

The set of solutions  $\mathcal{F} \subseteq \mathbb{R}_d[X]$  is an affine subspace which is either empty or of dimension at least  $d + 1 - |F|$ . It cannot be empty since by adding  $|E| - |F|$  constraints we can obtain an interpolation problem with a solution space of dimension  $d + 1 - |E| \geq 0$ . For the same reason, it cannot be of dimension  $d + 2 - |F|$  or more. In this case, by adding  $|E| - |F|$  constraints we would obtain an interpolation problem with a solution space of dimension at least  $(d + 2 - |F|) - (|E| - |F|) = d + 2 - |E|$ . This is impossible since  $(E, X)$  is regular.  $\square$

Another somewhat more succinct way of phrasing the above proof is to consider the matrix of the linear system defining the affine subset  $\mathcal{F}$ . Anticipating on Section 4, let us denote this matrix by  $A(E, X)$ . The pair  $(E, X)$  is regular iff  $A(E, X)$  is of full row rank. The rows of  $A(F, X)$  are also rows of  $A(E, X)$ , so  $A(F, X)$  must be of full row rank if  $A(E, X)$  is.

For an interpolation matrix, the notions of *regularity* and *order regularity* are classically defined [18] in the case  $|E| = d + 1$ , but the extension to the general case  $|E| \leq d + 1$  is straightforward:

**Definition 6.** *The interpolation matrix  $E$  is regular if  $(E, X)$  is regular for any choice of  $m$  knots  $x_1, \dots, x_m$ . It is order regular if  $(E, X)$  is regular for any choice of  $m$  ordered knots  $x_1 < x_2 < \dots < x_m$ .*

As an immediate corollary of Proposition 5 we have:

**Corollary 7.** *Let  $E, F$  be two interpolation matrices with  $F \subseteq E$ . If  $E$  is regular (respectively, order regular) then  $F$  is also regular (respectively, order regular).*

We will give in Theorem 8 a sufficient condition for order regularity, but we first need some additional definitions. Say that an interpolation matrix  $E$  satisfies the *upper Pólya condition* if for  $r = 1, \dots, d + 1$  there are at most  $r$  1's in the last  $r$  columns of  $E$ . If  $|E| = d + 1$  this is equivalent to the *Pólya condition*: there are at least  $r$  1's in the first  $r$  columns of  $E$  for  $r = 1, \dots, d + 1$ .

Consider a row of an interpolation matrix  $E$ . By *sequence* we mean a maximal sequence of consecutive 1's in this row. A sequence containing an odd number of 1's is naturally called an *odd sequence*. A sequence of the  $i$ th row is *supported* if there are 1's in  $E$  both to the northwest and southwest of the first element of the row. More precisely, if  $(i, k)$  is the position of the first 1 of the sequence,  $E$  should contain 1's in positions  $(i_1, k_1)$  and  $(i_2, k_2)$  where  $i_1 < i < i_2$ ,  $k_1 < k$  and  $k_2 < k$ . The following important result (Theorem 1.5 in [18]) is due to Atkinson and Sharma [3].

**Theorem 8.** *Let  $E$  be an  $m \times (d + 1)$  interpolation matrix with  $|E| = d + 1$ . If  $E$  satisfies the Pólya condition and contains no odd supported sequence then  $E$  is order regular.*



As an example, the interpolation problem corresponding to the polynomial identity  $(x + 1)^2 - (x - 1)^2 - 4x = 0$  is:

$$g(-1) = 0, \quad g'(0) = 0, \quad g(1) = 0$$

where  $g \in \mathbb{R}_2[X]$ . It admits  $g(x) = x^2 - 1$  as a nontrivial solution. The corresponding interpolation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

satisfies the Pólya condition but contains an odd supported sequence in its second row.

**Corollary 9.** *Let  $E$  be an  $m \times (d + 1)$  interpolation matrix with  $|E| = d + 1$ . If  $E$  satisfies the Pólya condition, then:*

(i) *if every odd sequence of  $E$  belongs to the first row, to the last row, or begins in the first column then  $E$  is order regular.*

(ii) *if every odd sequence of  $E$  begins in the first column then  $E$  is regular.*

*Proof.* Part (i) follows from the fact that a sequence which belongs to the first row, to the last row, or begins in the first column cannot be supported.

For part (ii), assume that every odd sequence of  $E$  begins in the first column and fix  $m$  distinct nodes  $x_1, \dots, x_m$ . By reordering the  $x_i$ 's we obtain an increasing sequence  $x'_1 < x'_2 < \dots < x'_m$ . Applying the same permutation on the rows of  $E$ , we obtain an interpolation matrix  $E'$ ; clearly, the pair  $(E, X)$  is regular if and only if  $(E', X')$  is. The latter pair is regular because  $X'$  is ordered and (by part (i))  $E'$  is order regular.  $\square$

We can now prove the main result of this section.

**Theorem 10.** *Let  $F$  be an  $m \times (d + 1)$  interpolation matrix. We denote by  $N_r$  the number of 1's in the last  $r$  columns of  $F$ . If  $N_1 \leq 1$  and  $N_r + N_{r-1} \leq r$  for  $r = 2, \dots, d + 1$  then  $F$  is regular.*

Note that the conditions on  $N_r$  are a strengthening of the upper Pólya condition  $N_r \leq r$ .

*Proof.* We will add 1's to  $F$  so as to obtain a matrix  $E$  satisfying the hypothesis of Corollary 9, part (ii). Corollary 7 will then imply that  $F$  is regular.

In order to construct  $E$  we proceed as follows. First, for every odd sequence of  $F$  which does not begin in the first column we add a 1 in the cell immediately to the left of its first 1. All odd sequences of the resulting matrix  $F'$  begin in the first column. Moreover, we have added at most  $N_{r-1}$

1's in the last  $r$  columns of  $F$  (note that we can add exactly  $N_{r-1}$  1's when the last  $r-1$  columns contain  $N_{r-1}$  sequences of length 1). Since  $N_1 \leq 1$  and  $N_r + N_{r-1} \leq r$ ,  $F'$  satisfies the upper Pólya condition. If  $|F'| = d+1$  we set  $E = F'$ . This matrix satisfies the Pólya condition and its odd sequences all begin in the first column, so we can indeed apply Corollary 9 to get that  $E$  is regular and. Since  $F \subseteq E$ , by Corollary 7, we conclude that  $F$  is also regular.

If  $|F'| < d+1$  we need to add more 1's. It suffices to add  $d+1 - |F'|$  new rows to  $F'$  with a 1 in the first column and 0's everywhere else. Denoting by  $E$  the resulting matrix we clearly have that  $E$  satisfies the Pólya condition,  $|E| = d+1$  and its odd sequences begin in the first column, so Corollary 9 and Corollary 7 apply here to conclude that  $F$  is regular. Note that  $E$  and  $F$  do not have the same format since  $E$  has more rows, but we can apply Corollary 7 if we first expand  $F$  with  $d+1 - |F'|$  empty rows.  $\square$

### Proof of Theorem 1

At this point we have enough knowledge of Birkhoff interpolation to prove Theorem 1. In view of Proposition 2 we need to show that the interpolation problem

$$g^{(d-e_i)}(a_i) = 0 \text{ for } i = 1, \dots, k$$

has a solution space of dimension  $d+1 - k$ . Let  $F$  be the corresponding interpolation matrix. This matrix contains  $d+1 - k$  1's and is of size  $m \times (d+1)$  for some  $m \leq k$  (we have  $m = k$  only when the  $a_i$ 's are all distinct). The hypothesis on the  $n_j$ 's implies that  $F$  satisfies the hypothesis of Theorem 10, and the result follows from the regularity of  $F$ .

## 4 Lower bounds

System (6) is a linear system of equations in the coefficients of  $g$ . Following [18], to set up this system it is convenient to work in the basis  $(x^j/j!)_{0 \leq j \leq d}$  instead of the standard basis  $(x^j)_{0 \leq j \leq d}$ . We denote by  $A(E, X)$  the matrix of the system in that basis, where as in the previous section  $E$  denotes the corresponding interpolation matrix and  $X$  the set of knots. As already pointed out after Proposition 5, the pair  $(E, X)$  is regular if and only if  $A(E, X)$  is of rank  $|E|$ . In our chosen basis, an interpolation constraint of the form (6) reads:

$$\sum_{j=0}^d \frac{x_i^{j-k}}{(j-k)!} g_j = c_{i,k}$$

where the coefficients  $g_0, \dots, g_d$  are the unknowns and we choose as in [18] to interpret  $1/r!$  as 0 for  $r < 0$ .

**Proposition 11.** Consider a pair  $(E, X)$  where  $E$  is an interpolation matrix of size  $m \times (d + 1)$  and  $X$  a set of  $m$  knots. Let  $E_1$  be the matrix formed of the first  $r + 1$  columns of  $E$  and  $E_2$  the matrix formed of the remaining  $d - r$  columns.

Suppose that  $E_1$  contains at most  $r + 1$  1's and  $E_2$  at most  $d - r$  1's. If both pairs  $(E_1, X)$ ,  $(E_2, X)$  are regular then  $(E, X)$  is regular.

*Proof.* The case where  $|E_1| = r + 1$  and  $|E_2| = d - r$  is treated in Theorem 1.4 of [18]. Their argument extends to the general case. Indeed, as shown in [18] the rank of  $A(E, X)$  is at least equal to the sum of the ranks of  $A(E_1, X)$  and  $A(E_2, X)$ . For the reader's convenience, we recall from [18] that this inequality is due to the fact that  $A(E, X)$  can be transformed by a permutation of rows into a matrix of the form<sup>1</sup>

$$\begin{pmatrix} A(E_1, X) & * \\ 0 & A(E_2, X) \end{pmatrix}.$$

The two matrices  $A(E_1, X)$ ,  $A(E_2, X)$  are respectively of rank  $|E_1|$  and  $|E_2|$  since the corresponding pairs are assumed to be regular. Thus,  $A(E, X)$  is of rank at least  $|E| = |E_1| + |E_2|$ . This matrix must in fact be of rank exactly  $|E|$  since it has  $|E|$  rows, and we conclude that the pair  $(E, X)$  is regular.  $\square$

**Lemma 12.** For any finite sequence  $(u_i)_{0 \leq i \leq n}$  of real numbers with  $n \geq 1$  there is an index  $s \in \{0, \dots, n - 1\}$  such that  $(u_{s+t} - u_s)/t \leq (u_n - u_0)/n$  for every  $t = 1, \dots, n - s$ .

*Proof.* Let  $\alpha = \min_{0 \leq i \leq n-1} (u_n - u_i)/(n - i)$  and let  $s$  be an index where the minimum is achieved. We have  $(u_n - u_s)/(n - s) = \alpha \leq (u_n - u_0)/n$ .

For every  $t = 1, \dots, n - s$  we also have  $(u_n - u_s)/(n - s) \leq (u_n - u_{s+t})/(n - s - t)$ , which implies  $(u_{s+t} - u_s)/t \leq (u_n - u_s)/(n - s) \leq (u_n - u_0)/n$ .  $\square$

Our lower bound results are easily derived from the following theorem.

**Theorem 13.** Consider a polynomial identity of the form:

$$\sum_{i=1}^k \alpha_i (x + x_i)^d = \sum_{i=1}^l \beta_i (x + y_i)^{e_i} \quad (7)$$

where the  $x_i$  are distinct real constants, the constants  $\alpha_i$  are not all zero, the  $\beta_i$  and  $y_i$  are arbitrary real constants, and  $e_i < d$  for every  $i$ . Then we must have  $k + l > (d + 2)/2$ .

*Proof.* We assume without loss of generality that the  $l$  polynomials  $(x + y_i)^{e_i}$  are linearly independent. Indeed, the right-hand side of (7) could otherwise be rewritten as a linear combination of an independent subfamily, and this

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<sup>1</sup>We give an example in the appendix.

would only decrease  $l$ . Let us also assume that  $k + l \leq (d + 2)/2$ . Then we will show that the  $k + l$  polynomials  $(x + x_i)^d, (x + y_i)^{e_i}$  must be linearly independent. This is clearly in contradiction with (7).

In view of Proposition 2, to show that our  $k + l$  polynomials are linearly independent we need to show that the corresponding interpolation problem has a solution space of dimension  $d + 1 - k - l$ . Let  $E$  be the corresponding interpolation matrix and  $X$  the set of knots:  $|X| = m$  where  $m$  is the number of distinct points in  $x_1, \dots, x_k, y_1, \dots, y_l$ ; moreover,  $E$  is a matrix of size  $m \times (d + 1)$  which contains  $k + l$  1's. We need to show that the pair  $(E, X)$  is regular.

Let  $N_t$  be the number of 1's in the last  $t$  columns of  $E$ . We must have  $N_1 \leq 1$ , or else the independent family  $(x + y_i)^{e_i}$  would contain more than one constant polynomial. We can now complete the proof of the theorem in the special case where  $E$  satisfies the conditions  $N_t + N_{t-1} \leq t$  for every  $t = 2, \dots, d + 1$ . Indeed, in this case  $E$  is regular by Theorem 10 (remember that this is how we proved Theorem 1, our main linear independence result). For the general case, the idea of the proof is to:

- (i) Split vertically  $E$  in two matrices  $E_1, E_2$ .
- (ii) Apply the same argument (Theorem 10) to  $E_1$ .
- (iii) Obtain the regularity of the pair  $(E_2, X)$  from the linear independence of the  $(x + y_i)^{e_i}$ .
- (iv) Conclude from Proposition 11 that the pair  $(E, X)$  is regular.

We now explain how to carry out these four steps. For the first one, note that  $N_{d+1} = |E| = k + l \leq (d + 2)/2$ . Let us apply Lemma 12 to the sequence  $(N_i)_{0 \leq i \leq d+1}$  beginning with  $N_0 = 0$ . The lemma shows the existence of an index  $s \in \{0, \dots, d\}$  such that

$$\frac{N_{s+t} - N_s}{t} \leq \frac{N_{d+1}}{d+1} \leq \frac{d+2}{2(d+1)} = \frac{1}{2} + \frac{1}{2(d+1)} \leq \frac{1}{2} + \frac{1}{2t}$$

for every  $t = 1, \dots, d + 1 - s$ . Let  $E_1$  be the matrix formed of the first  $r + 1$  columns of  $E$ , where  $r = d - s$ . The number of 1's in the last  $t$  columns of  $E_1$  is  $N'_t = N_{s+t} - N_s \leq (t + 1)/2$ . In particular,  $N'_1 \leq 1$  and, since  $N'_t, N'_{t-1}$  are integers, we get that  $N'_t + N'_{t-1} \leq \lfloor (2t + 1)/2 \rfloor = t$  for all  $t \in \{2, \dots, r + 1\}$ . This matrix therefore satisfies the hypotheses of Theorem 10, and we conclude that  $E_1$  is regular. This completes step (ii). For step (iii), we note that since  $E_2$  has  $s < d + 1$  columns the Birkhoff interpolation problem corresponding to the polynomials  $(x + y_i)^{e_i}$  with  $e_i \leq s - 1$  admits  $(E_2, X)$  as its pair. Since these polynomials are assumed to be linearly independent,  $E_2$  must contain at most  $s$  1's and  $(E_2, X)$  must indeed be a regular pair. Finally, we conclude from Proposition 11 that  $(E, X)$  is regular as well.  $\square$

**Theorem 14** (First lower bound). *Consider a polynomial of the form*

$$H_1(x) = \sum_{i=1}^k \alpha_i (x + x_i)^d \quad (8)$$

where the  $x_i$  are distinct real constants, the  $\alpha_i$  are nonzero real constants, and  $k \leq (d+2)/4$ . If  $H_1$  is written under the form

$$H_1(x) = \sum_{i=1}^l \beta_i (x + y_i)^{e_i} \quad (9)$$

with  $e_i \leq d$  for every  $i$  then we must have  $l \geq k$ .

*Proof.* Assume first that  $e_i < d$  for all  $i$ . By Theorem 13 we must have  $k + l > (d+2)/2$ , so  $l > (d+2)/2 - k \geq k$ . Consider now the general case and assume that  $l < k$ . We reduce to the previous case by moving on the side of (8) the  $k'$  polynomials in (9) of degree  $e_i = d$ . On the second side remains a sum of  $l - k'$  terms of degree less than  $d$ . We have on the first side a sum of terms of degree  $d$ . Taking possible cancellations into account, the number of such terms is at least  $k - k' > 0$ , and at most  $k + k'$ . We must therefore have  $(k + k') + (l - k') > (d+2)/2$ , so  $l > (d+2)/2 - k \geq k$  after all.  $\square$

In other words, writing  $H_1$  under form (8) is exactly optimal when  $k \leq (d+2)/4$ . We can give another lower bound of order  $d$  (with an improved constant) for a polynomial of a different form.

**Theorem 15** (Second lower bound). *Let  $H_2 \in \mathbb{R}_d[X]$  be the polynomial  $H_2(x) = (x+1)^{d+1} - x^{d+1}$ . If  $H_2$  is written under the form*

$$H_2(x) = \sum_{i=1}^l \beta_i (x + y_i)^{e_i}$$

with  $e_i \leq d$  for every  $i$  then we must have  $l > (d-1)/2$ .

*Proof.* This follows directly from Theorem 13 after replacing  $d$  by  $d+1$  in (7). Since  $k = 2$  we must have  $2 + l > (d+3)/2$ , i.e.,  $l > (d-1)/2$ .  $\square$

This result shows that allowing exponents  $e_i > d$  can drastically decrease the ‘‘complexity’’ of a polynomial since  $H_2$  can be expressed as the difference of only two  $(d+1)$ -st powers. Such savings cannot be obtained for all polynomials. Indeed, the next result, which subsumes Theorem 14, shows that no improvement is possible for  $H_1$  even if arbitrarily large powers are allowed.

**Theorem 16** (Third lower bound). *Consider a polynomial of the form*

$$H_1(x) = \sum_{i=1}^k \alpha_i (x + x_i)^d \quad (10)$$

where the  $x_i$  are distinct real constants, the  $\alpha_i$  are nonzero real constants, and  $k \leq (d+2)/4$ . If  $H_1$  is written under the form

$$H_1(x) = \sum_{i=1}^l \beta_i (x + y_i)^{e_i} \quad (11)$$

then we must have  $l \geq k$ .

Note that the exponents  $e_i$  may be arbitrarily large.

*Proof.* Let  $n$  be the largest exponent  $e_i$  which occurs with a coefficient  $\beta_i \neq 0$ . The case  $n \leq d$  is covered by Theorem 14, so we assume here that  $n > d$ . In equation (11), let us move all the  $n$ -th powers from the right hand side to the left hand side, and the  $k$  degree- $d$  terms of  $H_1$  from the left hand side to the right hand side. Applying Theorem 13 to this identity shows that  $k + l > (n+2)/2$ . Hence  $l > (n+2)/2 - k \geq k$ .  $\square$

**Remark 17.** *The lower bound for  $H_2$  in Theorem 15 is essentially optimal. More concretely, it is optimal up to one unit when  $d$  is even, and exactly optimal when  $d$  is odd.*

*Note indeed that by a change of variable, representing  $H_2$  is equivalent to representing the polynomial  $H_3(x) = (x+1)^{d+1} - (x-1)^{d+1}$ . If we expand the two binomials in  $H_3$  the monomials of degree  $d+1-j$  with even  $j$  cancel, and we obtain a sum of  $\lceil \frac{d+1}{2} \rceil$  monomials. See Proposition 19 for a generalization of this observation. In fact, with the same number of terms we can represent not only  $H_2$  but all polynomials of degree  $d$ : see Proposition 18 below.*

*The consideration of  $H_3$  also shows that Theorem 1 is optimal up to one unit when  $d$  is even, and exactly optimal when  $d$  is odd. Indeed, we have just observed that there is a linear dependence between the  $2 + \lceil \frac{d+1}{2} \rceil$  polynomials  $(x+1)^{d+1}, (x-1)^{d+1}, x^d, x^{d-2}, x^{d-4}, \dots$*

*If  $d$  is odd, the number of polynomials of degree less than  $j$  in this sequence is  $n_j = \lfloor j/2 \rfloor$  for  $j \leq d+1$ ; moreover,  $n_{d+2} = 2 + (d+1)/2 = (d+5)/2$ . Hence  $n_j + n_{j+1} = j$  for  $j \leq d$ ; moreover,  $n_{d+1} + n_{d+2} = d+3$ .*

*If  $d$  is even, the number of polynomials of degree less than  $j$  in this sequence is  $n_j = \lceil j/2 \rceil$  for  $j \leq d+1$ ; moreover,  $n_{d+2} = 2 + (d+2)/2 = (d+6)/2$ . Hence  $n_j + n_{j+1} = j+1$  for  $j \leq d$ ; moreover,  $n_{d+1} + n_{d+2} = d+4$ .*

A simple construction shows that all polynomials of degree  $d$  can be written as a linear combination of  $\lceil (d+1)/2 \rceil$  powers.

**Proposition 18.** *Every polynomial of degree  $d$  can be expressed as  $\sum_{i=1}^l \beta_i(x + y_i)^{e_i}$  with  $l \leq \lceil (d+1)/2 \rceil$ .*

*Proof.* We use induction on  $d$ . Since the result is obvious for  $d = 0, 1$  we consider a polynomial  $f = \sum_{i=0}^d a_i x^i$  of degree  $d \geq 2$ , and we assume that the Proposition holds for polynomials of degree  $d-2$ . We observe that  $g := f - a_d(x + (a_{d-1}/da_d))^d$  has degree  $\leq d-2$ . Applying the induction hypothesis to  $g$  we get that  $g = \sum_{i=1}^{l'} \beta_i(x + y_i)^{e_i}$ , with  $l' \leq \lceil (d-1)/2 \rceil$ . Hence, setting  $l = l' + 1$ ,  $\beta_l = a_d$ ,  $y_l = a_{d-1}/(da_d)$  and  $e_l = d$ , we conclude that  $f = \sum_{i=1}^l \beta_i(x + y_i)^{e_i}$  and  $l \leq 1 + \lceil (d-1)/2 \rceil = \lceil (d+1)/2 \rceil$ .  $\square$

Theorem 15 therefore shows that of all polynomials of degree  $d$ ,  $H_2$  is essentially (up to a small additive constant) the hardest one.

## 5 Changing Fields

Some of the proof techniques used in this paper, and even the results themselves, are specific to the field of real numbers. This is due to the fact that certain linear dependence relations which cannot occur over  $\mathbb{R}$  may occur if we change the base field. For instance, over a field of characteristic  $p > 0$  we have  $(X+1)^{p^k} - X^{p^k} - 1 = 0$  for any  $k$  (compare with Theorem 1 for the real case). The remainder of this section is devoted to a discussion of the complex case. We begin with an identity which generalizes the identity  $(x+1)^2 - (x-1)^2 - 4x = 0$ .

**Proposition 19.** *Take  $k \in \mathbb{Z}^+$  and let  $\xi$  be a  $k$ -th primitive root of unity. Then, for all  $d \in \mathbb{Z}^+$  and all  $\mu \in \mathbb{C}$  the following equality holds:*

$$\sum_{j=1}^k \xi^j (x + \xi^j \mu)^d = \sum_{\substack{i \equiv -1 \pmod{k} \\ 0 \leq i \leq d}} k \binom{d}{i} \mu^i x^{d-i}.$$

*Proof.* We observe that

$$\sum_{j=1}^k \xi^j (x + \xi^j \mu)^d = \sum_{i=0}^d \binom{d}{i} \mu^i x^{d-i} \left( \sum_{j=1}^k \xi^{ji+j} \right).$$

To deduce the result it suffices to prove that  $\sum_{j=1}^k \xi^{ji+j}$  equals  $k$  if  $i \equiv -1 \pmod{k}$ , or 0 otherwise. Whenever  $i \equiv -1 \pmod{k}$  we have that  $\xi^{ji+j} = (\xi^{i+1})^j = 1$  for all  $j \in \{1, \dots, k\}$ . For  $i \not\equiv -1 \pmod{k}$ , the summation of the geometric series shows that

$$\sum_{j=1}^k \xi^{j(i+1)} = \xi^{i+1} \cdot \frac{\xi^{k(i+1)-1} - 1}{\xi^{i+1} - 1} = 0.$$

$\square$

For any  $d, k \in \mathbb{Z}^+$ , Proposition 19 yields an identity of the form

$$\sum_{j=1}^k \alpha_j (x + x_j)^d = \sum_{j=1}^l \beta_j x^{e_j} \quad (12)$$

where the  $x_j$  are distinct complex constants, the  $\alpha_j, \beta_j$  are nonzero complex numbers,  $l = \lfloor \frac{d+1}{k} \rfloor$  and  $e_j < d$  for all  $j$ . Note the sharp contrast with theorems 13 and 14. In particular, Theorem 14 gives an  $\Omega(d)$  lower bound for polynomials of the form  $\sum_{j=1}^k \alpha_j (x + x_j)^d$  over the field of real numbers (the implied constant in the  $\Omega$  notation is equal to  $1/4$ ). But in (12) we have  $k \cdot l \leq d + 1$ , so  $k \leq \sqrt{d+1}$  or  $l \leq \sqrt{d+1}$ . We conclude that no better lower bound than  $\Omega(\sqrt{d})$  can possibly hold over  $\mathbb{C}$  for the same family of polynomials, at least for arbitrary distinct  $x_i$ 's and arbitrary nonzero  $\alpha_i$ . Such a  $\Omega(\sqrt{d})$  lower bound was recently obtained for the more general model of sums of power of bounded degree polynomials: see Theorem 2 in [11].

We leave it as an open problem to close this quadratic gap between lower bounds over  $\mathbb{R}$  and  $\mathbb{C}$ : find an explicit polynomial  $f \in \mathbb{C}[X]$  of degree  $d$  which requires at least  $k = \Omega(d)$  terms to be represented under the form

$$f(x) = \sum_{i=1}^k \alpha_i (x + x_i)^{e_i}.$$

With the additional requirements  $e_i \leq d$  for all  $i$ , the ‘‘target polynomial’’  $H_2(x) = (x+1)^{d+1} - x^{d+1}$  from Theorem 15 looks like a plausible candidate.

## Acknowledgments

P.K. acknowledges useful discussions with member of the Aric research group in the initial stages of this work.

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## Appendix: a worked out example

We illustrate the proof method of Theorem 13 (more than the result itself) on a small example: we show with this method that there is no identity of the form

$$\alpha_1 x^5 + \alpha_2 (x+1)^5 + \alpha_3 (x+3)^5 = \beta_1 x^2 + \beta_2 (x+1) + \beta_3 (x+3)^2 \quad (13)$$

except if the coefficients  $\alpha_i, \beta_i \in \mathbb{R}$  are all 0. The corresponding interpolation problem is:

$$g(0) = 0, \quad g(1) = 0, \quad g(3) = 0, \quad g^{(3)}(0) = 0, \quad g^{(4)}(1) = 0, \quad g^{(3)}(3) = 0 \quad (14)$$

where  $g \in \mathbb{R}_5[X]$ . The set of knots is  $X = \{x_1, x_2, x_3\} = \{0, 1, 3\}$  and the interpolation matrix is

$$E = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

This matrix is not order regular. Indeed, the pair  $(E, Y)$  where  $Y = \{-1, 0, 1\}$  is not regular. This follows from the identity

$$(x+1)^2 - (x-1)^2 - 4x = 0$$

which was pointed out earlier in the paper. We will nonetheless show that the pair  $(E, X)$  is regular. Toward this, let us split  $E$  in the middle to obtain the two matrices

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The first matrix is regular since all its 1's are in the first column. The second matrix fails to be order regular for the same reason that  $E$  does, but it is easy to check that the interpolation problem  $h(0) = 0, h'(1) = 0, h(3) = 0$  has no nontrivial solution in  $\mathbb{R}_2[X]$ . Hence the pair  $(E_2, X)$  is regular. It follows from Proposition 11 that  $(E, X)$  is a regular pair, and the 6 polynomials in (13) are indeed linearly independent.

We conclude with a remark about Proposition 11. In the proof of this result, we used the fact that the matrix  $A(E, X)$  of the linear system can be transformed by a permutation of rows into a matrix of the form

$$\begin{pmatrix} A(E_1, X) & * \\ 0 & A(E_2, X) \end{pmatrix}.$$

We point out that when the 6 interpolation constraints are listed in the same order as in (14),  $A(E, X)$  is already in this form. In particular, the equations for the last 3 constraints are:

$$g_3 + x_1 g_4 + x_1^2 g_5 / 2 = 0,$$

$$g_4 + x_2 g_5 = 0,$$

$$g_3 + x_3 g_4 + x_3^2 g_5 / 2 = 0$$

and the matrix of this subsystem is indeed  $A(E_2, X)$ . As to  $A(E_1, X)$ , consider for instance the third constraint  $g(x_3) = 0$ . The corresponding equation is  $\sum_{j=0}^5 x_3^j g_j / j! = 0$ . The first 3 coefficients are  $1, x_3, x_3^2/2$  and this is the last row of  $A(E_1, X)$ .