The number of topological generators for full groups of ergodic equivalence relations

François Le Maître

The number of topological generators for full groups of ergodic equivalence relations

François Le Maître*

February 11, 2013

Abstract

We completely elucidate the relationship between two invariants associated with an ergodic probability measure-preserving (pmp) equivalence relation, namely its cost and the minimal number of topological generators of its full group. It follows that for any free pmp ergodic action of the free group on n generators, the minimal number of topological generators for the full group of the action is $n+1$, answering a question of Kechris.

Let Γ be a countable group of measure-preserving automorphisms on a standard probability space (X, μ), in other words a countable subgroup of $\mathrm{Aut}(X, \mu)$. Classical ergodic theory is concerned with the conjugacy class of Γ, especially when $\Gamma = \mathbb{Z}$. A natural invariant is the partition of X into orbits induced by Γ, that is, the orbit equivalence relation $x \,\mathcal{R}_\Gamma\, y$ iff there exists an element $\gamma \in \Gamma$ such that $x = \gamma y$. This probability measure-preserving (pmp) equivalence relation is entirely captured by its full group $[\mathcal{R}_\Gamma]$, defined to be the group of measure-preserving automorphisms φ such that $\varphi(x) \,\mathcal{R}_\Gamma\, x$ for all $x \in X$. So while classical ergodic theory studies the conjugacy class of Γ in $\mathrm{Aut}(X, \mu)$, orbit equivalence theory considers the conjugacy class of the much bigger group $[\mathcal{R}_\Gamma]$ in $\mathrm{Aut}(X, \mu)$.

This motivates the study of $[\mathcal{R}_\Gamma]$ as a topological group in its own right. Let d_u be the bi-invariant complete metric on $\mathrm{Aut}(X, \mu)$ defined by

$$d_u(T, U) = \mu(\{x \in X : T(x) \neq U(x)\}).$$

Then $[\mathcal{R}_\Gamma]$ is closed and separable for this metric, in other words it is a Polish group. Some of its topological properties are relevant to the study of $\mathcal{R}_\Gamma$. For instance, a theorem of Dye [Dye63, prop. 5.1] asserts that closed normal subgroups of $[\mathcal{R}_\Gamma]$ are in a natural bijection with Γ-invariant subsets of X. In particular, $[\mathcal{R}_\Gamma]$ is topologically simple iff $\mathcal{R}_\Gamma$ is ergodic. Another example is given by amenability: Giordano and Pestov showed that $[\mathcal{R}_\Gamma]$ is extremely amenable iff $\mathcal{R}_\Gamma$ is amenable [GP07, thm. 5.7].

If G is a separable topological group, a natural invariant of G is its minimal number of topological generators $t(G)$, that is, the least $n \in \mathbb{N} \cup \{\infty\}$ such that there are $g_1, ..., g_n \in G$ which generate a dense subgroup of G. For full groups of

*Research partially supported by ANR AGORA (ANR-09-BLAN-0059)

pmp equivalence relations, the investigations on this number were started by a question of Kechris [Kec10, 4.(D)]. Kittrell and Tsankov provided a partial answer to this question, using the theory of cost initiated by Levitt [Lev95] and developped by Gaboriau in the seminal article [Gab00]. They showed that if $\mathcal{R}$ is a pmp ergodic equivalence relation, then the cost of $\mathcal{R}$ is finite iff $[\mathcal{R}]$ is topologically finitely generated [KT10, thm. 1.7]. Using a theorem of Matui [Mat06, ex. 6.2], they also showed that for an ergodic $\mathbb{Z}$-action, the associated orbit equivalence relation $\mathcal{R}_{\mathbb{Z}}$ satisfies $2 \leqslant t([\mathcal{R}_{\mathbb{Z}}]) \leqslant 3$, and deduced the explicit bounds

$$\lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1 \leqslant t([\mathcal{R}]) \leqslant 3(\lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1)$$

for an arbitrary pmp ergodic equivalence relation $\mathcal{R}$. These bounds were later refined by Matui [Mat11, thm. 3.2] who showed that the number of topological generators for the full group of an ergodic equivalence relation generated by a $\mathbb{Z}$-action is in fact equal to two, which implies that

$$\lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1 \leqslant t([\mathcal{R}]) \leqslant 2(\lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1).$$

In this paper, we show that the minimal number of topological generators for the full group of $\mathcal{R}$ is actually completely determined by the cost of $\mathcal{R}$.

Theorem 1. *Let $\mathcal{R}$ be a pmp ergodic equivalence relation. Then the minimal number $t([\mathcal{R}])$ of topological generators for the full group of $\mathcal{R}$ is related to the integer part of the cost of $\mathcal{R}$ by the formula*

$$t([\mathcal{R}]) = \lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1.$$

Using a theorem of Gaboriau (cf. theorem 4), we get the following corollary:

Corollary 2. *Let $\mathbb{F}_n \curvearrowright (X, \mu)$ be a free pmp ergodic action. Then the minimal number of topological generators for the full group of this action is $n+1$.*

Moreover, if $\mathcal{R}$ is an ergodic pmp equivalence relation, one can fully recover the cost of $\mathcal{R}$ as the infimum over topological generators of $[\mathcal{R}]$ of the sums of the measures of their supports (cf. theorem 10).

Note that in 2011 Marks showed that pmp modular actions of $\mathbb{F}_n$ have a full group with $n+1$ topological generators (cf. corrections and updates of [Kec10]). The question of the number of topological generators for non ergodic aperiodic equivalence relations will be treated in a forthcoming paper, where it will be proved for instance that for free pmp actions of $\mathbb{F}_n$, one still has $t([\mathcal{R}_{\mathbb{F}_n}]) = n+1$.

The paper is organised as follows: section 1 contains standard definitions and notations used throughout the paper as well as a rough review of the orbit equivalence theory we need, section 2 introduces the notion of pre-p-cycle, which is then used to prove theorem 1 in section 3.

1 Definitions and notations

Everything will be understood "modulo sets of measure zero". We now review some standard notation and definitions.

If (X, μ) is a standard probability space, and A, B are Borel subsets of X, a **partial isomorphism** of (X, μ) of **domain** A and **range** B is a Borel bijection $f : A \to B$ which is measure-preserving for the measures induced by μ on A and B respectively. We denote by $\operatorname{dom} f = A$ its domain, and by $\operatorname{rng} f = B$ its range. Note that in particular, $\mu(\operatorname{dom} f) = \mu(\operatorname{rng} f)$. A **graphing** is a countable set of partial isomorphisms of (X, μ), denoted by $\Phi = \{\varphi_1, ..., \varphi_k, ...\}$ where the φ_k's are partial isomorphisms. It **generates** a **measure-preserving equivalence relation** $\mathcal{R}_\Phi$, defined to be the smallest equivalence relation containing $(x, \varphi(x))$ for every $\varphi \in \Phi$ and $x \in \operatorname{dom} \varphi$. The **cost** of a graphing Φ is the sum of the measures of the domains of the partial isomorphisms it contains. The **cost** of a measure-preserving equivalence relation $\mathcal{R}$ is the infimum of the costs of the graphings that generate it, we denote it by $\operatorname{Cost}(\mathcal{R})$. The **full group** of $\mathcal{R}$ is the group $[\mathcal{R}]$ of automorphisms of (X, μ) which induce permutations in the $\mathcal{R}$-classes, that is

$$[\mathcal{R}] = \{\varphi \in \operatorname{Aut}(X, \mu) : \forall x \in X, \varphi(x) \,\mathcal{R}\, x\}.$$

It is a separable group when equipped with the complete metric d_u defined by

$$d_u(T, U) = \mu(\{x \in X : T(x) \neq U(x)\}.$$

One also defines the **pseudo full group** of $\mathcal{R}$, denoted by $[[\mathcal{R}]]$, which consists of all partial isomorphisms φ such that $\varphi(x) \,\mathcal{R}\, x$ for all $x \in \operatorname{dom} \varphi$.

Say that a measure-preserving equivalence relation $\mathcal{R}$ is **ergodic** when every Borel $\mathcal{R}$-saturated set has measure 0 or 1. We will use without mention the following standard fact about ergodic measure-preserving equivalence relations:

Proposition 3 (see e.g. [KM04], lemma 7.10.)**.** *Let $\mathcal{R}$ be an ergodic measure-preserving equivalence relation on (X, μ), let A and B be two Borel subsets of X such that $\mu(A) = \mu(B)$. Then there exists $\varphi \in [[\mathcal{R}]]$ of domain A and range B.*

We now give a very sparse outline of orbit equivalence theory (for a survey, see [Gab10]). For ergodic $\mathbb{Z}$-actions, it turns out that there is only one orbit equivalence relation: a theorem of Dye [Dye59, thm. 5] states that if (X, μ) is a standard probability space and T, U are ergodic automorphisms of (X, μ) then their full groups are conjugated by an automorphism of (X, μ). This was later generalized by Ornstein and Weiss [OW80] who proved that all amenable groups induce the same ergodic orbit equivalence relation.

Actually, inducing only one orbit equivalence relation characterizes amenable groups: if Γ is a non amenable group, then it has continuum many non orbit equivalent actions, as was shown by Epstein [Eps08], building on earlier work by Gaboriau, Lyons and Ioana ([GL09] and [Ioa11]).

Last but not least, Gaboriau has computed the cost for free pmp actions of free groups, and deduced that two free pmp actions of $\mathbb{F}_n$ and $\mathbb{F}_m$ cannot be orbit equivalent if $n \neq m$.

Theorem 4 ([Gab00], corollaire 1)**.** *If $\mathbb{F}_n \curvearrowright (X,\mu)$ is a free measure-preserving action, then the associated orbit equivalence relation has cost n.*

The proof of our theorem will rely on a result of Matui which gives two topological generators for the full group of some ergodic $\mathbb{Z}$-action. His construction actually yields the following stronger statement.

Theorem 5 ([Mat11], theorem 3.2)**.** *For every $\epsilon > 0$, there exists an ergodic automorphism T of (X,μ) and an involution $U \in [\mathcal{R}_T]$ with support of measure less than ϵ such that*

$$\overline{\langle T, U\rangle} = [\mathcal{R}_T].$$

We also borrow the following result from Kittrell and Tsankov.

Theorem 6 ([KT10], theorem 4.7)**.** *Let $\mathcal{R}_1$, $\mathcal{R}_2$,... be measure-preserving equivalence relations on (X,μ), and let $\mathcal{R}$ be their join (i.e. the smallest equivalence relation containing all of them). Then $\left\langle \bigcup_{n\in\mathbb{N}}[\mathcal{R}_n]\right\rangle$ is dense in $[\mathcal{R}]$.*

2 Pre-p-cycles, p-cycles and associated full groups

Let us give a way to construct pmp equivalence relations whose nontrivial classes have cardinality p.

Definition 7. Let $p \in \mathbb{N}$. A **pre-p-cycle** is a graphing $\Phi = \{\varphi_1, ..., \varphi_{p-1}\}$ such that the following two conditions are satisfied:

(i) $\forall i \in \{1, ..., p-1\}, \operatorname{rng} \varphi_i = \operatorname{dom} \varphi_{i+1}$.

(ii) The following sets are all disjoint:

$$\operatorname{dom} \varphi_1, \operatorname{dom} \varphi_2, ..., \operatorname{dom} \varphi_{p-1}, \operatorname{rng} \varphi_{p-1}.$$

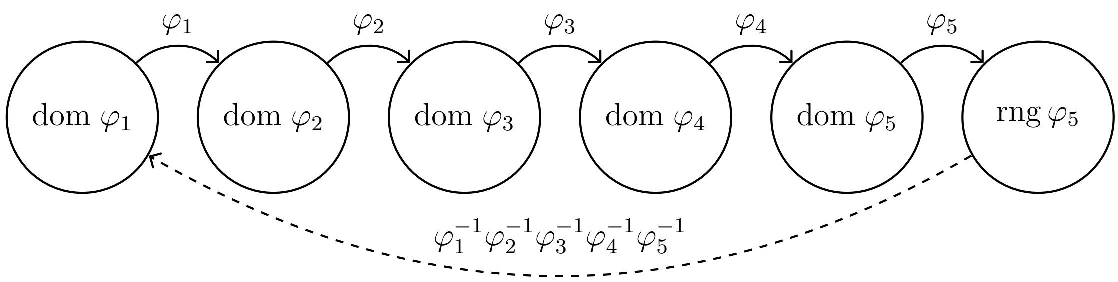

Figure 1: A pre-6-cycle $\Phi = \{\varphi_1, ..., \varphi_5\}$. The dashed arrow shows how to get the corresponding 6-cycle C_Φ (cf. the paragraph following definition 8).

Remark. Given a graphing Φ, the fact that it is a pre-p-cycle is witnessed by a unique enumeration of its elements. When we have a pre-p-cycle Φ and write it as $\Phi = \{\varphi_1, ..., \varphi_{p-1}\}$, we will always assume that conditions (i) and (ii) above are satisfied.

Definition 8. A p**-cycle** is an element $C \in \mathrm{Aut}(X, \mu)$ whose orbits have cardinality 1 or p.

Given a pre-p-cycle $\Phi = \{\varphi_1, ..., \varphi_{p-1}\}$, we define a p-cycle $C_\Phi \in \mathrm{Aut}(X, \mu)$ as follows:

$$C_\Phi(x) = \begin{cases} \varphi_i(x) & \text{if } x \in \mathrm{dom}\,\varphi_i \text{ for some } i < p, \\ \varphi_1^{-1}\varphi_2^{-1} \cdots \varphi_{p-1}^{-1}(x) & \text{if } x \in \mathrm{rng}\,\varphi_{p-1}, \\ x & \text{otherwise.} \end{cases}$$

Note that $\{C_\Phi\}$ generates the pmp equivalence relation $\mathcal{R}_\Phi$, whose classes have cardinality either 1 or p. We now give a useful generating set for the full group of $\mathcal{R}_\Phi$.

Proposition 9. *If $\Phi = \{\varphi_1, ..., \varphi_{p-1}\}$ is a pre-p-cycle, then for all $i \in \{1, ..., p-1\}$, the full group of $\mathcal{R}_\Phi$ is topologically generated by $[\mathcal{R}_{\{\varphi_i\}}] \cup \{C_\Phi\}$.*

Proof. By definition, the equivalence relation $\mathcal{R}_\Phi$ is the join of the equivalence relations $\mathcal{R}_{\{\varphi_1\}}, ..., \mathcal{R}_{\{\varphi_{p-1}\}}$. Then by theorem 6 we only need to show that for all $j \in \{1, .., p-1\}$, $[\mathcal{R}_{\{\varphi_j\}}]$ is contained in the group generated by $[\mathcal{R}_{\{\varphi_i\}}] \cup \{C_\Phi\}$. But for all $j \in \{1, ..., p-2\}$, we have the conjugation relation

$$C_\Phi \varphi_j C_\Phi^{-1} = \varphi_{j+1},$$

which in turn yields $C_\Phi[\mathcal{R}_{\{\varphi_j\}}]C_\Phi^{-1} = [\mathcal{R}_{\{\varphi_{j+1}\}}]$. So the group generated by $[\mathcal{R}_{\{\varphi_i\}}] \cup \{C_\Phi\}$ contains all $[\mathcal{R}_{\{\varphi_j\}}]$ for $1 \leqslant j \leqslant p-1$. □

3 Proof of the theorem

We now get to the proof of our main result. Fix an ergodic measure-preserving equivalence relation $\mathcal{R}$ on (X, μ).

Let us first show that $t([\mathcal{R}]) \geqslant \lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1$, following an argument of Miller [KT10, cor. 4.12]. This inequality is equivalent to $t([\mathcal{R}]) > \mathrm{Cost}(\mathcal{R})$. Observe that topological generators of $[\mathcal{R}]$ must also generate the equivalence relation $\mathcal{R}$, so the definition of cost yields $t([\mathcal{R}]) \geqslant \mathrm{Cost}(\mathcal{R})$. Assume, towards a contradiction, that $t([\mathcal{R}]) = \mathrm{Cost}(\mathcal{R})$ and fix $\mathrm{Cost}(\mathcal{R}) = n$ topological generators of $[\mathcal{R}]$. Then again, these elements also generate $\mathcal{R}$ and thus realize the cost of $\mathcal{R}$. By proposition I.11 in [Gab00], they induce a free action of $\mathbb{F}_n$ on (X, μ). In particular, the countable group they generate is discrete in $[\mathcal{R}]$, hence closed, which contradicts its density.

The other inequality requires more work. Suppose that for a fixed $n \in \mathbb{N}$, $\mathrm{Cost}\,\mathcal{R} < n+1$. We want to find $n+1$ topological generators of $[\mathcal{R}]$. It is a standard fact that we can find a pmp ergodic equivalence relation $\mathcal{R}_0 \subseteq \mathcal{R}$ generated by a single automorphism (cf. for instance [Kec10, prop. 3.5]).

Let Φ_0 be a graphing of cost 1 which generates $\mathcal{R}_0$. Lemma III.5 in [Gab00] provides a graphing Φ such that $\mathrm{Cost}(\Phi) < n$ and $\Phi_0 \cup \Phi$ generates $\mathcal{R}$. Let

$$c = \frac{\mathrm{Cost}(\Phi)}{n} < 1,$$

and fix some odd $p \in \mathbb{N}$ such that $(\frac{p+2}{p})c < 1$. Splitting the domains of the partial automorphisms in Φ, we find Φ_1,...,Φ_n of cost c such that $\Phi = \Phi_1 \cup \cdots \cup \Phi_n$.

By theorem 5 for $\epsilon = 1 - (1 + \frac{p}{2})c$, there is an ergodic automorphism T_0 of (X, μ) and an involution $U_0 \in [\mathcal{R}_{T_0}]$, whose support has measure less than ϵ, such that T_0 and U_0 topologically generate $[\mathcal{R}_{T_0}]$. Dye's theorem [Dye59, thm. 5] yields $\varphi \in \mathrm{Aut}(X, \mu)$ which conjugates $[\mathcal{R}_{T_0}]$ and $[\mathcal{R}_0]$, so up to conjugation by φ we can assume that $\mathcal{R}_{T_0} = \mathcal{R}_0$.

Let $A_1, ..., A_{p+2}$ be disjoint subsets of $X \setminus \operatorname{supp} U_0$, of measure $\frac{c}{p}$ each. Recall proposition 3, and note that up to pre- and post-composition of the partial isomorphisms of each Φ_i by elements in $[[\mathcal{R}_0]]$, we can assume that each Φ_i is a pre-$(p+1)$-cycle $\Phi_i = \{\varphi_1^i, \varphi_2^i, ..., \varphi_p^i\}$ such that $\varphi_j^i : A_j \to A_{j+1}$.

Now choose $\psi \in [[\mathcal{R}_0]]$ with domain A_{p+1} and range A_{p+2}, and add it to every Φ_i. We get n pre-$(p+2)$-cycles $\tilde{\Phi}_i = \Phi_i \cup \{\psi\}$, and $\Phi_0 \cup \tilde{\Phi}_1 \cup \cdots \cup \tilde{\Phi}_n$ still generates $\mathcal{R}$. Consider the associated $(p+2)$-cycles $C_{\tilde{\Phi}_i}$.

Claim. The $(n+2)$ elements $T_0, U_0, C_{\tilde{\Phi}_1}, ..., C_{\tilde{\Phi}_n}$ topologically generate the full group of $\mathcal{R}$.

Proof. Let G be the closed group generated by $T_0, U_0, C_{\tilde{\Phi}_1}, ..., C_{\tilde{\Phi}_n}$. Recall that T_0 and U_0 have been chosen so that they topologically generate the full group of $\mathcal{R}_0$, so G contains $[\mathcal{R}_0]$. Because ψ is a partial isomorphism of $\mathcal{R}_0$, we get $[\mathcal{R}_{\{\psi\}}] \subseteq [\mathcal{R}_0] \subseteq G$. Then, since for all $i \in \{1, ..., n\}$ we have $\psi \in \tilde{\Phi}_i$ and $C_{\tilde{\Phi}_i} \in G$, proposition 9 implies that G contains $[\mathcal{R}_{\tilde{\Phi}_i}]$. But $\mathcal{R}$ is the join of $\mathcal{R}_0$, $\mathcal{R}_{\tilde{\Phi}_1}, ..., \mathcal{R}_{\tilde{\Phi}_n}$, so by theorem 6 we are done. □

Then observe that the involution U_0 and the $(p+2)$-cycle $C_{\tilde{\Phi}_1}$ have disjoint support, so they commute and the product $U_1 = U_0 C_{\tilde{\Phi}_1}$ satisfies

$$(U_0 C_{\tilde{\Phi}_1})^n = U_0^n C_{\tilde{\Phi}_1}^n \qquad \forall n \in \mathbb{N}.$$

In particular,

$$\begin{aligned} U_1^{p+2} &= (U_0 C_{\tilde{\Phi}_1})^{p+2} = U_0^{p+2} = U_0 \qquad \text{since } p \text{ is odd and} \\ U_1^{p+3} &= (U_0 C_{\tilde{\Phi}_1})^{p+3} = U_0^{p+3} C_{\tilde{\Phi}_1}^{p+3} = C_{\tilde{\Phi}_1}, \end{aligned}$$

so that in fact $T_0, U_1, C_{\tilde{\Phi}_2}, ..., C_{\tilde{\Phi}_n}$ topologically generate $[\mathcal{R}]$.

Note that the construction yields the following statement, which says that the cost is an invariant of the metric group $([\mathcal{R}], d_u)$.

Theorem 10. *Let $\mathcal{R}$ be a pmp ergodic equivalence relation. Then we have the formula*

$$\mathrm{Cost}(\mathcal{R}) = \inf \left\{ \sum_{i=1}^{t([\mathcal{R}])} d_u(T_i, \mathrm{id}) : \overline{\langle T_1, ..., T_{t([\mathcal{R}])} \rangle} = [\mathcal{R}] \right\}.$$

Acknowledgments. I am very grateful to Damien Gaboriau and Julien Melleray for their constant help and support. Theorem 1 would certainly have read

$$t([\mathcal{R}]) \leqslant \lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 2$$

without the enlightening conversations I have had with Damien, and the proof would have been quite different. Finally, I thank Damien, Julien and Adriane Kaïchouh for proofreading an undisclosed amount of preliminary versions of this article.

References

[Dye59] H. A. Dye. On groups of measure preserving transformation. I. *Amer. J. Math.*, 81:119–159, 1959.

[Dye63] H. A. Dye. On groups of measure preserving transformations. II. *Amer. J. Math.*, 85:551–576, 1963.

[Eps08] Inessa Epstein. *Some results on orbit inequivalent actions of non-amenable groups*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–University of California, Los Angeles.

[Gab00] Damien Gaboriau. Coût des relations d'équivalence et des groupes. *Invent. Math.*, 139(1):41–98, 2000.

[Gab10] Damien Gaboriau. Orbit equivalence and measured group theory. In *Proceedings of the International Congress of Mathematicians. Volume III*, pages 1501–1527, New Delhi, 2010. Hindustan Book Agency.

[GL09] Damien Gaboriau and Russell Lyons. A measurable-group-theoretic solution to von Neumann's problem. *Invent. Math.*, 177(3):533–540, 2009.

[GP07] Thierry Giordano and Vladimir Pestov. Some extremely amenable groups related to operator algebras and ergodic theory. *J. Inst. Math. Jussieu*, 6(2):279–315, 2007.

[Ioa11] Adrian Ioana. Orbit inequivalent actions for groups containing a copy of $\mathbb{F}_2$. *Invent. Math.*, 185(1):55–73, 2011.

[Kec10] Alexander S. Kechris. *Global aspects of ergodic group actions*, volume 160 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[KM04] Alexander S. Kechris and Benjamin D. Miller. *Topics in orbit equivalence*, volume 1852 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.

[KT10] John Kittrell and Todor Tsankov. Topological properties of full groups. *Ergodic Theory Dynam. Systems*, 30(2):525–545, 2010.

[Lev95] Gilbert Levitt. On the cost of generating an equivalence relation. *Ergodic Theory Dynam. Systems*, 15(6):1173–1181, 1995.

[Mat06] Hiroki Matui. Some remarks on topological full groups of Cantor minimal systems. *Internat. J. Math.*, 17(2):231–251, 2006.

[Mat11] Hiroki Matui. Some remarks on topological full groups of Cantor minimal systems II. *to appear in Ergodic Theory Dynam. Systems*, 2011.

[OW80] Donald S. Ornstein and Benjamin Weiss. Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc. (N.S.)*, 2(1):161–164, 1980.