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A note on ergodicity and indistinguishability

Sébastien Martineau

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Abstract

The purpose of this note is to explore the link between the ergodicity of the cluster equivalence relation restricted to the infinite locus and the indistinguishability of infinite clusters. It is an important element of the dictionary connecting orbit equivalence and percolation theory. This paper starts with a short exposition (as self-contained as possible) of some standard material of these theories. Then, the classical correspondence between ergodicity and indistinguishability is presented. Finally, we introduce a notion of strong indistinguishability that corresponds to strong ergodicity, and obtain that this strong indistinguishability holds in the Bernoulli case.

Introduction

Orbit equivalence is a branch of ergodic theory that focuses on the dynamical properties of equivalence relations. Its fruitful interactions with other mathematical fields are numerous: operator algebra theory, foliation theory, descriptive set theory... Among the many concepts of the field, a fundamental one is the notion of *ergodicity*: an equivalence relation defined on a probability space is ergodic if every saturated set has measure 0 or 1. It is striking to see how a definition that is usually given in the group action context can be easily stated in the seemingly static framework of equivalence relations.

The other fundamental notion considered in this note, *indistinguishability*, belongs to percolation theory, a branch of statistical physics. Percolation is concerned with the study of random subgraphs of a given graph. These subgraphs are generally far from connected, and one is naturally interested in its infinite connected components (or infinite clusters). A difficult theorem due to Lyons and Schramm ([18]) states that, under some hypotheses, if several infinite clusters are produced, they all "look alike". This is the indistinguishability theorem (theorem 2.9 in this note).

Its connection to some form of ergodicity should not be surprising: in both cases, when one asks a (nice) question, all the objects — in one case, the points of the space lying under the relation, in the other one, the infinite

clusters — give the same answer. This connection is well-understood; see [10], [11] and proposition 3.1.

Besides, in the orbit equivalence world, a hard theorem due to Chifan and Ioana implies that, *in certain cases*, ergodicity is equivalent to a *stronger* form of ergodicity; see [7] and theorem 1.11.

The purpose of this note is the following. We have seen that indistinguishability holds, and that it is equivalent to some ergodicity; besides, using Chifan-Ioana theorem, one can, in the correct context, strengthen this ergodicity. We will define a notion of strong indistinguishability and prove its equivalence to strong ergodicity; this is theorem 3.2. As a byproduct, one will obtain strong indistinguishability for percolations where Lyons-Schramm theorem and Chifan-Ioana theorem can both be applied, which includes Bernoulli percolation. This is corollary 3.3.

Since this work lies at the interface between percolation and orbit equivalence theories, I have made this note self-contained, so that the orbit equivalence part can be read without prerequisite by a percolation theorist and vice versa. The first section presents what will be needed of orbit equivalence theory. The second one deals with percolation theory. The third and last section recalls the classic correspondence between ergodicity and indistinguishability and explores the correspondence between strong ergodicity and the notion of strong indistinguishability defined in this note.

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Terminology

If R is an equivalence relation defined on a set X, the *R-class* of x is

$$[x]_R := \{y \in X : xRy\}$$

A subset A of X is said to be *R-saturated*, or *R-invariant*, if

$$\forall x \in A, [x]_R \subset A$$

The *R-saturation* of a subset A of X is the smallest subset R-saturated subset of X that contains A. Concretely, it is $\bigcup_{x\in A} [x]_R$.

1 Orbit equivalence theory

This section presents standard definitions and theorems from orbit equivalence theory (except for lemma 1.13). For details relative to subsection 1.0, one can refer to [14]. For subsequent parts of this section, a possible reference is [15].

1.0 Generalities on the standard Borel space

A measurable space X is called a *standard Borel space* if it can be endowed with a Polish topology inducing its σ-algebra. For instance, $\{0,1\}^{\mathbb{N}}$ endowed with the product σ-algebra is a standard Borel space. A measurable subset of a standard Borel space is called a *Borel subset*.

The following general results on standard Borel spaces will be used without explicit mention.

THEOREM 1.1. *Any Borel subset of a standard Borel space is itself a standard Borel space.*

Let X and Y be two measurable spaces. A bijection $f : X \to Y$ is a *Borel isomorphism* if f and f^{-1} are measurable. If $X = Y$, we speak of *Borel automorphism*.

THEOREM 1.2. *Let X and Y be standard Borel spaces. If $f : X \to Y$ is a measurable bijection, then f^{-1} is automatically measurable, hence a Borel isomorphism.*

THEOREM 1.3. *Every non-countable standard Borel space is isomorphic to $[0,1]$. (In particular, we have a form of continuum hypothesis for standard Borel spaces.)*

1.1 Countable Borel equivalence relations

Let Γ be a countable group and $\Gamma \curvearrowright X$ be a Borel action of it on a standard Borel space. By *Borel action*, we mean that every $\gamma \in \Gamma$ induces a Borel automorphism of X. Such an action induces a partition of X into orbits. Let us consider R (or $R_{\Gamma \curvearrowright X}$) the relation "being in the same orbit" and call it the *orbit equivalence relation* of $\Gamma \curvearrowright X$. It is a subset of X^2. Since Γ is countable, the following assertions hold:

- R is *countable*, i.e. every R-class is (finite or) countable ;
- R is Borel, as a subset of X^2.

The following theorem provides a reciprocal:

THEOREM 1.4 (FELDMAN-MOORE, [8]). *Every countable Borel equivalence relation on a standard Borel space is induced by a Borel action of some countable group.*

In other words, every countable Borel equivalence relation on a standard Borel space is an orbit equivalence relation. This is why the theory of "countable Borel equivalence relations" is called "orbit equivalence theory".

1.2 Measure invariance

When dealing with a Borel action of Γ on a probability space, it makes sense to speak of invariance of the probability measure. The purpose of this subsection is to define this notion for countable Borel equivalence relations. To begin with, one needs to know how the standard Borel space behaves when it is endowed with a probability measure.

DEFINITION. A *standard probability space* is a standard Borel space endowed with a probability measure.

THEOREM 1.5. *Every atomless standard probability space (X, μ) is isomorphic to $[0, 1]$ endowed with its Borel σ-algebra and the Lebesgue measure, i.e. there exists a measure-preserving Borel isomorphism between (X, μ) and $([0, 1], \mathrm{d}x)$.*

THROUGHOUT THIS PAPER, STANDARD PROBABILITY SPACES WILL IMPLICITLY BE ASSUMED ATOMLESS.

Having a nice measured space to work on is not enough to provide a notion of invariance of the measure; to do so, one needs relevant transformations, presented below.

DEFINITION. If R is a countable Borel equivalence relation, $[R]$ is the group of the Borel automorphisms of X whose graph is included in R. A *partial Borel automorphism* of X is a Borel isomorphism between two Borel subsets of X. One denotes by $[[R]]$ the set of partial Borel automorphisms whose graph is included in R.

Remark. In the literature, X is often equipped with a "nice" probability measure[1] and the notations $[R]$ and $[[R]]$ are also used to denote the objects defined above, quotiented out by almost everywhere agreement. In this paper, we will stick to the definition we gave, that can be found in [15].

Remark. As exemplified by the theorem below, these Borel automorphisms allow us to mimic intrinsically the "group action" definitions in the "orbit equivalence" setting.

[1] Here, "nice" means "R-invariant", which will be defined using $[R]$ (as defined above).

THEOREM 1.6. *Let R be a countable Borel equivalence relation on a standard probability space (X, μ). The following assertions are equivalent:*

- *there exist Γ a countable group and $\Gamma \curvearrowright X$ a measure-preserving Borel action of it such that $R = R_{\Gamma \curvearrowright X}$;*
- *every Borel action of a countable group that induces R preserves μ;*
- *every element of $[R]$ preserves μ;*
- *every element of $[[R]]$ preserves μ.*

When any of these equivalent properties is satisfied, we say that the measure μ is preserved *by R, or that it is R-*invariant.

HENCEFORTH, X WILL ALWAYS BE AN ATOMLESS STANDARD PROBABILITY SPACE AND THE EQUIVALENCE RELATIONS WE WILL CONSIDER ON X WILL ALWAYS BE MEASURE-PRESERVING COUNTABLE BOREL EQUIVALENCE RELATIONS.

Remark. There is no uniqueness theorem (analogous to 1.3 or 1.5) for the object (X, μ, R); this is why orbit equivalence theory is not empty. Another fact to keep in mind is that the space X/R essentially never bears a natural standard Borel structure (even though R is Borel).

1.3 Amenability and hyperfiniteness

Amenability of a group can be defined in many equivalent ways; for our purpose, the following characterization will be enough.

THEOREM 1.7. *A countable group Γ is amenable if and only if there exists a Reiter sequence, i.e. $f_n \in \ell^1(\Gamma)$ such that:*

- $\forall n, f_n \geq 0$ *and* $\|f_n\|_1 = 1$;
- $\forall \gamma \in \Gamma, \|f_n - \gamma \cdot f_n\|_1 \underset{n \to \infty}{\longrightarrow} 0$.

In the theorem above, Γ acts on $\ell^1(\Gamma)$ via $\gamma \cdot f(\eta) := f(\gamma^{-1}\eta)$. Taking the inverse of γ guarantees that this defines a left action. Besides, the action it induces on indicator functions corresponds to the natural action $\Gamma \curvearrowright \text{Subsets}(\Gamma)$, i.e. we have $\gamma \cdot 1_A = 1_{\gamma A}$.

This theorem in mind, the following definition of the amenability of an equivalence relation is natural.

DEFINITION. Let R be a countable Borel equivalence relation on (X, μ). One says that R is *μ-amenable* if and only if there exists a sequence of *Borel* functions $f_n : R \to \mathbb{R}^+$ such that, if we set $f_{n,x} : y \mapsto f_n(x, y)$,

- $\forall x \in X,\ \sum_{y \in [x]_R} f_{n,x}(y) = 1$;
- there exists a full-measure R-invariant Borel subset $A \subset X$ such that

$$\forall (x, y) \in (A \times A) \cap R, \|f_{n,x} - f_{n,y}\|_1 \underset{n \to \infty}{\longrightarrow} 0$$

Comment. In the definition above (and in others), one can indifferently impose A to be R-invariant or not. Indeed, it can be deduced from theorem 1.4 that the R-saturation of a μ-negligible set is still μ-negligible. (Recall that all considered equivalence relations are tacitly assumed to preserve the measure.)

Proposition 1.8 shows that this definition is a nice extension of the classic notion of amenability (for countable groups) to equivalence relations.

NOTATION. Let $\Gamma \curvearrowright X$ be a Borel action of a countable group on a standard Borel space. If X is endowed with an atomless probability measure μ that is Γ-invariant, we will write $\Gamma \curvearrowright (X, \mu)$.

PROPOSITION 1.8. *Let $\Gamma \curvearrowright (X, \mu)$ be a measure-preserving action. If Γ is amenable, then $R_{\Gamma \curvearrowright X}$ is μ-amenable. Besides, if $\Gamma \curvearrowright X$ is free, then the converse holds.*

It is easy to see that *finite* equivalence relations (i.e. whose classes are finite) are amenable: one just needs to set $f_{n,x}(y) = \frac{1}{|[x]_R|}$. The proof naturally extends to hyperfinite equivalence relations, defined below.

DEFINITION. An equivalence relation R on a standard Borel space X is said to be *hyperfinite* if it is a countable increasing union of finite Borel equivalence subrelations. (No measure appears in this definition.) If μ is an R-invariant probability measure on X, R is *hyperfinite μ-almost everywhere* if there exists a full-measure Borel subset $A \subset X$ such that $R \cap (A \times A)$ is hyperfinite.

Example. The group $\Gamma_\infty := \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ is the increasing union of the subgroups $\Gamma_N := \bigoplus_{n \leq N} \mathbb{Z}/2\mathbb{Z}$. Hence, any $R_{\Gamma_\infty \curvearrowright X}$ is hyperfinite. Besides, Γ_∞ is amenable: set $f_n = \frac{1_{\Gamma_n}}{|\Gamma_n|}$. Hence, any $R_{\Gamma_\infty \curvearrowright (X,\mu)}$ is μ-amenable.

THEOREM 1.9 (CONNES-FELDMAN-WEISS, [6]). *Let R be a Borel countable equivalence relation on (X, μ). It is μ-amenable if and only if it is hyperfinite μ-a.e. In other words, μ-amenability and "hyperfiniteness μ-a.e." are the same notion.*

1.4 Ergodicity and strong ergodicity

1.4.1 Ergodicity

DEFINITION. Let $\Gamma \curvearrowright (X, \mu)$ be a measure-preserving action. It is said to be *ergodic* if, for every Γ-invariant Borel subset B of X, either $\mu(B) = 0$ or $\mu(B) = 1$.

DEFINITION. An equivalence relation R on a standard probability space (X, μ) is said to be *ergodic* (or *μ-ergodic*) if, for every R-invariant Borel subset B of X, either $\mu(B) = 0$ or $\mu(B) = 1$.

Remark. Let $\Gamma \curvearrowright (X, \mu)$ be a measure-preserving group action. Let A be a subset of X. Notice that it is the same for A to be Γ-invariant or $R_{\Gamma \curvearrowright X}$-invariant. This means that the following assertions are equivalent:

- $\forall \gamma \in \Gamma, \gamma \cdot A = A$
- $\forall x \in A, \forall y \in X, x R_{\Gamma \curvearrowright X} y \implies y \in A$

In particular, $\Gamma \curvearrowright X$ is ergodic if and only if $R_{\Gamma \curvearrowright X}$ is ergodic.

The Bernoulli example. Let Γ be an infinite countable group and (Σ, ν) denote either $([0,1], \text{Leb})$ or $(\{0,1\}, \text{Ber}(p)) = (\{0,1\}, (1-p)\delta_0 + p\delta_1)$. Let A denote either Γ or the edge-set of a Cayley graph of Γ. (The definition of Cayley graphs is recalled in subsection 2.1.) Let $\mathcal{S}$ be the equivalence relation induced by the shift action of Γ on $\left(\Sigma^A, \nu^{\otimes A}\right)$ defined by

$$\gamma \cdot (\sigma_a)_{a \in A} = (\sigma_{\gamma^{-1} a})_{a \in A}$$

This equivalence relation preserves $\nu^{\otimes A}$ and is ergodic.

The following theorem states that the amenable world shrinks to a point from the orbital point of view.

THEOREM 1.10 (DYE). *Every countable Borel equivalence relation that is ergodic and hyperfinite μ-a.e. is isomorphic to the orbit equivalence relation of $\left(\mathbb{Z} \curvearrowright \left(\{0,1\}^{\mathbb{Z}}, \text{Ber}(1/2)^{\otimes \mathbb{Z}}\right)\right)$. This means that if R is such a relation on a standard probability space (X, μ), there exist*

- *a full-measure R-invariant Borel subset A of X,*
- *a full-measure $\mathbb{Z}$-invariant Borel subset B of $\{0,1\}^{\mathbb{Z}}$,*
- *a measure-preserving Borel isomorphism $f : A \to B$*

such that $\forall x, y \in A, xRy \Leftrightarrow f(x) R_{\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}} f(y)$.

1.4.2 Strong ergodicity

The notion of strong ergodicity, presented in this paragraph, is due to Schmidt ([19]).

DEFINITION. If $\Gamma \curvearrowright (X, \mu)$ and if (B_n) is a sequence of Borel subsets of X, it is *asymptotically Γ-invariant (with respect to μ)* if

$$\forall \gamma \in \Gamma,\ \mu((\gamma \cdot B_n) \triangle B_n) \underset{n\to\infty}{\longrightarrow} 0.$$

The action $\Gamma \curvearrowright (X, \mu)$ is said to be *strongly ergodic* if, for every asymptotically Γ-invariant sequence of Borel sets (B_n),

$$\mu(B_n)(1 - \mu(B_n)) \underset{n\to\infty}{\longrightarrow} 0$$

Making use of $[R]$, one can extend this notion to equivalence relations.

DEFINITION. Let R be an equivalence relation on a standard probability space (X, μ). A sequence (B_n) of Borel subsets of X is said to be *asymptotically R-invariant (with respect to μ)* if

$$\forall \phi \in [R], \mu(\phi(B_n) \triangle B_n) \underset{n\to\infty}{\longrightarrow} 0$$

DEFINITION. The equivalence relation R is said to be *strongly ergodic* if, for every asymptotically R-invariant sequence of Borel sets (B_n),

$$\mu(B_n)(1 - \mu(B_n)) \underset{n\to\infty}{\longrightarrow} 0$$

Remark. It can be checked that, if $\Gamma \curvearrowright (X, \mu)$ is a measure-preserving action, (B_n) is asymptotically Γ-invariant (in the "group action" sense) if and only if it is asymptotically $R_{\Gamma \curvearrowright X}$-invariant (in the "relation" sense). In particular, $\Gamma \curvearrowright (X, \mu)$ is strongly ergodic if and only if $R_{\Gamma \curvearrowright X}$ is strongly ergodic.

Remark. It is clear that strong ergodicity implies ergodicity (if B is invariant, set $B_n := B$ for all n and apply strong ergodicity). What may be less clear is that the converse does not hold. In fact, it is not hard to see that the unique ergodic amenable relation is not strongly ergodic: consider an ergodic measure-preserving action of $\Gamma_\infty := \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ on a standard probability space (X, μ), for example the Bernoulli shift. For $N \in \mathbb{N}$, set as previously $\Gamma_N := \bigoplus_{n\leq N} \mathbb{Z}/2\mathbb{Z}$. Since Γ_N is finite, the restricted action $\Gamma_N \curvearrowright (X, \mu)$ admits a fundamental domain D, that is a Borel subset that intersects each orbit in exactly one point[2]. One can find a Borel subset of D of measure $\frac{\mu(D)}{2}$. Then, define B_N as the $R_{\Gamma_N \curvearrowright X}$-saturation of D. Each B_N has measure $\frac{1}{2}$ and is Γ_M-invariant for $M \leq N$, which ends the proof.

[2]To get such a fundamental domain, one can think of X as $[0, 1]$ and keep a point iff it is the smallest in its orbit for the usual ordering of the interval.

The following theorem will be crucial in section 3 because it allows, under conditions, to deduce strong ergodicity from ergodicity. In its statement, $\mathcal{S}$ stands for the relation introduced in the Bernoulli example of 1.4.1 and (X, μ) for its underlying standard probability space.

THEOREM 1.11 (CHIFAN-IOANA). *Let B be a Borel subset of X and R be a non-amenable ergodic equivalence subrelation of $\left(\mathcal{S}_{|B}, \frac{\mu}{\mu(B)}\right)$. The relation R is automatically strongly ergodic.*

Comment. In fact, [7] proves a lot more. But since we do not need the full result of Chifan and Ioana, we will stick to the stated version.

1.5 Graphings

A *graphing* of a relation R on X is an at most countable family (φ_i) of partial Borel automorphisms of X that generate R as an equivalence relation, i.e. such that the smallest equivalence relation that contains their graphs is R. In particular, the Borel partial automorphisms that appear in a graphing belong to $[[R]]$. The notion of graphing generalizes to relations the notion of generating system.

Notice that the data of a graphing endows each R-class with a structure of connected graph: put an edge from x to x' if there is an i such that x belongs to the domain of φ_i and $x' = \varphi_i(x)$. One can do this with multiplicity.

Example. Let Γ be a finitely generated group and S a finite generating system of Γ. Let $\Gamma \curvearrowright X$ be a Borel action on a standard Borel space. For $s \in S$, let φ_s denote the Borel automorphism implementing the action of s^{-1}. Then, $(\varphi_s)_{s\in S}$ is a graphing of $R_{\Gamma\curvearrowright X}$. Let us take a closer look at the graph structure.

Let $\mathcal{G} = (V, E) = (\Gamma, E)$ denote the Cayley graph of Γ relative to S (see subsection 2.1 for the definition); for this example, we will use the concrete definition of Cayley graphs and take the vertex-set to be Γ. If the action is free, then, for every x, $\gamma \mapsto \gamma^{-1} \cdot x$ is a graph isomorphism between $\mathcal{G}$ and the graphed orbit of x. The only point to check is that the graph structure is preserved. For all $(\gamma, \eta, x) \in \Gamma \times \Gamma \times X$,

$$\begin{aligned}
(\gamma, \eta) \in E &\Leftrightarrow \exists s \in S, \eta = \gamma s \\
&\Leftrightarrow \exists s \in S, \eta^{-1} = s^{-1}\gamma^{-1} \\
&\Leftrightarrow \exists s \in S, \eta^{-1} \cdot x = s^{-1}\gamma^{-1} \cdot x \\
&\Leftrightarrow (\eta^{-1} \cdot x, \gamma^{-1} \cdot x) \text{ is an edge}
\end{aligned}$$

The point in putting all these inverses is that, this way, we only work with Cayley graphs on which the group acts from the left. If the action is not assumed to be free, $\gamma \mapsto \gamma^{-1} \cdot x$ is only a graph-covering.

THEOREM 1.12. *Let R be a countable Borel equivalence relation on X that preserves the atomless probability measure μ. If it admits a graphing such that μ-a.e. x has an orbit that has two ends, seen as a graph, then R is hyperfinite μ-a.e. If it admits a graphing such that μ-almost every x has an orbit that has infinitely many ends, seen as a graph, then R is not "hyperfinite μ-a.e.".*

This theorem is the corollaire IV.24 of [9]. It is a statement among several of the kind (see [1], [12]).

1.6 A lemma on asymptotic invariance

We end this section with the statement and proof of a lemma that will be needed in subsection 3.2.

LEMMA 1.13. *A sequence (B_n) of Borel subsets of X is μ-asymptotically R-invariant if and only if for every Borel (not necessarily bijective) map $\phi : X \to X$ whose graph is included in R, $\mu(\phi^{-1}(B_n)\triangle B_n) \underset{n\to\infty}{\longrightarrow} 0$.*

Remark. This result is false if we replace $\phi^{-1}(B_n)$ with $\phi(B_n)$. Indeed, a Borel map whose graph is included in R may have a range of small measure. For instance, take the "first-return in $[0,\epsilon[$ map" for an action of $\mathbb{Z}$ on $\mathbb{R}/\mathbb{Z} \simeq [0,1[$ by irrational translation.

Proof. One implication is tautological. To establish the other, assume that (B_n) is asymptotically invariant and take ϕ a Borel map from X to X whose graph is included in R.

There exist

- a partition $X = \bigsqcup_{i\in\mathbb{N}} X_i$ of X into countably many Borel subsets,
- countably many $\varphi_i \in [R]$,

such that for all i, ϕ and φ_i coincide on X_i (this can be proved using theorem 1.4). Let ϵ be a positive real number. Take N such that $\mu\left(\bigsqcup_{i>N} X_i\right) < \epsilon$.

For every i and n, we have,

$$\begin{aligned}
\phi^{-1}(B_n)\triangle B_n &\overset{\epsilon}{\simeq} \bigsqcup_{i\leq N} X_i \cap (\phi^{-1}(B_n)\triangle B_n) \\
&= \bigsqcup_{i\leq N} X_i \cap ((\varphi_i)^{-1}(B_n)\triangle B_n) \\
&\subset \bigcup_{i\leq N} (\varphi_i)^{-1}(B_n)\triangle B_n,
\end{aligned}$$

where $A \overset{\epsilon}{\simeq} B$ means that $\mu(A\triangle B) \leq \epsilon$.

Since $\mu\left(\bigcup_{i\leq N}(\varphi_i)^{-1}(B_n)\triangle B_n\right)$ goes, by hypothesis, to 0 as n goes to infinity, the lemma is established. □

2 Percolation

Percolation is a topic coming originally from statistical mechanics ([13]). After a foundational paper by Benjamini and Schramm ([5]), strong connections with group theory have developed. This section presents the objects and theorems that will be needed in the section 3. For more information about this material, one can refer to [10], [16] and [17].

2.1 General definitions

From here on, Γ will be assumed to be finitely generated.

Let S be a finite generating set of Γ. Define a graph by taking Γ as vertex-set and putting, for each $\gamma \in \Gamma$ and $s \in S$, an edge from γ to γs. This defines a locally finite connected graph $\mathcal{G} = (V, E)$ that is called the *Cayley graph* of Γ relative to S. The action of Γ on itself by multiplication from the left induces a (left) action on $\mathcal{G}$ by graph automorphisms; it is free and transitive as an action on the vertex-set. In fact, a locally finite connected graph $\mathcal{G}$ is a Cayley graph of Γ if and only if Γ admits an action on $\mathcal{G}$ that is free and transitive on the vertex-set.

We have defined $\mathcal{G}$ explicitly to prove that Γ admits Cayley graphs, but further reasonings shall be clearer if one forgets that $V = \Gamma$ and just remembers that $\mathcal{G}$ is endowed with a free vertex-transitive action of Γ. Thus, in order to get an element of Γ from a vertex, one will need a reference point. Let ρ be a vertex of $\mathcal{G}$, that we shall use as such a reference or anchor point. Any vertex $v \in V$ can be written uniquely in the form $\gamma \cdot \rho$.

The action $\Gamma \curvearrowright E$ induces a shift action $\Gamma \curvearrowright \Omega := \{0,1\}^E$. A *(bond) percolation* will be a probability measure on Ω. It is said to be Γ-*invariant* if it is as a probability measure on Ω.

In what follows, all considered percolations will be assumed to be Γ-invariant. Besides, for simplicity, we will work under the implicit assumption that $\mathbb{P}$ is atomless, so that $(\Omega, \mathbb{P})$ will always be a standard probability space.

A point ω of Ω is seen as a subgraph of $\mathcal{G}$ the following way: V is its set of vertices, and $\omega^{-1}(\{1\})$ its set of edges. In words, keep all edges whose label is 1 and throw away the others. The connected components of this graph are called the *clusters* of ω. If $v \in V$, its ω-cluster will be denoted by $\mathcal{C}(\omega, v)$. For $v \in V$, $\omega \mapsto \mathcal{C}(\omega, v)$ is a Borel map, since the set of finite paths in $\mathcal{G}$ is countable.

For $\omega \in \Omega$, we set $V_\infty(\omega) := \{v \in V : |\mathcal{C}(\omega, v)| = \infty\}$. If $(u, v) \in V^2$, we will use $u \underset{\omega}{\longleftrightarrow} v$ as an abbreviation for "u and v are in the same ω-cluster".

$N_\infty(\omega)$ denotes the number of infinite clusters of ω. All these functions are Borel.

2.2 Independent percolation

The simplest interesting example of percolation is the product measure $\mathrm{Ber}(p)^{\otimes E}$, for $p \in [0,1]$; it will be denoted by $\mathbb{P}_p$. Such percolations are called *independant* or *Bernoulli* percolations.

We are interested in the emergence of an infinite cluster when p increases. Let $\theta_{\mathcal{G}} : p \mapsto \mathbb{P}_p[|\mathcal{C}(\omega, \rho)| = \infty]$, the *percolation function* of $\mathcal{G}$.

Endow $[0,1]^E$ with the probability $\mathbb{P}_{[0,1]} := \mathrm{Leb}([0,1])^{\otimes E}$. Notice that $\mathbb{P}_p$ is the push-forward of $\mathbb{P}_{[0,1]}$ by the following map

$$\begin{array}{rccc} \pi_p : & [0,1]^E & \longrightarrow & \{0,1\}^E \\ & x & \longmapsto & (1_{x(e)<p})_{e\in E} \end{array}$$

Realizing probability measures as distributions of random variables suitably defined on a same probability space is called a *coupling*. A fundamental property of this coupling is that, when $x \in [0,1]^E$ is fixed, $p \mapsto \pi_p(x)$ is non-decreasing for the product order. One deduces the following proposition.

Proposition 2.1. *The function $\theta_{\mathcal{G}}$ is non-decreasing.*

Corollary 2.2. *There exists a unique real number $p_c(\mathcal{G}) \in [0,1]$ such that the following two conditions hold:*

- $\forall p < p_c(\mathcal{G}),\ \theta_{\mathcal{G}}(p) = 0$
- $\forall p > p_c(\mathcal{G}),\ \theta_{\mathcal{G}}(p) > 0$

One calls $p_c(\mathcal{G})$ the critical probability *of $\mathcal{G}$.*

Remark. When $p_c(\mathcal{G})$ is not trivial (neither 0 nor 1), this result establishes the existence of a *phase transition*. One cannot have $p_c(\mathcal{G}) = 0$, but $p_c(\mathcal{G}) = 1$ may occur (e.g. it does for $\mathbb{Z}$).

The following theorems describe almost totally the phase transitions related to the number of infinite clusters.

Proposition 2.3. *For all $p \in [0,1]$, N_∞ takes a $\mathbb{P}_p$-almost deterministic value, which is 0, 1 or ∞. In particular, this value is 0 if $p < p_c(\mathcal{G})$ and 1 or ∞ if $p > p_c(\mathcal{G})$.*

Theorem 2.4 (Häggström-Peres). *There exists a unique real number $p_u(\mathcal{G}) \in [p_c(\mathcal{G}), 1]$ such that the following two conditions hold:*

- $\forall p < p_u(\mathcal{G}), \mathbb{P}_p[N_\infty = 1] = 0$

- $\forall p > p_u(\mathcal{G}), \mathbb{P}_p[N_\infty = 1] = 1$

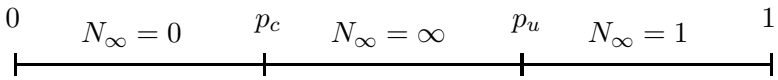
One calls $p_u(\mathcal{G})$ *the* uniqueness probability *of* $\mathcal{G}$.

Remark. If Γ is amenable, proposition 2.7 gives $p_c(\mathcal{G}) = p_u(\mathcal{G})$. The converse is conjectured.

PROPOSITION 2.5 ([3]). *If* Γ *is non-amenable,* $\mathbb{P}_{p_c(\mathcal{G})}$*-almost surely, there is no infinite cluster.*

CONJECTURE 2.6. *If* $p_c(\mathcal{G}) < 1$*, then,* $\mathbb{P}_{p_c(\mathcal{G})}$*-almost surely, there is no infinite cluster.*

The phase transition theorems are roughly summarized in the picture below. Remember that the quantities p_c, p_u and 1 may coincide.



2.3 Generalized percolation

The notion of generalized percolation that is presented in this subsection is due to Gaboriau ([10]).

Let $\Gamma \curvearrowright (X, \mu)$ be a Borel action on a standard probability space. Assume that it is provided together with a Γ-equivariant map $\pi : X \to \Omega = \{0,1\}^E$. This will be called a *generalized (Γ-invariant) percolation*. As for percolations, we will omit the "Γ-invariant".

To begin with, let us see how this notion is connected to the one presented in subsection 2.1. If a generalized percolation is given (we will stick to the notations above), then $\pi_\star\mu$, the pushforward of μ by π, is a (Γ-invariant) percolation (that *may* have atoms). Conversely, if $\mathbb{P}$ is a (Γ-invariant atomless) percolation, one can consider the Bernoulli shift action $\Gamma \curvearrowright X = \Omega$ together with $\pi : X \to \Omega$ the identity. Via this procedure, one can redefine in the percolation setting any notion introduced in the generalized framework.

Notice that the π_p's of the standard coupling, introduced at the beginning of subsection 2.2, provide interesting examples of such generalized percolations.

This setting provides the same atomless measures on Ω as the previous one, but it allows more flexibility in our way to speak of them. In the next subsection, we will discuss properties of clusters. The usual setting allows to speak of properties such that "being infinite", "having three ends", "being transient for simple random walk". The generalized one will allow us, if we consider $\Gamma \curvearrowright [0,1]^E$ together with π_{p_1}, to speak of "the considered p_1-cluster contains an infinite p_0-cluster".

2.4 Cluster indistinguishability

In this subsection, we work on a given generalized percolation. The action is denoted by $\Gamma \curvearrowright (X, \mathbb{P})$ and the equivariant map by π.

NOTATION. We call *vertex property*, or *property*, a Borel Γ-INVARIANT Boolean function on $X \times V$, i.e. a Borel function

$$P : X \times V \to \{\text{true}, \text{false}\}$$

which is invariant under the diagonal action of Γ. If $S \subset V$, $P^+(x, S)$ will mean "all the vertices in S satisfy $P(x, .)$". More formally, we define

$$P^+(x, S) := \text{“}\forall v \in S, P(x, v)\text{”}$$

We also set

- $P^-(x, S) := \text{“}\forall v \in S, \neg P(x, v)\text{”}$
- $P^\pm(x, S) := \text{“}P^+(x, S) \vee P^-(x, S)\text{”}$

$P^\pm(x, S)$ means "all the vertices in S agree on $P(x, .)$". Finally, one will write V_∞^π for $V_\infty \circ \pi$.

Example. The degree of a vertex in a graph is its number of neighbors. "The vertex v has degree 4 in $\pi(x)$ seen as a subgraph of $\mathcal{G}$" is a property.

DEFINITION. We call *cluster property* a property P such that $P(x, v) \Leftrightarrow P(x, u)$ as soon as $u \underset{\pi(x)}{\longleftrightarrow} v$. In words, it is a vertex property such that, for any x, $P(x, .)$ is constant on $\pi(x)$-clusters.

Example. The previous example is (usually) not a cluster property, since, usually, there exist subgraphs of $\mathcal{G}$ where some component has some vertices of degree 4 and others of other degree. "The $\pi(x)$-cluster of v is infinite", "the $\pi(x)$-cluster of v is transient", "the $\pi(x)$-cluster of v has a vertex of degree 4" are cluster properties.

Counter-example. "The $\pi(x)$-cluster of v contains ρ" is *not* a cluster property, because of the lack of Γ-invariance. This is to avoid such "properties" that Γ-invariance is required in the definition of vertex properties; allowing them would automatically make any indistinguishability theorem false, since they make the distinction between the cluster of the origin and others.

Example. Here is another example of cluster property: this one can be (directly) considered only in the generalized setting. Consider $X = [0, 1]^E$ and $0 < p_0 < p_1 < 1$. We take $\pi = \pi_{p_1}$ (see subsection 2.2). The property "the $\pi_{p_1}(x)$-cluster of v contains an infinite $\pi_{p_0}(x)$-cluster" is a cluster property. It has been considered by Lyons and Schramm in [18] to derive the Häggström-Peres theorem from indistinguishability.

DEFINITION. The considered generalized percolation will be said to satisfy *(infinite cluster) indistinguishability* (or one will say that its infinite clusters are indistinguishable) if, for every cluster property P,

$$\mathbb{P}[P^{\pm}(x, V_{\infty}^{\pi}(x))] = 1.$$

Of course, this notion is empty as soon as $\mathbb{P}[N_{\infty}(\pi(x)) \leq 1] = 1$, e.g. for $\mathbb{P}_p$ when Γ is amenable.

Remark. Assume momentarily that $\Gamma \curvearrowright (X, \mathbb{P})$ is ergodic and that the infinite clusters are indistinguishable. Then for every cluster property P, by indistinguishability,

$$\mathbb{P}[P^{+}(x, V_{\infty}^{\pi}(x)) \text{ or } P^{-}(x, V_{\infty}^{\pi}(x))] = 1$$

Besides, by ergodicity, $\mathbb{P}[P^{+}(x, V_{\infty}^{\pi}(x))]$ and $\mathbb{P}[P^{-}(x, V_{\infty}^{\pi}(x))]$ are 0 or 1. Altogether, these identities guarantee that

$$\mathbb{P}[P^{+}(x, V_{\infty}^{\pi}(x))] = 1 \quad \text{or} \quad \mathbb{P}[P^{-}(x, V_{\infty}^{\pi}(x))] = 1$$

To state the indistinguishability theorem in its natural form, we need to introduce the notion of insertion-tolerance.

2.5 Insertion-tolerance

In this subsection, we work with non-generalized percolations.

DEFINITION. If $(\omega, e) \in \Omega \times E$, one denotes by ω^e the unique element of Ω equal to ω on $E \backslash \{e\}$ taking the value 1 at e. One sets $\Pi^e : \omega \mapsto \omega^e$. A percolation is said to be *insertion-tolerant* if for every Borel subset $B \subset \Omega$, for every edge e,

$$\mathbb{P}[B] > 0 \Rightarrow \mathbb{P}[\Pi^e(B)] > 0.$$

Example. For $p > 0$, $\mathbb{P}_p$ is always insertion-tolerant.

PROPOSITION 2.7. *If Γ is amenable and if $\mathbb{P}$ is an insertion-tolerant percolation on $\mathcal{G}$, then $\mathbb{P}[N_{\infty}(\omega) \leq 1] = 1$.*

PROPOSITION 2.8 ([18], PROPOSITION 3.10). *If $\mathbb{P}$ is an insertion-tolerant percolation on $\mathcal{G}$ that produces a.s. at least two infinite clusters, then, a.s., it produces infinitely many infinite clusters and each of them has infinitely many ends.*

Now that insertion-tolerance has been introduced, we can state the indistinguishability theorem of Lyons and Schramm ([18]).

THEOREM 2.9 (LYONS-SCHRAMM,[18]). *Any insertion-tolerant percolation has indistinguishable infinite clusters.*

2.6 Percolation and orbit equivalence

In this subsection, we work with a generalized percolation, where the action is denoted by $\Gamma \curvearrowright (X, \mathbb{P})$ and the equivariant map by π.

The *cluster equivalence relation* is defined on X the following way. Two configurations x and x' in X are said to be R_{cl}*-equivalent* if there exists $\gamma \in \Gamma$ such that $\gamma^{-1} \cdot x = x'$ and $\gamma \cdot \rho \underset{\pi(x)}{\longleftrightarrow} \rho$. In words, an R_{cl}-class is a configuration up to Γ-translation, with a distinguished cluster, the one of the root ρ.

Every generalized percolation is R_{cl}-invariant, since R_{cl} is a subrelation of $R_{\Gamma \curvearrowright X}$.

Let S denote the generating set associated to the choice of the Cayley graph $\mathcal{G}$. For $s \in S$, let $\tilde{\varphi}_s : x \mapsto s^{-1} \cdot x$, defined only where the edge $(\rho, s \cdot \rho)$ is $\pi(x)$-open. This graphing induces on $[x]_{R_{cl}}$ the graph structure of the $\pi(x)$-cluster of the anchor point ρ. This remark, together with theorem 1.12 and proposition 2.8, provides the following proposition.

Proposition 2.10. *For any insertion-tolerant classic percolation such that N_∞ is almost surely infinite, R_{cl} is non-amenable.*

3 Ergodicity and indistinguishability

Throughout this section, we will work with a generalized percolation. The underlying standard probability space will be denoted by $(X, \mathbb{P})$ and the equivariant map by π.

3.1 Classic connection

The *infinite locus* is defined as

$$X_\infty := \{x \in X : |[x]_{R_{cl}}| = \infty\} = \{x \in X : |\mathcal{C}(\pi(x), \rho)| = \infty\}$$

Remember that if there is no π in the first description, it is because it is hidden in R_{cl}. Let R denote the restriction of R_{cl} to $X_\infty \times X_\infty$.

Proposition 3.1 (Gaboriau-Lyons, [11]). *Consider a generalized percolation defined by $\Gamma \curvearrowright (X, \mathbb{P})$ and $\pi : X \to \Omega$ a Γ-equivariant map. Assume that $\Gamma \curvearrowright (X, \mathbb{P})$ is ergodic and $\mathbb{P}[X_\infty] > 0$. Then the considered generalized percolation has indistinguishable clusters iff R is $\frac{\mathbb{P}}{\mathbb{P}[X_\infty]}$-ergodic.*

As a preliminary for the next subsection, we detail the proof of this theorem, which can be found in [11]. With the following reading grid in mind, the formal proof should not be obscure. In the following "dictionary", the bijection $\psi : \Gamma \backslash (X \times V) \to X$ induced by $\psi_0 : (x, \gamma \cdot \rho) \mapsto \gamma^{-1} \cdot x$ is the translator.

Orbit equivalence		Percolation
X	$\overset{\psi}{\longleftrightarrow}$	$\Gamma\backslash(X \times V)$
$\gamma^{-1} \cdot x$		$[(x, \gamma \cdot \rho)]$
Borel subset		vertex property
R_{cl}-class		cluster
R_{cl}-invariant		cluster property
ergodicity of R		indistinguishability
graphing		graph structure

New items will be added to this "dictionary" in subsection 3.2.

Proof. Assume that R is ergodic. Let us prove that $\mathbb{P}$ has indistinguishable infinite clusters. Let P be a cluster property. Consider

$$A := \{x \in X_\infty : P(x, \rho)\}$$

It is an R-invariant Borel subset of X_∞. Assume that $\mathbb{P}[P^+(x, V_\infty^\pi(x))] < 1$; we want to show that $\mathbb{P}[P^-(x, V_\infty^\pi(x))] = 1$. By assumption, there must exist $v = \gamma \cdot \rho \in V$ such that

$$\mathbb{P}[\{x : v \in V_\infty^\pi(x) \wedge \neg P(x, v)\}] > 0.$$

Since P and $\mathbb{P}$ are Γ-invariant,

$$\begin{aligned} \mathbb{P}[X_\infty \backslash A] &= \mathbb{P}[\{x : \rho \in V_\infty^\pi(x) \wedge \neg P(x, \rho)\}] \\ &= \mathbb{P}\left[\{x : \rho \in V_\infty^\pi(\gamma^{-1} \cdot x) \wedge \neg P(\gamma^{-1} \cdot x, \rho)\}\right] \\ &= \mathbb{P}[\{x : v \in V_\infty^\pi(x) \wedge \neg P(x, v)\}] > 0 \end{aligned}$$

Since A is R-invariant and R is ergodic, $\mathbb{P}[X_\infty \backslash A] = \mathbb{P}[X_\infty]$. Hence, $\mathbb{P}[A] = 0$. By invariance and countability, $\mathbb{P}\left[\bigcup_\gamma \gamma \cdot A\right] = 0$. This means that

$$\mathbb{P}[\{x : \exists v \in V, v \in V_\infty^\pi(x) \wedge P(x, v)\}] = 0$$

so that $\mathbb{P}[P^-(x, V_\infty^\pi(x))] = 1$. This ends the proof of the first implication.

Now, assume that $\mathbb{P}$ has indistinguishable infinite clusters. We want to show that R is $\frac{\mathbb{P}}{\mathbb{P}[X_\infty]}$-ergodic. Let A be a Borel R-invariant subset of X_∞. Since our assumption is about properties, we need to build a property out of A. Set

$$P(x, v) := \text{“}\psi_0(x, v) \in A\text{”}$$

Since ψ_0 is constant on Γ-orbits, P is a property (it is Γ-invariant). Besides, A being R-invariant, P is a cluster property. Using indistinguishability of infinite clusters, the ergodicity of $\Gamma \curvearrowright (X, \mathbb{P})$ and the first remark of subsection 2.4, we have

$$\mathbb{P}[P^+(x, V_\infty^\pi(x))] = 1 \text{ or } \mathbb{P}[P^-(x, V_\infty^\pi(x))] = 1.$$

By symmetry, we can assume that the first case occurs. By definition of P and X_∞,

$$\{x \in X : P^+(x, V_\infty^\pi(x))\} \cap X_\infty \subset A$$

(In words, if ρ is connected to infinity and $P(x,.)$ holds for every vertex connected to infinity, then $P(x,\rho)$ holds.) In particular, since $\mathbb{P}[P^+(x, V_\infty^\pi(x))] = 1$, we have $\mathbb{P}[A \cap X_\infty] = \mathbb{P}[X_\infty]$, so that R is ergodic. □

3.2 Strong version

Consider $\mathbb{P}_p$ for $p \in (p_c(\mathcal{G}), p_u(\mathcal{G}))$. By theorems 1.11, 2.9 and 2.10 and proposition 3.1, its cluster equivalence relation is strongly ergodic on the infinite locus. One would like to deduce from this information a strong form of indistinguishability of $\mathbb{P}_p$. This idea is due to Damien Gaboriau.

Another way to describe our goal is to say that we look for a proposition similar to proposition 3.1 for strong notions. This is achieved in theorem 3.2.

Again, everything will be stated for a generalized percolation, with the same notations as previously.

DEFINITION. We call *re-anchoring*, or *rerooting*, a Borel map

$$\begin{array}{rccc} \alpha : & X \times V & \longrightarrow & V \\ & (x, v) & \longmapsto & u_{x,v}^\alpha \end{array}$$

which is Γ-equivariant under the diagonal action and such that

$$\forall (x, v) \in X \times V,\ u_{x,v}^\alpha \underset{\pi(x)}{\longleftrightarrow} v$$

It is said to be *vertex-bijective* if for every x, $v \mapsto u_{x,v}^\alpha$ is bijective.

In words, a re-anchoring is a Γ-equivariant way of changing of position within one's cluster.

Example. If $\gamma \in \Gamma$, setting

$$u_{x,v}^\alpha := \begin{cases} \gamma \cdot v & \text{if } v \underset{\pi(x)}{\longleftrightarrow} \gamma \cdot v \\ v & \text{otherwise} \end{cases}$$

defines a re-anchoring.

DEFINITION. Let (P_n) be a sequence of vertex properties. Let $\mathbb{P}$ be a percolation. We will say that (P_n) is an *asymptotic cluster property* (for $\mathbb{P}$) if, for any rerooting α,

$$\forall v \in V, \mathbb{P}\left[\left\{x \in X : P_n(x, v) \Leftrightarrow P_n\left(x, u_{x,v}^\alpha\right)\right\}\right] \underset{n\to\infty}{\longrightarrow} 1 \qquad (1)$$

Remark. For a given rerooting, the convergence (1) holds for all v as soon as it holds for one, by Γ-invariance and -equivariance.

Remark. This definition of "depending asymptotically only on one's cluster[3]" is quite natural if one looks for a translation of strong ergodicity, but it may not be the clearest definition from a probabilistic point of view. For a probabilistically more natural definition, see subsection 3.4.

NOTATION. In what follows, $A \Subset B$ means that A is a finite subset of B.

DEFINITION. We will say that $\mathbb{P}$ satisfies the *strong indistinguishability property* if, for every $\mathbb{P}$-asymptotic cluster property (P_n) and every $\Lambda \Subset V$,

$$\mathbb{P}[P_n^{\pm}(x, V_\infty^\pi(x) \cap \Lambda)] \underset{n\to\infty}{\longrightarrow} 1.$$

Remark. Subsection 3.4 makes the definition of asymptotic cluster property look like the conclusion of strong indistinguishability.

THEOREM 3.2. *Consider a generalized percolation such that $\Gamma \curvearrowright (X, \mathbb{P})$ is strongly ergodic and $\mathbb{P}[X_\infty] > 0$. It satisfies the strong indistinguishability property iff R is $\frac{\mathbb{P}}{\mathbb{P}[X_\infty]}$-strongly ergodic.*

The few lines at the beginning of the current subsection allow us to derive the following corollary from this theorem (even for $p = p_u(\mathcal{G})$, if the assumption of the corollary is satisfied for this parameter).

COROLLARY 3.3. *As soon as $\mathbb{P}_p$ produces infinitely many infinite clusters, it satisfies the strong indistinguishability property.*

As for proposition 3.1, a reading grid may help getting through the proof of theorem 3.2:

Orbit equivalence		Percolation
X	$\overset{\psi}{\longleftrightarrow}$	$\Gamma\backslash(X \times V)$
$\gamma^{-1} \cdot x$		$[(x, \gamma \cdot \rho)]$
Borel subset		vertex property
R_{cl}-class		cluster
R_{cl}-invariant		cluster property
ergodicity of R		indistinguishability
ϕ s.t. $\mathrm{gr}(\phi) \subset R_{cl}$		rerooting
$\phi \in [R]$		vertex-bijective rerooting
asymptotically R_{cl}-invariant		asymptotic cluster property
strong ergodicity of R		strong indistinguishability
graphing		graph structure

[3]more precisely, one should say "depending $\mathbb{P}$-asymptotically only on x and one's $\pi(x)$-cluster "

Isolating the two following lemmas will also lighten the proof of theorem 3.2.

LEMMA 3.4. *Let (B_n) be an asymptotically R-invariant sequence of Borel subsets of X_∞. For $n \in \mathbb{N}$, set*

$$P_n(x, \gamma \cdot \rho) := \text{“}\gamma^{-1} \cdot x \in B_n\text{”}$$

Then (P_n) is an asymptotic cluster property.

Remark. The set $\{(x, v) : P_n(x, v)\}$ is the union of the elements of $\psi^{-1}(B_n)$.

Proof. The P_n's are Γ-invariant Borel Boolean functions defined on $X \times V$. Let α be a rerooting. Since $(x, v) \mapsto (x, u^\alpha_{x,v})$ is Γ-equivariant, it induces a map $\overline{\alpha} : \Gamma\backslash(X \times V) \to \Gamma\backslash(X \times V)$. Set

$$\phi := \psi \circ \overline{\alpha} \circ \psi^{-1}.$$

More explicitly, we have $\phi : x \mapsto \gamma_x^{-1} \cdot x$, where γ_x is defined by

$$u^\alpha_{x,\rho} = \gamma_x \cdot \rho$$

The graph of this Borel map is a subset of R. By lemma 1.13, $\mathbb{P}[B_n \triangle \phi^{-1}(B_n)]$ goes to 0 as n goes to infinity.

For $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}[B_n \triangle \phi^{-1}(B_n)] &= \mathbb{P}\left[\{x \in X_\infty : P_n(x, \rho)\} \triangle \left\{x \in X_\infty : P_n\left(x, u^\alpha_{x,\rho}\right)\right\}\right] \\
&= \mathbb{P}\left[\left\{x \in X_\infty : P_n(x, \rho) \not\Leftrightarrow P_n\left(x, u^\alpha_{x,\rho}\right)\right\}\right] \\
&= \mathbb{P}\left[\left\{x \in X : P_n(x, \rho) \not\Leftrightarrow P_n\left(x, u^\alpha_{x,\rho}\right)\right\}\right]
\end{aligned}$$

The last equality is proved as follows. Since $B_n \subset X_\infty$, $P_n(x, v)$ cannot hold if $\mathcal{C}(\pi(x), v)$ is finite. Besides, if $x \notin X_\infty$, then $\mathcal{C}(\pi(x), \rho) = \mathcal{C}\left(\pi(x), u^\alpha_{x,\rho}\right)$ is finite. Thus, if $x \notin X_\infty$, both considered properties are unsatisfied.

As a consequence, (P_n) is an asymptotic cluster property. □

Remark. In the previous proof, the use of lemma 1.13 allows us to obtain the asymptotic-cluster-property condition for all rerootings, while a “literal translation” would have given it only for the vertex-bijective ones. From the percolation point of view, vertex-bijective rerootings are absolutely non-natural objects; the use of such a lemma was unavoidable.

LEMMA 3.5. *Let (P_n) be a $\mathbb{P}$-asymptotic cluster property. Set*

$$B_n := \{x \in X_\infty : P_n(x, \rho)\}$$

Then (B_n) is $\frac{\mathbb{P}}{\mathbb{P}[X_\infty]}$-asymptotically R-invariant.

Proof. Let $\phi \in [R]$. Since $R_{cl} \subset R_{\Gamma \curvearrowright X}$, one can define a Borel map

$$\left| \begin{array}{ccc} X_\infty & \longrightarrow & \Gamma \\ x & \longmapsto & \gamma_x \end{array} \right.$$

such that $\forall x \in X_\infty, \phi(x) = {\gamma_x}^{-1} \cdot x$. Define α by $u^\alpha_{x,\eta\cdot\rho} := \eta \cdot \gamma_{\eta^{-1}x}$. This is a rerooting. We have

$$\begin{array}{rcll} \phi^{-1}(B_n) & = & \{x \in X_\infty : P_n(\phi(x), \rho)\} & \\ & = & \left\{x \in X_\infty : P_n(\gamma_x^{-1} \cdot x, \rho)\right\} & \\ & = & \{x \in X_\infty : P_n(x, \gamma_x \cdot \rho)\} & \text{by } \Gamma\text{-invariance of } P_n \\ & = & \left\{x \in X_\infty : P_n(x, u^\alpha_{x,\rho})\right\} & \end{array}$$

Since (P_n) is an asymptotic cluster property, we deduce from this that the probability of $B_n \triangle \phi^{-1}(B_n)$ tends to 0. Since this holds for every $\phi \in [R]$, (B_n) is $\frac{\mathbb{P}}{\mathbb{P}[X_\infty]}$-asymptotically R-invariant. □

Proof of theorem 3.2. Assume that R is strongly ergodic. Let (P_n) be a $\mathbb{P}$-asymptotic cluster property. Set $B_n := \{x \in X_\infty : P_n(x, \rho)\}$. By lemma 3.5, (B_n) is $\frac{\mathbb{P}}{\mathbb{P}[X_\infty]}$-asymptotically R-invariant. By strong ergodicity of R, 0 and $\mathbb{P}[X_\infty]$ are the only possible accumulation points for $\mathbb{P}[B_n]$. For $n \in \mathbb{N}$, set

- $Y_n^+ := (X \backslash X_\infty) \cup B_n$,
- $Y_n^- := (X \backslash X_\infty) \cup (X_\infty \backslash B_n)$,
- $\epsilon_n := 1 - \max\{\mathbb{P}[Y_n^+], \mathbb{P}[Y_n^-]\}$.

We have just established that ϵ_n converges to 0.

Let now Λ be a finite subset of V. We want to show that the probability of the following event

$$\mathrm{Ind}_{\Lambda,n} := \{x \in X : P_n^\pm(x, V_\infty^\pi(x) \cap \Lambda)\}$$

tends to 1 when n goes to infinity. Notice that

$$\mathrm{Ind}_{\Lambda,n} = \left(\bigcap_{\gamma \in \Lambda} \gamma^{-1} \cdot Y_n \right) \cup \left(\bigcap_{\gamma \in \Lambda} \gamma^{-1} \cdot Y_n' \right).$$

Since $\Gamma \curvearrowright (X, \mathbb{P})$ is measure-preserving, we have $\mathbb{P}[\mathrm{Ind}_{\Lambda,n}] \geq 1 - |\Lambda|\epsilon_n$. Consequently, $\mathbb{P}[\mathrm{Ind}_{\Lambda,n}]$ converges to 1 and $\mathbb{P}$ satisfies the strong indistinguishability property.

Now, assume that the considered generalized percolation satisfies the strong indistinguishability property. Let (B_n) be an asymptotically R-invariant sequence of Borel subsets of X_∞. For $n \in \mathbb{N}$, set

$$P_n(x, \gamma \cdot \rho) := \text{“}\gamma^{-1} \cdot x \in B_n\text{”}$$

By lemma 3.4, (P_n) is an asymptotic cluster property. By strong indistinguishability, for every $\Lambda \Subset V$,

$$\mathbb{P}\left[\{x : P_n^{\pm}(x, V_\infty^\pi(x) \cap \Lambda)\}\right] \underset{n\to\infty}{\longrightarrow} 1. \tag{2}$$

We want to deduce from this that some sequence of Borel subsets is $\mathbb{P}$-asymptotically $R_{\Gamma\curvearrowright X}$-invariant. Indeed, from this and the strong ergodicity of $\Gamma \curvearrowright X$, we will get that this sequence is asymptotically of trivial mass. To this end, we fix $(k_n)_{n\in\mathbb{Z}_+}$ a sequence of integers and $(\Lambda_n)_{n\in\mathbb{Z}_+}$ an increasing sequence of finite subsets of V such that the following conditions are satisfied:

i. $\Lambda_0 = \{\rho\}$,

ii. $k_n \underset{n\to\infty}{\longrightarrow} \infty$,

iii. $\bigcup_n \Lambda_n = V$,

iv. $\mathbb{P}\left[\left\{x : P_n^{\pm}\left(x, V_\infty^\pi(x) \cap \Lambda_n^{k(n)}\right)\right\}\right] \underset{n\to\infty}{\longrightarrow} 1$, where $\Lambda_n^{k(n)}$ denotes the k_n-neighborhood of Λ_n.

The existence of such sequences stems from the fact that convergence (2) holds for every finite subset of V. Set

- $A_n^+ := \{x \in X : P_n^+(x, V_\infty^\pi(x) \cap \Lambda_n)\}$,
- $A_n^- := \{x \in X : P_n^-(x, V_\infty^\pi(x) \cap \Lambda_n)\}$.

The relation (iv) implies that $\mathbb{P}[A_n^+ \cup A_n^-] \underset{n\to\infty}{\longrightarrow} 1$; this will be used later.

We shall prove that A_n^+ is $\mathbb{P}$-asymptotically $R_{\Gamma\curvearrowright X}$-invariant. We will also need this property for A_n^-, which is proved following the very same lines. Let $\gamma \in \Gamma$. Consider only integers n such that $k_n > d_{\mathcal{G}}(\rho, \gamma \cdot \rho)$ (they form a neighborhood of infinity by (ii)). Set

- $A_{n,k(n)}^+ := \left\{x \in X : P_n^+\left(x, V_\infty^\pi(x) \cap \Lambda_n^{k(n)}\right)\right\} \subset A_n^+$,
- $A_{n,k(n)}^- := \left\{x \in X : P_n^-\left(x, V_\infty^\pi(x) \cap \Lambda_n^{k(n)}\right)\right\} \subset A_n^-$.

Claim. *We have*

$$A_n^+ \backslash A_{n,k(n)}^+ \subset X \backslash \left(A_{n,k(n)}^+ \cup A_{n,k(n)}^-\right) \cup \{x \in X :\ V_\infty^\pi(x) \cap \Lambda_n = \emptyset\}$$

This amounts to saying that if, for x, an infinite cluster touches Λ_n and all infinite clusters touching $\Lambda_n^{k(n)}$ agree[4] on P_n, then x cannot belong to $A_n^+ \backslash A_{n,k(n)}^+$. This is proved as follows. Assume that for x, an infinite cluster touches Λ_n and all infinite clusters touching $\Lambda_n^{k(n)}$ agree on P_n. Besides, assume, for contradiction, that $x \in A_n^+ \backslash A_{n,k(n)}^+$, which means that every infinite cluster touching Λ_n satisfies P_n but that this is not the case for $\Lambda_n^{k(n)}$. Since there exists an infinite cluster touching Λ_n and every infinite cluster touching Λ_n satisfies P_n, there exists an infinite cluster that touches Λ_n and satisfies P_n. It also touches $\Lambda_n^{k(n)} \supset \Lambda_n$. Since all infinite clusters touching $\Lambda_n^{k(n)}$ must agree, they must all satisfy P_n, which contradicts our last assumption.

By (iv), the probability of

$$X \backslash (A_{n,k(n)}^+ \cup A_{n,k(n)}^-)$$

goes to 0 as n goes to infinity. By (iii) and because $\mathbb{P}$-a.s. there exists an infinite cluster[5], the probability of $\{x \in X : V_\infty^\pi(x) \cap \Lambda_n = \emptyset\}$ also goes to 0. Thus, $\mathbb{P}[A_n^+ \backslash A_{n,k(n)}^+] \underset{n\to\infty}{\longrightarrow} 0$.

Now, remember that $k_n > d_{\mathcal{G}}(\rho, \gamma \cdot \rho)$. It results from this that

$$A_{n,k(n)}^+ \subset \gamma \cdot A_n^+$$

so that $\mathbb{P}[A_n^+ \backslash \gamma \cdot A_n^+] \underset{n\to\infty}{\longrightarrow} 0$. By symmetry, $\mathbb{P}[A_n^+ \triangle \gamma \cdot A_n^+] \underset{n\to\infty}{\longrightarrow} 0$. This holding for all γ, (A_n^+) is $\mathbb{P}$-asymptotically $R_{\Gamma \curvearrowright X}$-invariant. By strong ergodicity of $\Gamma \curvearrowright X$, the only possible accumulation points of $(\mathbb{P}[A_n^+])$ are 0 and 1 (the same holds for A_n^-). By (i), $B_n \supset A_n^+ \cap X_\infty$ and $X_\infty \backslash B_n \supset A_n^- \cap X_\infty$. It is consequently enough, in order to end the proof of the strong ergodicity, to show that

$$\max(\mathbb{P}[A_n^+ \cap X_\infty], \mathbb{P}[A_n^- \cap X_\infty]) \underset{n\to\infty}{\longrightarrow} \mathbb{P}[X_\infty].$$

But this is a consequence of the following facts at our disposal:

- 0 and $\mathbb{P}[X_\infty]$ are the only possible accumulation points of the sequences $(\mathbb{P}[A_n^+ \cap X_\infty])$ and $(\mathbb{P}[A_n^- \cap X_\infty])$,
- $\mathbb{P}[A_n^+ \cup A_n^-] \underset{n\to\infty}{\longrightarrow} 1$, which is implied by (iv).

□

[4] In the proof of the claim, we shall use, for convenience, "all infinite clusters touching Λ satisfy P_n" or "agree on P_n" as *abusive* shortcuts for $P_n^+(x, \Lambda \cap V_\infty^\pi(x))$ and $P_n^\pm(x, \Lambda \cap V_\infty^\pi(x))$. Recall that the P_n's are *not* cluster properties (but form an *asymptotic* cluster property).

[5] Indeed, $\Gamma \curvearrowright (X, \mathbb{P})$ is ergodic and $\mathbb{P}[\text{there exists an infinite } \pi(x)\text{-cluster}] \geq \mathbb{P}[X_\infty]$, which is positive.

Remark. In the first part of the proof, some sequence (k_n) is introduced. Then, some γ is considered and we take n large enough so that

$$k_n > d_{\mathcal{G}}(\rho, \gamma \cdot \rho)$$

In fact, we could have taken γ in S, so that we did not really need k_n to go to infinity. Though, we have chosen to present the proof this way because it highlights the fact that the "finitely generated assumption" is not crucial here, and exists only to stick to the usual framework of percolation theory. For example, if one is interested in random partitions of a countable non-finitely generated group, one can adjust the theorem to this setting.

3.3 Classic and strong indistinguishability do not coincide

Obviously, strong indistinguishability implies the classic one (take $P_n = P$ for all n). The following proposition proves that the converse does not hold.

PROPOSITION 3.7. *There exist generalized percolations that satisfy indistinguishability but not strong indistinguishability.*

Proof. Let $\Gamma_0 \curvearrowright (X_0, \mu_0)$ be a weakly mixing action of a finitely generated group that is not strongly ergodic (e.g. $\Gamma_0 \curvearrowright \left(\{0,1\}^{\Gamma_0}, \mathrm{Ber}(1/2)^{\otimes \Gamma_0}\right)$ with Γ_0 an infinite amenable group). Recall that $\Gamma_0 \curvearrowright (X_0, \mu_0)$ being *weakly mixing* (see e.g. [2]) means that the diagonal action

$$\Gamma_0 \curvearrowright (X_0 \times X_0, \mu_0 \otimes \mu_0)$$

is ergodic. Consider $\Gamma := \Gamma_0 \times \mathbb{Z}/2\mathbb{Z}$. To avoid problems of additive/multiplicative notations, we will think of $\mathbb{Z}/2\mathbb{Z}$ as $\langle a | a^2 = 1 \rangle$.

The action of Γ is defined as follows: we take $(X, \mathbb{P}) := (X_0 \times X_0, \mu_0 \otimes \mu_0)$, let Γ_0 act diagonally and let $\mathbb{Z}/2\mathbb{Z}$ act by permutation of the coordinates. Since these two actions commute, this defines an action of Γ on $(X, \mathbb{P})$.

Let S be a finite generating system of Γ_0. We will define a generalized percolation on the Cayley graph $\mathcal{G} = (V, E)$ of Γ associated to the generating system

$$S \times \{1\} \cup \{(1_{\Gamma_0}, a)\}$$

Say that the edges of the first kind are blue and the other ones are red. Let $\pi : X \to \Omega$ be the constant map whose unique value is

$$e \mapsto 1_{e \text{ is blue}}$$

This map is Γ-equivariant, since its unique value is left invariant by the Γ-action. In a nutshell, we consider the deterministic percolation with two clusters, $\Gamma_0 \times \{1\}$ and $\Gamma_0 \times \{a\}$, but properties will be allowed to depend on the information X contains.

First, let us prove that this generalized percolation satisfies the indistinguishability condition. It results from the choice of π that a cluster property is precisely a Γ_0-invariant property. Consequently, if P is a cluster property, by ergodicity of $\Gamma_0 \curvearrowright (X, \mathbb{P})$, the Boolean value of $P(x, (1_{\Gamma_0}, 1))$ must be almost deterministic. By $\mathbb{Z}/2\mathbb{Z}$-invariance, it is the same as the one of $P(x, (1_{\Gamma_0}, a))$, and indistinguishability is established.

Now, let us prove that this generalized percolation does not satisfy the strong indistinguishability condition. Take (B_n) a sequence of Borel subsets of X_0 that is μ_0-asymptotically Γ_0-invariant but such that $\mu_0(B_n)$ stays away from 0 and 1: it exists, since we assumed $\Gamma_0 \curvearrowright (X_0, \mu_0)$ not to be strongly ergodic. Set

$$P_n((x_0, x_1), (\gamma_0, b)) := \begin{cases} \text{“}\gamma_0^{-1} \cdot x_0 \in B_n\text{”} & \text{if } b = 1 \\ \text{“}\gamma_0^{-1} \cdot x_1 \in B_n\text{”} & \text{if } b = a \end{cases}$$

Since (B_n) is asymptotically Γ_0-invariant and by definition of π, (P_n) is an asymptotic cluster property. Besides, for any n, $P_n(x, (1_{\Gamma_0}, 1))$ and $P_n(x, (1_{\Gamma_0}, a))$ are independent and of measure bounded away from 0 and 1, hence the probability of “$P_n(x, (1_{\Gamma_0}, 1)) \neq P_n(x, (1_{\Gamma_0}, a))$” stays away from 0. This proves that the considered generalized percolation does not satisfy the strong indistinguishability condition. □

Let us take a closer look at what happens when we take

$$\mathbb{Z} \curvearrowright (\{0, 1\}^{\mathbb{Z}}, \mathrm{Ber}(1/2)^{\otimes \mathbb{Z}})$$

In this case, one can set $P_n(x, (k, b))$ to be

“there are more of x-bits ‘1’ than ‘0’ in $\{k - n, \dots, k + n\} \times \{b\}$”

Indeed, if k and k' are fixed, for n big enough, the probability that

$$P_n(x, (k, b)) \neq P_n(x, (k', b))$$

is less than the probability that a simple random walk on $\mathbb{Z}$ that takes $2n + 1 - |k - k'|$ steps ends up in $\{-2|k - k'|, \dots, 2|k - k'|\}$. This is known to go to zero as n goes to infinity (as $n^{-1/2}$).

Remark. This counterexample can be turned into a classic-percolation counterexample as follows. Take $\Gamma := \mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Endow it with the generating system $\{(1, 0), (0, 1)\}$. Toss a fair coin to decide which one of the following edge-sets you definitely erase:

- $\{[(k, 0); (k, 1)]; k \in \mathbb{Z}\} \cup \{[(k, 2); (k, 3)]; k \in \mathbb{Z}\}$,
- $\{[(k, 1); (k, 2)]; k \in \mathbb{Z}\} \cup \{[(k, 3); (k, 0)]; k \in \mathbb{Z}\}$.

Decide to keep automatically all edges of the form $[(k,z);(k+1,z)]$. For all remaining edges, toss independent fair coins to decide if they are kept or destroyed. It is not hard to deduce from what precedes this remark that we have defined a classic percolation that satisfies indistinguishability but not strong indistinguishability.

3.4 Complements on asymptotic cluster properties

This subsection provides equivalent definitions of asymptotic cluster properties. We stick to the usual notations for generalized percolations.

NOTATION. If $x \in X$, denote by $\mathfrak{C}^\pi(x)$ the set of the clusters of $\pi(x)$.

PROPOSITION 3.8. *Let (P_n) be a sequence of properties. The following assertions are equivalent*

i. *(P_n) is a $\mathbb{P}$-asymptotic cluster property,*

ii. $\forall \Lambda \Subset V, \mathbb{P}\left[\forall C \in \mathfrak{C}^\pi(x), P_n^\pm(x, C \cap \Lambda)\right] \underset{n\to\infty}{\longrightarrow} 1$,

iii. $\exists u \in V, \forall v \in V, \mathbb{P}[P_n^\pm(x,\{u,v\}) | u \underset{\pi(x)}{\longleftrightarrow} v] \underset{n\to\infty}{\longrightarrow} 1$,

iv. $\forall u \in V, \forall v \in V, \mathbb{P}[P_n^\pm(x,\{u,v\}) | u \underset{\pi(x)}{\longleftrightarrow} v] \underset{n\to\infty}{\longrightarrow} 1$.

Remark. Above, we set $P[A|B] := 1$ when $\mathbb{P}[B] = 0$.

Proof. The assertions (iii) and (iv) are equivalent by Γ-invariance.

Rewriting (ii) as follows

$$\forall \Lambda \Subset V, \mathbb{P}\left[\forall (u,v) \in \Lambda^2, \left(u \underset{\pi(x)}{\longleftrightarrow} v\right) \Rightarrow P_n^\pm(x,\{u,v\})\right] \underset{n\to\infty}{\longrightarrow} 1$$

clarifies its equivalence[6] with (iv) (one way, take $\Lambda := \{u,v\}$, the other way, write Λ as the *finite* union of the pairs it contains).

Now assume (i) and establish (iii); we will do so for $u = \rho$. Let $v = \gamma \cdot \rho$ be a vertex. Define

$$\alpha : (x, v') \longmapsto \begin{cases} \gamma \cdot v' & \text{if } v' \underset{\pi(x)}{\longleftrightarrow} \gamma \cdot v' \\ v' & \text{otherwise} \end{cases}$$

Applying (i), one gets

$$\mathbb{P}\left[\left\{x \in X : P_n(x,\rho) = P_n\left(x, u_{x,\rho}^\alpha\right)\right\}\right] \underset{n\to\infty}{\longrightarrow} 1$$

[6]Remember that $\mathbb{P}[Q_n|Q] \underset{n\to\infty}{\longrightarrow} 1$ is equivalent to $\mathbb{P}[Q \Rightarrow Q_n] \underset{n\to\infty}{\longrightarrow} 1$.

Hence, if $A := \{x \in X : \rho \underset{\pi(x)}{\longleftrightarrow} \gamma \cdot v\}$,

$$\mathbb{P}\left[\left\{x \in A : P_n(x,\rho) = P_n\left(x, u^\alpha_{x,\rho}\right)\right\}\right] \underset{n\to\infty}{\longrightarrow} \mathbb{P}[A]$$

But, on A, "$P_n(x,\rho) = P_n\left(x, u^\alpha_{x,\rho}\right)$" means that "$P_n(x,\rho) = P_n(x,v)$", so that (iii) is established.

It is now enough to show that (ii) implies (i). Assume (ii). Let α be a rerooting. Set $w(x) := u^\alpha_{x,\rho}$ and take $\epsilon > 0$. Let $\Lambda \Subset V$ be such that $\mathbb{P}[w \notin \Lambda] < \epsilon$. We have

$$(w \in \Lambda) \wedge (\forall C \in \mathfrak{C}^\pi(x), P^\pm_n(x, \Lambda \cap C)) \Rightarrow P^\pm_n(x, \{\rho, w\})$$

(Take C to be the common cluster of ρ and w.)

The condition on the left hand side being satisfied with probability asymptotically larger than $1 - 2\epsilon$ (by (ii) and choice of Λ),

$$\liminf_n \mathbb{P}\left[P^\pm_n(x, \{\rho, w\})\right] \geq 1 - 2\epsilon$$

Since this holds for all ϵ, the proof is over. □

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