

# Spatiotemporal chaos: the microscopic perspective

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Extended nonequilibrium systems can be studied in the framework of field theory or from dynamical systems perspective. Here we report numerical evidence that the sum of a well-defined number of instantaneous Lyapunov exponents for the complex Ginzburg-Landau equation is given by a simple function of the space average of the square of the macroscopic field. This relationship follows from an explicit formula for the time-dependent values of almost all the exponents.

The search for connections between theories and quantities defined at different scales is the essence of statistical physics and the backbone of condensed matter theory. It nourishes the field of turbulence and spatiotemporal chaos where there is interest in finding connections between dynamical characteristics such as fractal dimensions and Lyapunov exponents, and statistics of macroscopic quantities such as correlation lengths. Such connections have not only a theoretical value but also important practical consequences because it is much easier to study macroscopic quantities than to obtain dynamical characteristics, especially in experiments [1, 2].

In the last decade, statistical mechanics community has also been interested in relating dynamical characteristics of the system, e.g. Lyapunov exponents, KS entropy and fractal dimensions, with the macroscopic properties, such as transport coefficients or entropy production, both in the classical and quantum systems [3–11]. All deterministic systems studied within this perspective were finite dimensional. A natural question then arises if similar results can be obtained for spatially extended systems. For instance, one would like to know the statistical properties of the fluctuations of phase space contraction rate and of the entropy production in driven fluid systems. Infinite dimensionality of Navier-Stokes equations makes such inquiries a challenge, although some interesting conjectures have been proposed [12].

These considerations prompted us to consider the complex Ginzburg-Landau equation (CGL) and study the fluctuation properties of its phase space contraction and the connections to macroscopic quantities. CGL is a paradigmatic model of spatiotemporal chaos which in certain sense is intermediate between thermostated molecular dynamics models and realistic fluid systems. Due to its strong dissipative properties infinite dimensional CGL has a finite-dimensional attractor which can be appropriately described in terms of low spatial frequency Fourier modes [13].

In this Letter we show that, even though the phase space contraction rate in CGL is infinite, one can consider contraction rate of volumes restricted to the inertial

manifold, which is finite dimensional. This rate is equal to the sum of a finite number of instantaneous Lyapunov exponents. It turns out to be proportional to the macroscopic mass of the field. Thus we have found out a direct relation between the “microscopic” sum of a finite number of instantaneous Lyapunov exponents and “macroscopic” mass of the field. We explore the structure of the spectrum of Lyapunov exponents and instantaneous Lyapunov exponents and show an approximate formula for large part of the spectrum of instantaneous Lyapunov exponents. The statistical properties of the fluctuations of phase space contraction rates and its relations to other macroscopic entropy-like quantities will be reported in a follow-up article [14].

We consider one-dimensional cubic complex Ginzburg-Landau equation on an interval of length  $L$  with periodic boundary conditions:

$$A_t = \varepsilon A + (1 + ic_1)\Delta A - (1 + ic_2)|A|^2 A, \quad (1)$$

where all the coefficients  $\varepsilon, c_1, c_2$  are real numbers. For convenience, let us restrict to a finite dimensional truncation in Fourier base with  $N = 2K$  modes and write

$$A(x, t) = \sum_{n=-K}^K A_n(t) e^{i2\pi n x/L}.$$

From eq.(1) we obtain

$$\begin{aligned} \dot{A}_n = & \varepsilon A_n - \left(\frac{2\pi n}{L}\right)^2 (1 + ic_1) A_n \\ & - (1 + ic_2) \sum_{k+l-m=n} A_k A_l A_m^*. \end{aligned} \quad (2)$$

Note that  $A_K = A_{-K}$  due to periodicity. Writing  $A_n = B_n + iC_n$  where  $B_n$  and  $C_n$  are real we derive a formula for the phase space contraction rate  $\sigma = \text{div}_A \dot{A} = \sum_n \frac{\partial \dot{B}_n}{\partial B_n} + \frac{\partial \dot{C}_n}{\partial C_n}$  as well as the normalized phase space contraction rate  $\tilde{\sigma} := \sigma/N_{\text{modes}}$ , where  $N_{\text{modes}} = 2N = 4K$  is the number of real modes under examination:

$$\tilde{\sigma} = \frac{\sigma}{2N} = \varepsilon - 2\langle \varrho \rangle - \left(\frac{2\pi}{L}\right)^2 \frac{N^2 + 1}{12}, \quad (3)$$

where  $\langle \varrho \rangle = (1/L) \int_0^L dx |A|^2 = \sum_k |A_k|^2$ . Using  $a = L/N$  we get

$$\tilde{\sigma} = \varepsilon - 2\langle \varrho \rangle - \frac{\pi^2}{3a^2} \left(1 + \frac{1}{N^2}\right) \approx \varepsilon - 2\langle \varrho \rangle - \frac{\pi^2}{3a^2}.$$

The beauty of this result connecting the average macroscopic field  $\langle \varrho \rangle$  to the microscopic normalized phase space contraction rate  $\tilde{\sigma}$  is jeopardized by the last term that diverges when the spatial resolution  $N$  is increases. However, increasing the resolution only adds high frequency modes which are strongly damped. We show below that their contribution can be isolated and removed, as it is the case for zero-temperature entropy in spin systems.

We conjecture that there is a distinguished dimension such that the contraction rate of volumes restricted to this dimension are always finite and connected to the space averaged  $\varrho$  in a simple manner. These volumes are defined by the sum of an appropriate number of instantaneous Lyapunov exponents. Before supporting these claims let us recall the definitions of Lyapunov exponents and instantaneous Lyapunov exponents, and show how they connect to the volume contraction rates.

Consider a continuous time dynamical system defined by a set of differential equations  $\dot{x} = F(x)$ ,  $x \in \mathbb{R}^n$ . The solution of the system is given by the flow  $x_t = \Phi^t(x_0)$ ,  $t \in \mathbb{R}$ . Then the growth of an infinitesimal perturbation  $\delta x_0$  around  $x_0$  is governed by the linearization of the flow  $\delta x_t = D_{x_0} \Phi^t \cdot \delta x_0 = M(t, x_0) \cdot \delta x_0$ . The *fundamental matrix*  $M(t, x_0)$  governing this growth is the solution of the equation  $\dot{M}(t, x_0) = J(t, x_0) \cdot M(t, x_0)$ , where  $J(t, x_0) = \frac{\partial F}{\partial x}(\Phi^t(x_0))$  is the Jacobi matrix of partial derivatives of the field velocity. *Oseledec matrix*  $(M(x_0, t)^\dagger M(x_0, t))^{1/2t}$  has  $n$  positive eigenvalues  $\Lambda_i(x_0, t)$  which we order by size  $\Lambda_1 \geq \Lambda_2 \geq \dots \geq \Lambda_n$ . *Lyapunov exponents*  $\lambda_i(x_0)$  are defined as logarithms of eigenvalues of long time limit of Oseledec matrix  $\lambda_i(x_0) := \lim_{t \rightarrow \infty} \ln \Lambda_i(x_0, t)$ . For an ergodic system, Lyapunov exponents are the same for almost every initial point [15, 16].

To define *instantaneous Lyapunov exponents* [17]  $\mu_i$  consider volume  $V_k(t)$  of a parallelogram  $u_1(x_0, t) \wedge u_2(x_0, t) \wedge \dots \wedge u_k(x_0, t)$ , spanned initially by  $k$  orthogonal vectors  $\tilde{u}_i$  attached at  $x_0$ , travelling along the trajectory;  $u_i \in \mathbb{R}^n$ . Its evolution is given by the fundamental matrix, i.e.  $u_i(x_0, t) = M(x_0, t)\tilde{u}_i$ . Then the  $k \times n$  matrix  $U = [u_1, \dots, u_k]$  can be uniquely decomposed into a product of  $k \times n$  orthogonal matrix  $Q$  and upper-diagonal  $k \times k$  matrix  $R$  (*QR decomposition*)

$$U = QR = [Q_1, \dots, Q_k] \begin{bmatrix} R_{11} & R_{12} & \dots & R_{1k} \\ 0 & R_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & R_{kk-1} \\ 0 & \dots & 0 & R_{kk} \end{bmatrix}.$$

The product of the diagonal elements of  $R$  gives the vol-

ume spanned by  $u_i$ . Its contraction rate is

$$\sigma_k(t) := \lim_{dt \rightarrow 0} \frac{1}{dt} \ln \frac{V_k(t+dt)}{V_k(t)} = \frac{\dot{V}_k(t)}{V_k(t)}.$$

We define instantaneous exponents by  $\mu_k(t) := \sigma_k(t) - \sigma_{k-1}(t)$ . They depend on the initial point and on the initial vectors  $\tilde{u}_i$ . However, for almost all initial vectors, the first vector with time aligns along the most unstable direction, the first two vectors span the fastest stretching 2d volumes, and so on. Therefore, after some time the vectors become almost independent of the initial directions modulo degeneracy, and consequently the instantaneous Lyapunov exponents characterize the trajectory.

In practice, we propagate the vectors by finite time steps at each time reorthogonalizing the set. Thus starting from  $Q_0 \equiv U$  we move to  $U_1 = M(dt)Q_0 \equiv Q_1 R_1$ . Then  $U_{n+1} = M(dt, x(ndt, x_0))Q_n \equiv Q_{n+1} R_{n+1}$ . Thus we have  $R(n \cdot dt) = \tilde{R}_n \cdot \dots \cdot \tilde{R}_1$  and  $\mu_k(n \cdot dt) \approx \frac{1}{dt} \ln[\tilde{R}_n]_{kk}$ . Time averages of  $\mu_i$  are sorted in decreasing order and equal to the usual Lyapunov exponents  $\lambda_i$  [15, 16].

To estimate the values of the instantaneous Lyapunov exponents, we consider an initial perturbation  $\delta A_n$  tangent to a single mode  $A_n^0$ . Inserting  $A_n = A_n^0 + \delta A_n$  into eq. (1), we obtain

$$\begin{aligned} \partial \delta A_n / \partial t &= (\varepsilon - (1 + ic_1)q^2 - 2(1 + ic_2)\langle \varrho \rangle) \delta A_n \\ &+ (1 + ic_2)(\alpha_n + i\beta_n) A_n^* + f(\delta A) \end{aligned} \quad (4)$$

where  $q := 2\pi n/L$ ,  $f(\delta A)$  is a linear function of  $\{\delta A\}$  not depending on  $\delta A_n$  or  $\delta A_n^*$ , and  $\alpha_n, \beta_n \in \mathbb{R}$  stand for the real and imaginary part of the time dependent sum

$$\alpha_n + i\beta_n = \sum_{j=-(K-|n|)}^{K-|n|} A_{n-j} A_{n+j}.$$

Rewriting the equation for real and imaginary parts of  $A_n = B_n + iC_n$ , we can obtain short time evolution of tangent vectors

$$\begin{bmatrix} \delta B_n(dt) \\ \delta C_n(dt) \end{bmatrix} = [a_0 I + a_i \sigma_i] \begin{bmatrix} \delta B_n(0) \\ \delta C_n(0) \end{bmatrix},$$

where  $\sigma_i$  are the Pauli matrices [18] and  $a_0 = 1 + (\varepsilon - q^2 - 2\langle \varrho \rangle)dt$ ,  $a_x = \beta_n + c_2 \alpha_n$ ,  $a_y = i(c_1 q^2 + 2c_2 \langle \varrho \rangle)$ ,  $a_z = \alpha_n - c_2 \beta_n$ . Then the eigenvalues of  $M^\dagger M(dt)$  are  $\Lambda_\pm = 1 + 2(\varepsilon - q^2 - 2\langle \varrho \rangle) \pm \sqrt{1 + c_2^2} \left| \sum_{j=-(K-|n|)}^{K-|n|} A_{n-j} A_{n+j} \right|$  which gives extremum possible values of instantaneous Lyapunov exponents

$$\mu_{n\pm} = \varepsilon - q^2 - 2\langle \varrho \rangle \pm \sqrt{1 + c_2^2} \left| \sum_{j=-(K-|n|)}^{K-|n|} A_{n-j} A_{n+j} \right|.$$

Observed values depend on initial vector  $[\delta B_n \ \delta C_n]^T$  and are between  $\mu_{n\pm}$ . To find out what is the contraction rate of the 2d volumes in  $\delta A_n$  plane consider the action of

$M(dt)$  restricted to  $\delta A_n$  on a pair of initially orthogonal vectors  $[v_1, v_2]$  in this plane. The volume of  $M(dt)[v_1, v_2]$  is given by the determinant, and since  $\det[v_1, v_2] = 1$ , we have

$$\frac{1}{2}(\mu_{n1} + \mu_{n2}) = \lim_{dt \rightarrow 0} \frac{1}{dt} \ln \det M(dt) = \varepsilon - q^2 - 2\langle \varrho \rangle. \quad (6)$$

Therefore, at any time we predict that the sum of the two instantaneous Lyapunov exponents for perturbations in the plane tangent to any Fourier mode should be given by above formula.

It is not *a priori* obvious that this prediction holds for *any* exponents calculated for volumes evolved over long time span, since we have considered the evolution along  $(A_n, A_n^*)$  only. In fact, realistic evolution mixes all the modes via the nonlinear term in (1), and vectors align arbitrarily in phase space along their evolution. However, our numerical simulations show that there is only a finite number of modes  $W$ , which we call “active”, behaving in apparently random (though smooth) fashion, which disobey the above prediction (Figure 1). The remaining exponents in their time course oscillate around (6) and at every instant the sum of the four modes for  $\pm n$  is given by (6). Sorting the instantaneous Lyapunov exponents according to their time averages (i.e. Lyapunov exponents  $\lambda_i$ ), the “active” instantaneous Lyapunov exponents are the first  $W$  curves. The “active” modes include those tangent to the inertial manifold, the remaining exponents describe the decay towards the attractor.

There remains the question of obtaining  $W$ . Our numerical results show that  $W$  is the smallest number of the form  $4n + 2$  greater or equal to  $L$ , i.e.  $W = 2 + 4\lceil (L - 2)/4 \rceil$ , where  $\lceil n \rceil$  is  $n$  rounded upwards. This is larger than the Kaplan-Yorke dimension of the attractor, and sometimes much larger. Therefore we believe our procedure probe the fluctuation of volumes on the inertial manifold, not on the attractor, which is a subset of it.  $W$  correspond in real space to a given size, coherent objects of size larger then  $L(W - 2)/(4.2\pi)$  drives the dynamics while structures of lower size are slaved to them.

We have studied the spectra of Lyapunov exponents and instantaneous Lyapunov exponents for CGLe for several parameter values and truncations to 32, 64, 128 and 256 Fourier modes or equivalent numbers of spatial points. Figure 1 shows the time dependence of the first 46 instantaneous Lyapunov exponents computed along a trajectory for CGLe with  $L = 10\pi$  ( $q = 0.2$ ),  $c_1 = 4$ ,  $c_2 = -4$  [19] for a computation with 128 Fourier modes; both groups of exponents are clearly visible. There is a constant difference between theoretical prediction and the numerical value of the order 0.15 for the first inactive exponents ( $n = 9$  in this case), dropping to around 0.01 for  $n = 25$ .

Figure 2 shows the spectrum of average instantaneous Lyapunov exponents, i. e. of Lyapunov exponents, for

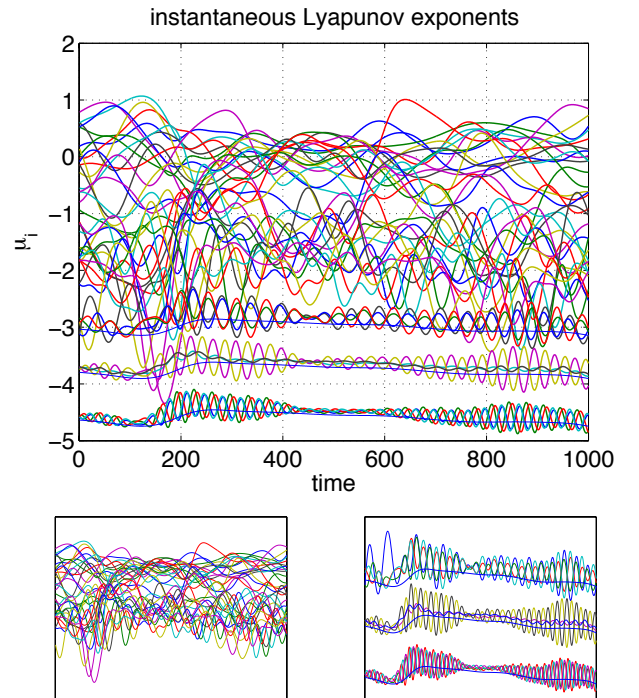


FIG. 1: Time dependence of all the 34 “active” and the largest 12 “inactive” instantaneous Lyapunov exponents for CGLe with  $L = 10\pi$ ,  $c_1 = 4$ ,  $c_2 = -4$ . The bottom figures show two subsets of curves from the upper plot, the first 34 (left) and the next 12 exponents (right). Prediction (6) is also plotted.

CGLe with the same parameter values. Crosses are theoretically predicted values (6), circles are numerical values. The staircase structure in the spectrum is well approximated by (6).

To separate the changes of volumes on the inertial manifold from the trivial contraction of infinite dimensional phase space onto finite dimensional inertial manifold we consider contraction of  $W$  dimensional volumes given by the sum of all the nontrivial instantaneous Lyapunov exponents. Since the sum of all the instantaneous Lyapunov exponents in the Galerkin representation is equal to the phase space contraction rate (3), and since the “inactive” exponents on the average follow the average field (6), the relevant value is

$$\tilde{\sigma}_{\text{active}} := \frac{1}{W} \sum_{i=1}^W \mu_i = \varepsilon - 2\langle \varrho \rangle - \frac{\pi^2(W^2 - 4)}{12L^2}. \quad (7)$$

This is approximately  $\tilde{\sigma}_{\text{active}} \approx \varepsilon - 2\langle \varrho \rangle - \frac{\pi^2(L-2)^2}{12L^2} \approx -2\langle \varrho \rangle$ . Figure 3 compares the evolution of  $0.15 + \frac{1}{W} \sum_{i=1}^W \mu_i$  with  $\varepsilon - 2\langle \varrho \rangle - \frac{\pi^2(W^2-4)}{12L^2}$ .

To summarize, we have shown that the phase space contraction rate in Galerkin approximation to the complex Ginzburg-Landau equation is given by a simple function of the spatial average of the squared modulus  $\varrho$  of the solution. The divergence occurring when increasing

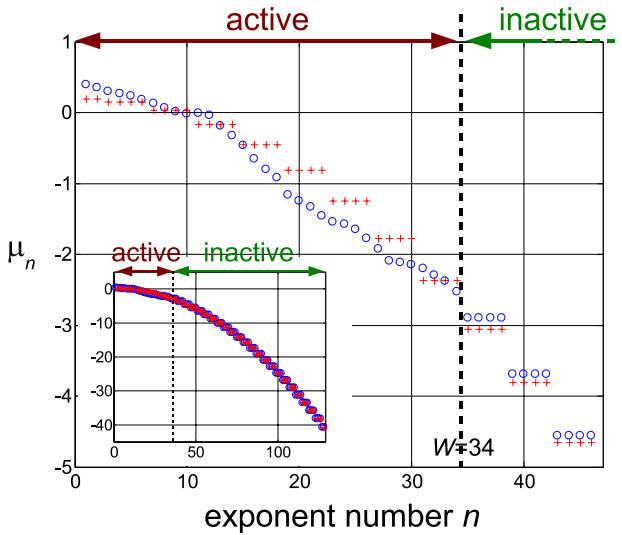


FIG. 2: Spectrum of Lyapunov exponents for CGLE with  $L = 10\pi$ ,  $c_1 = 4$ ,  $c_2 = -4$ .  $N = 64$  modes, there are 128 exponents (all shown in the inset). The quadruplet structure of the lower exponents comes from the real and complex parts of corresponding positive and negative Fourier modes. Crosses are theoretical prediction (6), circles are numerical values. There are  $W = 34$  “active” exponents.

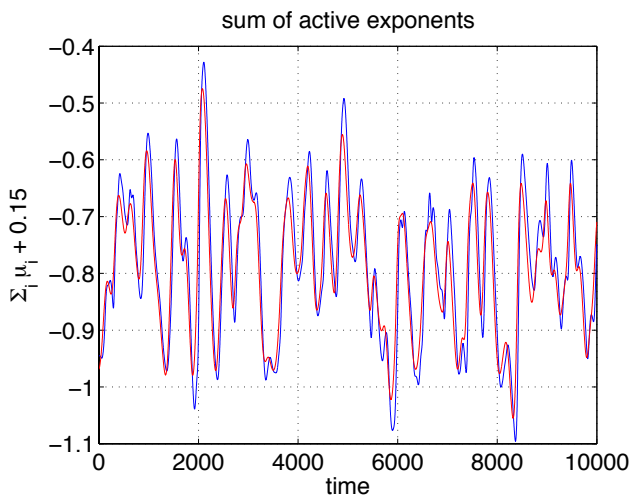


FIG. 3: Numerical test of formula (7). The time evolution of the sum of the first  $W=34$  instantaneous Lyapunov exponents (active modes; in red) are compared with the time evolution of  $\varepsilon - 2\langle\rho\rangle - \frac{\pi^2(W^2-4)}{12L^2}$  (in blue). The second term has been shifted by  $-0.15$ .

spatial resolution can be removed by restricting the contracting volumes to a finite number of dimensions. The corresponding volume contraction rate is given by the sum of a finite number of Lyapunov exponents which time-behavior is non-trivial. We have identified a natural division of the spectrum into the part corresponding to the dynamics on the inertial manifold and the other part corresponding to the modes decaying towards the

attractor. Instantaneous Lyapunov exponents in the second part are approximately given by simple functions of the square of the Ginzburg-Landau field (6). We have found a formula for the volume space contraction rate on the inertial manifold (7). The formula bridges the gap between the dynamical systems picture of the CGLE (volumes contracting in the phase space and instantaneous Lyapunov exponents) and the macroscopic picture (spatio-temporal solution).

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