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# On the critical value function in the divide and color model

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**Abstract.** The divide and color model on a graph  $G$  arises by first deleting each edge of  $G$  with probability  $1 - p$  independently of each other, then coloring the resulting connected components (*i.e.*, every vertex in the component) black or white with respective probabilities  $r$  and  $1 - r$ , independently for different components. Viewing it as a (dependent) site percolation model, one can define the critical point  $r_c^G(p)$ .

In this paper, we mainly study the continuity properties of the function  $r_c^G$ , which is an instance of the question of locality for percolation. Our main result is the fact that in the case  $G = \mathbb{Z}^2$ ,  $r_c^G$  is continuous on the interval  $[0, 1/2)$ ; we also prove continuity at  $p = 0$  for the more general class of graphs with bounded degree. We then investigate the sharpness of the bounded degree condition and the monotonicity of  $r_c^G(p)$  as a function of  $p$ .

## 1. Introduction

The divide and color (DaC) model is a natural dependent site percolation model introduced by Häggström (2001). It has been studied directly in Häggström (2001); Garett (2001); Bálint et al. (2009), and as a member of a more general family of models in Kahn and Weininger (2007); Bálint et al. (2009); Bálint (2010); Graham and Grimmett (2011). This model is defined on a multigraph  $G = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{E}$  is a multiset (*i.e.*, it may contain an element more than once), thus allowing parallel edges between pairs of vertices. For simplicity, we will imprecisely call  $G$  a *graph*

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and  $\mathcal{E}$  the *edge set*, even if  $G$  contains self-loops or multiple edges. The DaC model with parameters  $p, r \in [0, 1]$ , on a general (finite or infinite) graph  $G$  with vertex set  $\mathcal{V}$  and edge set  $\mathcal{E}$ , is defined by the following two-step procedure:

- First step: Bernoulli bond percolation. We independently declare each edge in  $\mathcal{E}$  to be open with probability  $p$ , and closed with probability  $1 - p$ . We can identify a bond percolation configuration with an element  $\eta \in \{0, 1\}^{\mathcal{E}}$ : for each  $e \in \mathcal{E}$ , we define  $\eta(e) = 1$  if  $e$  is open, and  $\eta(e) = 0$  if  $e$  is closed.
- Second step: Bernoulli site percolation on the resulting cluster set. Given  $\eta \in \{0, 1\}^{\mathcal{E}}$ , we call *p-clusters* or *bond clusters* the connected components in the graph with vertex set  $\mathcal{V}$  and edge set  $\{e \in \mathcal{E} : \eta(e) = 1\}$ . The set of *p-clusters* of  $\eta$  gives a partition of  $\mathcal{V}$ . For each *p-cluster*  $\mathcal{C}$ , we assign the same color to all the vertices in  $\mathcal{C}$ . The chosen color is black with probability  $r$  and white with probability  $1 - r$ , and this choice is independent for different *p-clusters*.

These two steps yield a site percolation configuration  $\xi \in \{0, 1\}^{\mathcal{V}}$  by defining, for each  $v \in \mathcal{V}$ ,  $\xi(v) = 1$  if  $v$  is black, and  $\xi(v) = 0$  if  $v$  is white. The connected components (via the edge set  $\mathcal{E}$ ) in  $\xi$  of the same color are called (black or white) *r-clusters*. The resulting measure on  $\{0, 1\}^{\mathcal{V}}$  is denoted by  $\mu_{p,r}^G$ .

Let  $E_{\infty}^b \subset \{0, 1\}^{\mathcal{V}}$  denote the event that there exists an infinite black *r-cluster*. By standard arguments (see Proposition 2.5 in Häggström (2001)), for each  $p \in [0, 1]$ , there exists a *critical coloring value*  $r_c^G(p) \in [0, 1]$  such that

$$\mu_{p,r}^G(E_{\infty}^b) \begin{cases} = 0 & \text{if } r < r_c^G(p), \\ > 0 & \text{if } r > r_c^G(p). \end{cases}$$

The *critical edge parameter*  $p_c^G \in [0, 1]$  is defined as follows: the probability that there exists an infinite bond cluster is 0 for all  $p < p_c^G$ , and positive for all  $p > p_c^G$ . The latter probability is in fact 1 for all  $p > p_c^G$ , whence  $r_c^G(p) = 0$  for all such  $p$ . Kolmogorov's 0 - 1 law shows that in the case when all the bond clusters are finite,  $\mu_{p,r}^G(E_{\infty}^b) \in \{0, 1\}$ ; nevertheless it is possible that  $\mu_{p,r}^G(E_{\infty}^b) \in (0, 1)$  for some  $r > r_c^G(p)$  (e.g. on the square lattice, as soon as  $p > p_c = 1/2$ , one has  $\mu_{p,r}^G(E_{\infty}^b) = r$ ).

*Statement of the results.* Our main goal in this paper is to understand how the critical coloring parameter  $r_c^G$  depends on the edge parameter  $p$ . Since the addition or removal of self-loops obviously does not affect the value of  $r_c^G(p)$ , we will assume that all the graphs  $G$  that we consider are without self-loops. On the other hand,  $G$  is allowed to contain multiple edges.

Our first result, based on a stochastic domination argument, gives bounds on  $r_c^G(p)$  in terms of  $r_c^G(0)$ , which is simply the critical value for Bernoulli site percolation on  $G$ . By the *degree* of a vertex  $v$ , we mean the number of edges incident on  $v$  (counted with multiplicity).

**Proposition 1.1.** *For any graph  $G$  with maximal degree  $\Delta$ , for all  $p \in [0, 1]$ ,*

$$1 - \frac{1 - r_c^G(0)}{(1 - p)^{\Delta}} \leq r_c^G(p) \leq \frac{r_c^G(0)}{(1 - p)^{\Delta}}.$$

As a direct consequence, we get continuity at  $p = 0$  of the critical value function:

**Proposition 1.2.** *For any graph  $G$  with bounded degree,  $r_c^G(p)$  is continuous in  $p$  at 0.*

One could think of an alternative approach to the question, as follows: the DaC model can be seen as Bernoulli site percolation of the random graph  $G_p = (V_p, E_p)$  where  $V_p$  is the set of bond clusters and two bond clusters are connected by a bond of  $E_p$  if and only if they are adjacent in the original graph. The study of how  $r_c^G(p)$  depends on  $p$  is then a particular case of a more general question known as the *locality problem*: is it true in general that the critical points of site percolation on a graph and a small perturbation of it are always close? Here, for small  $p$ , the graphs  $G$  and  $G_p$  are somehow very similar, and their critical points are indeed close.

Dropping the bounded-degree assumption allows for the easy construction of graphs for which continuity does not hold at  $p = 0$ :

**Proposition 1.3.** *There exists a graph  $G$  with  $p_c^G > 0$  such that  $r_c^G$  is discontinuous at 0.*

In general, when  $p > 0$ , the graph  $G_p$  does not have bounded degree, even if  $G$  does; this simple remark can be exploited to construct bounded degree graphs for which  $r_c^G$  has discontinuities below the critical point of bond percolation (though of course not at 0):

**Theorem 1.4.** *There exists a graph  $G$  of bounded degree satisfying  $p_c^G > 1/2$  and such that  $r_c^G(p)$  is discontinuous at  $1/2$ .*

*Remark 1.5.* The value  $1/2$  in the statement above is not special: in fact, for every  $p_0 \in (0, 1)$ , it is possible to generalize our argument to construct a graph with a critical bond parameter above  $p_0$  and for which the discontinuity of  $r_c$  occurs at  $p_0$ .

Our main results concerns the case  $G = \mathbb{Z}^2$ , for which the above does not occur:

**Theorem 1.6.** *The critical coloring value  $r_c^{\mathbb{Z}^2}(p)$  is a continuous function of  $p$  on the whole interval  $[0, 1/2)$ .*

The other, perhaps more anecdotal question we investigate here is whether  $r_c^G$  is monotonic below  $p_c$ . This is the case on the triangular lattice (because it is constant equal to  $1/2$ ), and appears to hold on  $\mathbb{Z}^2$  in simulations (see the companion paper [Bálint et al. \(2013\)](#)).

In the general case, the question seems to be rather delicate. Intuitively the presence of open edges would seem to make percolation easier, leading to the intuition that the function  $p \mapsto r_c(p)$  should be nonincreasing. Theorem 2.9 in [Häggström \(2001\)](#) gives a counterexample to this intuition. It is even possible to construct quasi-transitive graphs on which any monotonicity fails:

**Proposition 1.7.** *There exists a quasi-transitive graph  $G$  such that  $r_c^G$  is not monotone on the interval  $[0, p_c^G)$ .*

A brief outline of the paper is as follows. We set the notation and collect a few results from the literature in Section 2. In Section 3, we stochastically compare  $\mu_{p,r}^G$  with Bernoulli site percolation (Theorem 3.1), and show how this result implies Proposition 1.1. We then turn to the proof of Theorem 1.6 in Section 4, based on a finite-size argument and the continuity of the probability of cylindrical events.

In Section 5, we determine the critical value function for a class of tree-like graphs, and in the following section we apply this to construct most of the examples of graphs we mentioned above.

## 2. Definitions and notation

We start by explicitly constructing the model, in a way which will be more technically convenient than the intuitive one given in the introduction.

Let  $G$  be a connected graph  $(\mathcal{V}, \mathcal{E})$  where the set of vertices  $\mathcal{V} = \{v_0, v_1, v_2, \dots\}$  is countable. We define a total order “ $<$ ” on  $\mathcal{V}$  by saying that  $v_i < v_j$  if and only if  $i < j$ . In this way, for any subset  $V \subset \mathcal{V}$ , we can uniquely define  $\min(V) \in V$  as the minimal vertex in  $V$  with respect to the relation “ $<$ ”. For a set  $S$ , we denote  $\{0, 1\}^S$  by  $\Omega_S$ . We call the elements of  $\Omega_{\mathcal{E}}$  *bond configurations*, and the elements of  $\Omega_{\mathcal{V}}$  *site configurations*. As defined in the Introduction, in a bond configuration  $\eta$ , an edge  $e \in \mathcal{E}$  is called *open* if  $\eta(e) = 1$ , and *closed* otherwise; in a site configuration  $\xi$ , a vertex  $v \in \mathcal{V}$  is called *black* if  $\xi(v) = 1$ , and *white* otherwise. Finally, for  $\eta \in \Omega_{\mathcal{E}}$  and  $v \in \mathcal{V}$ , we define the *bond cluster*  $\mathcal{C}_v(\eta)$  of  $v$  as the maximal connected induced subgraph containing  $v$  of the graph with vertex set  $\mathcal{V}$  and edge set  $\{e \in \mathcal{E} : \eta(e) = 1\}$ , and denote the vertex set of  $\mathcal{C}_v(\eta)$  by  $C_v(\eta)$ .

For  $a \in [0, 1]$  and a set  $S$ , we define  $\nu_a^S$  as the probability measure on  $\Omega_S$  that assigns to each  $s \in S$  value 1 with probability  $a$  and 0 with probability  $1 - a$ , independently for different elements of  $S$ . We define a function

$$\begin{aligned} \Phi : \Omega_{\mathcal{E}} \times \Omega_{\mathcal{V}} &\rightarrow \Omega_{\mathcal{E}} \times \Omega_{\mathcal{V}}, \\ (\eta, \kappa) &\mapsto (\eta, \xi), \end{aligned}$$

where  $\xi(v) = \kappa(\min(C_v(\eta)))$ . For  $p, r \in [0, 1]$ , we define  $\mathbb{P}_{p,r}^G$  to be the image measure of  $\nu_p^{\mathcal{E}} \otimes \nu_r^{\mathcal{V}}$  by the function  $\Phi$ , and denote by  $\mu_{p,r}^G$  the marginal of  $\mathbb{P}_{p,r}^G$  on  $\Omega_{\mathcal{V}}$ . Note that this definition of  $\mu_{p,r}^G$  is consistent with the one in the Introduction.

Finally, we give a few definitions and results that are necessary for the analysis of the DaC model on the square lattice, that is the graph with vertex set  $\mathbb{Z}^2$  and edge set  $\mathcal{E}^2 = \{(v, w) : v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{Z}^2, |v_1 - w_1| + |v_2 - w_2| = 1\}$ . The *matching graph*  $\mathbb{Z}_*^2$  of the square lattice is the graph with vertex set  $\mathbb{Z}^2$  and edge set  $\mathcal{E}_*^2 = \{(v, w) : v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{Z}^2, \max(|v_1 - w_1|, |v_2 - w_2|) = 1\}$ . In the same manner as in the Introduction, we define, for a color configuration  $\xi \in \{0, 1\}^{\mathbb{Z}^2}$ , (black or white) *\*-clusters* as connected components (via the edge set  $\mathcal{E}_*^2$ ) in  $\xi$  of the same color. We denote by  $\Theta^*(p, r)$  the  $\mathbb{P}_{p,r}^{\mathbb{Z}^2}$ -probability that the origin is contained in an infinite black *\*-cluster*, and define

$$r_c^*(p) = \sup\{r : \Theta^*(p, r) = 0\}$$

for all  $p \in [0, 1]$  — note that this value may differ from  $r_c^{\mathbb{Z}_*^2}(p)$ . The main result in [Bálint et al. \(2009\)](#) is that for all  $p \in [0, 1/2)$ , the critical values  $r_c^{\mathbb{Z}^2}(p)$  and  $r_c^*(p)$  satisfy the duality relation

$$r_c^{\mathbb{Z}^2}(p) + r_c^*(p) = 1. \tag{2.1}$$

We will also use exponential decay result for subcritical Bernoulli bond percolation on  $\mathbb{Z}^2$ . Let  $\mathbf{0}$  denote the origin in  $\mathbb{Z}^2$ , and for each  $n \in \mathbb{N} = \{1, 2, \dots\}$ , let us define  $S_n = \{v \in \mathbb{Z}^2 : \text{dist}(v, \mathbf{0}) = n\}$  (where *dist* denotes graph distance), and the event  $M_n = \{\eta \in \Omega_{\mathcal{E}^2} : \text{there is a path of open edges in } \eta \text{ from } \mathbf{0} \text{ to } S_n\}$ . Then we have the following result:

**Theorem 2.1** ([Kesten \(1980\)](#)). *For  $p < 1/2$ , there exists  $\psi(p) > 0$  such that for all  $n \in \mathbb{N}$ , we have that*

$$\nu_p^{\mathcal{E}^2}(M_n) < e^{-n\psi(p)}.$$

### 3. Stochastic domination and continuity at $p = 0$

In this section, we prove Proposition 1.1 via a stochastic comparison between the DaC measure and Bernoulli site percolation. Before stating the corresponding result, however, let us recall the concept of stochastic domination.

We define a natural partial order on  $\Omega_{\mathcal{V}}$  by saying that  $\xi' \geq \xi$  for  $\xi, \xi' \in \Omega_{\mathcal{V}}$  if, for all  $v \in \mathcal{V}$ ,  $\xi'(v) \geq \xi(v)$ . A random variable  $f : \Omega_{\mathcal{V}} \rightarrow \mathbb{R}$  is called *increasing* if  $\xi' \geq \xi$  implies that  $f(\xi') \geq f(\xi)$ , and an event  $E \subset \Omega_{\mathcal{V}}$  is increasing if its indicator random variable is increasing. For probability measures  $\mu, \mu'$  on  $\Omega_{\mathcal{V}}$ , we say that  $\mu'$  is *stochastically larger* than  $\mu$  (or, equivalently, that  $\mu$  is *stochastically smaller* than  $\mu'$ , denoted by  $\mu \leq_{st} \mu'$ ) if, for all bounded increasing random variables  $f : \Omega_{\mathcal{V}} \rightarrow \mathbb{R}$ , we have that

$$\int_{\Omega_{\mathcal{V}}} f(\xi) d\mu'(\xi) \geq \int_{\Omega_{\mathcal{V}}} f(\xi) d\mu(\xi).$$

By Strassen’s theorem (1965), this is equivalent to the existence of an appropriate coupling of the measures  $\mu'$  and  $\mu$ ; that is, the existence of a probability measure  $\mathbb{Q}$  on  $\Omega_{\mathcal{V}} \times \Omega_{\mathcal{V}}$  such that the marginals of  $\mathbb{Q}$  on the first and second coordinates are  $\mu'$  and  $\mu$  respectively, and  $\mathbb{Q}(\{(\xi', \xi) \in \Omega_{\mathcal{V}} \times \Omega_{\mathcal{V}} : \xi' \geq \xi\}) = 1$ .

**Theorem 3.1.** *For any graph  $G = (\mathcal{V}, \mathcal{E})$  whose maximal degree is  $\Delta$ , at arbitrary values of the parameters  $p, r \in [0, 1]$ ,*

$$\nu_{r(1-p)^\Delta}^{\mathcal{V}} \leq_{st} \mu_{p,r}^G \leq_{st} \nu_{1-(1-r)(1-p)^\Delta}^{\mathcal{V}}.$$

Before turning to the proof, we show how Theorem 3.1 implies Proposition 1.1. It follows from Theorem 3.1 and the definition of stochastic domination that for the increasing event  $E_\infty^b$  (which was defined in the Introduction), we have  $\mu_{p,r}^G(E_\infty^b) > 0$  whenever  $r(1-p)^\Delta > r_c^G(0)$ , which implies that  $r_c^G(p) \leq r_c^G(0)/(1-p)^\Delta$ . The derivation of the lower bound for  $r_c^G(p)$  is analogous.  $\square$

Now we give the proof of Theorem 3.1, which bears some resemblance with the proof of Theorem 2.3 in Häggström (2001). Fix  $G = (\mathcal{V}, \mathcal{E})$  with maximal degree  $\Delta$ , and parameter values  $p, r \in [0, 1]$ . We will use the relation “ $<$ ” and the minimum of a vertex set with respect to this relation as defined in Section 2. In what follows, we will define several random variables; we will denote the joint distribution of all these variables by  $\mathbb{P}$ .

First, we define a collection  $(\eta_{x,y}^e : x, y \in \mathcal{V}, e = \langle x, y \rangle \in \mathcal{E})$  of i.i.d. Bernoulli( $p$ ) random variables (*i.e.*, they take value 1 with probability  $p$ , and 0 otherwise); one may imagine having each edge  $e \in \mathcal{E}$  replaced by two directed edges, and the random variables represent which of these edges are open. We define also a set  $(\kappa_x : x \in \mathcal{V})$  of Bernoulli( $r$ ) random variables. Given a realization of  $(\eta_{x,y}^e : x, y \in \mathcal{V}, e = \langle x, y \rangle \in \mathcal{E})$  and  $(\kappa_x : x \in \mathcal{V})$ , we will define an  $\Omega_{\mathcal{V}} \times \Omega_{\mathcal{E}}$ -valued random configuration  $(\eta, \xi)$  with distribution  $\mathbb{P}_{p,r}^G$ , by the following algorithm.

- (1) Let  $v = \min\{x \in \mathcal{V} : \text{no } \xi\text{-value has been assigned yet to } x \text{ by this algorithm}\}$ . (Note that  $v$  and  $V, v_i, H_i$  ( $i \in \mathbb{N}$ ), defined below, are running variables, *i.e.*, their values will be redefined in the course of the algorithm.)
- (2) We explore the “directed open cluster”  $V$  of  $v$  iteratively, as follows. Define  $v_0 = v$ . Given  $v_0, v_1, \dots, v_i$  for some integer  $i \geq 0$ , set  $\eta(e) = \eta_{v_i,w}^e$  for every edge  $e = \langle v_i, w \rangle \in \mathcal{E}$  incident to  $v_i$  such that no  $\eta$ -value has been assigned yet to  $e$  by the algorithm, and write  $H_{i+1} = \{w \in \mathcal{V} \setminus \{v_0, v_1, \dots, v_i\} : w \text{ can be reached from any of } v_0, v_1, \dots, v_i \text{ by using only those edges } e \in \mathcal{E}$

such that  $\eta(e) = 1$  has been assigned to  $e$  by this algorithm}. If  $H_{i+1} \neq \emptyset$ , then we define  $v_{i+1} = \min(H_{i+1})$ , and continue exploring the directed open cluster of  $v$ ; otherwise, we define  $V = \{v_0, v_1, \dots, v_i\}$ , and move to step 3.

(3) Define  $\xi(w) = \kappa_v$  for all  $w \in V$ , and return to step 1.

It is immediately clear that the above algorithm eventually assigns a  $\xi$ -value to each vertex. Note also that a vertex  $v$  can receive a  $\xi$ -value only after all edges incident to  $v$  have already been assigned an  $\eta$ -value, which shows that the algorithm eventually determines the full edge configuration as well. It is easy to convince oneself that  $(\eta, \xi)$  obtained this way indeed has the desired distribution.

Now, for each  $v \in \mathcal{V}$ , we define  $Z(v) = 1$  if  $\kappa_v = 1$  and  $\eta_{w,v}^e = 0$  for all edges  $e = \langle v, w \rangle \in \mathcal{E}$  incident on  $v$  (i.e., all directed edges towards  $v$  are closed), and  $Z(v) = 0$  otherwise. Note that every vertex with  $Z(v) = 1$  has  $\xi(v) = 1$  as well, whence the distribution of  $\xi$  (i.e.,  $\mu_{p,r}^G$ ) stochastically dominates the distribution of  $Z$  (as witnessed by the coupling  $\mathbb{P}$ ).

Notice that  $Z(v)$  depends only on the states of the edges pointing to  $v$  and on the value of  $\kappa_v$ ; in particular the distribution of  $Z$  is a product measure on  $\Omega_{\mathcal{V}}$  with parameter  $r(1-p)^{d(v)}$  at  $v$ , where  $d(v) \leq \Delta$  is the degree of  $v$ , whence  $\mu_{p,r}^G$  stochastically dominates the product measure on  $\Omega_{\mathcal{V}}$  with parameter  $r(1-p)^\Delta$ , which gives the desired stochastic lower bound. The upper bound can be proved analogously; alternatively, it follows from the lower bound by exchanging the roles of black and white.  $\square$

#### 4. Continuity of $r_c^{\mathbb{Z}^2}(p)$ on the interval $[0, 1/2)$

In this section, we will prove Theorem 1.6. Our first task is to prove a technical result valid on more general graphs stating that the probability of any event  $A$  whose occurrence depends on a finite set of  $\xi$ -variables is a continuous function of  $p$  for  $p < p_c^G$ . The proof relies on the fact that although the color of a vertex  $v$  may be influenced by edges arbitrarily far away, if  $p < p_c^G$ , the corresponding influence decreases to 0 in the limit as we move away from  $v$ . Therefore, the occurrence of the event  $A$  depends essentially on a finite number of  $\eta$ - and  $\kappa$ -variables, whence its probability can be approximated up to an arbitrarily small error by a polynomial in  $p$  and  $r$ .

Once we have proved Proposition 4.1 below, which is valid on general graphs, we will apply it on  $\mathbb{Z}^2$  to certain “box-crossing events,” and appeal to results in Bálint et al. (2009) to deduce the continuity of  $r_c^{\mathbb{Z}^2}(p)$ .

**Proposition 4.1.** *For every site percolation event  $A \subset \{0, 1\}^{\mathcal{V}}$  depending on the color of finitely many vertices,  $\mu_{p,r}^G(A)$  is a continuous function of  $(p, r)$  on the set  $[0, p_c^G) \times [0, 1]$ .*

*Proof.* In this proof, when  $\mu$  is a measure on a set  $S$ ,  $X$  is a random variable with law  $\mu$  and  $F : S \rightarrow \mathbb{R}$  is a bounded measurable function, we write abusively  $\mu[F(X)]$  for the expectation of  $F(X)$ . We show a slightly more general result: for any  $k \geq 1$ ,  $\mathbf{x} = (x_1, \dots, x_k) \in \mathcal{V}^k$  and  $f : \{0, 1\}^k \rightarrow \mathbb{R}$  bounded and measurable,  $\mu_{p,r}^G[f(\xi(x_1), \dots, \xi(x_k))]$  is continuous in  $(p, r)$  on the product  $[0, p_c^G) \times [0, 1]$ . Proposition 4.1 will follow by choosing an appropriate family  $\{x_1, \dots, x_k\}$  such that the states of the  $x_i$  suffices to determine whether  $A$  occurs, and take  $f$  to be the indicator function of  $A$ .

To show the previous affirmation, we condition on the vector

$$\mathbf{m}_{\mathbf{x}}(\eta) = (\min C_{x_1}(\eta), \dots, \min C_{x_k}(\eta))$$

which takes values in the finite set

$$\mathbf{V} = \{(v_1, \dots, v_k) \in \mathcal{V}^k : \forall i v_i \leq \max\{x_1, \dots, x_k\}\},$$

and we use the definition of  $\mathbb{P}_{p,r}^G$  as an image measure. By definition,

$$\begin{aligned} \mu_{p,r}^G [f(\xi(x_1), \dots, \xi(x_k))] &= \sum_{\mathbf{v} \in \mathbf{V}} \mathbb{P}_{p,r}^G [f(\xi(x_1), \dots, \xi(x_k)) | \{\mathbf{m}_{\mathbf{x}} = \mathbf{v}\}] \mathbb{P}_{p,r}^G [\{\mathbf{m}_{\mathbf{x}} = \mathbf{v}\}] \\ &= \sum_{\mathbf{v} \in \mathbf{V}} \nu_p^\varepsilon \otimes \nu_r^\nu [f(\kappa(v_1), \dots, \kappa(v_k)) | \{\mathbf{m}_{\mathbf{x}} = \mathbf{v}\}] \nu_p^\varepsilon [\{\mathbf{m}_{\mathbf{x}} = \mathbf{v}\}] \\ &= \sum_{\mathbf{v} \in \mathbf{V}} \nu_r^\nu [f(\kappa(v_1), \dots, \kappa(v_k))] \nu_p^\varepsilon [\{\mathbf{m}_{\mathbf{x}} = \mathbf{v}\}]. \end{aligned}$$

Note that  $\nu_r^\nu [f(\kappa(v_1), \dots, \kappa(v_k))]$  is a polynomial in  $r$ , so to conclude the proof we only need to prove that for any fixed  $\mathbf{x}$  and  $\mathbf{v}$ ,  $\nu_p^\varepsilon (\{\mathbf{m}(\mathbf{x}) = \mathbf{v}\})$  depends continuously on  $p$  on the interval  $[0, p_c^G]$ .

For  $n \geq 1$ , write  $F_n = \{|C_{x_1}| \leq n, \dots, |C_{x_k}| \leq n\}$ . It is easy to verify that the event  $\{\mathbf{m}_{\mathbf{x}} = \mathbf{v}\} \cap F_n$  depends on the state of finitely many edges. Hence,  $\nu_p^\varepsilon [\{\mathbf{m}_{\mathbf{x}} = \mathbf{v}\} \cap F_n]$  is a polynomial function of  $p$ .

Fix  $p_0 < p_c^G$ . For all  $p \leq p_0$ ,

$$\begin{aligned} 0 \leq \nu_p^\varepsilon [\{\mathbf{m}(\mathbf{x}) = \mathbf{v}\}] - \nu_p^\varepsilon [\{\mathbf{m}_{\mathbf{x}} = \mathbf{v}\} \cap F_n] &\leq \nu_p^\varepsilon [F_n^c] \\ &\leq \nu_{p_0}^\varepsilon [F_n^c] \end{aligned}$$

where  $\lim_{n \rightarrow \infty} \nu_{p_0}^\varepsilon [F_n^c] = 0$ , since  $p_0 < p_c^G$ . So,  $\nu_p^\varepsilon [\{\mathbf{m}(\mathbf{x}) = \mathbf{v}\}]$  is a uniform limit of polynomials on any interval  $[0, p_0]$ ,  $p_0 < p_c^G$ , which implies the desired continuity.  $\square$

*Remark 4.2.* In the proof we can see that, for fixed  $p < p_c^G$ ,  $\mu_{p,r}^G(A)$  is a polynomial in  $r$ .

*Remark 4.3.* If  $G$  is a graph with uniqueness of the infinite bond cluster in the supercritical regime, then it is possible to verify that  $\nu_p^\varepsilon [\{\mathbf{m}(\mathbf{x}) = \mathbf{v}\}]$  is continuous in  $p$  on the whole interval  $[0, 1]$ . In this case, the continuity given by the Proposition 4.1 can be extended to the whole square  $[0, 1]^2$ .

*Proof of Theorem 1.6.* In order to simplify our notations, we write  $\mathbb{P}_{p,r}, \nu_p, r_c(p)$ , for  $\mathbb{P}_{p,r}^{\mathbb{Z}^2}, \nu_p^{\varepsilon^2}$  and  $r_c^{\mathbb{Z}^2}(p)$  respectively. Fix  $p_0 \in (0, 1/2)$  and  $\varepsilon > 0$  arbitrarily. We will show that there exists  $\delta = \delta(p_0, \varepsilon) > 0$  such that for all  $p \in (p_0 - \delta, p_0 + \delta)$ ,

$$r_c(p) \geq r_c(p_0) - \varepsilon, \tag{4.1}$$

and

$$r_c(p) \leq r_c(p_0) + \varepsilon. \tag{4.2}$$

Note that by equation (2.1), for all small enough choices of  $\delta > 0$  (such that  $0 \leq p_0 \pm \delta < 1/2$ ), (4.1) is equivalent to

$$r_c^*(p) \leq r_c^*(p_0) + \varepsilon. \tag{4.3}$$



Below we will show how to find  $\delta_1 > 0$  such that we have (4.2) for all  $p \in (p_0 - \delta_1, p_0 + \delta_1)$ . One may then completely analogously find  $\delta_2 > 0$  such that (4.3) holds for all  $p \in (p_0 - \delta_2, p_0 + \delta_2)$ , and take  $\delta = \min(\delta_1, \delta_2)$ .

Fix  $r = r_c(p_0) + \varepsilon$ , and define the event  $V_n = \{(\xi, \eta) \in \Omega_{\mathbb{Z}^2} \times \Omega_{\mathcal{E}_2} : \text{there exists a vertical crossing of } [0, n] \times [0, 3n] \text{ that is black in } \xi\}$ . By “vertical crossing,” we mean a self-avoiding path of vertices in  $[0, n] \times [0, 3n]$  with one endpoint in  $[0, n] \times \{0\}$ , and one in  $[0, n] \times \{3n\}$ . Recall also the definition of  $M_n$  in Theorem 2.1. By Lemma 2.10 in Bálint et al. (2009), there exists a constant  $\gamma > 0$  such that the following implication holds for any  $p, a \in [0, 1]$  and  $L \in \mathbb{N}$ :

$$\left. \begin{aligned} (3L + 1)(L + 1)\nu_a(M_{\lfloor L/3 \rfloor}) &\leq \gamma, \\ \text{and } \mathbb{P}_{p,a}(V_L) &\geq 1 - \gamma \end{aligned} \right\} \Rightarrow a \geq r_c(p).$$

As usual,  $\lfloor x \rfloor$  for  $x > 0$  denotes the largest integer  $m$  such that  $m \leq x$ . Fix such a  $\gamma$ .

By Theorem 2.1, there exists  $N \in \mathbb{N}$  such that

$$(3n + 1)(n + 1)\nu_{p_0}(M_{\lfloor n/3 \rfloor}) < \gamma$$

for all  $n \geq N$ . On the other hand, since  $r > r_c(p_0)$ , it follows from Lemma 2.11 in Bálint et al. (2009) that there exists  $L \geq N$  such that

$$\mathbb{P}_{p_0,r}(V_L) > 1 - \gamma.$$

Note that both  $(3L + 1)(L + 1)\nu_p(M_{\lfloor L/3 \rfloor})$  and  $\mathbb{P}_{p,r}(V_L)$  are continuous in  $p$  at  $p_0$ . Indeed, the former is simply a polynomial in  $p$ , while the continuity of the latter follows from Proposition 4.1. Therefore, there exists  $\delta_1 > 0$  such that for all  $p \in (p_0 - \delta_1, p_0 + \delta_1)$ ,

$$\left. \begin{aligned} (3L + 1)(L + 1)\nu_p(M_{\lfloor L/3 \rfloor}) &\leq \gamma, \\ \text{and } \mathbb{P}_{p,r}(V_L) &\geq 1 - \gamma. \end{aligned} \right\}$$

By the choice of  $\gamma$ , this implies that  $r \geq r_c(p)$  for all such  $p$ , which is precisely what we wanted to prove.

Finding  $\delta_2 > 0$  such that (4.3) holds for all  $p \in (p_0 - \delta_2, p_0 + \delta_2)$  is analogous: one only needs to substitute  $r_c(p_0)$  by  $r_c^*(p_0)$  and “crossing” by “\*-crossing,” and the exact same argument as above works. It follows that  $\delta = \min(\delta_1, \delta_2) > 0$  is a constant such that both (4.2) and (4.3) hold for all  $p \in (p_0 - \delta, p_0 + \delta)$ , completing the proof of continuity on  $(0, 1/2)$ . Right-continuity at 0 may be proved analogously; alternatively, it follows from Proposition 1.2.  $\square$

*Remark 4.4.* It follows from Theorem 1.6 and equation (2.1) that  $r_c^*(p)$  is also continuous in  $p$  on  $[0, 1/2)$ .

### 5. The critical value functions of tree-like graphs

In this section, we will study the critical value functions of graphs that are constructed by replacing edges of an infinite tree by a sequence of finite graphs. We will then use several such constructions in the proofs of our main results in Section 6.

Let us fix an arbitrary sequence  $D_n = (\mathcal{V}_n, \mathcal{E}_n)$  of finite connected graphs and, for every  $n \in \mathbb{N}$ , two distinct vertices  $a_n, b_n \in \mathcal{V}_n$ . Let  $\mathbb{T}_3 = (V_3, E_3)$  denote the (infinite) regular tree of degree 3, and fix an arbitrary vertex  $\rho \in V_3$ . Then, for each edge  $e \in E_3$ , we denote the end-vertex of  $e$  which is closer to  $\rho$  by  $f(e)$ , and

the other end-vertex by  $s(e)$ . Let  $\Gamma_D = (\tilde{V}, \tilde{E})$  be the graph obtained by replacing every edge  $e$  of  $\Gamma_3$  between levels  $n - 1$  and  $n$  (i.e., such that  $\text{dist}(s(e), \rho) = n$ ) by a copy  $D_e$  of  $D_n$ , with  $a_n$  and  $b_n$  replacing respectively  $f(e)$  and  $s(e)$ . Each vertex  $v \in V_3$  is replaced by a new vertex in  $\tilde{V}$ , which we denote by  $\tilde{v}$ . It is well known that  $p_c^{\Gamma_3} = r_c^{\Gamma_3}(0) = 1/2$ . Using this fact and the tree-like structure of  $\Gamma_D$ , we will be able to determine bounds for  $p_c^{\Gamma_D}$  and  $r_c^{\Gamma_D}(p)$ .

First, we define  $h^{D_n}(p) = \nu_p^{\mathcal{E}^n}(a_n \text{ and } b_n \text{ are in the same bond cluster})$ , and prove the following, intuitively clear, lemma.

**Lemma 5.1.** *For any  $p \in [0, 1]$ , the following implications hold:*

- a) *if  $\limsup_{n \rightarrow \infty} h^{D_n}(p) < 1/2$ , then  $p \leq p_c^{\Gamma_D}$ ;*
- b) *if  $\liminf_{n \rightarrow \infty} h^{D_n}(p) > 1/2$ , then  $p \geq p_c^{\Gamma_D}$ .*

*Proof.* We couple Bernoulli bond percolation with parameter  $p$  on  $\Gamma_D$  with inhomogeneous Bernoulli bond percolation with parameters  $h^{D_n}(p)$  on  $\mathbb{T}_3$ , as follows. Let  $\eta$  be a random variable with law  $\nu_p^{\tilde{E}}$ , and define, for each edge  $e \in E_3$ ,  $W(e) = 1$  if  $\tilde{f}(e)$  and  $\tilde{s}(e)$  are connected by a path consisting of edges that are open in  $\eta$ , and  $W(e) = 0$  otherwise. The tree-like structure of  $\Gamma_D$  implies that  $W(e)$  depends only on the state of the edges in  $D_e$ , and it is clear that if  $\text{dist}(s(e), \rho) = n$ , then  $W(e) = 1$  with probability  $h^{D_n}(p)$ .

It is easy to verify that there exists an infinite open self-avoiding path on  $\Gamma_D$  from  $\tilde{\rho}$  in the configuration  $\eta$  if and only if there exists an infinite open self-avoiding path on  $\mathbb{T}_3$  from  $\rho$  in the configuration  $W$ . Now, if we assume  $\limsup_{n \rightarrow \infty} h^{D_n}(p) < 1/2$ , then there exists  $t < 1/2$  and  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $h^{D_n}(p) \leq t$ . Therefore, the distribution of the restriction of  $W$  on  $L = \{e \in E_3 : \text{dist}(s(e), \rho) \geq N\}$  is stochastically dominated by the projection of  $\nu_t^{E_3}$  on  $L$ . This implies that, a.s., there exists no infinite self-avoiding path in  $W$ , whence  $p \leq p_c^{\Gamma_D}$  by the observation at the beginning of this paragraph. The proof of b) is analogous.  $\square$

We now turn to the DaC model on  $\Gamma_D$ . Recall that for a vertex  $v$ ,  $C_v$  denotes the vertex set of the bond cluster of  $v$ . Let  $E_{a_n, b_n} \subset \Omega_{\mathcal{E}^n} \times \Omega_{\mathcal{V}^n}$  denote the event that  $a_n$  and  $b_n$  are in the same bond cluster, or  $a_n$  and  $b_n$  lie in two different bond clusters, but there exists a vertex  $v$  at distance 1 from  $C_{a_n}$  which is connected to  $b_n$  by a black path (which also includes that  $\xi(v) = \xi(b_n) = 1$ ). This is the same as saying that  $C_{a_n}$  is *pivotal* for the event that there is a black path between  $a_n$  and  $b_n$ , i.e., that such a path exists if and only if  $C_{a_n}$  is black. It is important to note that  $E_{a_n, b_n}$  is independent of the color of  $a_n$ . Define  $f^{D_n}(p, r) = \mathbb{P}_{p, r}^{D_n}(E_{a_n, b_n})$ , and note also that, for  $r > 0$ ,  $f^{D_n}(p, r) = \mathbb{P}_{p, r}^{D_n}(\text{there is a black path from } a_n \text{ to } b_n \mid \xi(a_n) = 1)$ .

**Lemma 5.2.** *For any  $p, r \in [0, 1]$ , we have the following:*

- a) *if  $\limsup_{n \rightarrow \infty} f^{D_n}(p, r) < 1/2$ , then  $r \leq r_c^{\Gamma_D}(p)$ ;*
- b) *if  $\liminf_{n \rightarrow \infty} f^{D_n}(p, r) > 1/2$ , then  $r \geq r_c^{\Gamma_D}(p)$ .*

*Proof.* We couple here the DaC model on  $\Gamma_D$  with inhomogeneous Bernoulli site percolation on  $\mathbb{T}_3$ . For each  $v \in V_3 \setminus \{\rho\}$ , there is a unique edge  $e \in E_3$  such that  $v = s(e)$ . Here we denote  $D_e$  (i.e., the subgraph of  $\Gamma_D$  replacing the edge  $e$ ) by  $D_{\tilde{v}}$ , and the analogous event of  $E_{a_n, b_n}$  for the graph  $D_{\tilde{v}}$  by  $E_{\tilde{v}}$ . Let  $(\eta, \xi)$  with values in  $\Omega_{\tilde{E}} \times \Omega_{\tilde{V}}$  be a random variable with law  $\mathbb{P}_{p, r}^{\Gamma_D}$ . We define a random variable  $X$

with values in  $\Omega_{V_3}$ , as follows:

$$X(v) = \begin{cases} \xi(\tilde{\rho}) & \text{if } v = \rho, \\ 1 & \text{if the event } E_{\tilde{v}} \text{ is realized by the restriction of } (\eta, \xi) \text{ to } D_{\tilde{v}}, \\ 0 & \text{otherwise.} \end{cases}$$

As noted after the proof of Lemma 5.1, if  $u = f(\langle u, v \rangle)$ , the event  $E_{\tilde{v}}$  is independent of the color of  $\tilde{u}$ , whence  $(E_{\tilde{v}})_{v \in V_3 \setminus \{\rho\}}$  are independent. Therefore, as  $X(\rho) = 1$  with probability  $r$ , and  $X(v) = 1$  is realized with probability  $f^{D_n}(p, r)$  for  $v \in V_3$  with  $\text{dist}(v, \rho) = n$  for some  $n \in \mathbb{N}$ ,  $X$  is inhomogeneous Bernoulli site percolation on  $\mathbb{T}_3$ .

Our reason for defining  $X$  is the following property: it holds for all  $v \in V_3 \setminus \{\rho\}$  that

$$\tilde{\rho} \overset{\xi}{\leftrightarrow} \tilde{v} \quad \text{if and only if} \quad \rho \overset{X}{\leftrightarrow} v, \tag{5.1}$$

where  $x \overset{Z}{\leftrightarrow} y$  denotes that  $x$  and  $y$  are in the same *black* cluster in the configuration  $Z$ . Indeed, assuming  $\tilde{\rho} \overset{\xi}{\leftrightarrow} \tilde{v}$ , there exists a path  $\rho = x_0, x_1, \dots, x_k = v$  in  $\Gamma_3$  such that, for all  $0 \leq i < k$ ,  $\tilde{x}_i \overset{\xi}{\leftrightarrow} \tilde{x}_{i+1}$  holds. This implies that  $\xi(\tilde{\rho}) = 1$  and that all the events  $(E_{\tilde{x}_i})_{0 < i \leq k}$  occur, whence  $X(x_i) = 1$  for  $i = 0, \dots, k$ , so  $\rho \overset{X}{\leftrightarrow} v$  is realized. The proof of the other implication is similar. It follows in particular from (5.1) that  $\tilde{\rho}$  lies in an infinite black cluster in the configuration  $\xi$  if and only if  $\rho$  lies in an infinite black cluster in the configuration  $X$ .

Lemma 5.2 presents two scenarios when it is easy to determine (via a stochastic comparison) whether the latter event has positive probability. For example, if we assume that  $\liminf_{n \rightarrow \infty} f^{D_n}(p, r) > 1/2$ , then there exists  $t > 1/2$  and  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $f^{D_n}(p, r) \geq t$ . In this case, the distribution of the restriction of  $X$  on  $K = \{v \in V_3 : \text{dist}(v, \rho) \geq N\}$  is stochastically larger than the projection of  $\nu_t^{E_3}$  on  $K$ . Let us further assume that  $r > 0$ . In that case,  $X(\rho) = 1$  with positive probability, and  $f^{D_n}(p, r) > 0$  for every  $n \in \mathbb{N}$ . Therefore, under the assumptions  $\liminf_{n \rightarrow \infty} f^{D_n}(p, r) > 1/2$  and  $r > 0$ ,  $\rho$  is in an infinite black cluster in  $X$  (and, hence,  $\tilde{\rho}$  is in an infinite black cluster in  $\xi$ ) with positive probability, which can only happen if  $r \geq r_c^{\Gamma_D}(p)$ . On the other hand, if  $\liminf_{n \rightarrow \infty} f^{D_n}(p, 0) > 1/2$ , then it is clear that  $\liminf_{n \rightarrow \infty} f^{D_n}(p, r) > 1/2$  (whence  $r \geq r_c^{\Gamma_D}(p)$ ) for all  $r > 0$ , which implies that  $r_c^{\Gamma_D}(p) = 0$ . The proof of part a) is similar.  $\square$

### 6. Counterexamples

In this section, we study two particular graph families and obtain examples of non-monotonicity and non-continuity of the critical value function.

6.1. *Non-monotonicity.* The results in Section 5 enable us to prove that (a small modification of) the construction considered by Häggström in the proof of Theorem 2.9 in Häggström (2001) is a graph whose critical coloring value is non-monotone in the subcritical phase.

Proof of Proposition 1.7. Define for  $k \in \mathbb{N}$ ,  $D^k$  to be the complete bipartite graph with the vertex set partitioned into  $\{z_1, z_2\}$  and  $\{a, b, v_1, v_2, \dots, v_k\}$  (see Figure 6.1). We call  $e_1, e'_1$  and  $e_2, e'_2$  the edges incident to  $a$  and  $b$  respectively, and for  $i = 1, \dots, k$ ,  $f_i, f'_i$  the edges incident to  $v_i$ . Consider  $\Gamma_k$  the quasi-transitive graph obtained by replacing each edge of the tree  $\mathbb{T}_3$  by a copy of  $D_k$ .  $\Gamma_k$  can be seen

as the tree-like graph resulting from the construction described at beginning of the section, when we start with the constant sequence  $(D_n, a_n, b_n) = (D^k, a, b)$ .

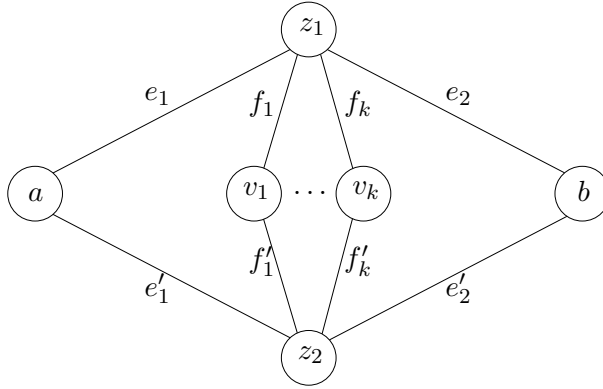


FIGURE 6.1. The graph  $D^k$ .

We will show below that it holds for all  $k \in \mathbb{N}$  that

$$p_c^{\Gamma^k} > 1/3, \tag{6.1}$$

$$r_c^{\Gamma^k}(0) < 2/3, \quad \text{and} \tag{6.2}$$

$$r_c^{\Gamma^k}(1/3) < 2/3. \tag{6.3}$$

Furthermore, there exists  $k \in \mathbb{N}$  and  $p_0 \in (0, 1/3)$  such that

$$r_c^{\Gamma^k}(p_0) > 2/3. \tag{6.4}$$

Proving (6.1)–(6.4) will finish the proof of Proposition 1.7 since these inequalities imply that the quasi-transitive graph  $\Gamma_k$  has a non-monotone critical value function in the subcritical regime.

Throughout this proof, we will omit superscripts in the notation when no confusion is possible. For the proof of (6.1), recall that  $h^{D^k}$  is strictly increasing in  $p$ , and  $h^{D^k}(p_{D^k}) = 1/2$ . Since  $1 - h^{D^k}(p)$  is the  $\nu_p$ -probability of  $a$  and  $b$  being in two different bond clusters, we have that

$$1 - h^{D^k}(1/3) \geq \nu_{1/3}(\{e_1 \text{ and } e'_1 \text{ are closed}\} \cup \{e_2 \text{ and } e'_2 \text{ are closed}\}).$$

From this, we get that  $h^{D^k}(1/3) \leq 25/81$ , which proves (6.1).

To get (6.2), we need to remember that for fixed  $p < p_{D^k}$ ,  $f^{D^k}(p, r)$  is strictly increasing in  $r$ , and  $f^{D^k}(p, r_{D^k}(p)) = 1/2$ . One then easily computes that  $f(0, 2/3) = 16/27 > 1/2$ , whence (6.2) follows from Lemma 5.2.

Now, define  $A$  to be the event that at least one edge out of  $e_1, e'_1, e_2$  and  $e'_2$  is open. Then

$$\begin{aligned} f^{D^k}(1/3, 2/3) &\geq \mathbb{P}_{1/3, 2/3}(E_{a,b} \mid A) \mathbb{P}_{1/3, 2/3}(A) \\ &\geq \mathbb{P}_{1/3, 2/3}(C_b \text{ black} \mid A) \cdot 65/81, \end{aligned}$$

which gives that  $f^{D^k}(1/3, 2/3) \geq 130/243 > 1/2$ , and implies (6.3) by 5.2.

To prove (6.4), we consider  $B_k$  to be the event that  $e_1, e'_1, e_2$  and  $e'_2$  are all closed and that there exists  $i$  such that  $f_i$  and  $f'_i$  are both open. One can easily compute that

$$\mathbb{P}_{p,r}(B_k) = (1 - p)^4 \left( 1 - (1 - p^2)^k \right),$$

which implies that we can choose  $p_0 \in (0, 1/3)$  (small) and  $k \in \mathbb{N}$  (large) such that  $\mathbb{P}_{p_0,r}(B_k) > 17/18$ . Then,

$$\begin{aligned} f^{D^k}(p_0, 2/3) &= \mathbb{P}_{p_0,r}(E_{a,b} \mid B_k) \mathbb{P}_{p_0,r}(B_k) + \mathbb{P}_{p_0,r}(E_{a,b} \mid B_k^c) (1 - \mathbb{P}_{p_0,r}(B_k)) \\ &< (2/3)^2 \cdot 1 + 1 \cdot 1/18 (= 1/2), \end{aligned}$$

whence inequality (6.4) follows with these choices from Lemma 5.2, completing the proof.  $\square$

6.2. *Graphs with discontinuous critical value functions.*

Proof of Proposition 1.3. For  $n \in \mathbb{N}$ , let  $D_n$  be the graph depicted in Figure 6.2, and let  $G$  be  $\Gamma_D$  constructed with this sequence of graphs as described at the beginning of Section 5.

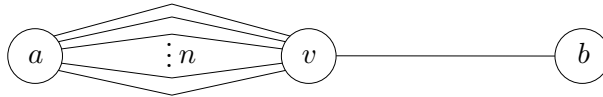


FIGURE 6.2. The graph  $D_n$ .

It is elementary that  $\lim_{n \rightarrow \infty} h^{D_n}(p) = p$ , whence  $p_c^G = 1/2$  follows from Lemma 5.1, thus  $p = 0$  is subcritical. Since  $\lim_{n \rightarrow \infty} f^{D_n}(0, r) = r^2$ , Lemma 5.2 gives that  $r_c^G(0) = 1/\sqrt{2}$ . On the other hand,  $\lim_{n \rightarrow \infty} f^{D_n}(p, r) = p + (1 - p)r$  for all  $p > 0$ , which implies by Lemma 5.2 that for  $p \leq 1/2$ ,

$$r_c^G(p) = \frac{1/2 - p}{1 - p} \rightarrow 1/2$$

as  $p \rightarrow 0$ , so  $r_c^G$  is indeed discontinuous at  $0 < p_c^G$ .  $\square$

In the rest of this section, for vertices  $v$  and  $w$ , we will write  $v \leftrightarrow w$  to denote that there exists a path of open edges between  $v$  and  $w$ . Our proof of Theorem 1.4 will be based on the Lemma 2.1 in Peres et al. (2009), that we rewrite here:

**Lemma 6.1.** *There exists a sequence  $G_n = (V^n, E^n)$  of graphs and  $x_n, y_n \in V^n$  of vertices ( $n \in \mathbb{N}$ ) such that*

- (1)  $\nu_{1/2}^{E^n}(x_n \leftrightarrow y_n) > \frac{2}{3}$  for all  $n$ ;
- (2)  $\lim_{n \rightarrow \infty} \nu_p^{E^n}(x_n \leftrightarrow y_n) = 0$  for all  $p < 1/2$ , and
- (3) there exists  $\Delta < \infty$  such that, for all  $n$ ,  $G_n$  has degree at most  $\Delta$ .

Lemma 6.1 provides a sequence of bounded degree graphs that exhibit sharp threshold-type behavior at  $1/2$ . We will use such a sequence as a building block to obtain discontinuity at  $1/2$  in the critical value function in the DaC model.

Proof of Theorem 1.4. We first prove the theorem in the case  $p_0 = 1/2$ . Consider the graph  $G_n = (V^n, E^n)$ ,  $x_n, y_n$  ( $n \in \mathbb{N}$ ) as in Lemma 6.1. We construct  $D_n$  from  $G_n$  by adding to it one extra vertex  $a_n$  and one edge  $\{a_n, x_n\}$ . More precisely  $D_n$  has vertex set  $V^n \cup \{a_n\}$  and edge set  $E^n \cup \{a_n, x_n\}$ . Set  $b_n = y_n$  and let  $G$  be the graph  $\Gamma_D$  defined with the sequence  $(D_n, a_n, b_n)$  as in Section 5.

We will show below that there exists  $r_0 > r_1$  such that the graph  $G$  verify the following three properties:

- (i)  $1/2 < p_c^G$
- (ii)  $r_c^G(p) \geq r_0$  for all  $p < 1/2$ .
- (iii)  $r_c^G(1/2) \leq r_1$ .

It implies a discontinuity of  $r_c^G$  at  $1/2 < p_c^G$ , finishing the proof.

One can easily compute  $h^{D_n}(p) = p\nu_p^{E^n}(x_n \leftrightarrow y_n)$ . Since the graph  $G_n$  has degree at most  $\Delta$  and the two vertices  $x_n, y_n$  are disjoint, the probability  $\nu_p^{E^n}(x_n \leftrightarrow y_n)$  cannot exceed  $1 - (1-p)^\Delta$ . This bound guarantees the existence of  $p_0 > 1/2$  independent of  $n$  such that  $h^{D_n}(p_0) < 1/2$  for all  $n$ , whence Lemma 5.1 implies that  $1/2 < p_0 \leq p_c^G$ .

For all  $p \in [0, 1]$ , we have

$$f^{D_n}(p, r) \leq (p + r(1-p)) \left( \nu_p^{E^n}(x_n \leftrightarrow y_n) + r(1 - \nu_p^{E^n}(x_n \leftrightarrow y_n)) \right)$$

which gives that  $\lim_{n \rightarrow \infty} f^{D_n}(p, r) < \left(\frac{r+1}{2}\right)r$ . Denoting by  $r_0$  the positive solution of  $r(1+r) = 1$ , we get that  $\lim_{n \rightarrow \infty} f^{D_n}(p, r_0) < 1/2$  for all  $p < 1/2$ , which implies by Lemma 5.2 that  $r_c^G(p) \geq r_0$ .

On the other hand,  $f^{D_n}(1/2, r) \geq \nu_p^{E^n}(x_n \leftrightarrow y_n) \left(\frac{1+r}{2}\right)$ , which gives by Lemma 6.1 that  $\lim_{n \rightarrow \infty} f^{D_n}(1/2, r) > \frac{2}{3} \cdot \frac{1+r}{2}$ . Writing  $r_1$  such that  $\frac{2}{3}(1+r_1) = 1$ , it is elementary to check that  $r_1 < r_0$  and that  $\lim_{n \rightarrow \infty} f^{D_n}(1/2, r_1) > 1/2$ . Then, using Lemma 5.2, we conclude that  $r_c(1/2) \leq r_1$ .  $\square$

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