On counting untyped lambda terms

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Abstract

Despite λ-calculus is now three quarters of a century old, no formula counting λ-terms has been proposed yet, and the combinatorics of λ-calculus is considered a hard problem. The difficulty lies in the fact that the recursive expression of the numbers of terms of size n with at most m free variables contains the number of terms of size $n-1$ with at most $m+1$ variables. This leads to complex recurrences that cannot be handled by classical analytic methods. Here based on de Bruijn indices (another presentation of λ-calculus) we propose several results on counting untyped lambda terms, i.e., on telling how many terms belong to such or such class, according to the size of the terms and/or to the number of free variables. We extend the results to normal forms.

Keywords Combinatorics, lambda calculus, functional programming, randomization, Catalan numbers

1 Introduction

This paper presents several results on counting untyped lambda terms, i.e., on telling how many terms belong to such or such class, according to the size of the terms and/or to the number of free variables. In addition to the inherent interest of these results from the mathematical point of view, we expect that a knowledge on the distribution of terms will improve the implementation of reduction [12] and that results on asymptotic distributions of terms will give a better insight of the lambda calculus. For counting more easily lambda terms we adopt de Bruijn indices that are a well-known coding of bound variables by natural numbers. First we give recurrence formulas for the number of terms (and of normal forms) of size n containing at most m distinct free variables. These recurrence formulas are not familiar in combinatorics and not amenable to a classical analytic treatment by generating functions. In this paper, we examine the formulas for terms and normal forms when n is fixed and m varies, which are polynomials. We give the expressions of the first coefficients of those polynomials since an expression for the generic coefficients seems out of reach and no regularity appears. However this shows that these expressions are clearly connected to Catalan numbers C_n which count the number of binary trees having

n internal nodes. If we would find an explicit expression for the last coefficients of the polynomials, this would be an explicit expression for the closed terms. In the last section, we give formulas for the generating functions showing the difficulty of a mathematical treatment. The results presented here are a milestone in describing probabilistic properties of lambda terms with answers to questions like: How does a random lambda term look like? How does a random normal form look like? How to generate a random lambda term (a random normal form)?

Related works

Previous works on counting lambda terms were by O. Bodini et al. [2], R. David et al. [4] and J. Wang [13]. Related works are on counting types and/or counting tautologies [14, 8, 5, 9]. Complexity of rewriting was studied by Choppy et al. [3].

2 Untyped lambda terms with de Bruijn indices

> *I am dedicating this book to N. G. "Dick" de Bruijn, because his influence can be felt on every page. Ever since the 1960s he has been my chief mentor, the main person who would answer my question when I was stuck on a problem that I had not been taught how to solve.*
>
> Donald Knuth in preface of [10]

The λ-calculus [1] is a logic formalism to describe functions, for instance, the function $f \mapsto (x \mapsto f(f(x))$, which takes a function f and applies it twice. For historical reason, this function is written $\lambda f.\lambda x.f(f x)$, which contains the two variables f and x, bounded by λ.

In this paper we represent terms by de Bruijn indices [6], this means that variables are represented by numbers $\underline{1}, \underline{2}, ..., \underline{m}, ...$, where an index, for instance $\underline{k}$, is the number of λ's, above the location of the index and below the λ that binds the variable, in a representation of λ-terms by trees. For instance, the term with variables $\lambda x.\lambda y.x\, y$ is represented by the term with de Bruijn indices $\lambda\lambda\underline{2}\underline{1}$. The variable x is bound by the head λ. Above the occurrence of x, there are two λ's, therefore x is represented by $\underline{2}$ and from the occurrence of y, we count just the λ that binds y; so y is represented by $\underline{1}$. In what follows we will call *terms*, the untyped terms[1] with de Bruijn indices. A *term* is either an index or, an abstraction or an application, hence the recursive definition:

$$\mathcal{T} ::= \mathbb{N} \mid \lambda\mathcal{T} \mid \mathcal{T}\,\mathcal{T}$$

and terms with indices up to m, i.e., with indices in $\mathcal{I}(m) = \{\underline{1}, \underline{2}, ..., \underline{m}\}$:

$$\mathcal{T}_m ::= \mathcal{I}(m) \mid \lambda\mathcal{T}_{m+1} \mid \mathcal{T}_m\,\mathcal{T}_m.$$

[1] Roughly speaking, typed terms are terms consistent with properties of the domain and the codomain of the function they represent.

Let us define a few functions on terms. To give the connection between λ-terms with de Bruijn indices and standard λ-terms with explicit variables, let us define two functions: $\Lambda 2\mathsf{db}$ and $\mathsf{db}2\Lambda$. Each function uses a list of variables.[2] In addition, the function $\Lambda 2\mathsf{db}$ (from standard lambda λ-terms to de Bruijn terms) needs a function index which returns the position of the given variable in the list[3]

$$\begin{array}{lcl} \Lambda 2\mathsf{db}(lv, x) & = & \mathsf{index}(lv, x) \\ \Lambda 2\mathsf{db}(lv, \lambda x.M) & = & \lambda(\Lambda 2\mathsf{db}(x :: lv, M)) \\ \Lambda 2\mathsf{db}(lv, M_1\, M_2) & = & \Lambda 2\mathsf{db}(lv, M_1)\, \Lambda 2\mathsf{db}(lv, M_2) \end{array}$$

The function $\mathsf{db}2\Lambda$ (from de Bruijn terms to standard λ-terms) use a list lv with a function nth ($\mathsf{nth}(lv, k)$ returns the k^{th} variable of the list lv).

$$\begin{array}{lcl} \mathsf{db}2\Lambda(lv, \underline{\mathsf{k}}) & = & \mathsf{nth}(lv, k) \\ \mathsf{db}2\Lambda(lv, \lambda t) & = & \lambda x.\mathsf{db}2\Lambda(x :: lv, t) \quad \text{where } x \text{ is a fresh variable } x \notin lv \\ \mathsf{db}2\Lambda(lv, t_1\, t_2) & = & \mathsf{db}2\Lambda(lv, t_1)\, \mathsf{db}2\Lambda(lv, t_2) \end{array}$$

Applying $\Lambda 2\mathsf{db}$ on a empty list and a standard closed term returns a term in $\mathcal{T}_0$. Reciprocally applying $\mathsf{db}2\Lambda$ on an empty list and a term in $\mathcal{T}_0$ returns a standard closed λ-term. The function size defines the size of a term. It assigns a size 1 to indices (in other words to variables):

$$\begin{array}{rcl} \mathsf{size}(\underline{\mathsf{k}}) & = & 1 \\ \mathsf{size}(\lambda t) & = & \mathsf{size}(t) + 1 \\ \mathsf{size}(t_1\, t_2) & = & \mathsf{size}(t_1) + \mathsf{size}(t_2). \end{array}$$

A *head* λ of a term t is a λ that occurs on the top of the term t or recursively on the top of the term below the head λ. We are interested by the number of head λ's given by the function $\sharp\mathsf{head\_}\lambda$:

$$\begin{array}{rcl} \sharp\mathsf{head\_}\lambda(\underline{\mathsf{k}}) & = & 0 \\ \sharp\mathsf{head\_}\lambda(\lambda t) & = & \sharp\mathsf{head\_}\lambda(t) + 1 \\ \sharp\mathsf{head\_}\lambda(t_1\, t_2) & = & 0. \end{array}$$

Let us call $\mathcal{T}_{n,m}$, the set of terms of size n, with at most m de Bruijn indices, i.e., with indices in $\mathcal{I}(m) = \{\underline{\mathsf{1}}, \underline{\mathsf{2}}, ..., \underline{\mathsf{m}}\}$. We can write, using @ as the application symbol,[4]

$$\mathcal{T}_{n+1,m} = \lambda \mathcal{T}_{n,m+1} \uplus \biguplus_{k=0}^{n} \mathcal{T}_{n-k,m} @ \mathcal{T}_{k,m}.$$

[2] The position of the variable in the list is another view of the de Bruijn index.

[3] $\mathsf{index}(x :: lv, x) = \underline{1}$, $\mathsf{index}(x :: lv, y) = \underline{\mathsf{index}(lv, y) + 1}$ where $x \neq y$. We assume there is no failure. In other words, when we invoke $\mathsf{index}(l, z)$, we assume that z belongs to l.

[4] We write $t_1 @ t_2$ instead of $t_1\, t_2$ to make explicit the presence of the binary operator *application*.

Moreover terms of size 1 are only made of de Bruijn indices, therefore

$$\mathcal{T}_{1,m} \quad = \quad \mathcal{I}(m).$$

There is no term of size 0:

$$\mathcal{T}_{0,m} \quad = \quad \emptyset.$$

From this we get:

$$\begin{aligned} T_{n+1,m} &= T_{n,m+1} + \sum_{k=0}^{n} T_{n-k,m}.T_{k,m} \\ T_{1,m} &= m \\ T_{0,m} &= 0 \end{aligned}$$

$\mathcal{T}_{n,0}$ is the set of closed terms (terms with no non bound indices) of size n. Notice that

$$T_{n+1,m} \quad = \quad T_{n,m+1} + \sum_{k=1}^{n-1} T_{n-k,m}.T_{k,m}$$

Let us illustrate this result by the array of closed terms up to size 5:

n	terms	$T_{n,0}$
1	none	0
2	$\lambda\underline{1}$	1
3	$\lambda\lambda\underline{1}$, $\lambda\lambda\underline{2}$,	2
4	$\lambda\lambda\lambda\underline{1}$, $\lambda\lambda\lambda\underline{2}$, $\lambda\lambda\lambda\underline{3}$, $\lambda(\underline{1}.\underline{1})$	4
5	$\lambda\lambda\lambda\lambda\underline{1}$, $\lambda\lambda\lambda\lambda\underline{2}$, $\lambda\lambda\lambda\lambda\underline{3}$, $\lambda\lambda\lambda\lambda\underline{4}$, $\lambda\lambda(\underline{1}.\underline{1})$, $\lambda\lambda(\underline{1}.\underline{2})$, $\lambda\lambda(\underline{2}.\underline{1})$, $\lambda\lambda(\underline{2}.\underline{2})$, $\lambda(\underline{1}.\lambda\underline{1})$, $\lambda(\underline{1}.\lambda\underline{2})$, $\lambda((\lambda\underline{1}).\underline{1})$, $\lambda((\lambda\underline{2}).\underline{1})$, $\lambda\underline{1}.\lambda\underline{1}$	13

The equation that defines $T_{n,m}$ allows us to compute it, since it relies on entities $T_{i,j}$ where either $i < n$ or $j < m$. Figure 1 is a table of the first values of $T_{n,m}$ up to $T_{18,7}$. We are mostly interested by the sequence of sizes of the closed terms, namely $T_{n,0}$, in other words the first column of the table.

Terms with explicit variables

The values of $T_{n,0}$ correspond to sequence **A135501** (see `http://www.research.att.com/~nudges/sequences/A135501`) due to Christophe Raffalli, which is defined as the *number of closed lambda-terms of size n*. His recurrence formula for those numbers is more complex. Actually he counts the number of lambda-terms with exactly m free variables. Raffalli considers the values of the double sequence $f_{n,m}$, which is up to α-conversion the number of λ-terms of size n with exactly m free variables, whereas $T_{n,m}$ is the number of λ-terms with at most m free variables. On closed terms (terms with no free variable, that correspond to the case $m = 0$) the number of terms with exactly m free variables (Raffalli's) coincides with the number of terms with at most m free variables (ours). $T_{n,m}$ and $f_{n,m}$ coincide for $m = 0$ which means $T_{n,0} = f_{n,0}$.

$$
\begin{aligned}
f_{1,1} &= 1 \\
f_{0,m} &= 0 \\
f_{n,m} &= 0 \;\text{ if } m > 2n-1 \\
f_{n,m} &= f_{n-1,m} + f_{n-1,m+1} + \sum_{p=1}^{n-2}\sum_{c=0}^{m}\sum_{l=0}^{m-c}\binom{m}{c}\binom{m-c}{l} f_{p,l+c} f_{n-p-1,m-l}.
\end{aligned}
$$

3 Bounding the $T_{n,0}$'s

Here we give a rough lower bound of the $T_{n,0}$'s. We can show easily that Motzkin numbers[5] are a lower bound of the $T_{n,0}$'s. More precisely we get the following proposition.

Proposition 1 *If M_n are the Motzkin numbers, $M_n < T_{n+1,0}$.*

> Proof: There is a one-to-one correspondance between unary-binary trees and lambda terms of the form λM in which all the indices are $\underline{1}$. Hence the results, since Motzkin numbers count unary-binary trees. $\Box$

We conclude that the asymptotic behavior of the $T_{n,0}$'s are at least 3^n since the Motzkin numbers are asymptotically equivalent to $\sqrt{\frac{3}{4\pi n^3}}\, 3^n$ ([7], Example VI.3). Noticed that David et al. [4] have exhibited a lower bound and a upper bound, but they give size 0 to variables (or de Bruijn's indices). Their *size* function, which we write size_D to distinguish from ours, is:

$$
\begin{aligned}
\mathsf{size}_D(\underline{\mathsf{k}}) &= 0 \\
\mathsf{size}_D(\lambda t) &= \mathsf{size}_D(t) + 1 \\
\mathsf{size}_D(t_1\, t_2) &= \mathsf{size}_D(t_1) + \mathsf{size}_D(t_2).
\end{aligned}
$$

size_D differs from size by the fact that size_D is 0 on variables or indices. In other words, David et al. consider the following induction, for the number $D_{n,m}$ of terms on size n with at most m free variables and variables sized as 0:

$$
\begin{aligned}
D_{0,m} &= m \\
D_{n+1,m} &= D_{n,m+1} + \sum_{k=0}^{n} D_{n-k,m}.D_{k,m}
\end{aligned}
$$

[5]Motzkin numbers M_n count the number of unary-binary trees of size n.

Proposition 2 (David et al.) *For any $\varepsilon \in (0,4)$, one has*[6]

$$\left(\frac{(4-\varepsilon)n}{\ln(n)}\right)^{n-\frac{n}{\ln(n)}} \lesssim D_{n,0} \lesssim \left(\frac{(12+\varepsilon)n}{\ln(n)}\right)^{n-\frac{n}{3\ln(n)}}.$$

4 The functions $m \mapsto T_{n,m}$

In this section, we study in more detail the $T_{n,m}$'s. We assume the reader familiar with generating functions. Otherwise he is advised to the read the reference book *Analytic Combinatorics*, by Ph. Flajolet and R. Sedgewick [7].

Due to properties of the generating function (see Section 6) we are not able to give a simple expression for the function $n \mapsto T_{n,m}$, so we focus on the function $m \mapsto T_{n,m}$. These functions are polynomials P_n^T, defined recursively as follows:

$$\begin{aligned} P_0^T(m) &= 0 && (1)\\ P_1^T(m) &= m && (2)\\ P_{n+1}^T(m) &= P_n^T(m+1) + \sum_{k=1}^{n-1} P_k^T(m)\, P_{n-k}^T(m). && (3) \end{aligned}$$

See Figure 2 for the first 18 polynomials. The table below gives the coefficients of the polynomials P_n^T up to 16.

$n\backslash m^i$	m^8	m^7	m^6	m^5	m^4	m^3	m^2	m	1
1								1	0
2								*1*	*1*
3							1	1	2
4							*3*	*5*	*4*
5						2	6	17	13
6						*10*	*26*	*49*	*42*
7					5	30	111	179	139
8					*35*	*134*	*405*	*683*	*506*
9				14	140	652	1658	2629	1915
10				*126*	*676*	*2812*	*7122*	*10725*	*7558*
11			42	630	3610	12760	30783	45195	31092
12			*462*	*3334*	*17670*	*60240*	*138033*	*196355*	*132170*
13		132	2772	19218	87850	285982	635178	880379	580466
14		*1716*	*16108*	*104034*	*449290*	*1390246*	*2991438*	*4052459*	*2624545*
15	429	12012	99386	560854	2308173	6895122	14436365	19144575	12190623
16	*6435*	*76444*	*584878*	*3076878*	*12039895*	*34815210*	*71170791*	*92631835*	*58083923*

The degrees of those polynomials increase two by two and we can describe their leading coefficients, their second leading coefficients and the third leading coefficients of the odd polynomials.

Proposition 3 *$deg(P_{2p-1}^T) = deg(P_{2p}^T) = p$.*

Proof: This is true for $P_1^T = m$ and $P_2^T = m+1$ which have degree 1. Assume the property true up to p. Note that all the coefficients of the P_n^T's are positive. In

$$P_n^T(m+1) + \sum_{k=1}^{n-1} P_k^T(m)\, P_{n-k}^T(m),$$

[6] $f \lesssim g$ iff there exists a function $h : \mathbb{N} \to \mathbb{R}$ such that $h \sim g$ and there exists $N \in \mathbb{N}$ such that $f(n) \geq h(n)$ for $n \geq N$.

the degree of $P^T_{n+1}(m)$ comes from the $P^T_k(m)\,P^T_{n-k}(m)$'s. Indeed, par induction the degree of $P^T_n(m+1)$ is $(n-1) \div 2 + 1$ which is smaller than $n \div 2 + 1$, therefore we can consider that $P^T_n(m+1)$ does not contribute to the degree of $P^T_{n+1}(m)$. Consider the degree of $P^T_k(m)\,P^T_{n-k}(m)$ according to the parity of n and k.

$n = 2p+1$ **and** $k = 2h-1$**.** In this case, $p \geq h \geq 1$ and the degree of $P^T_{2h-1}(m)$ is h and the degree of $P^T_{2p+1-2h+1}(m)$ is $p-h+1$, hence the degree of $P^T_{2h-1}(m)\,P^T_{2p+1-2h+1}(m)$ is $p+1$.

$n = 2p+1$ **and** $k = 2h$**.** In this case, $p \geq h \geq 1$ and the degree of $P^T_{2h}(m)$ is h and the degree of $P^T_{2p+1-2h}(m)$ is $p-h+1$, hence the degree of $P^T_{2h}(m)\,P^T_{2p+1-2h}(m)$ is $p+1$.

$n = 2p$ **and** $k = 2h-1$**.** In this case, $p+1 \geq h \geq 1$ and the degree of $P^T_{2h-1}(m)$ is h and the degree of $P^T_{2p-2h+1}(m)$ is $p-h+1$, hence the degree of $P^T_{2h-1}(m)\,P^T_{2p-2h+1}(m)$ is $p+1$.

$n = 2p$ **and** $k = 2h$**.** In this case, $p+1 \geq h \geq 1$ and the degree of $P^T_{2h}(m)$ is h and the degree of $P^T_{2p-2h}(m)$ is $p-h$, hence the degree of $P^T_{2h}(m)\,P^T_{2p-2h}(m)$ is p. These products $P^T_{2h}(m)\,P^T_{2p-2h}(m)$ do not contribute to the degree of $P^T_{2p+1}(m)$.

□

In what follows, for short, we write θ_{2q+1} and θ_{2q} the leading coefficients of $P^T_{2q+1}(m)$ and $P^T_{2q}(m)$, τ_{2q+1} and τ_{2q} the second leading coefficients of $P^T_{2q+1}(m)$ and $P^T_{2q}(m)$, and δ_{2q+1} the third leading coefficients of $P^T_{2q+1}(m)$. We also write, as usual, C_n the n^{th} Catalan number.

We define five generating functions.

$$\mathcal{O}d(z) = \sum_{i=0}^{\infty} \theta_{2i+1} z^i \qquad \mathcal{E}v(z) = \sum_{i=0}^{\infty} \theta_{2i} z^i$$

$$\mathcal{S}od(z) = \sum_{i=0}^{\infty} \tau_{2i+1} z^i \qquad \mathcal{S}ev(z) = \sum_{i=0}^{\infty} \tau_{2i} z^i$$

$$\mathcal{T}od(z) = \sum_{i=0}^{\infty} \delta_{2i+1} z^i.$$

Proposition 4 *The leading coefficients of P^T_{2q+1} are $\frac{1}{q+1}\binom{2q}{q}$, i.e., the Catalan numbers C_q.*

Proof: From Equation (3) and the last two steps of the proof of Proposition 3, we deduce the following relation :

$$\begin{aligned}\theta_{2q+1} &= \sum_{h=0}^{q-1} \theta_{2h+1}\,\theta_{2q-2h-1} \qquad \text{for } q \geq 1\\ \theta_1 &= 1.\end{aligned}$$

which says that the leading coefficient of an odd polynomial comes only from the leading coefficients in the products of odd polynomials. We get:

$$\mathcal{O}d(z) = 1 + z\,\mathcal{O}d(z)^2.$$

which shows that

$$\mathcal{O}d(z) = \frac{1-\sqrt{1-4z}}{2z}.$$

and $\mathcal{O}d(z) = C(z)$, the generating function of the Catalan numbers. $\square$

Proposition 5 *The leading coefficients of P^T_{2q} are $\binom{2q-1}{q}$, for $q \geq 1$.*

Proof: Without lost of generality, we assume that $\theta_0 = 0$. From Equation (3), we get, for $q \geq 1$,

$$\begin{aligned}
\theta_{2q+2} &= \theta_{2q+1} + \sum_{k=0}^{2q+1} \theta_k\, \theta_{2q+1-k} \\
&= \theta_{2q+1} + \sum_{h=0}^{q} \theta_{2h}\, \theta_{2q+1-2h} + \sum_{h=0}^{q} \theta_{2h+1}\, \theta_{2q-2h} \\
&= \theta_{2q+1} + 2\sum_{h=0}^{q} \theta_{2h}\, \theta_{2q-2h+1}.
\end{aligned}$$

which says that the leading coefficient of an even polynomial comes from the leading coefficient of the preceding odd polynomial and of the products of the leading coefficients of the products of the smaller polynomials. We get:

$$\mathcal{E}v(z) = z\mathcal{O}d(z) + 2\,z\mathcal{E}v(z)\mathcal{O}d(z),$$

hence

$$\mathcal{E}v(z) = \frac{z\mathcal{O}d(z)}{1-2z\mathcal{O}d(z)} = \frac{1-\sqrt{1-4z}}{2\sqrt{1-4z}} = \frac{\sqrt{1-4z}}{2(1-4z)} - \frac{1}{2}$$

which is the generating function of the sequence $\binom{2q-1}{q}$. $\square$

Proposition 6 *The second leading coefficients of P^T_{2q+1} are $(2q-1)\binom{2(q-1)}{q-1}$.*

Proof: From the proof of Proposition 3, we see that the monomial of second highest degree of P_{2q+1} is made as the sum:

- of the monomial of highest degree of P_{2q},
- of the products of the monomials of highest degree from the P_i's with even indices and

- the products of monomials of highest degree with monomials of second highest degree from the P_i's with odd indices.

We get for $q \geq 1$:

$$\tau_{2q+1} = \theta_{2q} + \sum_{h=0}^{q} \theta_{2h}\theta_{2q-2h} + \sum_{h=0}^{q-1} \theta_{2h+1}\tau_{2q-2h-1} + \sum_{h=0}^{q-1} \tau_{2h+1}\theta_{2q-2h-1}.$$

We notice that $\tau_1 = 0$. Therefore we get:

$$\mathcal{S}od(z) = \mathcal{E}v(z) + \mathcal{E}v(z)^2 + 2z\mathcal{O}d(z)\,\mathcal{S}od(z).$$

Then

$$\mathcal{S}od(z) = \frac{\mathcal{E}v(z) + \mathcal{E}v(z)^2}{1 - 2z\mathcal{O}d(z)} = \frac{z}{\sqrt{1-4z}(1-4z)} = \frac{z\sqrt{1-4z}}{(1-4z)^2}$$

which is the generating function of $(2q-1)\binom{2(q-1)}{q-1}$. $\square$

Proposition 7 *The second leading coefficients of P^T_{2q} are $\tau_0 = 0$, $\tau_1 = 1$, $\tau_2 = 5$ and for $q \geq 3$,*

$$\tau_{2q} = 4^{q-1} + \frac{2(2q-5)(2q-3)(2q-1)}{3(q-2)}\binom{2(q-3)}{q-3}.$$

Proof: From Equation (3),we get

$$\begin{cases} \tau_{2q+2} = (q+1)\theta_{2q+1} + \tau_{2q+1} \\ \qquad +2\sum_{i=1}^{q}\theta_{2i-1}\tau_{2q-2i+2} + 2\sum_{i=1}^{q}\theta_{2i}\tau_{2q-2i+1} \\ \tau_0 = 0 \end{cases}$$

The second leading coefficient of an even polynomial P^T_{2m+2} is made of four components:

- the coefficient of degree q in $\theta_{2q+1}(m+1)^{q+1}$, namely $(q+1)\theta_{2q+1}$,
- the coefficient of degree q in $\tau_{2q+1}(m+1)^q$, namely τ_{2q+1},
- the sum of the products of the leading coefficients of the odd polynomials and the second leading coefficients of the even polynomials (this occurs twice, once in product $P_{2i-1}(m)\,P_{2q-2i+2}(m)$ and once in product $P_{2i}(m)\,P_{2q-2i+1}(m)$),
- the sum of the products of the leading coefficients of the even polynomials and the second leading coefficients of the odd polynomials (twice).

From the above induction, $\mathcal{S}ev$ fulfils the following functional equation:

$$\mathcal{S}ev(z) = z\mathcal{O}d(z) + z^2\mathcal{O}d'(z) + z\mathcal{S}od(z) + 2z\mathcal{O}d(z)\mathcal{S}ev(z) + 2z\mathcal{E}v(z)\mathcal{S}od(z).$$

Therefore

$$
\begin{aligned}
\mathcal{S}ev(z) &= \frac{z\mathcal{O}d(z) + z^2\mathcal{O}d'(z) + z\mathcal{S}od(z) + 2z\mathcal{E}v(z)\mathcal{S}od(z)}{\sqrt{1-4z}} \\
&= \frac{(1-\sqrt{1-4z})}{2\sqrt{1-4z}} \\
&\quad + \frac{z}{1-4z} - \frac{1-\left(\sqrt{1-4z}\right)}{2\sqrt{1-4z}} \\
&\quad + \frac{z^2}{(1-4z)^2} \\
&\quad + \frac{z(1-\sqrt{1-4z})}{(1-4z)^2\sqrt{1-4z}} \\
&= \frac{z}{1-4z} + \frac{z^2}{(1-4z)^2} + \frac{z^2(1-\sqrt{1-4z})}{(1-4z)^2\sqrt{1-4z}} \\
&= \sum_{q=1}^{\infty} 4^{q-1}z^q + \sum_{q=2}^{\infty}(q-1)4^{q-2}z^q + \sum_{q=3}^{\infty} 2a_{q-3}z^q
\end{aligned}
$$

where $(a_n)_{n\in\mathbf{N}}$ is sequence A029887 of the *On-Line Encyclopedia of Integer Sequences* whose value is:

$$\frac{(2n+1)(2n+3)(2n+5)}{3}C_n - (n+2)2^{2n+1}.$$

Hence

$$
\begin{aligned}
\mathcal{S}ev(z) &= \sum_{q=1}^{\infty} 4^{q-1}z^q + \sum_{q=3}^{\infty} \frac{2(2q-5)(2q-3)(2q-1)}{3}C_{q-3}z^q \\
&= \frac{z}{1-4z} + \frac{z^2}{(1-4z)^2\sqrt{1-4z}}.
\end{aligned}
$$

□

Proposition 8 *The third leading coefficients of P^T_{2q+1} are*

$$q\ 2^{2q-1} + \frac{q(q-1)(q-2)}{120}\binom{2q}{q} + \frac{(q+1)q(q-1)}{120}\binom{2(q+1)}{q+1}.$$

Proof: Since $deg(P^T_{2n}) = deg(P^T_{2n+1}) - 1$, the third coefficient is the sum of seven items:

- the second coefficient of $\theta_{2q}\,(m+1)^q$, namely $q\theta_{2q}$,
- the first coefficient of $(m+1)^{q-1}$, namely τ_{2q},
- the sum of products of leading coefficients and second leading coefficients for even polynomials (twice),

- the sum of leading coefficients and third leading coefficients for odd polynomials (twice),
- the sum of second leading coefficients with second leading coefficients.

The formula for δ_{2q+1} is:

$$\begin{aligned}\delta_{2q+1} &= q\,\theta_{2q} + \tau_{2q} + \sum_{i=0}^{q} \tau_{2i}\theta_{2q-2i} + \sum_{i=0}^{q} \theta_{2i}\tau_{2q-2i} + \\ & \sum_{i=0}^{q-1} \theta_{2i+1}\delta_{2q-2i-1} + \sum_{i=0}^{q-1} \delta_{2i+1}\theta_{2q-2i-1} + \sum_{i=0}^{q-1} \tau_{2i+1}\tau_{2q-2i-1},\end{aligned}$$

which give the following equation on generating functions:

$$\begin{aligned}\mathcal{T}od(z) &= z\,\mathcal{E}v'(z) + \mathcal{S}ev(z) + 2z\mathcal{E}v(z)\mathcal{S}ev(z) + \\ & 2z\,\mathcal{O}d(z)\mathcal{T}od(z) + z\,\mathcal{S}ev(z)^2.\end{aligned}$$

which yields:

$$\begin{aligned}\mathcal{T}od(z) &= \frac{z\,\mathcal{E}v'(z) + \mathcal{S}ev(z) + 2z\mathcal{E}v(z)\mathcal{S}ev(z) + z\,\mathcal{S}ev(z)^2}{1 - 2z\,\mathcal{O}d(z)} \\ &= \frac{1}{\sqrt{1-4z}} \left(\frac{z}{(1-4z)\sqrt{1-4z}} + \right. \\ & \qquad \frac{z}{1-4z} + \frac{z^2}{(1-4z)^2\sqrt{1-4z}} + \\ & \qquad \frac{1-\sqrt{1-4z}}{\sqrt{1-4z}} \left(\frac{z}{1-4z} + \frac{z^2}{(1-4z)^2\sqrt{1-4z}} \right) + \\ & \qquad \left. \frac{z^3}{(1-4z)^3} \right) \\ &= \frac{2z}{(1-4z)^2} + \frac{z^2+z^3}{(1-4z)^3\sqrt{1-4z}}.\end{aligned}$$

The first part corresponds to sequence A002699 which expression is $q\,2^{2q-1}$. $1/(1-4z)^3\sqrt{1-4z}$ corresponds to sequence A144395. Therefore the second part yields the expression

$$\frac{q(q-1)(q-2)}{120}\binom{2q}{q} + \frac{(q+1)q(q-1)}{120}\binom{2(q+1)}{q+1}.$$

□

Hence typically if we pose

$$\begin{aligned}\tau_{2q} &= 4^{q-1} + \frac{2(2q-5)(2q-3)(2q-1)}{3} C_{q-3} \\ \delta_{2q+1} &= q\;2^{2q-1} + \frac{(q+1)q(q-1)(q-2)}{120} C_q + \frac{(q+2)(q+1)q(q-1)}{120} C_{q+1}\end{aligned}$$

we have in general:

$$\begin{aligned} P^T_{2q}(m) &= (2q-1)C_{q-1}m^q + \tau_{2q}m^{q-1} + \ldots + T_{2q,0} \\ P^T_{2q+1}(m) &= C_q m^{q+1} + \frac{2q(2q-1)}{2}C_{q-1}m^q + \delta_{2q+1}m^{q-1} + \ldots + T_{2q+1,0} \end{aligned}$$

showing the prominent role of Catalan numbers. The relations for the other coefficients are more convoluted[7] and have not been computed.

It should be interesting to study the connection with the derivatives of the generating function $C(z)$ of the Catalan numbers [11].

5 Normal forms

Normal forms are important in λ-calculus. They are terms containing no sub-term of the form $(\lambda t_1)\, t_2$. We study in detail the expression giving the number of normal forms of size n with at most m variables. Let us call $\mathcal{F}_m$ the set of normal forms with $\{\underline{1}, .., \underline{m}\}$ de Bruijn indices and $\mathcal{G}_m$ the sets of normal forms with no head λ and de Bruijn indices in $\{\underline{1}, .., \underline{m}\}$. The combinatorial structure equations are

$$\begin{aligned} \mathcal{G}_m &= \mathcal{I}(m) \uplus \mathcal{G}_m @ \mathcal{F}_m \\ \mathcal{F}_m &= \lambda\, \mathcal{F}_{m+1} \uplus \mathcal{G}_m \end{aligned}$$

Let $G_{n,m}$ be the number of normal forms of size n with no head λ and with de Bruijn indices in $\mathcal{I}(m)$ and let $F_{n,m}$ be the number of normal forms of size n with de Bruijn indices in $\mathcal{I}(m)$. The relations between $G_{n,m}$ and $F_{n,m}$ are

$$\begin{aligned} G_{0,m} &= 0 \\ G_{1,m} &= m \\ G_{n+1,m} &= \sum_{k=0}^{n} G_{n-k,m} F_{k,m} \\ F_{0,m} &= 0 \\ F_{1,m} &= m \quad = \quad G_{1,m} \\ F_{n+1,m} &= F_{n,m+1} + G_{n+1,m} \end{aligned}$$

whereas the relations between generating functions are

$$\begin{aligned} G_m(z) &= m\, z + z\, G_m(z)\, F_m(z) \\ F_m(z) &= z\, F_{m+1}(z) + G_m(z). \end{aligned}$$

The coefficients $F_{n,m}$ are given in Figure 3.

[7] Like τ_{2q} and δ_{2q+1}, they correspond to non studied sequences according to the *On-Line Encyclopedia of Integer Sequences*.

The functions $m \mapsto F_{n,m}$

Like for $m \mapsto T_{n,m}$, the functions $m \mapsto F_{n,m}$ are polynomials of degree $(n-1) \div 2 + 1$, which we write P_n^{NF} and which we give in Figure 4. The coefficients of polynomials P_n^{NF} enjoy properties somewhat similar to those proved for polynomials P_n^T. In this section, we write $P_n(m)$ the polynomial $P_n^{NF}(m)$, $Q_n(m)$ the polynomial associated with $G_{n,m}$, φ_n the leading coefficient of P_n, $\overline{\varphi}_n$ the leading coefficient of Q_n, ψ_n the second leading coefficient of P_n and $\overline{\psi}_n$ the second leading coefficient of Q_n. We have the equations

$$\begin{aligned} P_{n+1}(m) &= P_n(m+1) + Q_{n+1}(m) && (4) \\ Q_{n+1}(m) &= \sum_{k=0}^{n} Q_{n-k}(m) P_k(m) && (5) \end{aligned}$$

Proposition 9 *$deg(P_{2p-1}) = deg(P_{2p}) = deg(Q_{2p-1}) = deg(Q_{2p}) = p$.*

> Proof: Here also the coefficients are positive. The degree of P_n is the degree of Q_n by (4). One notices that $deg\ P_0 = deg\ Q_0 = 0$ and $deg\ P_1 = deg\ Q_1 = 1$. The general step can be mimicked from this of Proposition 3. □

We define eight generating functions:

$$\begin{aligned} \mathcal{F}ev(z) &= \sum_{i=0}^{\infty} \varphi_{2i} z^i & \overline{\mathcal{F}ev}(z) &= \sum_{i=0}^{\infty} \overline{\varphi}_{2i} z^i \\ \mathcal{F}od(z) &= \sum_{i=0}^{\infty} \varphi_{2i+1} z^i & \overline{\mathcal{F}od}(z) &= \sum_{i=0}^{\infty} \overline{\varphi}_{2i+1} z^i \\ \mathcal{SF}ev(z) &= \sum_{i=0}^{\infty} \psi_{2i} z^i & \overline{\mathcal{SF}ev}(z) &= \sum_{i=0}^{\infty} \overline{\psi}_{2i} z^i \\ \mathcal{SF}od(z) &= \sum_{i=0}^{\infty} \psi_{2i+1} z^i & \overline{\mathcal{SF}od}(z) &= \sum_{i=0}^{\infty} \overline{\psi}_{2i+1} z^i \end{aligned}$$

Proposition 10 *The leading coefficients of P_{2q+1}^{NF} are Catalan numbers.*

> Proof: We see easily that $\varphi_{2q+1} = \overline{\varphi}_{2q+1}$ by (4). By (5), we see that
>
> $$\begin{aligned} \varphi_{2q+1} &= \sum_{h=0}^{q-1} \overline{\varphi}_{2q+1} \varphi_{2q-2h-}1 \\ \varphi_1 &= \overline{\varphi}_1 = 1. \end{aligned}$$
>
> Hence the result $\mathcal{F}od(z) = \overline{\mathcal{F}od}(z) = C(z)$ (see proof of Proposition 4). □

Proposition 11 *The leading coefficients of the P_n^{NF}'s for n even, are $P_0^{NF}=0$, $P_2^{NF}=1$ and $P_{2q+4}^{NF}=2\binom{2q+1}{q}$, i.e., $P_{2q+4}^{NF}=2P_{2q+2}^{T}$.*

Proof: From Equations (4) and (5),we get:

$$\begin{aligned}
\varphi_{2(q+1)} &= \varphi_{2q+1}+\overline{\varphi}_{2(q+1)}\\
\overline{\varphi}_{2(q+1)} &= \sum_{i=0}^{q}\overline{\varphi}_{2q+1-2i}\varphi_{2i}+\sum_{i=0}^{q}\overline{\varphi}_{2q-2i}\varphi_{2i+1}
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{F}ev(z) &= z\mathcal{F}od(z)+\overline{\mathcal{F}ev}(z)\\
\overline{\mathcal{F}ev}(z) &= z\overline{\mathcal{F}od}(z)\mathcal{F}ev(z)+z\overline{\mathcal{F}ev}(z)\mathcal{F}od(z)
\end{aligned}$$

from which we get

$$\overline{\mathcal{F}ev}(z) = \frac{z\mathcal{F}od(z)\mathcal{F}ev(z)}{1-z\mathcal{F}od(z)} \tag{6}$$

then

$$\mathcal{F}ev(z) = z\mathcal{F}od(z)+\frac{z\mathcal{F}od(z)\mathcal{F}ev(z)}{1-z\mathcal{F}od(z)}$$

and

$$\mathcal{F}ev(z)-z\,\mathcal{F}od(z)\mathcal{F}ev(z) = z\,\mathcal{F}od(z)-z^2\mathcal{F}od(z)^2+z\,\mathcal{F}od(z)\mathcal{F}ev(z)$$

and

$$\begin{aligned}
\mathcal{F}ev(z) &= \frac{z\mathcal{F}od(z)-z^2\mathcal{F}od(z)^2}{1-2z\mathcal{F}od(z)}\\
&= \frac{z}{\sqrt{1-4z}} = \frac{z\sqrt{1-4z}}{1-4z}\\
&= \frac{z}{1-2zC(z)}.
\end{aligned}$$

Hence $\mathcal{F}ev(z)$ is the generating function of the sequence $\varphi_0=0$, $\varphi_2=1$ and $\varphi_{2q+4}=2\binom{2q+1}{q}$. □

Corollary 1 $\overline{\mathcal{F}ev}(z)=\frac{1-2z-\sqrt{1-4z}}{2\sqrt{1-4z}}$

Proof:

$$\begin{aligned}
\overline{\mathcal{F}ev}(z) &= \mathcal{F}ev(z)-zC(z)\\
&= \frac{z}{\sqrt{1-4z}}-\frac{1-\sqrt{1-4z}}{2}=z^2C'(z).
\end{aligned}$$

□

Proposition 12 *The second leading coefficients of the* P^{NF}_{2q+1}*'s are* $\psi_0 = 0$, $\psi_3 = 1$ *and* $\psi_{2q+5} = (q+3)\binom{2q+1}{q}$.

Proof: From the proof of Proposition 9,

$$\begin{aligned}
\psi_{2q+1} &= \varphi_{2q} + \overline{\psi}_{2q+1} \\
\overline{\psi}_1 &= 0 \\
\overline{\psi}_{2q+3} &= \sum_{i=0}^{q+1} \overline{\varphi}_{2i}\varphi_{2q-2i} + \sum_{i=0}^{q} \overline{\psi}_{2i+1}\varphi_{2q-2i+1} + \sum_{i=0}^{q} \psi_{2i+1}\overline{\varphi}_{2q-2i+1},
\end{aligned}$$

from which we get

$$\begin{aligned}
\mathcal{SF}od(z) &= \mathcal{F}ev(z) + \overline{\mathcal{SF}od}(z) \\
\overline{\mathcal{SF}od}(z) &= \overline{\mathcal{F}ev}(z)\mathcal{F}ev(z) + z\,\overline{\mathcal{SF}od}(z)\mathcal{F}od(z) + z\,\mathcal{SF}od(z)\overline{\mathcal{F}od}(z).
\end{aligned}$$

Then we get

$$\overline{\mathcal{SF}od}(z)(1 - z\mathcal{F}od(z)) = \mathcal{F}ev(z)\overline{\mathcal{F}ev}(z) + z\mathcal{SF}od(z)\overline{\mathcal{F}od}(z).$$

We know that $1 - z\mathcal{F}od(z) = 1 - zC(z) = 1/C(z)$, then

$$\overline{\mathcal{SF}od}(z) = \frac{z}{\sqrt{1-4z}}\, z^2 C'(z)C(z) + z\,\mathcal{SF}od(z)C(z)^2$$

and

$$\mathcal{SF}od(z) = \mathcal{F}ev(z) + \frac{z^3\, C(z)C'(z)}{\sqrt{1-4z}} + z\mathcal{SF}od(z)C(z)^2.$$

We know $1 - z\,C(z)^2 = C(z)\sqrt{1-4z}$, then

$$\begin{aligned}
\mathcal{SF}od(z) &= \left(\frac{z}{\sqrt{1-4z}} + \frac{z^3 C(z)C'(z)}{\sqrt{1-4z}}\right)\frac{1}{C(z)\,\sqrt{1-4z}} \\
&= \frac{z}{C(z)\,(1-4z)} + \frac{z^3\, C'(z)}{1-4z} \\
&= \frac{z^2}{(1-4z)\sqrt{1-4z}} + \frac{z}{\sqrt{1-4z}}.
\end{aligned}$$

which is the generating function of the sequence $0, 1$ followed by $(q+3)\binom{2q+1}{q}$. $\square$

Corollary 2 $\overline{\mathcal{SF}od}(z) = \frac{z^2}{(1-4z)\sqrt{1-4z}}$

Proof:

$$\overline{\mathcal{SF}od}(z) = \mathcal{SF}od(z) - \mathcal{F}ev(z) = \frac{z^2}{(1-4z)\sqrt{1-4z}}.$$

Notice that $\overline{\mathcal{SF}od}(z) = z\mathcal{S}od(z)$. $\square$

Proposition 13 *The second leading coefficients of the P_{2q}^{NF}'s are $\psi_0 = 0$, $\psi_2 = 1$, $\psi_4 = 4$, $\psi_6 = 15$ and for $q \geq 4$*

$$\begin{aligned}
\psi_{2q} &= \binom{2q-3}{q-2} + 2^{2q-3} + (q-2)\binom{2q-2}{q-2} + \\
&\quad 2\binom{2q-5}{q-3} + \frac{(q-3)(q-2)}{3}\binom{2q-5}{q-3}.
\end{aligned}$$

Proof: We have

$$\begin{aligned}
\psi_{2q+2} &= (q+1)\varphi_{2q+1} + \psi_{2q+1} + \overline{\psi}_{2q+2} \\
\overline{\psi}_{2q+2} &= \sum_{i=1}^{q} \psi_{2i-1}\overline{\varphi}_{2q-2i+2} + \sum_{i=1}^{q} \varphi_{2i-1}\overline{\psi}_{2q-2i+2} + \\
&\quad \sum_{i=1}^{q} \overline{\varphi}_{2i-1}\psi_{2q-2i+2} + \sum_{i=1}^{q} \overline{\psi}_{2i-1}\varphi_{2q-2i+2}.
\end{aligned}$$

This gives the equations on generic functions.

$$\begin{aligned}
\mathcal{SF}ev(z) &= z\mathcal{F}od(z) + z^2\mathcal{F}od'(z) + z\mathcal{SF}od(z) + \overline{\mathcal{SF}ev}(z) \\
\overline{\mathcal{SF}ev}(z) &= z\mathcal{SF}od(z)\overline{\mathcal{F}ev}(z) + z\mathcal{F}od(z)\overline{\mathcal{SF}ev}(z) + \\
&\quad z\mathcal{SF}ev(z)\overline{\mathcal{F}od}(z) + z\mathcal{F}ev(z)\overline{\mathcal{SF}od}(z).
\end{aligned}$$

Hence

$$\overline{\mathcal{SF}ev}(z) = \frac{z\mathcal{SF}od(z)\overline{\mathcal{F}ev}(z) + z\mathcal{SF}ev(z)\overline{\mathcal{F}od}(z) + z\mathcal{F}ev(z)\overline{\mathcal{SF}od}(z)}{1 - zC(z)}$$

which yields

$$\begin{aligned}
\mathcal{SF}ev(z) &= \mathcal{F}od(z) + z^2\mathcal{F}od'(z) + z\mathcal{SF}od(z) + \\
&\quad C(z)(z\mathcal{SF}od(z)\overline{\mathcal{F}ev}(z) + z\mathcal{F}ev(z)\overline{\mathcal{SF}od}(z)) \\
&\quad zC(z)\mathcal{SF}ev(z)\overline{\mathcal{F}od}(z).
\end{aligned}$$

and

$$\begin{aligned}
SFev(z) &= \frac{\mathcal{F}od(z) + z^2\mathcal{F}od'(z) + z\mathcal{SF}od(z)}{1 - zC(z)^2} + \\
&\quad \frac{zC(z)\mathcal{SF}od(z)\overline{\mathcal{F}ev}(z) + zC(z)\mathcal{F}ev(z)\overline{\mathcal{SF}od}(z)}{1 - zC(z)^2} \\
&= \frac{z}{\sqrt{1-4z}} + \frac{z^2C'(z)}{C(z)\sqrt{1-4z}} + \\
&\quad \frac{z}{C(z)\sqrt{1-4z}}\left(\frac{z^2}{(1-4z)\sqrt{1-4z}} + \frac{z}{\sqrt{1-4z}}\right) + \\
&\quad \left(\frac{z}{\sqrt{1-4z}} - \frac{1-\sqrt{1-4z}}{2}\right)\left(\frac{z^2}{(1-4z)^2} + \frac{z^2}{1-4z}\right) + \\
&\quad \frac{z^4}{(1-4z)^2\sqrt{1-4z}}.
\end{aligned}$$

Notice that

$$\frac{z^2C'(z)}{C(z)\sqrt{1-4z}} \quad = \quad \frac{z}{2(1-4z)} - \frac{z}{2\sqrt{1-4z}}.$$

and

$$\begin{aligned}&\frac{z}{C(z)\sqrt{1-4z}}\left(\frac{z^2}{(1-4z)\sqrt{1-4z}}+\frac{z}{\sqrt{1-4z}}\right)+\\ &\left(\frac{z}{\sqrt{1-4z}}-\frac{1-\sqrt{1-4z}}{2}\right)\left(\frac{z^2}{(1-4z)^2}+\frac{z^2}{1-4z}\right) \quad = \quad \frac{2z^3}{\sqrt{1-4z}(1-4z)}+\frac{z^2}{\sqrt{1-4z}}+\\ &\qquad\qquad\frac{z^4}{\sqrt{1-4z}(1-4z)^2}\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{SF}ev(z) \quad = \quad & \frac{z}{2\sqrt{1-4z}}+\frac{z}{2(1-4z)}+\\ & \frac{2z^3}{\sqrt{1-4z}(1-4z)}+\frac{z^2}{\sqrt{1-4z}}+\frac{2z^4}{\sqrt{1-4z}(1-4z)^2}\end{aligned}$$

We summarize the result in the following table.

gen. fonct.	*coefficients*	*up to*	*why?*
$\frac{z}{2\sqrt{1-4z}}$	$\binom{2q-3}{q-2}$	$q \geq 2$	Proposition 11
$\frac{z}{2(1-4z)}$	2^{2q-3}	$q \geq 2$	
$\frac{2z^3}{\sqrt{1-4z}(1-4z)}$	$(q-2)\binom{2q-2}{q-2}$	$q \geq 2$	
$\frac{z^2}{\sqrt{1-4z}}$	$2\binom{2q-5}{q-3}$	$q \geq 3$	
$\frac{2z^4}{\sqrt{1-4z}(1-4z)^2}$	$\frac{(q-3)(q-2)}{3}\binom{2q-5}{q-3}$	$q \geq 4$	A002802

Hence we have for $q \geq 4$:

$$\begin{aligned}\psi_q \quad = \quad & \binom{2q-3}{q-2}+2^{2q-3}+(q-2)\binom{2q-2}{q-2}+\\ & 2\binom{2q-5}{q-3}+\frac{(q-3)(q-2)}{3}\binom{2q-5}{q-3}.\end{aligned}$$

□

Recall what we have computed for *plain terms*:

coefficients	*generating functions*		*values*	*equivalents*
$P^T_{2q+1,q+1}$	$\mathcal{O}d(z)$	$\frac{1-\sqrt{1-4z}}{2z}$	C_q	$4^q\sqrt{\frac{1}{\pi q^3}}$
$P^T_{2q+1,q}$	$\mathcal{S}od(z)$	$\frac{z}{(1-4z)\sqrt{1-4z}}$	$(2q-1)\binom{2(q-1)}{q-1}$	$4^q\,\frac{1}{2}\,\sqrt{\frac{q}{\pi}}$
$P^T_{2q+1,q-1}$	$\mathcal{T}od(z)$	$\frac{2z}{(1-4z)^2}+\frac{z^2+z^3}{(1-4z)^3\sqrt{1-4z}}$	$q\;2^{2q-1}+\frac{q(q-1)(q-2)}{120}\binom{2q}{q}+\frac{(q+1)q(q-1)}{120}\binom{2(q+1)}{q+1}$	$4^q\,\frac{1}{24}\,\sqrt{\frac{q^5}{\pi}}$
$P^T_{2q,q}$	$\mathcal{E}v(z)$	$\frac{4z-1+\sqrt{1-4z}}{2(1-4z)}$	$\binom{2q-1}{q}$	$4^q\,\frac{1}{2}\,\sqrt{\frac{1}{\pi q}}$
$P^T_{2q,q-1}$	$\mathcal{S}ev(z)$	$\frac{z}{1-4z}+\frac{z^2}{(1-4z)^2\sqrt{1-4z}}$	$4^{q-1}+\frac{2(2q-5)(2q-3)(2q-1)}{3(q-2)}\binom{2(q-3)}{q-3}$	$4^q\,\frac{1}{12}\,\sqrt{\frac{q^3}{\pi}}$

and for *normal forms*

coefficients	*generating functions*		*values*	*equivalents*
$P^{NF}_{2q+1,q+1}$	$\mathcal{F}od(z)$	$\frac{1-\sqrt{1-4z}}{2z}$	C_q	$4^q\sqrt{\frac{1}{\pi q^3}}$
$P^{NF}_{2q+1,q}$	$\mathcal{SF}od(z)$	$\frac{z}{\sqrt{1-4z}}+\frac{z^2}{(1-4z)\sqrt{1-4z}}$	$(q+1)\binom{2q-3}{q-2}$	$4^q\frac{1}{8}\sqrt{\frac{q}{\pi}}$
$P^{NF}_{2q,q}$	$\mathcal{F}ev(z)$	$\frac{z}{\sqrt{1-4z}}$	$2\binom{2q-3}{q-2}$	$4^q\frac{1}{4}\sqrt{\frac{1}{\pi q}}$
$P^{NF}_{2q,q-1}$	$\mathcal{SF}ev(z)$	$\frac{z}{2\sqrt{1-4z}}+\frac{z}{2(1-4z)}+\frac{2z^3}{(1-4z)\sqrt{1-4z}}+\frac{z^2}{\sqrt{1-4z}}+\frac{2z^4}{(1-4z)^2\sqrt{1-4z}}$	$\binom{2q-3}{q-2}+2^{2q-3}+(q-2)\binom{2q-2}{q-2}+2\binom{2q-5}{q-3}+\frac{(q-3)(q-2)}{3}\binom{2q-5}{q-3}$	$4^q\,\frac{1}{96}\,\sqrt{\frac{q^3}{\pi}}$

We notice that the coefficients of the P^{NF}_n's have the same asymptotic behavior as the coefficients of P^T_n's, with a slightly smaller constant, 1/8 or 1/4 for 1/2 and 1/96 for 1/12. Notice, in particular, that the results $P^{NF}_{2q,q} \sim \frac{1}{2}\,P^T_{2q,q}$ and $P^{NF}_{2q+1,q} \sim \frac{1}{4}\,P^T_{2q+1,q}$ comes from the identities.

$$\begin{aligned} 2\binom{2q-3}{q-2} &= \frac{q}{2q-1}\binom{2q-1}{q} \\ (q+1)\binom{2q-3}{q-2} &= \frac{q+1}{2(2q-1)}\,(2q-1)\binom{2(q-1)}{q-1}. \end{aligned}$$

6 Generating functions for terms

$T_{n,m}$ is associated with a *bivariate generating* function (see [7] Section III.1):

$$\mathcal{T}(z,u) \;=\; \sum_{n,m} T_{n,m} z^n u^m.$$

There is no current analytic method to study it. The function:

$$T^{\langle m \rangle}(z) = \sum_{n=0}^{\infty} T_{n,m} z^n$$

is called the *vertical generating function*. It gives the $T_{n,m}$'s for each value of m.

Vertical generating functions

We see that

$$T_{n,m+1} = T_{n+1,m} - \sum_{k=0}^{n} T_{n-k,m} T_{k,m}.$$

Hence

$$T^{\langle m \rangle}(0) \quad = \quad 0$$

and

$$\begin{aligned} T^{\langle m+1 \rangle}(z) &= \sum_{n=0}^{\infty} T_{n,m+1} z^n \\ &= \sum_{n=0}^{\infty} T_{n+1,m} z^n - \sum_{n=0}^{\infty} \sum_{k=0}^{n} T_{n-k,m} T_{k,m} z^n \\ &= \frac{T^{\langle m \rangle}(z)}{z} - (T^{\langle m \rangle}(z))^2. \end{aligned}$$

In other words

$$z(T^{\langle m \rangle}(z))^2 - T^{\langle m \rangle}(z) + zT^{\langle m+1 \rangle}(z) = 0.$$

Hence

$$T^{\langle m \rangle}(z) = \frac{1 - \sqrt{1 - 4z^2 T^{\langle m+1 \rangle}(z)}}{2z}.$$

Moreover

$$[z]T^{\langle m \rangle}(z) = \frac{d\,T^{\langle m \rangle}}{d\,z}(0) = m.$$

We see that $T^{\langle m \rangle}$ is defined from $T^{\langle m+1 \rangle}$. $T^{\langle m \rangle}(z)$ is difficult to study, because we have $T^{\langle m \rangle}$ defined in term of $T^{\langle m+1 \rangle}$.

7 Conclusion

We have given several parameters on numbers of untyped lambda terms and untyped normal forms and proved or conjectured facts about them. On another direction, it could be worth to study typed lambda terms, whereas we have only analyzed untyped lambda terms in this paper.

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$n\backslash m$	0	1	2	3	4	5	6	7
1	0	1	2	3	4	5	6	7
2	1	2	3	4	5	6	7	8
3	2	4	8	14	22	32	44	58
4	4	12	26	46	72	104	142	186
5	13	38	87	172	305	498	763	1112
6	42	127	324	693	1294	2187	3432	5089
7	139	464	1261	2890	5831	10684	18169	29126
8	506	1763	5124	12653	27254	52671	93488	155129
9	1915	7008	21709	57070	130863	269260	508513	896634
10	7558	29019	94840	265129	646458	1406983	2791564	5136885
11	31092	124112	427302	1264362	3262352	7502892	15703602	30429782
12	132170	548264	1977908	6168242	16811366	40776020	89671904	181746638
13	580466	2491977	9384672	30755015	88253310	225197061	520076012	1104714147
14	2624545	11629836	45585471	156409882	471315501	1263116040	3058077451	6789961206
15	12190623	55647539	226272369	810506769	2558249963	7184911623	18208806189	42244969589
16	58083923	272486289	1146515237	4275219191	14098296495	41417170373	109721440529	265618096347
17	283346273	1363838742	5923639803	22933607180	78832280277	241776779298	668513708207	1686996660888
18	1413449148	6968881025	31177380822	125027527671	446961983408	1428444131853	4116538065930	10816530842627

Figure 1: Values of $T_{n,m}$ up to $(18,7)$

n	$P_n^T(m)$
1	m
2	$m+1$
3	m^2+m+2
4	$3m^2+5m+4$
5	$2m^3+6m^2+17m+13$
6	$10m^3+26m^2+49m+42$
7	$5m^4+30m^3+111m^2+179m+139$
8	$35m^4+134m^3+405m^2+683m+506$
9	$14m^5+140m^4+652m^3+1658m^2+2629m+1915$
10	$126m^5+676m^4+2812m^3+7122m^2+10725m+7558$
11	$42m^6+630m^5+3610m^4+12760m^3+30783m^2+45195m+31092$
12	$462m^6+3334m^5+17670m^4+60240m^3+138033m^2+196355m+132170$
13	$132m^7+2772m^6+19218m^5+87850m^4+285982m^3+635178m^2+880379m+580466$
14	$1716m^7+16108m^6+104034m^5+449290m^4+1390246m^3+2991438m^2+4052459m+2624545$
15	$429m^8+12012m^7+99386m^6+560854m^5+2308173m^4+6895122m^3+14436365m^2+19144575m+12190623$
16	$6435m^8+76444m^7+584878m^6+3076878m^5+12039895m^4+34815210m^3+71170791m^2+92631835m+58083923$
17	$1430m^9+51480m^8+502384m^7+3389148m^6+16925916m^5+63753310m^4+179178860m^3+358339416m^2+458350525m+283346273$
18	$24310m^9+357256m^8+3176112m^7+19799164m^6+93981244m^5+342274990m^4+938333964m^3+1840448776m^2+2317036061m+1413449148$

Figure 2: The polynomials P_n^T for the function $m \mapsto T_{n,m}$

$n \backslash m$	0	1	2	3	4	5	6	7	8
1	0	1	2	3	4	5	6	7	8
2	1	2	3	4	5	6	7	8	9
3	2	4	8	14	22	32	44	58	74
4	4	10	20	34	52	74	100	130	164
5	10	25	58	121	226	385	610	913	1306
6	25	72	185	400	753	1280	2017	3000	4265
7	72	223	614	1497	3244	6347	11418	19189	30512
8	223	728	2195	5716	12863	25688	46723	78980	125951
9	728	2549	8108	22745	56360	125093	253004	473753	832280
10	2549	9254	31253	93734	244997	564854	1173029	2237558	3983189
11	9254	35168	124778	395720	1109222	2770904	6261818	12999728	25130630
12	35168	138606	512898	1720040	5097660	13347978	31308206	66902388	132274680
13	138606	563907	2174894	7645095	23948550	66818531	167837142	384821079	816168830
14	563907	2369982	9459993	34771380	114618495	335857722	880524117	2092596528	4571548155
15	2369982	10231830	42221886	161568762	558056526	1723895502	4785906510	12073186866	28016723742
16	10231830	45381558	192944940	765787548	2764390146	8947158690	25962816408	68135021640	163627733358
17	45381558	206266797	901441688	3701763855	13912595562	47127027713	143678500332	397091138883	1005324501470
18	206266797	959283300	4302919895	18223902654	71123969121	251343711032	799893538635	2302171013970	6046781201429

Figure 3: Values of $F_{n,m}$ up to $(18, 8)$

n	$P_n^{NF}(m)$
1	m
2	$m+1$
3	m^2+m+2
4	$2m^2+4m+4$
5	$2m^3+3m^2+10m+10$
6	$6m^3+15m^2+26m+25$
7	$5m^4+12m^3+49m^2+85m+72$
8	$20m^4+62m^3+155m^2+268m+223$
9	$14m^5+50m^4+240m^3+589m^2+928m+728$
10	$70m^5+263m^4+870m^3+2146m^2+3356m+2549$
11	$42m^6+210m^5+1153m^4+3658m^3+8351m^2+12500m+9254$
12	$252m^6+1128m^5+4658m^4+14838m^3+33575m^2+48987m+35168$
13	$132m^7+882m^6+5446m^5+21198m^4+63138m^3+137695m^2+196810m+138606$
14	$924m^7+4862m^6+24086m^5+93748m^4+275898m^3+587814m^2+818743m+563907$
15	$429m^8+3696m^7+25372m^6+117120m^5+429435m^4+1223102m^3+2558090m^2+3504604m+2369982$
16	$3432m^8+20996m^7+121286m^6+556920m^5+2011411m^4+5601948m^3+11448828m^2+15384907m+10231830$
17	$1430m^9+15444m^8+116892m^7+624768m^6+2717670m^5+9524196m^4+26064412m^3+52459126m^2+69361301m+45381558$
18	$12870m^9+90683m^8+598120m^7+3162562m^6+13513606m^5+46329205m^4+124109404m^3+245453736m^2+319746317m+206266797$

Figure 4: The polynomials P_n^{NF} for the function $m \mapsto F_{n,m}$