# The set of realizations of a max-plus linear sequence is semi-polyhedral

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**Abstract**

We show that the set of realizations of a given dimension of a max-plus linear sequence is a finite union of polyhedral sets, which can be computed from any realization of the sequence. This yields an (expensive) algorithm to solve the max-plus minimal realization problem. These results are derived from general facts on rational expressions over idempotent commutative semirings: we show more generally that the set of values of the coefficients of a commutative rational expression in one letter that yield a given max-plus linear sequence is a finite union of polyhedral sets.

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## 1. Introduction and Statement of Results

A *realization* of a sequence $S_0, S_1, \dots$ of elements of a semiring $K$ is a triple $(c, A, b)$, where $c \in K^{1\times N}$, $A \in K^{N\times N}, b \in K^{N\times 1}$, and $S_0 = cb$, $S_1 = cAb$, $S_2 = cA^2b$, ... The integer $N$ is the *dimension* of the realization. A sequence $S$ is $K$-*recognizable* (or $K$-linear) if it has a realization $(c, A, b)$, and then, we say that $S$ is *recognized* by $(c, A, b)$.

In this paper, we consider the max-plus semiring $K = \mathbb{Q}_{\max}$, which is the set $\mathbb{Q} \cup \{-\infty\}$, equipped with the addition $(a, b) \mapsto a \oplus b = \max(a, b)$ and the multiplication $(a, b) \mapsto a \otimes b = a + b$. We address the following realization problem, which was raised as an open problem in several works [CMQV85, Ols86, BCOQ92, ODS99]: *does a $\mathbb{Q}_{\max}$-recognizable sequence have a realization of a given dimension?*

As observed by the first and third authors [BP02], it follows from an old result of Stockmeyer and Meyer [SM73] that this problem is co-NP-hard. In this paper, we show that it is decidable and we show how one can effectively construct the set of realizations. Our results are also valid for other tropical semirings [Pin98], like the semiring of max-plus integers $\mathbb{Z}_{\max} = (\mathbb{Z} \cup \{-\infty\}, \max, +)$, or the semiring $\mathbb{N}_{\min} = (\mathbb{N} \cup \{+\infty\}, \min, +)$, hence, it is convenient to consider more generally a semiring $K$, whose addition, multiplication, zero element, and unit elements will be denoted by $\oplus, \otimes, \mathbb{0}, \mathbb{1}$, respectively. We shall assume that $K$ is commutative, i.e., that $u \otimes v = v \otimes u$. We shall use the familiar algebraic notation, with the obvious changes (e.g., $a^2b = a \otimes a \otimes b$).

We say that a semiring is *idempotent* when $u \oplus u = u$, we say that an idempotent semiring is *linearly ordered* when the relation $u \leq v \iff u \oplus v = v$ is a linear order, and that it is *archimedian* if $u\lambda^k \geq v\mu^k$ for all $k \geq 0$ implies $v = \mathbb{0}$ or $\lambda \geq \mu$. Finally, we say that $K$ is *cancellative* if $uv = u'v \implies v = \mathbb{0}$ or $u = u'$.

A *monomial* in the $n$ variables $x_1, \dots, x_n$ is of the form $m(x) = ux_1^{\alpha_1} \cdots x_n^{\alpha_n}$, for some $u \in K$ and $\alpha_1, \dots, \alpha_n \in \mathbb{N}$. We call *half-space* of $K^n$ a set of the form $\{x \in K^n \mid m(x) \geq m'(x)\}$, where $m$ and $m'$ are monomials. (In $\mathbb{Q}_{\max}$, a monomial can be rewritten with the conventional notation as $m(x) = u + \sum_{i=1}^n \alpha_i x_i$, which accounts for the terminology "half-space"). A *polyhedron* is a finite intersection of half-spaces. A set is *semi-polyhedral* if it is a finite union of polyhedra.

A realization of dimension $N$, $(c, A, b)$, can be seen as an element of the set $K^{2N+N^2}$. We will prove:

**Theorem 1.** *Let $K$ denote an idempotent linearly ordered archimedian cancellative commutative semiring. Then, the set of realizations of dimension $n$ of a $K$-recognizable series is a semi-polyhedral subset of $K^{2N+N^2}$, which can be effectively constructed.*

We get as a consequence of Theorem 1:

**Corollary 1.** *When $K = \mathbb{Q}_{\max}$, $\mathbb{Z}_{\max}$, or $\mathbb{N}_{\min}$, the existence of a realization of dimension $n$ of a $K$-recognizable sequence is decidable.*

Indeed, when $K = \mathbb{Q}_{\max}$, the non-emptiness of a semi-polyhedral set is decidable, because the first order theory of $(\mathbb{Q}, +, \leq)$ is decidable, or, to use a perhaps more elementary argument, because the non-emptiness of an ordinary polyhedron can be checked by linear programming (see e.g. [Sch86]). When $K = \mathbb{Z}_{\max}$ or $\mathbb{N}_{\min}$, the corollary follows from the decidability of Presburger's arithmetics (see e.g. [End72]).

It follows from Corollary 1 that there is an algorithm to compute max-plus minimal realizations, a problem which arose from the beginning of the development of the max-plus modelling of discrete event systems [CMQV89], which was mentioned in the book [BCOQ92] and was stated by Olsder and De Schutter [ODS99] as one of the open problems of [BSVW99]. In fact, the algorithm is very expensive (see the discussion in §5), so our result only implies that we can solve the realization problem in "small" dimension. A Caml implementation by G. Melquiond and P. Philipps is available [3]. It would be interesting to find a less expensive algorithm.

Before proving Theorem 1, it is instructive to show why classical arguments fail to prove these result. A natural idea, would be to show that if two sequences $S$ and $T$ have realizations of respective sizes $N$ and $M$, there is an integer $\nu(N, M)$ such that:

$$(S_k = T_k, \forall k \leq \nu(N, M)) \implies (S_k = T_k, \forall k \in \mathbb{N}) \ . \tag{1}$$

(Results of this kind are called "equality theorems" by Eilenberg, see [Eil74, Chap. 6, § 8].) Indeed, if the semiring $K$ satisfies property (1), then, the set of realizations of dimension $N$ of a sequence $T$ given by a realization of dimension $M$ is the set defined by the finite system of equations $cA^kb = T_k$, for $k = 0, \ldots, \nu(N, M)$. There are two classical cases where property (1) is true. First, if $K$ is a finite semiring (like the Boolean semiring), (1) is trivially true since the set of realizations of a given dimension is finite (and, of course, the minimal realization problem is decidable). A second, more interesting case, is when $K$ is a subsemiring of a commutative ring. Then, the Cayley-Hamilton theorem implies that (1) holds with $\nu(N, M) = N + M - 1$, by a standard argument (see [Eil74, Chap. 6, proof of Th. 8.1]). An interesting feature of the max-plus semiring is that (1) does not hold. For instance, the realization of dimension 2 over $\mathbb{Q}_{\max}$,

$$c = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -\infty \\ -\infty & -1 \end{pmatrix}, \quad b = \begin{pmatrix} \alpha \\ 0 \end{pmatrix},$$

[3] http://perso.ens-lyon.fr/natacha.portier/realisations-max-plus.tar.gz

where $\alpha$ is an element of $\mathbb{Q}_{\max}$, recognizes the sequence $S^\alpha : S^\alpha_k = \max(\alpha, -k)$. To distinguish between $S^\alpha$ and $S^\beta$, we need to consider values of $k \geq \min(-\alpha, -\beta)$, and this contradicts (1).

Our proof of Theorem 1 relies on a more general result, of independent interest. Let us first briefly recall some basic facts about rational series in one letter (see [BR88] for a detailed presentation). Let $X$ denote an indeterminate. A sequence $S_0, S_1, \ldots \in K$ can be identified to the formal series $S = S_0 \oplus S_1 X \oplus S_2 X^2 \oplus \cdots \in K[[X]]$ (in particular, the indeterminate $X$ corresponds to the sequence $\mathbb{0}, \mathbb{1}, \mathbb{0}, \mathbb{0}, \ldots$). The set of formal series $K[[X]]$, equipped with entrywise sum and Cauchy product, is a semiring. The Kleene's star of a series $S$, defined when $S$ has a zero constant coefficient, is $S^* = S^0 \oplus S \oplus S^2 \oplus \cdots$ The $k$-th coefficient of $S$ will sometimes be denoted by $\langle S, X^k \rangle$ instead of $S_k$. The Kleene-Schützenberger theorem states that $S$ is recognizable if, and only if, it is rational, i.e., if it can be represented by a well formed expression involving sums, products, stars, and monomials.

Consider now a finite set of commuting indeterminates, $\Sigma = \{\mathsf{d}_1, \ldots, \mathsf{d}_n\}$, and let $K[\Sigma]$ denote the semiring of polynomials in $\mathsf{d}_1, \ldots, \mathsf{d}_n$. To a vector $d = (d_1, \ldots, d_n) \in K^n$, we associate the evaluation morphism $K[\Sigma][[X]] \to K[[X]]$, which sends the series $\mathsf{S} \in K[\Sigma][[X]]$ to the series $[\mathsf{S}]_d$ obtained by replacing each indeterminate $\mathsf{d}_i$ by the value $d_i$. Borrowing the probabilist notation, we denote by $\{\mathsf{S} = S\}$ the set $\{d \in K^n \mid [\mathsf{S}]_d = S\}$. More generally, for $\mathsf{S}, \mathsf{T} \in K[\Sigma][[X]]$, we shall write for instance $\{\mathsf{S} \geq \mathsf{T}\}$ as an abbreviation of $\{d \in K^n \mid [\mathsf{S}]_d \geq [\mathsf{T}]_d\}$.

**Theorem 2.** *(Rational series synthesis) Let $K$ denote an idempotent linearly ordered archimedian cancellative commutative semiring. For all rational series $\mathsf{S} \in K[\Sigma][[X]]$ and for all rational series $S \in K[[X]]$, the set $\{\mathsf{S} = S\}$ is semi-polyhedral.*

This theorem will be proved in Section 3.

An intuitive way to state this result is to say that "the set of values of the coefficients of a rational expression which yield a given rational series is semi-polyhedral".

Theorem 1 is an immediate corollary of Theorem 2. Indeed, consider the set $\Sigma = \Sigma_N$ whose elements are the $2N + N^2$ indeterminates $\mathsf{c}_i, \mathsf{A}_{ij}, \mathsf{b}_j$, where $1 \leq i, j \leq N$. Let $\mathsf{c} = (\mathsf{c}_i) \in (K[\Sigma_N])^{1\times N}$, $\mathsf{A} = (\mathsf{A}_{ij}) \in (K[\Sigma_N])^{N\times N}$, $\mathsf{b} = (\mathsf{b}_j) \in (K[\Sigma_N])^{N\times 1}$, and consider the *universal series* $\mathsf{S}_N = \mathsf{c}(\mathsf{A}X)^*\mathsf{b} = \mathsf{cb} \oplus \mathsf{cAb}X \oplus \cdots \in K[\Sigma_N][[X]]$, which, by construction, is recognizable (or equivalently, rational). Since the set of realizations of dimension $n$ of a rational series $S \in K[[X]]$ is exactly $\{\mathsf{S}_N = S\}$, Theorem 2 implies Theorem 1.

We warn the reader that some apparently minor variants of $\{\mathsf{S} = S\}$ need not be semi-polyhedral. For instance, since $\{\mathsf{S} \leq S\} = \{\mathsf{S} \oplus S = S\}$, by Theorem 2, $\{\mathsf{S} \leq S\}$ is semi-polyhedral, but we shall see in §4 that $\{\mathsf{S} \geq S\}$ need not be semi-polyhedral.

In Section 5, we bound the complexity of the algorithm which is contained in the proof of Theorems 1 and 2. The details of this complexity analysis are lengthy, but its principle is simple: we need first to compute a star height one representation of the universal series $\mathsf{S}_N = \mathsf{c}(\mathsf{A}X)^*\mathsf{b}$. We give an explicit representation, which turns out to be of double exponential size. Then, we compute the semi-polyhedral set arising from this expression, which yields a simply exponential blow up, leading to a final triple exponential bound.

This high complexity implies that Theorem 1 is only of theoretical interest. However, it should be noted that Theorem 2 allows us to solve more generally the "structured realization problem", in which some coefficients of the realizations are constrained to be zero. Consider for instance the problem of computing all $N$ dimensional realizations $(\mathsf{c}, \mathsf{A}, \mathsf{b})$ of a linear sequence $S$, subject to the constraint that $\mathsf{A}$ is diagonal. The set of realizations becomes $\{\bigoplus_{1\leq i\leq N} \mathsf{c}_i(\mathsf{A}_{ii}X)^*\mathsf{b}_i = S\}$, and Theorem 2 shows that this set is semi-polyhedral. For such structured problems in which the universal series $\mathsf{S}_N$ is replaced by a polynomial size rational expression, the present approach leads only to a simply exponential complexity.

The algorithmic difficulties encountered here are consistent with the observation that algorithmic issues concerning linear systems over rings (and a fortiori over semirings) are generally harder than in the case over fields. In particular, the powerful "geometric approach" based on the computations of invariant spaces does carry over to the ring case [BM91], and even to the max-plus case [CGQ99a, Kat07, LGKL09], but then, the analogues of the classical fixed point algorithms do not always terminate (due to the lack of Artinian or Noetherian properties). The present algebraic approach, via rational series, yields alternative tools to the geometric approach: no termination issue arises, but the algorithms are subject to a curse of complexity.

It is also instructive to look at Theorem 2 in the light of the recent developments of tropical geometry [IMS07, RGST05]. The latter studies in particular the tropical analogues of algebraic sets. The tropical analogues of *semi*-algebraic sets could be considered as well: it seems reasonable to define them precisely as the special semi-polyhedral sets introduced here (recall that the exponents appearing in the monomials are required to be nonnegative integers). Then, Theorem 1 may be thought of as the max-plus analogue of a known result, that the set of *nonnegative* realizations of a given dimension of a linear sequence over the real numbers (equipped with the usual addition and multiplication) is semi-algebraic (this follows readily from the "equality theorem" mentioned above). Then, a comparison with the complexity of existing semi-algebraic algorithms [BPR96] suggests that the present triple exponential bound is probably suboptimal. To improve it, we would need to further exploit the tropical semi-algebraic structure. This raises further issues which are beyond the scope of this paper.

Let us complete this long introduction by pointing out a few relevant references about the minimal realization problem.

First, there are two not so well known theorems, which hold in arbitrary semirings. A result of Fliess [Fli75] characterizes the minimal dimension of realization as the minimal dimension of a semimodule stable by shift and containing the semimodule of rows of the Hankel matrix. (The result is stated there for the semiring $(\mathbb{R}^+, +, \times)$, but, as observed by Jacob [Jac75], the proof is valid in an arbitrary semiring.) Maeda and Kodama found independently closely related results [MK80]. As observed by Duchamp and Reutenauer (see Theorem 2 in [DR97]), Fliess's characterization is a third fundamental statement to add to the Kleene-Schützenberger theorem. The classical realization theorems over fields are immediate corollaries of this result. The results of Anderson, Deistler, Farina and Bevenuti [ADFB96] and Benvenuti and Farina [BF99] for nice applications of these ideas. We also refer the reader to the book [BR10] for a general discussion of minimization issues concerning noncommutative rational series. A second fundamental result, due to Eilenberg [Eil74, Ch. 16] (inspired by Kalman), extends the notion of recognizability and shows the existence of a minimal module which recognizes a sequence. The difficulty is that this module need not be free. (Eilenberg's theorem is stated for modules over rings, but, as noted in [CGQ99b], it can be extended to semimodules over semirings). The max-plus minimal realization problem was raised by Cohen, Moller, Quadrat and Viot [CMQV85], and by Olsder [Ols86] (see also [BCOQ92]). There are relatively few general results about this (hard) problem. Olsder [Ols86] showed some connections between max-plus realizations, and conventional realizations, via exponential asymptotics. Cuninghame-Green [CG91] gave a realization procedure, which yields, when it can be applied, an upper bound for the minimal dimension of realization. Some lower and upper bounds involving various notions of rank over the max-plus semiring were given in [Gau92, Chap. 6]. In particular, the cardinality of a minimal generating family of the row or column space of the Hankel matrix, which characterizes the minimal dimension of realization in the case of fields, is only a (possibly coarse) upper bound in the max-plus semiring. The lower bound of [Gau92, Chap. 6] (which involves max-plus determinants) also appears in [GBCG98], where it is used to extend to the convex case a theorem proved by Butkovič and Cuninghame-Green [CGB95] in the strictly convex case. De Schutter and De Moor [DSDM95] observed that the (much simpler) *partial* max-plus realization problem can be interpreted as an extended linear complementarity problem. This work was pursued by De Schutter in [DS96].

## 2. Max-plus Rational Expressions

In this section, we recall some basic results about max-plus rational expressions, which will be needed in the proof of Theorem 2.

The first step of the proof of Theorem 2 is the following well known star height one representation (some variants of which already appeared in particular in works of Moller [Mol88], of Bonnier-Rigny and Krob [BRK94], and in [Gau92, Gau94a]). All these results can be thought of as specializations, or refinements, of general results on commutative rational expressions [ES69, Con71].

In the sequel, $\mathsf{K}$ denotes a generic semiring (which may or may not coincide with the semiring $K$ of Theorem 2).

**Lemma 1.** *Let $\mathsf{K}$ be an idempotent commutative semiring. A rational series $\mathsf{S} \in \mathsf{K}[[X]]$ can be written as*

$$\mathsf{S} = \bigoplus_{1 \le i \le r} \mathsf{P}_i(\mathsf{q}_i X^c)^* \quad , \tag{2}$$

*where $\mathsf{P}_1, \dots, \mathsf{P}_r \in \mathsf{K}[X]$, $\mathsf{q}_1, \dots, \mathsf{q}_r \in \mathsf{K}$, and $c$ is a positive integer.*

*Proof.* It suffices to check that the set of series of the form (2) is closed by sum, Cauchy product, and Kleene's star. This follows easily from the following classical commutative rational identities (see e.g. [Con71]), which are valid for all $\mathsf{U}, \mathsf{V} \in \mathsf{K}[[X]]$ (with zero constant coefficient) and $k \geq 1$,

$$(\mathsf{U} \oplus \mathsf{V})^* = \mathsf{U}^*\mathsf{V}^* \quad , \tag{3}$$

$$(\mathsf{V}\mathsf{U}^*)^* = \mathbb{1} \oplus \mathsf{V}(\mathsf{U} \oplus \mathsf{V})^* \quad , \tag{4}$$

$$\mathsf{U}^* = (\mathbb{1} \oplus \mathsf{U} \oplus \cdots \oplus \mathsf{U}^{k-1})(\mathsf{U}^k)^* \tag{5}$$

(only in (3) we used the idempotency and commutativity of the semiring). $\square$

The representation (2) of $\mathsf{s}$ is far from being unique. In particular, thanks to the rational identity

$$\mathsf{U}^* = \mathbb{1} \oplus \mathsf{U} \oplus \cdots \oplus \mathsf{U}^{k-1} \oplus \mathsf{U}^k\mathsf{U}^* \quad , \tag{6}$$

which holds for all $k \geq 1$, we can always rewrite the series (2) as

$$\mathsf{S} = \mathsf{P} \oplus X^{\kappa c}\Big( \bigoplus_{1 \le i \le \rho} \mathsf{u}_i X^{\mu_i}(\mathsf{q}_i X^c)^* \Big) \tag{7}$$

where $0 \leq \mu_i \leq c-1$, $\mathsf{u}_i \in \mathsf{K}$, and $\mathsf{P} \in \mathsf{K}[X]$ has degree less than $\kappa c$. The interest of (7), by comparison with (2), is that the asymptotics of $\langle \mathsf{S}, X^k \rangle$ can be read directly from the rational expression. Indeed, for all $0 \leq j \leq c-1$ and $k \geq 0$,

$$\langle \mathsf{S}, X^{(k+\kappa)c+j} \rangle = \bigoplus_{\mu_i = j} \mathsf{u}_i \mathsf{q}_i^k \quad . \tag{8}$$

When $\mathsf{K}$ is the max-plus semiring, the representations (7) and (8) can be simplified thanks to the archimedian property. We say that a series $S \in K[[X]]$

is *ultimately geometric* if there is an integer $\kappa$ and a scalar $\gamma \in K$ such that $\langle S, X^{k+1}\rangle = \gamma\langle S, X^k\rangle$, for all $k \geq \kappa$. The *merge* of $c$ series $S^{(0)}, \ldots, S^{(c-1)}$ is the series $S^{(0)}(X^c) \oplus XS^{(1)}(X^c) \oplus \cdots X^{c-1}S^{(c-1)}(X^c)$, whose coefficient sequence is obtained by "merging" the coefficient sequences of $S^{(0)}, \ldots, S^{(c-1)}$. E.g., the merge of

$$S^{(0)} = X^* = 0 \oplus 0X \oplus 0X^2 \oplus \cdots \text{ and } S^{(1)} = 1(1X)^* = 1 \oplus 2X \oplus 3X^2 \oplus \cdots \quad (9)$$

is

$$T = (X^2)^* \oplus 1X(1X^2)^* = 0 \oplus 1X \oplus 0X^2 \oplus 2X^3 \oplus 0X^4 \oplus 3X^5 \oplus \cdots \quad (10)$$

The following elementary but useful consequence of Lemma 1 and of the archimedian condition characterizes the rational series over max-plus like semirings. This theorem, which is a series analogue of the max-plus cyclicity theorem for powers of max-plus matrices of Cohen, Dubois, Quadrat and Viot [CDQV83] (see also [CDQV85, BCOQ92, GP97, AGW05, HOvdW06]), was anticipated by Cohen, Moller, Quadrat and Viot in [CMQV89], where a result similar to Theorem 3 is proved in the special case of series with nondecreasing coefficient sequence. Moller [Mol88], and Bonnier-Rigny and Krob [BRK94], proved results which are essentially equivalent to Theorem 3, which is taken from [Gau92, Gau94a, Gau94b] (slightly more general assumptions are made on the semiring, in the last two references). Theorem 3 is in fact a max-plus analogue of a deeper result, Soittola's theorem [Soi76], which characterizes nonnegative rational series as merges of series with a dominant root (see also Perrin [Per92]).

**Theorem 3.** *Let $K$ denote an idempotent linearly ordered archimedian commutative semiring. A series $S \in K[[X]]$ is rational if, and only if, it is a merge of ultimately geometric series.*

*Proof.* We have to show that a rational series $S \in K[[X]]$ satisfies

$$\langle S, X^{(k+\kappa)c+j}\rangle = uq^k \quad , \quad \forall k \geq 0,\ 0 \leq j \leq c-1 \ , \quad (11)$$

for some $u, q \in K$, and for some integers $\kappa \geq 0, c \geq 1$. But $S$ has a representation of the form (8), i.e. $\langle S, X^{(k+\kappa_1)c+j}\rangle = \bigoplus_{i\in I_j} u_i q_i^k$, where $u_i, q_i \in K$, $I_j$ is a finite set, and $\kappa_1 \geq 0, c \geq 1$ are integers. Let $q = \bigoplus_{i\in I_j} q_i$. Since $K$ is linearly ordered and $\oplus$ coincides with the least upper bound, we can find an index $\ell$ such that $q_\ell = q$, and $u_\ell \geq u_m$ for all $m$ such that $q_m = q$. Then, $\langle S, X^{(k+\kappa_1)c+j}\rangle = \bigoplus_{i\in I_j,\ q_i<q} u_i q_i^k \oplus u_\ell q_\ell^k$. Using the archimedian property, we get $\langle S, X^{(k+\kappa_1)c+j}\rangle = u_\ell q_\ell^k$, for $k$ large enough, say for $k \geq k_2$. Setting $\kappa = \kappa_1 + k_2$, $q = q_\ell$, and $u = u_\ell q_\ell^{k_2 c}$, we get (11). $\square$

Equivalently, $S$ can be written as

$$S = P \oplus X^{\kappa c}\Big(\bigoplus_{0\le i\le c-1} u_i X^i (q_i X^c)^*\Big) \ , \tag{12}$$

where $P \in K[X]$ has degree less than $\kappa c$, and $u_i, q_i \in K$.

Finally, we shall need the inverse operation of merging, that we call undersampling. For each integer $0 \le j \le c-1$ and for all series $T \in K[[X]]$, we define the undersampled series:

$$T^{(j,c)} = \bigoplus_{k\in\mathbb{N}} \langle T, X^{kc+j}\rangle X^k \ .$$

For instance, when $T$ is as in (10), $T^{(0,2)}$ and $T^{(1,2)}$ respectively coincide with the series $S^{(0)}$ and $S^{(1)}$ of (9). Trivially, testing the equality of two series amounts to testing the equality of undersampled series:

**Lemma 2.** *Let $c \ge 1$. Two series $\mathsf{T}, \mathsf{T}' \in \mathsf{K}[[X]]$ coincide if, and only if, $\mathsf{T}^{(j,c)} = \mathsf{T}'^{(j,c)}$ for all $0 \le j \le c-1$.* □

A last, trivial, remark will allow us to split the test that $\mathsf{S} = S$ into transient and ultimate parts. Recall that $X^{-m}\mathsf{S}$ denotes the series $\mathsf{T}$ such that $\langle \mathsf{T}, X^k\rangle = \langle \mathsf{S}, X^{m+k}\rangle$.

**Lemma 3.** *Let $m \ge 0$. Two series $\mathsf{T}, \mathsf{T}' \in \mathsf{K}[[X]]$ coincide if, and only if, $\langle \mathsf{T}, X^k\rangle = \langle \mathsf{T}', X^k\rangle$ for $k \le m-1$, and $X^{-m}\mathsf{T} = X^{-m}\mathsf{T}'$.* □

## 3. Proof of Theorem 2

In the sequel, $K$ denotes a semiring that satisfies the assumptions of Theorem 2 and $\Sigma = \{\mathsf{d}_1, \dots, \mathsf{d}_n\}$ is a finite set of commuting indeterminates. We first prove a simple lemma.

**Lemma 4.** *For all $\mathsf{p} \in K[\Sigma]$ and $p \in K$, the sets $\{\mathsf{p} \le p\}$, $\{\mathsf{p} \ge p\}$, and $\{\mathsf{p} = p\}$, are semi-polyhedral.*

*Proof.* Since in an idempotent semiring $u \le v \iff u \oplus v = v$, it suffices to prove more generally that when $\mathsf{p}, \mathsf{q} \in K[\Sigma]$, $\{\mathsf{p} = \mathsf{q}\}$ is semi-polyhedral. When $\mathsf{p}$ or $\mathsf{q} = \mathbb{0}$, $\{\mathsf{p} = \mathsf{q}\}$ is trivially semi-polyhedral. Otherwise, we can write $\mathsf{p}$ and $\mathsf{q}$ as finite sums of monomials, $\mathsf{p} = \bigoplus_{i\in I} \mathsf{p}_i$, and $\mathsf{q} = \bigoplus_{j\in J} \mathsf{q}_j$, with $I, J \ne \emptyset$. For all $(i,j) \in I \times J$, consider the polyhedron $U_{ij} = \big(\cap_{k\in I}\{\mathsf{p}_i \ge \mathsf{p}_k\}\big) \cap \big(\cap_{l\in J}\{\mathsf{q}_j \ge \mathsf{q}_l\}\big) \cap \{\mathsf{p}_i = \mathsf{q}_j\}$. Since $K$ is linearly ordered, and since the sum $\oplus$ is the least upper bound for $\le$, $\{\mathsf{p} = \mathsf{q}\} = \cup_{i\in I, j\in J} U_{ij}$ is a semi-polyhedral set. □

The fact that $\{\mathsf{p} = p\}$ is semi-polyhedral was already noticed by De Schutter [DS96] (who derived this result by modelling $\mathsf{p} = p$ as an extended linear complementarity problem).

We now prove Theorem 2 (the proof will be illustrated by the examples in §4). The discussion following the proof of Lemma 1 shows that the rational series $\mathsf{S} \in K[\Sigma][[X]]$ can be represented as (7). Let $F(\mathsf{S})$ denote the set of couples of integers $(\kappa, c)$ for which $\mathsf{S}$ has such a representation. The rational identities (5),(6) imply that $(\kappa, c) \in F(\mathsf{S}) \implies (\kappa, ck) \in F(\mathsf{S})$ for all $k \geq 1$. Similarly, the rational identity (6) shows that $(\kappa, c) \in F(\mathsf{S}) \implies (k, c) \in F(\mathsf{S})$, for all $k \geq \kappa$. The same argument can be applied to the set $F'(S)$ of couples of integers $(\kappa, c)$ for which the rational series $S \in K[[X]]$ has a representation of the form (12). Hence, $F(\mathsf{S}) \cap F'(S) \neq \emptyset$, which allows us to assume that $\mathsf{S}$ and $S$ are given by (7) and (12), where $\kappa$ and $c$ are the same in both formulæ.

By Lemma 2, $\{\mathsf{S} = S\} = \cap_{0 \leq j \leq c-1} \{\mathsf{S}^{(j,c)} = S^{(j,c)}\}$. Since the intersection of semi-polyhedral sets is semi-polyhedral, and since the series $\mathsf{S}^{(j,c)}$ and $S^{(j,c)}$ have expressions of the form (7) and (12), respectively, but with $c = 1$, it suffices to show Theorem 2 when $c = 1$. Moreover, thanks to Lemma 3, $\{\mathsf{S} = S\} = \cap_{0 \leq k \leq \kappa-1} \{\langle \mathsf{S}, X^k \rangle = \langle S, X^k \rangle\} \cap \{X^{-\kappa}\mathsf{S} = X^{-\kappa}S\}$. By Lemma 4, the sets $\{\langle \mathsf{S}, X^k \rangle = \langle S, X^k \rangle\}$ are semi-polyhedral, hence, using again the closure of semi-polyhedral sets by intersection, it suffices to show that $\{X^{-\kappa}\mathsf{S} = X^{-\kappa}S\}$ is semi-polyhedral. The series $X^{-\kappa}\mathsf{S}$ and $X^{-\kappa}S$ again have expressions of the form (7) and (12), respectively, but with $\kappa = 0$, i.e. with $\mathsf{p} = \mathbb{0}$ and $p = \mathbb{0}$. Summarizing, it remains to prove Theorem 2 when

$$\mathsf{S} = \bigoplus_{1 \leq i \leq r} \mathsf{u}_i (\mathsf{q}_i X)^* \quad \text{and} \tag{13}$$

$$S = u(qX)^* \ . \tag{14}$$

It is easy to eliminate the case where $u = \mathbb{0}$. Then, by Lemma 4, $\{\mathsf{S} = S\} = \{\bigoplus_{1 \leq i \leq r} \mathsf{u}_i = \mathbb{0}\}$ is semi-polyhedral. When $q = \mathbb{0}$, $\{\mathsf{S} = S\} = \{\mathsf{S} = uX^0\} = \{\langle \mathsf{S}, X^0 \rangle = u\} \cap \{X^{-1}\mathsf{S} = \mathbb{0}\}$ is semi-polyhedral. Thus, in the sequel, we shall assume that $u, q \neq \mathbb{0}$.

The reduction to (13) leads us to studying special series of this form. We call *line* a series of the form $T = u(qX)^*$, where $u, q \in K \setminus \{0\}$, and we say that a series $T \in K[[X]]$ is *convex* if it is a finite sum of lines. When $K = \mathbb{Q}_{\max}$, $T$ is a line if, and only if, $k \mapsto \langle T, X^k \rangle$ is an ordinary (discrete, half-)line, and $T$ is convex if, and only if, $k \mapsto \langle T, X^k \rangle$ is a finite supremum of lines. Convex series already arose in [GBCG98] (where it was shown that the minimal dimension of realization of a convex series can be computed in polynomial time, but here, we must find *all* convex realizations of (14)). Lines can be easily compared:

**Lemma 5.** *Let $u, q, v, w \in K$. Then, $v(wX)^* \leq u(qX)^* \implies v = \mathbb{0}$ or ($v \leq u$ and $w \leq q$).*

*Proof.* The inequality $v(wX)^* \leq u(qX)^*$ means that $vw^k \leq uq^k$, for all $k \geq 0$. If $v \neq \mathbb{0}$, the archimedian property implies that $w \leq q$. Moreover, taking $k = 0$, we get $v \leq u$. $\square$

We shall need the following refinement of the archimedian condition.

**Lemma 6.** *For all $\alpha, \beta, \gamma, \delta \in K$,*

$$(\alpha < \beta \text{ and } \delta \neq \mathbb{0}) \implies \gamma\alpha^k < \delta\beta^k \text{ for } k \text{ large enough.} \tag{15}$$

*Proof.* Since $K$ is linearly ordered, the archimedian condition means precisely that

$$(\alpha < \beta \text{ and } \delta \neq \mathbb{0}) \implies \gamma\alpha^k < \delta\beta^k \text{ for some } k. \tag{16}$$

Multiplying the inequality $\gamma\alpha^k \leq \delta\beta^k$ by $\beta$, we get $\gamma\alpha^{k+1} \leq \gamma\alpha^k\beta \leq \delta\beta^{k+1}$. Moreover, since $K$ is cancellative and $\beta \neq \mathbb{0}$ (because $\beta > \alpha \geq \mathbb{0}$), $\gamma\alpha^k\beta = \delta\beta^{k+1}$ would imply $\gamma\alpha^k = \delta\beta^k$, which contradicts (16). Hence, $\gamma\alpha^{k+1} < \delta\beta^{k+1}$, and after an immediate induction on $k$, we get (15). $\square$

The final, critical, step of the proof of Theorem 2 is an observation, which, when specialized to $K = \mathbb{Q}_{\max}$, is a geometrically obvious fact about ordinary piecewise affine convex maps.

**Lemma 7.** *Let $u, q \in K$, $S = u(qX)^*$, $u_1, \ldots, u_r, q_1, \ldots, q_r \in K$, $T_i = u_i(q_iX)^*$, and $T = \bigoplus_{1 \leq i \leq r} T_i$. Then, $T = S$ if, and only if, $T_i \leq S$ for all $1 \leq i \leq r$, and $T_j = S$ for some $1 \leq j \leq r$.*

*Proof.* Since $\oplus$ is the least upper bound in $K[[X]]$, if $T = S$, we have for all $1 \leq i \leq r$, $T_i \leq S$, which, by Lemma 5, means either $u_i = \mathbb{0}$ or $((u_i \leq u)$ and $(q_i \leq q))$. Let $I = \{1 \leq i \leq r \mid u_i \neq \mathbb{0}\}$. We shall assume that $S \neq \mathbb{0}$, i.e, that $u = \mathbb{0}$ (otherwise the result is obvious). Since $T = S$ and $S \neq \mathbb{0}$, $I \neq \emptyset$. Now, let $\overline{q} = \bigoplus_{i \in I} q_i \leq q$, $J = \{i \in I \mid q_i = \overline{q}\}$, and $\overline{u} = \bigoplus_{j \in J} u_j$. Using (15), we get $\langle T, X^k \rangle = \overline{u}\,\overline{q}^k$, for $k$ large enough. Identifying this expression with $\langle S, X^k \rangle = uq^k$, and using the archimedian condition, we get $\overline{q} = q$. Cancelling $q^k$ in $\overline{u}q^k = uq^k$, we get $\overline{u} = u$, and since $K$ is linearly ordered, $\overline{u} = \bigoplus_{i \in J} u_i = u_j$ for some $j \in J$. Thus, $S = T_j$, which shows that the condition of the lemma is necessary. The condition is trivially sufficient. $\square$

We now complete the proof of Theorem 2. Let $\mathsf{S}_i = \mathsf{u}_i(\mathsf{q}_iX)^*$. By Lemma 5, both $\{\mathsf{S}_i \leq S\} = \{\mathsf{u}_i = \mathbb{0}\} \cup (\{\mathsf{u}_i \leq u\} \cap \{\mathsf{q}_i \leq q\})$ and $\{\mathsf{S}_i = S\} = \{\mathsf{u}_i = u\} \cap \{\mathsf{q}_i = q\}$ are semi-polyhedral. Hence, by Lemma 7, $\{\mathsf{S} = S\} = \cup_{1 \leq j \leq r}\big(\{\mathsf{S}_j = S\} \cap (\cap_{i \in I, i \neq j}\{\mathsf{S}_i \leq S\})\big)$ is semi-polyhedral, which concludes the proof of Theorem 2.

## 4. Examples

*4.1. First example*

Let us illustrate the algorithm of the proof of Theorem 2 by computing $\{\mathsf{S}=S\}$ when $K=\mathbb{Q}_{\max}$, $\mathsf{S}=\mathsf{u}_1(\mathsf{v}_1X)^*\oplus\mathsf{u}_2(\mathsf{v}_2X^2)^*$, $\Sigma=\{\mathsf{u}_1,\mathsf{u}_2,\mathsf{v}_1,\mathsf{v}_2\}$ and $S=0\oplus X(1X)^*=0\oplus 0X\oplus 1X^2\oplus 2X^3\oplus\cdots$ The first step of the proof consists in putting $\mathsf{S}$ and $S$ in the forms (7) and (12), respectively. Here,

$$\begin{aligned}\mathsf{S} &= \mathsf{u}_1(\mathbb{1}\oplus\mathsf{v}_1X)(\mathsf{v}_1^2X^2)^*\oplus\mathsf{u}_2(\mathsf{v}_2X^2)^*\\ S &= 0\oplus(X\oplus 1X^2)(2X^2)^* \ .\end{aligned}$$

Then, $\{\mathsf{S}=S\}=\{\mathsf{S}^{(0,2)}=S^{(0,2)}\}\cap\{\mathsf{S}^{(1,2)}=S^{(1,2)}\}$, where the undersampled series are given by

$$\begin{aligned}\mathsf{S}^{(0,2)} &= \mathsf{u}_1(\mathsf{v}_1^2X)^*\oplus\mathsf{u}_2(\mathsf{v}_2X)^*\\ \mathsf{S}^{(1,2)} &= \mathsf{u}_1\mathsf{v}_1(\mathsf{v}_1^2X)^*\\ S^{(0,2)} &= 0\oplus 1X\oplus 3X^2=0\oplus 1X(2X)^*\\ S^{(1,2)} &= (2X)^* \ .\end{aligned}$$

By Lemma 5, $\{\mathsf{S}^{(1,2)}=S^{(1,2)}\}=\{\mathsf{u}_1\mathsf{v}_1=0\}\cap\{\mathsf{v}_1^2=2\}$. In $\mathbb{Q}_{\max}$, the unique solution of the equation $v_1^2=2$ is $v_1=1$, and the unique solution of $u_1\otimes 1=0$ is $u_1=-1$. Hence,

$$\{\mathsf{S}^{(1,2)}=S^{(1,2)}\}=\{\mathsf{u}_1=-1\}\cap\{\mathsf{v}_1=1\} \ . \tag{17}$$

The series $S^{(0,2)}$ has an expression of the form (12) with $\kappa=1$. Let us give an expression (7) with the same $\kappa$ for $S^{(0,2)}$:

$$\mathsf{S}^{(0,2)} = \mathsf{u}_1\oplus\mathsf{u}_1\mathsf{v}_1^2X(\mathsf{v}_1^2X)^*\oplus\mathsf{u}_2\oplus\mathsf{u}_2\mathsf{v}_2X(\mathsf{v}_2X)^*$$

Thus, $\{\mathsf{S}^{(0,2)}=S^{(0,2)}\}=\{\langle\mathsf{S}^{(0,2)},X^0\rangle=\langle S^{(0,2)},X^0\rangle\}\cap\{X^{-1}\mathsf{S}^{(0,2)}=X^{-1}S^{(0,2)}\}=\{\mathsf{u}_1\oplus\mathsf{u}_2=0\}\cap\{\mathsf{u}_1\mathsf{v}_1^2(\mathsf{v}_1^2X)^*\oplus\mathsf{u}_2\mathsf{v}_2(\mathsf{v}_2X)^*=1(2X)^*\}$. Combining this with (17) and using Lemma 5, we see that $\{\mathsf{S}=S\}$ is the polyhedron defined by

$$\mathsf{u}_1=-1,\mathsf{v}_1=1 \quad \mathsf{u}_2=0,\ \mathsf{v}_2\leq 1 \ .$$

*4.2. Second example*

Let $\alpha,\beta\in\mathbb{Q}$, and let us look for the realizations of dimension 2 of the series

$$S=X^0\oplus\alpha X^2(\beta X)^* \ . \tag{18}$$

The proof of Theorem 1 requires to find a star height one representation for the universal rational series $\mathsf{S}_2=\mathsf{c}(\mathsf{A}X)^*\mathsf{b}$. Such a representation can be obtained for instance by using the McNaughton-Yamada algorithm (see [HU79], Proof of

Th. 2.4), together with the rational identities (5), or directly from the classical graph interpretation of $\mathsf{c}(\mathsf{A}X)^*\mathsf{b}$. Setting $\alpha_{ij} = \mathsf{A}_{ij}X$, we easily get:

$$\begin{aligned} \mathsf{S}_2 \quad = \quad & (\mathsf{c}_2\alpha_{21}\mathsf{b}_1 \oplus \mathsf{c}_1\alpha_{12}\mathsf{b}_2)(\alpha_{11} \oplus \alpha_{22})^* \oplus \mathsf{c}_2\alpha_{22}^*\mathsf{b}_2 \oplus \mathsf{c}_1\alpha_{11}^*\mathsf{b}_1 \oplus \\ & \alpha_{21}\alpha_{12}(\alpha_{11} \oplus \alpha_{22} \oplus \alpha_{12}\alpha_{21})^*(\mathsf{c}_2\alpha_{21}\mathsf{b}_1 \oplus \mathsf{c}_2\mathsf{b}_2 \oplus \mathsf{c}_1\alpha_{12}\mathsf{b}_2 \oplus \mathsf{c}_1\mathsf{b}_1) \end{aligned} \tag{19}$$

After applying the algorithm of the proof of Theorem 2 to (19) (we do not reproduce the computations, which are a bit lengthy, but straightforward), we get that if $\alpha \leq \beta^2$, all the realizations of $S$ are similar[4] to:

$$c = \begin{pmatrix} \mathbb{1} & \mathbb{0} \end{pmatrix} \quad A = \begin{pmatrix} \mathbb{0} & \alpha \\ \alpha & \beta \end{pmatrix} \quad b = \begin{pmatrix} \mathbb{1} \\ \mathbb{0} \end{pmatrix} \ .$$

If $\alpha > \beta$, then $S$ has no two dimensional realization. Surprisingly enough, realizing even a simple series like (18) is not immediate: we do not know a simpler way to compute the set of dimension 2 realizations of $S$.

*4.3. Counter Example*

Let $\Sigma = \{\mathsf{u}_1, \mathsf{u}_2, \mathsf{v}_1, \mathsf{v}_2\}$. We prove that the subset of $\mathbb{Q}^4_{\max}$

$$\begin{aligned} \mathscr{S} \quad &= \quad \{\mathsf{u}_1(\mathsf{v}_1X)^* \oplus \mathsf{u}_2(\mathsf{v}_2X)^* \geq (0X)^*\} \\ &= \quad \{(u_1, v_1, u_2, v_2) \mid \forall k \in \mathbb{N},\ \max(u_1 + kv_1, u_2 + kv_2) \geq 0\} \end{aligned} \tag{20}$$

is not semi-polyhedral. It suffices to show that the projection $A$ of $\mathscr{S} \cap \{\mathsf{v}_1 = \mathsf{u}_2 = -1\} \cap \{\mathsf{u}_1, \mathsf{v}_2 \geq 0\}$ on the coordinates $\mathsf{u}_1, \mathsf{v}_2$ is not semi-polyhedral. Let $f(k) = \max(u_1 - k, -1 + v_2k)$. Specializing (20), we see that $(u_1, v_2) \in A$ if, and only if, $u_1, v_2 \geq 0$ and $\min_{k\in\mathbb{N}} f(k) \geq 0$. The map $f$ is decreasing from 0 to $x = (u_1+1)/(v_2+1)$, and increasing from $x$ to $+\infty$, therefore, $\min_{k\in\mathbb{N}} f(k) \geq 0 \iff f(\lfloor x \rfloor) \geq 0$ and $f(\lceil x \rceil) \geq 0$, which gives[5]

$$A = \{(u_1, u_2) \mid\ u_1, v_2 \geq 0,\ \ u_1 - \left\lfloor \frac{u_1+1}{v_2+1} \right\rfloor \geq 0, -1 + v_2 \left\lceil \frac{u_1+1}{v_2+1} \right\rceil \geq 0\} \ .$$

The set $A$ is depicted by the grey region on Figure 1. Note that the border of this region contains an infinite number of vertices lying on the hyperbola $u_1v_2 = 1$. It is geometrically obvious that $A$ is not semi-polyhedral, but we can

[4]We say, as usual, that two representations $(c, A, b)$ and $(c', A', b')$ are *similar* if $c' = cP, A' = P^{-1}AP, b' = P^{-1}b$, for some invertible matrix $P$. In the max-plus semiring, an invertible matrix is the product of a diagonal matrix by a permutation matrix (see e.g. [BCOQ92] for this standard result). Unlike in conventional algebra, max-plus minimal realizations are in general not similar.

[5]We recall that $\lfloor x \rfloor$ stands for the integer part of $x$ and $\lceil x \rceil$ is equal to $-\lfloor -x \rfloor$ and is the rounding to the smallest bigger than $x$ integer

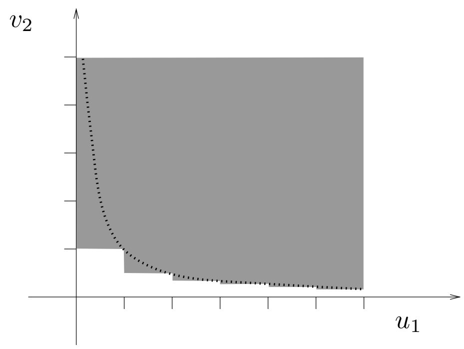



Figure 1: The set of realizations $(c, A, b)$ of dimension 2 such that $cA^k b \geq 0$ for all $k$ is not semi-polyhedral. A two dimensional section of this set is represented.

check it without appealing to the figure, as follows. For any integer $n$ the point $(n, 1/n)$ belongs to the set $A$. If $A$ was a finite union of polyhedra, then there would be a polyhedron $P \subseteq A$ with an infinite number of points of $(n, 1/n)$ in $P$, and the low borderline of $P$ would be the line $\{v_2 = 0\}$. This is not possible, because for $v_2 > 0$, the point $(u_1, v_2)$ is not in $A$, as soon as $v_2 < 1/(u_1 + 2)$.

## 5. Universal Commutative Rational Expressions and Complexity Analysis

In this section, we bound the complexity of the algorithm of the proofs of Theorem 1 and 2. Suppose we are looking for a realization of size $N$ of the series $S$ given as in (12):

$$S = P \oplus X^{\kappa_0 c_0} \Big( \bigoplus_{0 \leq i \leq c_0 - 1} u_i X^i (q_i X^c)^* \Big) \ , \tag{21}$$

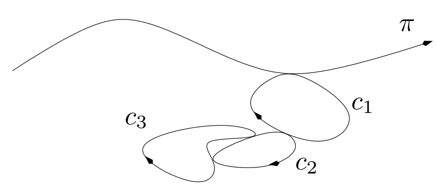
where $c_0 \geq 1$, $\kappa_0 \geq 0$, $P \in K[X]$ has degree less than $\kappa_0 c_0$, and $u_i, q_i \in K$. A critical step of our algorithm is to build, like we did in (19), a star height one representation of the form (7) for the universal series $\mathsf{S} = \mathsf{S}_N$:

$$\mathsf{S} = \mathsf{P} \oplus X^{\kappa_1 c_1} \Big( \bigoplus_{1 \leq i \leq \rho} \mathsf{u}_i X^{\mu_i} (\mathsf{q}_i X^{c_1})^* \Big) \tag{22}$$

where $0 \leq \mu_i \leq c_1 - 1$, $\mathsf{u}_i \in \mathsf{K}$, and $\mathsf{P} \in \mathsf{K}[X]$ has degree less than $\kappa_1 c_1$. In section §5.1, we shall give an explicit star height one representation for $\mathsf{S}_N$ which is of independent interest. This expression has a double exponential size. In §5.2, we shall bound the size of an expression of $\{\mathsf{S} = S\}$ as a union of intersection of half-spaces, when $\mathsf{S}$ and $S$ are given by (21) and (22), and show that the subproblem of computing $\{\mathsf{S} = S\}$ has a simply exponential complexity. Finally, in section §5.3, we shall combine the results of §5.1 and §5.2 to show that the method of Theorem 1 yields a triply exponential algorithm to compute the set of realizations of a max-plus rational series. This triply exponential bound is a coarse one: trying examples by hand suggests that the naive version of the algorithm that we analyse here could be made much more practicable by using linear programming and constraint programming techniques.

*5.1. Universal Commutative Rational Expressions*

Let us associate to the triple $\mathsf{c} \in \mathsf{K}^{1\times N}, \mathsf{A} \in \mathsf{K}^{N\times N}, \mathsf{b} \in \mathsf{K}^{N\times 1}$ a digraph $G_N$ composed of the nodes $1, \ldots, N$, together with an *input* node *in* and an *output* node *out*, arcs $j \to i$ with weights $\mathsf{A}_{ij}X$, for $1 \leq i, j \leq N$, input arcs $in \to i$ with weights $\mathsf{b}_i$, and output arcs $i \to out$ with weights $\mathsf{c}_i$. The *weight* of a path $\pi$, denoted by $w(\pi)$, is defined as the product of the weight of its arcs. We say that two circuits $\gamma$ and $\gamma'$ are *cyclic conjugates* if one is obtained from the other by a circular permutation. When $\mathsf{K}$ is commutative, $w(\gamma) = w(\gamma')$. We denote by $\mathscr{C}_N$ the set of conjugacy classes of elementary circuits of $G_N$. Let $C \subset \mathscr{C}_N$, and let $\pi$ denote a path of $G_N$. We say that $C$ is *accessible* from $\pi$ if the union of the circuits of $C$ and of the path $\pi$ is a connected subgraph (we use here the *undirected* notion of connectedness, not to be confused with strong connectedness). An accessible set $C$ for a path $\pi$ looks typically as follows:




We denote by $\mathscr{A}(\pi)$ the set of $C \subset \mathscr{C}_N$ accessible from $\pi$. We set $S^+ = SS^*$, for all series $s$ such that $S^*$ is well defined. We denote by $\mathscr{P}_N$ the set of elementary paths from *in* to *out*. The following result is Lemma 6.2.3 from [Gau92, Chap. VII].

**Proposition 1.** *Let* $\mathsf{K}$ *denote a commutative idempotent semiring,* $\mathsf{A} \in \mathsf{K}^{N\times N}$, $\mathsf{b} \in \mathsf{K}^{N\times 1}$, $\mathsf{c} \in \mathsf{K}^{1\times N}$, *and* $\mathsf{S}_N = \mathsf{c}(\mathsf{A}X)^*\mathsf{b}$. *We have*

$$\mathsf{S}_N = \bigoplus_{\pi \in \mathscr{P}_N} w(\pi) \left( \bigoplus_{C \in \mathscr{A}(\pi)} \bigotimes_{\gamma \in C} w(\gamma)^+ \right) . \tag{23}$$

(By convention, $\emptyset \in \mathscr{A}(\pi)$ for all paths $\pi$, and the products in (23) corresponding to $C = \emptyset$ are equal to $\mathbb{1}$.)

Before proving Proposition 1, it is instructive to consider the case when $N = 2$. Then, there are four paths in the sum (23), $\pi_1 = in \to 1 \to out$, $\pi_2 = in \to 2 \to out$, $\pi_3 = in \to 1 \to 2 \to out$, $\pi_4 = in \to 2 \to 1 \to out$, with respective weights $\mathsf{c}_1\mathsf{b}_1$, $\mathsf{c}_2\mathsf{b}_2$, $\mathsf{c}_2\alpha_{21}\mathsf{b}_1$, and $\mathsf{c}_1\alpha_{12}\mathsf{b}_2$. We have for instance $\mathscr{A}(\pi_1) = \{\{1 \to 1\}, \{1 \to 2 \to 1\}, \{1 \to 2 \to 1, 1 \to 1\}, \{1 \to 2 \to 1, 2 \to 2\}, \{1 \to 2 \to 1, 1 \to 1, 2 \to 2\}\}$. Thus, the contribution of $\pi_1$ in (23) is

$$\mathsf{c}_1\mathsf{b}_1(\mathbb{1} \oplus \alpha_{11}^+ \oplus (\alpha_{12}\alpha_{21})^+ \oplus \alpha_{11}^+(\alpha_{12}\alpha_{21})^+ \oplus (\alpha_{12}\alpha_{21})^+\alpha_{22}^+ \oplus \alpha_{11}^+(\alpha_{12}\alpha_{21})^+\alpha_{22}^+)$$

and, considering the similar contributions of $\pi_2, \pi_3, \pi_4$, it is easy to see that (23) coincides with (19).

*Proof of Proposition 1.* Let $B$ denote the right hand side of (23). We shall prove by induction on $k$ the following property: $(H_k)$ for all (possibly non elementary) paths $\pi$ from *in* to *out*, for all sets of $k$ circuits $C = \{\gamma_1, \dots, \gamma_k\} \in \mathscr{A}(\pi)$, and for all $n_1, \dots, n_k \geq 1$, $w(\pi)w(\gamma_1)^{n_1} \dots w(\gamma_k)^{n_k}$ is the weight of a path $\pi'$ from *in* to *out*. If $k = 1$, since the graph induced by $\pi \cup \gamma_1$ is connected, $\gamma_1$ must have a common node with $\pi$, say node $r$. Possibly after replacing $\gamma_1$ by a cyclic conjugate, we may assume that $r$ is the initial (and final) node of $\gamma$. We can write $\pi = \pi_{out,r}\pi_{r,in}$ (here, and in the sequel, composition of paths is denoted by concatenation), where $\pi_{r,in}$ is a path from *in* to $r$, and $\pi_{out,r}$ is a path from $r$ to *out*. Thus $w(\pi)w(\gamma_1)^{n_1} = w(\pi_{out,r}\gamma_1^{n_1}\pi_{r,in})$ is the weight of the path $\pi' = \pi_{out,r}\gamma_1^{n_1}\pi_{r,in}$ from *in* to *out*, which proves $(H_1)$. We now assume that $k \geq 2$. By definition of $\mathscr{A}(\pi)$, at least one of the circuits $\gamma_1, \dots, \gamma_k$, say $\gamma_1$, has a node in common with $\pi$. Arguing as in the proof of $(H_1)$, we see that $w(\pi)w(\gamma)^{n_1}$ is the weight of a path $\pi'$ from *in* to *out*, which is such that $\{\gamma_2, \dots, \gamma_k\} \in \mathscr{A}(\pi')$. Applying $(H_{k-1})$ to $\pi'$, we get $(H_k)$.

Since $(H_k)$ holds for all $k$, all the terms of the sum $B$ can be interpreted as weights of paths from *in* to *out*, but we know that $\mathsf{S}_N$ is the sum of the weights of all paths from *in* to *out*. Hence, $B \preceq \mathsf{S}_N$. Conversely, if $\pi$ is a path from *in* to *out*, we can write $\pi = \pi_1\gamma_1^{n_1}\pi_2 \dots \gamma_k^{n_k}\pi_{k+1}$, where $\pi_1\pi_2 \dots \pi_{k+1}$ is an elementary path from *in* to *out*, and $\gamma_1, \dots, \gamma_k$ are elementary circuits which form an accessible set for $\pi$. This implies that $w(\pi) \preceq B$, and since this holds for all $\pi$, $\mathsf{S}_N \preceq B$. $\square$

We tabulate the size of the sets determining the size of the expression (23), for further use. We denote by $\#X$ the cardinality of a set $X$. It is easy to check that

$$\#\mathscr{P}_N = \sum_{i=0}^{N} \frac{N!}{i!} \leq eN! = \mathcal{O}(N!) \ , \tag{24}$$

and that

$$\#\mathscr{C}_N = \sum_{i=1}^{N} \frac{N!}{(N-i)!i!} \ .$$

The $\mathscr{C}_N$ are called *logarithmic numbers*, their exponential generating function, $\sum_{N\geq 1}(N!)^{-1}\mathscr{C}_N z^N$, is equal to $-\log(1-z)\exp(z)$. Using for instance a singularity analysis [FO90, Th. 2], we get

$$\#\mathscr{C}_N = \mathcal{O}((N-1)!\log N) \ . \tag{25}$$

We have of course $\#C \leq \#\mathscr{C}_N$, and $\#\mathscr{A}(\pi) \leq 2^{\#\mathscr{C}_N}$, for all $C \in \mathscr{C}_N$ and $\pi \in \mathscr{P}_N$.

*5.2. Computing* $\{\mathsf{S} = S\}$

We now embark in the complexity analysis of the algorithm contained in the proof of Theorem 1. This analysis involves a tedious but conceptually simple bookeeping: we bound the number of polyhedral sets appearing when expressing that the star-height one rational expression (23) evaluates to a given rational series.

If $\mathsf{p} \in K[\Sigma]$, we denote by $|\mathsf{p}|$ the number of monomials which appear in $\mathsf{p}$ (for instance, if $K = \mathbb{Q}_{\max}$, $\Sigma = \{a, b\}$, $\mathsf{p} = \mathbb{1} \oplus 2a^2 \oplus ab \oplus 7b$, $|\mathsf{p}| = 4$). We consider the series $\mathsf{S}$ and $S$ given by (7) and (12), respectively, with $\mathsf{K} = K[\Sigma]$, and we set

$$m = \max(\max_{0\leq i<\kappa c} |\langle \mathsf{p}, X^i\rangle|, \max_{1\leq i\leq\rho} \max(|\mathsf{u}_i|, |\mathsf{q}_i|))$$

and $\rho_i = \#\{1 \leq j \leq \rho \mid \mu_j = i\}$.

**Proposition 2.** *Let* $\mathsf{S}$ *and* $S$ *be given by* (7) *and* (12)*, respectively. The set* $\{\mathsf{S} = S\}$ *can be expressed as the union of at most* $m^{\kappa c+2c}(\prod_{0\leq i<c}\rho_i)2^{\rho-c}$ *intersections of at most* $(m+1)\kappa c + 2c + 2\rho m$ *half-spaces.*

To show Proposition 2, we need to introduce some adapted notation. We say that a couple of positive integers $[n, k]$ is a *symbol* of a subset $\mathscr{S}$ of $K^\Sigma$, and we write $\mathscr{S} \in [n, k]$, if $\mathscr{S}$ can be written as the union of $n$ sets, $\mathscr{S} = \cup_{1\leq i\leq n}\mathscr{S}_i$, where each $\mathscr{S}_i$ is the intersection of at most $k$ half-spaces. Of course, $\mathscr{S} \in [n, k] \implies \mathscr{S} \in [n', k']$, for all $n' \geq n, k' \geq k$. For instance, taking $\mathsf{p} = \bigoplus_{i\in I}\mathsf{p}_i \in K[\Sigma]$ and $p \in K$ as in Lemma 4, and specializing the proof of Lemma 4, we have

$$\{\mathsf{p} = p\} = \bigcup_{i\in I}(\{\mathsf{p}_i \leq p\} \cap \{\mathsf{p}_i \geq p\} \cap \bigcap_{\substack{j\in I\\ j\neq i}}\{\mathsf{p}_j \leq p\}) \ . \tag{26}$$

Since, by definition, $|\mathsf{p}| = \#I$ we get from (26):

$$\{\mathsf{p} = p\} \in [|\mathsf{p}|, |\mathsf{p}| + 1] \ . \tag{27}$$

Similarly, since $\{\mathsf{p} \leq p\} = \cap_{i\in I}\{\mathsf{p}_i \leq p\}$,

$$\{\mathsf{p} \leq p\} \in [1, |\mathsf{p}|] \ . \tag{28}$$

It will be convenient to equip symbols with the binary laws $\sqcup$ and $\sqcap$, defined by:

$$[n,k] \sqcup [n',k'] = [n+n', \max(k,k')], \qquad [n,k] \sqcap [n',k'] = [nn', k+k'] \ .$$

This notation is motivated by the following rule, which holds for all subsets $\mathscr{S}, \mathscr{S}' \subset K^{\Sigma}$:

$$(\mathscr{S} \in [n,k] \text{ and } \mathscr{S}' \in [n',k']) \implies (\mathscr{S} \cup \mathscr{S}' \in [n,k] \sqcup [n',k'] \text{ and } \mathscr{S} \cap \mathscr{S}' \in [n,k] \sqcap [n',k']) \ . \tag{29}$$

*Proof of Proposition 2.* As a preliminary step, we compute symbols for the more elementary sets involved in the proof of Theorem 2.

First, if $\mathsf{u}, \mathsf{q} \in K[\Sigma]$ and $u, q \in K$, we get from Lemma 5, $\{\mathsf{u}(\mathsf{q}X)^* \le u(qX)^*\} = \{\mathsf{u} = \mathbb{0}\} \cup (\{\mathsf{u} \le u\} \cap \{\mathsf{q} \le q\})$, hence

$$\{\mathsf{u}(\mathsf{q}X)^* \le u(qX)^*\} \in [1, |\mathsf{u}|] \sqcup ([1,|\mathsf{u}|] \sqcap [1,|\mathsf{q}|]) = [2, |\mathsf{u}| + |\mathsf{q}|] \ . \tag{30}$$

Moreover, Lemma 5 shows that $\{\mathsf{u}(\mathsf{q}X)^* = u(qX)^*\} = \{\mathsf{u} = u\} \cap \{\mathsf{q} = q\}$, if $u \neq \mathbb{0}$. When $u = \mathbb{0}$, $\{\mathsf{u}(\mathsf{q}X)^* = u(qX)^*\} = \{\mathsf{u} = \mathbb{0}\} = \{\mathsf{u} \le \mathbb{0}\}$. Hence, using (27), (28), we get

$$\{\mathsf{u}(\mathsf{q}X)^* = u(qX)^*\} \in \begin{cases} [1, |\mathsf{u}|] & \text{if } u = \mathbb{0} \\ [|\mathsf{u}|, |\mathsf{u}|+1] \sqcap [|\mathsf{q}|, |\mathsf{q}|+1] = [|\mathsf{u}||\mathsf{q}|, |\mathsf{u}| + |\mathsf{q}| + 2] & \text{else} \end{cases} \tag{31}$$

Let us now take $\mathsf{u}_1, \dots, \mathsf{u}_r, \mathsf{q}_1, \dots, \mathsf{q}_r \in K[\Sigma]$, $u, q \in K$, $\mathsf{T}_i = \mathsf{u}_i(\mathsf{q}_i X)^*$, $S = u(qX)^*$. Lemma 7 shows that

$$\Big\{ \bigoplus_{1 \le i \le r} \mathsf{T}_i = S \Big\} = \bigcup_{1 \le i \le r} \big( \{\mathsf{T}_i = S\} \cap \bigcap_{\substack{1 \le j \le r \\ j \neq i}} \{\mathsf{T}_j \le S\} \big) \ , \tag{32}$$

hence, using (30) and (31)

$$\begin{aligned} \Big\{ \bigoplus_{1 \le i \le r} \mathsf{T}_i = S \Big\} &\in \bigsqcup_{1 \le i \le r} \big( [|\mathsf{u}_i||\mathsf{q}_i|, |\mathsf{u}_i| + |\mathsf{q}_i| + 2] \sqcap \bigsqcap_{\substack{1 \le j \le r \\ j \neq i}} [2, |\mathsf{u}_j| + |\mathsf{q}_j|] \big) \\ &= \big( \sum_{1 \le i \le r} |\mathsf{u}_i||\mathsf{q}_i| 2^{r-1}, 2 + \sum_{1 \le i \le r} (|\mathsf{u}_i| + |\mathsf{q}_i|) \big) \end{aligned} \tag{33}$$

The proof of Theorem 2, together with (33), (26), shows that

$$\begin{aligned} \{\mathsf{S} = S\} &= \bigcap_{0 \le i < \kappa c} \{\mathsf{p}_i = p_i\} \cap \bigcap_{0 \le i < c} \Big\{ \bigoplus_{\substack{1 \le j \le \rho \\ \mu_j = i}} \mathsf{u}_j(\mathsf{q}_j X)^* = u_i(q_i X)^* \Big\} \\ &\in \bigsqcap_{0 \le i < \kappa c} [m, m+1] \sqcap \bigsqcap_{0 \le i < c} [\rho_i m^2 2^{\rho_i - 1}, 2 + 2\rho_i m] \\ &= [m^{\kappa c + 2c} (\prod_{0 \le i < c} \rho_i) 2^{\rho - c}, (m+1)\kappa c + 2c + 2\rho m] \ , \end{aligned} \tag{34}$$

which concludes the proof. $\square$

*5.3. Final Complexity Analysis*

Let $\mathsf{K} = K[\Sigma]$ and $E$ be a formal expression of a polynomial $\mathsf{P} \in \mathsf{K}[X]$. We denote by $m(E)$ the maximum number of monomials of $K[\Sigma]$ arising as a coefficient of a power of $X$ in some polynomial expression of $E$. By abuse of notation we will write $m(\mathsf{P})$ instead of $m(E)$. For instance, with $\Sigma = \{a, b\}$ and $K = \mathbb{Q}_{\max}$, if $\mathsf{S} = 7aX \oplus 3abX \oplus X^2(\mathbb{1} \oplus 8a^2bX)(3X^2)^*$, $m(\mathsf{S}) = 2(= |\langle \mathsf{P}, X\rangle| = |7a \oplus 3ab|)$. If the expression is (7):

$$m(\mathsf{S}) = \max(\max_{0 \leq i < \kappa c} |\langle \mathsf{P}, X^i\rangle|, \max_{1 \leq i \leq \rho} \max(|\mathsf{u}_i|, |\mathsf{q}_i|)) \tag{35}$$

**Corollary 2.** *Let* $\mathsf{K} = K[\Sigma]$, $\mathsf{A} \in \mathsf{K}^{N \times N}$, $\mathsf{b} \in \mathsf{K}^{N \times 1}$, $\mathsf{c} \in \mathsf{K}^{1 \times N}$, *and* $\mathsf{S}_N = \mathsf{c}(\mathsf{A}X)^*\mathsf{b}$. *Then* $\mathsf{S}_N$ *can be written as in 7:*

$$\mathsf{S}_N = \mathsf{P} \oplus X^{\kappa_1 c_1}\Big(\bigoplus_{1 \leq i \leq \rho} \mathsf{u}_i X^{\mu_i}(\mathsf{q}_i X^{c_1})^*\Big) \tag{36}$$

*where* $c_1 = N!$, $\kappa_1 = \mathcal{O}(N!)$, $\rho = 2^{\mathcal{O}(N!)}$, $0 \leq \mu_i \leq c_1 - 1$, $\mathsf{u}_i \in \mathsf{K}$, $m(\mathsf{S}_N) = 2^{\mathcal{O}(N!)}$ *and* $\mathsf{P} \in \mathsf{K}[X]$ *has degree smaller than* $\kappa_1 c_1$.

*Proof.* We have:

$$\mathsf{S}_N = \bigoplus_{\pi \in \mathscr{P}_N, C \in \mathscr{A}(\pi)} w(\pi) \bigotimes_{\gamma \in C} w(\gamma) w(\gamma)^*$$

For every $\gamma$ (and $\pi$ also) the monomial $\mathsf{P} = w(\gamma)$ has degree at most $N$ and is equal to $\mathsf{q}X^\alpha$ where $\mathsf{q} \in \mathsf{K}$ is a monomial (i.e. $m(\mathsf{q} = 1)$ and $\alpha \leq N$. Let $\alpha'$ be the integer such that $\alpha\alpha' = N!$. Using the identity (5) we get:

$$\begin{aligned} \mathsf{P}(\mathsf{q}X^\alpha)^* &= \mathsf{P}(\mathbb{1} \oplus \mathsf{q}X^\alpha \oplus \ldots \oplus \mathsf{q}^{\alpha'-1}X^{\alpha(\alpha'-1)})(\mathsf{q}^{\alpha'}X^{N!}) \\ &= \mathsf{P}'(\mathsf{q}^{\alpha'}X^{N!})^* \end{aligned}$$

where the polynomial $\mathsf{P}'$ has degree $N + N! - \alpha = \mathcal{O}(N!)$ and $m(\mathsf{P}') = m(\mathsf{q}^{\alpha'}) = 1$. Using the identity (3) we have immediately:

$$\mathsf{S}_N = \bigoplus_{1 \leq j \leq r} \mathsf{P}_j(\mathsf{q}_j X^{N!})^*$$

where $\mathsf{P}_1, \ldots, \mathsf{P}_r \in \mathsf{K}[X]$, $\mathsf{q}_1, \ldots, \mathsf{q}_r \in \mathsf{K}$, $m(\mathsf{q}_j) \leq \#\mathscr{C}_N$ and $m(\mathsf{P}_j) = (N + N! - \alpha)^{\#\mathscr{C}_N - 1}$. To evaluate $r$ and the degrees of these polynomials, results from §5.1 are useful: the degree of each $\mathsf{P}_j$ is $\mathcal{O}((N!)^2)$, $r = 2^{\mathcal{O}(N!)}$, $m(\mathsf{q}_j) = \mathcal{O}(N!)$ and $m(\mathsf{P}_j) = 2^{\mathcal{O}(N!)}$.

Next step is to obtain an expression like (7) using the identity (6). Here is an example:

$$\begin{aligned}
s &= (0 \oplus X^3)(2X^2)^* \oplus (2 \oplus X^4)(3X^2)^* \\
&= (2X^2)^* \oplus XX^2(2X^2)^* \oplus 2(3X^2)^* \oplus X^2X^2(3X^2)^* \\
&= (0 \oplus 2X^2 \oplus 4X^4(2X^2)^*) \oplus XX^2(0 \oplus 2X^2(2X^2)^*) \\
&\quad \oplus 2(0 \oplus 3X^2 \oplus 6X^4(3X^2)^*) \oplus X^2X^2(0 \oplus 3X^2(3X^2)^*) \\
&= (2 \oplus 5X^2 \oplus X^3 \oplus X^4) \ \oplus \ X^4(4(2X^2)^* \oplus 2X(2X^2)^* \oplus 8(3X^2)^* \\
&\quad \oplus 3X^2(3X^2)^*)
\end{aligned}$$

Let $c_1 = N!$ and $\kappa_1$ be the smaller integer such that $c_1\kappa_1 + c_1 - 1$ is larger than the degrees of $P_j$. Then $\kappa_1 = \mathcal{O}(N!)$ and there are some polynomials $P_{j,0}, \ldots, P_{j,\kappa_1}$ of degrees at most $c_1 - 1$ such that:

$$P_j = P_{j,0} + X^{c_1}P_{j,1} + X^{2c_1}P_{j,2} + \ldots + X^{\kappa_1 c_1}P_{j,\kappa_1}$$

Using (6) we have:

$$\begin{aligned}
&P_j(Q_jX^{c_1})^* = \\
&P_{j,0}(\mathbb{1} \oplus Q_jX^{c_1} \oplus \ldots \oplus Q_j^{\kappa_1-1}X^{(\kappa_1-1)c_1} \oplus (Q_j^k)_1X^{\kappa_1c_1}(Q_jX^{c_1})^*) \\
&\oplus P_{j,1}X^{c_1}(\mathbb{1} \oplus Q_jX^{c_1} \oplus \ldots \oplus Q_j^{\kappa_1-2}X^{(\kappa_1-2)c_1} \oplus Q_j^{\kappa_1-1}X^{(\kappa_1-1)c_1}(Q_jX^{c_1})^*) \\
&\oplus \ldots \\
&\oplus P_{j,\kappa_1}X^{\kappa_1c_1}(Q_jX^{c_1})^*
\end{aligned}$$

and thus

$$P_j(Q_jX^{c_1})^* = R_j \oplus X^{\kappa_1c_1}\Big(\bigoplus_{0\le i\le c_1(\kappa_1+1)} \mathsf{u}_{i,j}X^{\mu_{i,j}}(Q_jX^{c_1})^*\Big)$$

where the degree of the polynomial $R_j$ is at most $\kappa_1c_1$, the $\mathsf{u}_{i,j}$'s are elements of $\mathsf{K}$, $\mu_{i,j} < c_1$, $m(R_j)$ and $m(\mathsf{u}_{i,j})$ are $2^{\mathcal{O}(N!)}$. At the end we have the equation (36) where $0 \le \mu_i \le c_1 - 1$, $\mathsf{u}_i \in \mathsf{K}$, $\mathsf{p} \in \mathsf{K}[X]$ has degree less than $\kappa_1c_1$ and $\rho = c_1(\kappa_1+1)r = 2^{\mathcal{O}(N!)}$.

□

**Corollary 3.** *Let $S$ be given by* (12), $\mathsf{A} \in \mathsf{K}^{N\times N}$, $\mathsf{b} \in \mathsf{K}^{N\times 1}$, $\mathsf{c} \in \mathsf{K}^{1\times N}$, *and* $\mathsf{S}_N = \mathsf{c}(\mathsf{A}X)^*\mathsf{b}$. *Then we have*

$$\begin{aligned}
\mathsf{S}_N &= \mathsf{P}_2 \oplus X^{k_2c_2}\Big(\bigoplus_{1\le i\le \rho_2} \mathsf{v}_iX^{\mu_i}(\mathsf{r}_iX^{c_2})^*\Big) \\
S &= P_2 \oplus X^{k_2c_2}\Big(\bigoplus_{0\le i\le c_2-1} v_iX^{i}(r_iX^{c_2})^*\Big)
\end{aligned}$$

*where* $c_2 = \mathrm{lcm}(N!, c)$, $k_2 = \mathcal{O}(N!)$ *and* $k_2 \le \kappa_0$, $\rho_2 = c_0 2^{\mathcal{O}(N!)}$, $0 \le \mu_i \le c_2 - 1$, $v_i \in K$, $\mathsf{v}_i \in \mathsf{K}$, $\mathsf{P}_2 \in \mathsf{K}[X]$ *and* $P_2 \in K[X]$ *have degree smaller than* $k_2 c_2$, $m(\mathsf{S}_N) = 2^{\mathcal{O}(N!)}$.

*Proof.* Considering the equation (36) of Corollary 2 let $c_2 = \mathrm{lcm}(c, c_1) = \alpha_0 c = \alpha_1 c_1$ and $k_2 = \max\{\lceil \kappa_0/\alpha_0 \rceil, \lceil \kappa_1/\alpha_1 \rceil\}$. Then we have two integers $h_0$ and $h_1$ such that $k_2 c_2 = \kappa_0 c + h_0 c = \kappa_1 c_1 + h_1 c_1$. Using the following equation

$$u_i X^i (q_i X^c)^* = u_i X^i (\mathbb{1} \oplus q_i X^c \oplus \ldots \oplus q_i^{h_0 - 1} X^{c(h_0-1)} \oplus q_i^{h_0} X^{c h_0} (q_i X^c)^*)$$

we have

$$S = P_2 \oplus X^{k_2 c_2} \Big( \bigoplus_{0 \le i \le c-1} u'_i X^i (q_i X^c)^* \Big)$$

with $u'_i = u_i q_i^{h_0}$ and $P_2 = P \oplus X^{\kappa_0 c} (\bigoplus_{0 \le i \le c-1} u_i X^i)(\mathbb{1} \oplus q_i X^c \oplus \ldots \oplus q_i^{h_0-1} X^{c(h_0-1)})$ is a polynomial of degree at most $k_2 c_2$. Similarly we have:

$$\mathsf{S}_N = \mathsf{P}_2 \oplus X^{k_2 c_2} \Big( \bigoplus_{0 \le \mu_i \le \rho} \mathsf{u}'_i X^{\mu_i} (\mathsf{q}_i X^{c_1})^* \Big)$$

where $\mathsf{P}_2$ is a polynomial of degree at most $k_2 c_2$ and $\mathsf{u}'_i$ are elements of $\mathsf{K}$. The last step is to use the equation (5)

$$(q_i X^c)^* = (\mathbb{1} \oplus q_i X^c \oplus \ldots \oplus q_i^{\alpha_0 - 1} X^{c(\alpha_0 - 1)})(q_i^{\alpha_0} X^{c_2})^*$$

which gives us

$$S \quad = \quad P_2 \oplus X^{k_2 c_2} \Big( \bigoplus_{0 \le i \le c_2 - 1} v_i X^i (r_i X^{c_2})^* \Big)$$

and similarly

$$\mathsf{S}_N \quad = \quad \mathsf{P}_2 \oplus X^{k_2 c_2} \Big( \bigoplus_{1 \le i \le \rho_2} \mathsf{v}_i X^{\mu_i} (\mathsf{r}_i X^{c_2})^* \Big)$$

where $r_i = q_i^{\alpha_0}$, $\mathsf{r}_i = \mathsf{q}_i^{\alpha_1}$, $v_i = \sum_{j=0}^{\inf\{i, c-1\}} u'_j q_i^{i-j} \in K$, $\mathsf{v}_i \in \mathsf{K}$ and $\rho_2 = \rho(1 + c_1(\alpha_1 - 1)) = \rho c_0 \mathcal{O}(N!)$. The bound for $m(\mathsf{S}_N)$ follows easily from Corollary 2. □

We say that a quantity $Q$ depending of parameters is *simply* (resp. *doubly*, *triply*) *exponential* if $Q$ can be bounded from above by a term of the form $2^P$ (resp. $2^{2^P}, 2^{2^{2^P}}$), where $P$ is a polynomial function of the parameters.

**Corollary 4.** *Let $S$ be given by* (12). *The set of realizations of dimension $N$ of $S$ can be written as the union of $n$ intersections of at most $k$ half-spaces, where $n$ is triply exponential in $N$ and simply exponential in $\kappa, c$, and $k$ is doubly exponential in $N$ and linear in $\kappa, c$. In particular, when $K = \mathbb{Q}_{\max}$, the existence of a realization of dimension $N$ of $S$ can be decided in triply exponential time in $N$ and simply exponential time in $\kappa, c$.*

*Proof.* The first statement of the corollary follows by applying Corollary 3, Proposition 2 and using Stirling's formula. The second statement follows from the first one together with the fact that linear programming has a polynomial time complexity (see e.g. [Sch86, Ch. 14 and 15]). □

## 6. Conclusion

We showed that the existence of a realization of a given dimension of a max-plus linear sequence is decidable, answering to a question which was raised from the beginning of the development of the max-plus modelling of discrete event systems, see [CMQV85, Ols86, BCOQ92, ODS99, BSVW99]. This decidability result is obtained as a byproduct of a general structural property: the set of realizations can be effectively written as a finite union of polyhedra, or as the max-plus analogue of a semi-algebraic set. The complexity analysis leads to a coarse triple exponential bound, but it also shows that some special structured instances of the problem can be solved in a more reasonable simply exponential time. A possible source of suboptimality of the present bound is that the underlying max-plus semi-algebraic structure is not exploited: this raises issues of an independent interest which we will examine further elsewhere.

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