On monochromatic arm exponents for 2D critical percolation

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Abstract

We investigate the so-called *monochromatic arm exponents* for critical percolation in two dimensions. These exponents, describing the probability of observing j disjoint macroscopic paths, are shown to exist and to form a different family from the (now well-understood) polychromatic exponents.

1 Introduction

Percolation is one of the most-studied discrete models in statistical physics. The usual setup is that of *bond percolation* on the square lattice $\mathbb{Z}^2$, where each bond is open (resp. closed) with probability $p \in (0,1)$ (resp. $1-p$), independently of the others. This model exhibits a phase transition at a critical point $p_c \in (0,1)$ (in this particular case, $p_c = 1/2$): For $p < p_c$, almost surely all connected components are finite, while for $p > p_c$ there exists a unique infinite component with density $\theta(p) > 0$. Site percolation is defined in a similar fashion, the difference being that the vertices are open or closed, instead of the edges; one can then see it as a random coloring of the lattice, and use the terms *black* and *white* in place of *open* and *closed*.

The behavior of percolation away from the critical point is well understood, however it is only recently that precise results have been obtained at and near criticality. For critical site percolation on the regular triangular lattice, the proof of conformal invariance in the scaling limit was obtained by Smirnov [15], and SLE processes, as introduced by Schramm [13] and further studied by Lawler, Schramm and Werner [8, 9], provide an explicit description of the interfaces (in the scaling limit) in terms of SLE(6) (see *e.g.* [17]).

This description allows for the derivation of the so-called *polychromatic arm exponents* [10, 16], describing the probability of observing connections across large modulus annuli by disjoint connected paths of different colors (with at least one arm of each color), and also the derivation of the one-arm exponent. Combined with Kesten's scaling relations [7], these exponents then provide the existence and the values of most of the other critical exponents, like *e.g.* the exponent $\beta = 5/36$ associated with the density of the infinite cluster: As $p \downarrow p_c$,

$$\theta(p) = (p - p_c)^{5/36 + o(1)}.$$

On the other hand, very little is known concerning the *monochromatic* arm exponents (*i.e.*, with all the connections of the same color — see below for a formal definition) with more than one arm. Here, the SLE approach does not seem to work and, correspondingly, there is no conjecture for the values of those exponents. One notable exception however is the 2-arm monochromatic exponent, for which an interpretation in terms of SLE(6) is proposed at the end of [10] — but again no explicit value has been computed. That particular exponent is actually of physical interest: Known as the *backbone exponent*, it describes the "skeleton" of a percolation cluster. Even the existence of these exponents is not clear, as there does not seem to be any direct sub-additivity argument.

In this paper, we prove that the monochromatic exponents do exist, and investigate how they are related to the polychromatic exponents. We show that they have different values than their polychromatic counterparts. As an illustration, our result implies that the backbone of a typical large percolation cluster at criticality is much "thinner" than its boundary.

Acknowledgments

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2 Background

2.1 The setting

We restrict ourselves here to site percolation on the triangular lattice, at criticality ($p = p_c = 1/2$). Recall that it can be obtained by coloring randomly the faces of the honeycomb lattice, each cell being black or white with probability 1/2, independently of the others. In the following, we denote by $\mathbb{P} = \mathbb{P}_{1/2}$ the corresponding probability measure on the set of configurations. Let us mention however that all the results of combinatorial nature based on Russo-Seymour-Welsh type estimates should also hold for bond percolation on $\mathbb{Z}^2$, due to the self-duality property of this lattice.

Let S_n denote the ball of radius n in the triangular lattice (*i.e.* the intersection of the triangular lattice with the Euclidean disc of radius n, though the specifics of the definition are of little relevance), seen as a set of vertices. We will denote by ∂S_n its outer boundary, *i.e.* the set of vertices in S_n that have at least one neighbor outside of S_n, and, for $n < N$, by

$$S_{n,N} := S_N \setminus S_n$$

the annulus of radii n and N. To describe critical and near-critical percolation, certain exceptional events play a central role: the arm events, referring to the existence of a number of crossings ("arms") of $S_{n,N}$, the color of each crossing (black or white) being prescribed.

Definition 1. *Let $j \geq 1$ be an integer and $\sigma = (\sigma_1, \ldots, \sigma_j)$ a sequence of colors (black or white). For any two positive integers $n < N$, a (j, σ)-*arm configuration *in the annulus $S_{n,N}$ is the data of j disjoint monochromatic paths $(r_i)_{1 \leqslant i \leqslant j}$ — the* arms *— connecting the inner boundary ∂S_n and the outer boundary ∂S_N, where the color of the arm r_i is given by σ_i. We denote by*

$$A_{j,\sigma}(n, N) := \left\{ \partial S_n \underset{j,\sigma}{\rightsquigarrow} \partial S_N \right\} \tag{2.1}$$

the corresponding event.

We will write down color sequences by abbreviating colors, using B and W for black and white respectively. To avoid the obvious combinatorial obstructions, we will also use the notation $n_0 = n_0(j)$ for the smallest integer such that j arms can possibly arrive on ∂S_{n_0} ($n_0(j)$ is of the order of j) and only consider annuli of inner radius larger than n_0. This restriction will be done implicitly in what follows.

The so-called *color exchange trick* (noticed in [1, 16]) shows that once fixed the number j of arms, prescribing the color sequence σ changes the probability only by a constant factor, *as long as both colors are present in σ* (an interface is needed to proceed). Their asymptotic behavior can be described precisely using SLE(6): It is possible to prove the existence of the (polychromatic) arm exponents, and to derive their values ([16]) which had been predicted in the physics literature (see e.g. [1]):

Theorem 2. *Fix $j \geqslant 2$. Then for any color sequence σ containing both colors,*

$$\mathbb{P}\left(A_{j,\sigma}(n_0(j), N)\right) = N^{-\alpha_j + o(1)} \tag{2.2}$$

as $N \to \infty$, with $\alpha_j = (j^2 - 1)/12$.

The value of the exponent for $j = 1$ (corresponding to the probability of observing one arm crossing the annulus) has also been established [10] and it is known to be equal to $5/48$ (oddly enough formally corresponding to $j = 3/2$ in the above formula ...).

For future reference, let us mention the following facts about critical percolation that we will use.

1. *A-priori bound for one arm:* There exist constants $C, \varepsilon > 0$ such that for all $n < N$,

$$\mathbb{P}\left(A_{1,B}(n, N)\right) = \mathbb{P}\left(A_{1,W}(n, N)\right) \leqslant C \left(\frac{n}{N}\right)^{\varepsilon}. \tag{2.3}$$

2. *Quasi-multiplicativity property:* For any $j \geqslant 1$ and any sequence σ, there exist constants $C_1, C_2 > 0$ such that for all $n_1 < n_2 < n_3$,

$$\begin{aligned} C_1 \, \mathbb{P}\left(A_{j,\sigma}(n_1, n_2)\right) \mathbb{P}\left(A_{j,\sigma}(n_2, n_3)\right) &\leqslant \mathbb{P}\left(A_{j,\sigma}(n_1, n_3)\right) \\ &\leqslant C_2 \, \mathbb{P}\left(A_{j,\sigma}(n_1, n_2)\right) \mathbb{P}\left(A_{j,\sigma}(n_2, n_3)\right). \end{aligned}$$

These two properties actually rely on the so-called Russo-Seymour-Welsh (RSW) lower bounds, that we will use extensively in various situations: Roughly speaking, these bounds state that the probability of crossing a given shape of fixed aspect ratio is bounded below independently of the scale. For instance, the probability of crossing a $3n \times n$ rectangle in its longer direction is bounded below, uniformly as $n \to \infty$. We refer the reader to [6] for more details.

2.2 A correlation inequality

A key ingredient in our proof will be a not-that-classic correlation inequality which is an intermediate step in the proof of the van-den-Berg-Kesten-Reimer (BKR) inequality, conjectured in [2] and proved in [12].

Let us first fix some notation. We follow here the lines of the review paper [3]. Consider an integer n, and $\Omega = \{0,1\}^n$. For any configuration $\omega \in \Omega$ and any set of indices $S \subseteq \{1,\dots,n\}$, we introduce the cylinder

$$[\omega]_S := \{\tilde{\omega} : \forall i \in S,\ \tilde{\omega}_i = \omega_i\},$$

and more generally for any $X \subseteq \Omega$, any $S : X \to \mathcal{P}(\{1,\dots,n\})$,

$$[X]_S := \bigcup_{\omega \in X} [\omega]_{S(\omega)}.$$

For any two $A, B \subseteq \Omega$, we denote as usual by $A \circ B$ the disjoint occurrence of A and B:

$$A \circ B := \{\omega \text{ s.t. for some } S(\omega) \subseteq \{1,\dots,n\},\ [\omega]_S \subseteq A \text{ and } [\omega]_{S^c} \subseteq B\}.$$

Recall that the BKR inequality states that

$$\mathbb{P}(A \circ B) \leqslant \mathbb{P}(A)\,\mathbb{P}(B). \tag{2.4}$$

We also denote by $\bar{\omega} = 1 - \omega$ the configuration obtained by "flipping" every bit of the configuration $\omega \in \Omega$, so that if $X \subseteq \Omega$, $\bar{X} := \{\bar{\omega}, \omega \in X\}$. We are now in a position to state the correlation inequality that will be a key ingredient in the following, referred to as *Reimer's main lemma* in [3]:

Theorem 3. *For any $A, B \subseteq \Omega$, we have*

$$|A \circ B| \leqslant |A \cap \bar{B}| = |\bar{A} \cap B|. \tag{2.5}$$

For the sake of completeness, let us just mention that this inequality is not stated explicitly in [3]. It can be deduced from Lemma 4.1 by taking $X = A \circ B$, and $S : X \to \mathcal{P}(\{1,\dots,n\})$ associated with A and B by the definition of disjoint occurrence, *i.e.* so as to satisfy $[\omega]_{S(\omega)} \subseteq A$ and $[\omega]_{S(\omega)^c} \subseteq B$ for all $\omega \in A \circ B$ (so that $[X]_S \subseteq A$ and $[X]_{S^c} \subseteq B$).

2.3 Statement of the results

In this paper, we will be interested in the asymptotic behavior of the probability $\mathbb{P}(A_{j,\sigma}(n_0(j), N))$ as $N \to \infty$ for a constant σ, say $\sigma = B \dots B$, so that $A_{j,\sigma}$ simply refers to the existence of j disjoint black arms. Our first result shows that this probability follows a power law, as in the case of a non-constant σ:

Theorem 4. *For any $j \geqslant 2$, there exists an exponent $\alpha'_j > 0$ such that*

$$\mathbb{P}\left(A_{j,B\ldots B}(n_0(j),N)\right) = N^{-\alpha'_j+o(1)} \tag{2.6}$$

as $N \to \infty$.

These exponents α'_j are known as the *monochromatic arm exponents*, and it is natural to try to relate them to the previously mentioned polychromatic exponents α_j.

Consider any $j \geqslant 2$; we start with a few easy remarks. On the one hand, the FKG inequality implies that

$$\begin{aligned}\mathbb{P}\left(A_{j+1,B\ldots BW}(n_0,N)\right) &= \mathbb{P}\left(A_{j,B\ldots B}(n_0,N) \cap A_{1,W}(n_0,N)\right)\\ &\leqslant \mathbb{P}\left(A_{j,B\ldots B}(n_0,N)\right)\,\mathbb{P}\left(A_{1,W}(n_0,N)\right),\end{aligned}$$

and by using item 1. above, we get that, for some constant C,

$$\mathbb{P}\left(A_{j+1,B\ldots BW}(n_0,N)\right) \leqslant CN^{-\varepsilon}\,\mathbb{P}\left(A_{j,B\ldots B}(n_0,N)\right), \tag{2.7}$$

or in other words that $\alpha'_j < \alpha_{j+1}$. On the other hand, inequality (2.5) directly implies that

$$\begin{aligned}\mathbb{P}\left(A_{j,B\ldots BB}(n_0,N)\right) &= \mathbb{P}\left(A_{j-1,B\ldots B}(n_0,N) \circ A_{1,B}(n_0,N)\right)\\ &\leqslant \mathbb{P}\left(A_{j-1,B\ldots B}(n_0,N) \cap A_{1,W}(n_0,N)\right)\\ &= \mathbb{P}\left(A_{j,B\ldots BW}(n_0,N)\right),\end{aligned}$$

hence $\alpha'_j \geqslant \alpha_j$. We will actually prove the following, stronger result:

Theorem 5. *For any $j \geqslant 2$, we have*

$$\alpha_j < \alpha'_j < \alpha_{j+1}. \tag{2.8}$$

The monochromatic exponents α'_j thus form a family of exponents different from the polychromatic exponents.

We would like to stress the fact that the case of half-plane exponents (or more generally, boundary exponents in any planar domain) is considerably different: Indeed, whenever a boundary is present, the color-exchange trick implies that the probability of observing j arms of prescribed colors is *exactly* the same for all color prescriptions, whether mono- or poly-chromatic. In particular there is no difference between the monochromatic and polychromatic boundary exponents. (For the reader's peace of mind, they can notice that the presence of the boundary provides for a canonical choice of a leftmost arm, the lack of which is precisely the core idea of the proof of our main result in the whole plane.)

We will first prove the inequality $\alpha_j < \alpha'_j$ (which is the main statement in the above theorem, the other strict inequality being the simple consequence of the FKG inequality we mentioned earlier), since its proof only requires combinatorial arguments, and postpone the proof of the existence of the exponents to the end of the paper.

In order not to refer to the α'_j's, we adopt the following equivalent formulation of the inequality: What we formally prove is that, for any $j \geqslant 2$, there exists $\varepsilon > 0$ such that for any N large enough,

$$\mathbb{P}\left(A_{j,B\ldots BB}(n_0, N)\right) \leqslant N^{-\varepsilon}\, \mathbb{P}\left(A_{j,B\ldots BW}(n_0, N)\right).$$

The proof of that inequality only relies on self-duality and RSW-type estimates, and hence it can be easily adapted to the case of bond percolation on $\mathbb{Z}^2$ — where the existence of the exponents, which strongly relies on the knowledge of the scaling limit, is still unproved.

3 The set of winding angles

3.1 Strict inequalities between the exponents

Our proof is based on an energy vs. entropy consideration. The difference between the monochromatic and the polychromatic j-arm exponents can be written in terms of the expected number of "really different" choices of j arms out of a percolation configuration with j arms: For a polychromatic configuration, this number is equal to 1, whereas for a monochromatic configuration, it grows at least like a positive power of the modulus, and the ratio between these two numbers behaves exactly like $(N/n)^{\alpha_j - \alpha'_j}$ because, for fixed disjoint arms $(r_1, \ldots, r_j)$ with respective lengths $(\ell_1, \ldots, \ell_j)$, the probability that they are present in the configuration with a prescribed coloring does not depend on that coloring (it is equal to $2^{-(\ell_1 + \cdots + \ell_j)}$).

More precisely, but still roughly speaking, the proof relies on the following observation: Given a configuration where j black arms are present, there are many ways to choose them, since by RSW there is a positive density of circuits around the origin (allowing "surgery" on the arms — see Figure 1), while if we consider a configuration with arms of both colors, then there is essentially only one way to select them. Of course the geometry of an arm is quite intricate and many local modifications — on every scale — are always possible: What we mean here is that this choice is unique from a macroscopic point of view. To formalize this intuition, we thus have to find a way of distinguishing two macroscopic choices of arms, and for this we will use the *set of winding angles* associated with a configuration.

Definition 6. *For any configuration of arms, one can choose a continuous determination of the argument along one of the arms; we call* winding angle of the arm *(or simply* angle *for short) the total variation of the argument along that arm.*

Clearly, the winding angles of the arms corresponding to a given (j, σ)-arm configuration differ by at most 2π. However, for the same percolation configuration, there might exist many different choices of a (j, σ)-arm configuration, corresponding to different angles: We denote by $I_{j,\sigma}(n, N)$ the set of all the angles which can be obtained from such a configuration; we omit the subscript from the notation whenever j and σ are clear from the context. For the sake of completeness, we also declare $I_{j,\sigma}(n, N)$ to be empty if the configuration does not contain j arms of the prescribed colors.

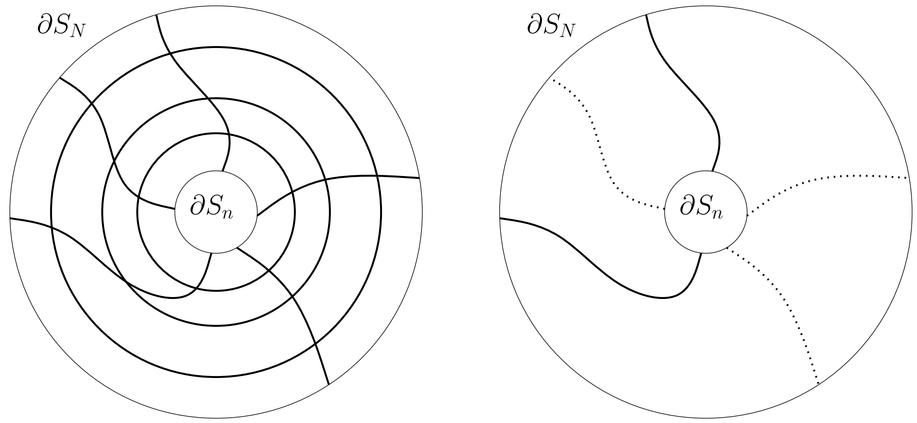

Figure 1: To a given monochromatic configuration correspond many different "macroscopic" ways to choose the arms, contrary to the polychromatic case.

We will actually rather use $\bar{I}_{j,\sigma}(n,N)$, the set of angles obtained by "completing" $I_{j,\sigma}(n,N)$:

$$\bar{I}_{j,\sigma}(n,N) := \bigcup_{\alpha \in I_{j,\sigma}(n,N)} (\alpha - \pi, \alpha + \pi].$$

It is an easy remark that in the polychromatic case (σ non-constant), we have for any $\alpha \in I_{j,\sigma}(n,N)$

$$I_{j,\sigma}(n,N) \subseteq (\alpha - 2\pi, \alpha + 2\pi),$$

so that $\bar{I}_{j,\sigma}(n,N)$ is an interval of length at most 6π. In the monochromatic case (σ constant), no such bound applies (and actually it is not obvious that $\bar{I}_{j,\sigma}(n,N)$ is an interval — this is proved as Proposition 7 below).

In the case of a polychromatic arm configuration, considering successive annuli of a given modulus as independent, one would expect a central limit theorem to hold on the angles, or at least fluctuations of order $\sqrt{\log N}$. On the other hand, for a monochromatic configuration, performing surgery using circuits in successive annuli should imply that every time one multiplies the outer radius by a constant, the expected largest available angle would increase by a constant, so that one would guess that, by a careful choice of arms, the total angle can be made of order $\pm \log N$.

Fix $\varepsilon > 0$, and let $A^{\varepsilon}_{1,B}$ (resp. $A^{\varepsilon}_{1,W}$, resp. $A^{\varepsilon}_{j,\sigma}$) be the event that there exists a black arm (resp. a white arm, resp. j arms with colors given by σ) with angle larger than $\varepsilon \log N$ between radii n_0 and N. Applying inequality (2.5) with $A = A_{j-1,B\dots B}$ and $B = A^{\varepsilon}_{1,B}$, this would imply:

$$\begin{aligned}\mathbb{P}(A_{j,B\dots BB}) &\asymp \mathbb{P}(A \circ B) \\ &\leqslant \mathbb{P}(A_{j-1,B\dots B} \cap A^{\varepsilon}_{1,W}) \\ &= \mathbb{P}(A^{\varepsilon}_{j,B\dots BW}),\end{aligned}$$

and we could expect

$$\mathbb{P}\left(A^{\varepsilon}_{j,B\ldots BW}\right) \leqslant N^{-\varepsilon'}\,\mathbb{P}\left(A_{j,B\ldots BW}\right)$$

by a large-deviation principle. However, proving this LDP seems to be difficult, and we propose here an alternative proof that relies on the same ideas, but bypasses some of the difficulties.

Proof of Theorem 5. **Step 1.** First, note that it suffices to prove that the ratio

$$\frac{\mathbb{P}\left(A_{j,B\ldots BB}(n,N)\right)}{\mathbb{P}\left(A_{j,B\ldots BW}(n,N)\right)}$$

can be made arbitrarily small as $n/N \to 0$, uniformly in n: Indeed, assuming that this is the case, then for any $\delta > 0$, there exists $\eta > 0$ such that this ratio is less than δ as soon as $n/N \leqslant \eta$. Then, as a direct consequence of the quasi-multiplicativity property (item 2. above), we have

$$\begin{aligned}
&\mathbb{P}\left(A_{j,B\ldots BB}(n,\eta^{-k}n)\right)\\
&\qquad\leqslant C_2^{k-1}\,\mathbb{P}\left(A_{j,B\ldots BB}(n,\eta^{-1}n)\right)\ldots\,\mathbb{P}\left(A_{j,B\ldots BB}(\eta^{-(k-1)}n,\eta^{-k}n)\right)\\
&\qquad\leqslant C_2^{k-1}\delta^k\,\mathbb{P}\left(A_{j,B\ldots BW}(n,\eta^{-1}n)\right)\ldots\,\mathbb{P}\left(A_{j,B\ldots BW}(\eta^{-(k-1)}n,\eta^{-k}n)\right)\\
&\qquad\leqslant C_2^{k-1}\delta^k(C_1^{-1})^{k-1}\,\mathbb{P}\left(A_{j,B\ldots BW}(n,\eta^{-k}n)\right),
\end{aligned}$$

and for $\delta = 1/(2C_2C_1^{-1})$ this gives

$$\mathbb{P}\left(A_{j,B\ldots BB}(n,\eta^{-k}n)\right) \leqslant 2^{-k}\,\mathbb{P}\left(A_{j,B\ldots BW}(n,\eta^{-k}n)\right), \tag{3.1}$$

which immediately implies that for some $\varepsilon > 0$,

$$\mathbb{P}\left(A_{j,B\ldots BB}(n,N)\right) \leqslant \left(\frac{N}{n}\right)^{-\varepsilon}\mathbb{P}\left(A_{j,B\ldots BW}(n,N)\right).$$

In particular, applying this for $n = n_0$ (and N large enough) leads to the inequality that we need.

Step 2. The key step of the proof is as follows. Given a configuration with j arms in an annulus of large modulus, we use RSW-type estimates to prove the existence of a large number of disjoint sub-annuli of it, in each of which one can find black paths topologically equivalent to those in Figure 2 (in the case $j = 2$) or its reflection. Every time this configuration appears, one has the possibility to replace the original arms (in plain lines on the figure) with modified — and still disjoint — arms, obtained by using one of the dashed spirals in each of them. The new arms then land at the same points on the outer circle, but with a winding angle differing by 2π. This allows us to show that, with high probability, the set of angles $\bar{I}(n,N)$ contains an interval of length at least $\varepsilon\log(N/n)$, for some $\varepsilon > 0$ (which can be written in terms of the RSW estimates). We now proceed to make the construction in detail.

Let $j \geqslant 2$, and let m be a positive integer. Define a *j-spiral* between radii m and $4m$ as the configuration pictured in Figure 3. More precisely, a j-spiral is the union of 4 families of j black paths in a percolation configuration, namely:

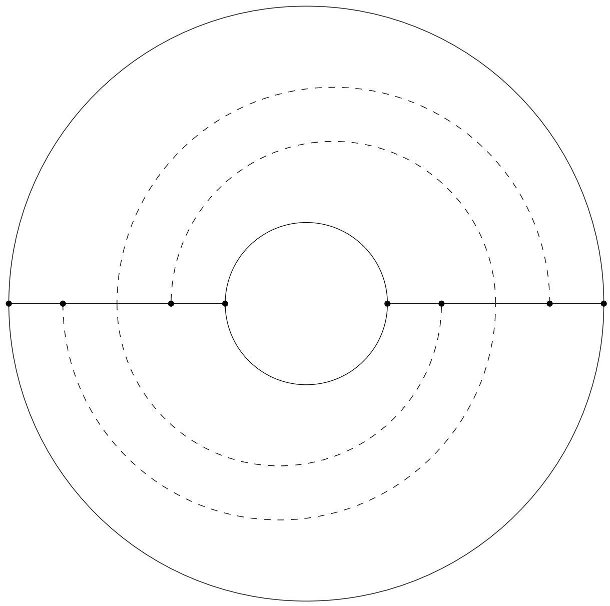

Figure 2: When they encounter this configuration, the arms can make an extra turn (or not).

- j disjoint rays between radii m and $4m$;
- j disjoint "spiraling paths" contained in the annulus $S_{2m,3m}$, each connecting two points of one of the rays and making one additional turn around the origin;
- j disjoint circuits around the origin, contained in the annulus $S_{m,2m}$;
- j disjoint circuits around the origin, contained in the annulus $S_{3m,4m}$.

RSW-type estimates directly show that, uniformly as $m \to \infty$, the probability of observing a j-spiral between radii m and $4m$ is bounded below by a positive constant (depending only on j).

The presence of j-spirals in disjoint annuli are independent events, each with positive probability, so that, for some $\varepsilon > 0$, the probability of the event $E_j^{(\varepsilon)}(n, N)$ of having at least $\varepsilon \log(N/n)$ disjoint j-spirals between radii n and N goes to 1 as N/n goes to infinity. The presence of j-spirals being an increasing event, the FKG inequality ensures the conditional probability of $E_j^{(\varepsilon)}(n, N)$, given the existence of j black arms between radii n and N, still goes to 1 as N/n goes to infinity.

We now explain how to use j-spirals to perform surgery on black arms. Assume that there are j arms between radii n and N, and a j-spiral Σ between radii m and $4m$. For each ray of Σ, we call inner (resp. outer) active point the last (resp. first) intersection point of the ray with ∂S_{2m} (resp. ∂S_{3m}), when starting from inside. Let Σ_1 be the connected component of Σ adjacent to ∂S_m when one removes the inner active points, and let Γ_1 be the union of Σ_1 and the j arms up to radius $2m$. It is easy to check that, whenever one marks $j-1$ points on Γ_1, there still exists a path completely contained in Γ_1 and avoiding the marked points, which connects the circle of radii n to one of the inner active points of Σ on the circle of radius $2m$. Indeed, at least one of the arms, one of the circuits and one of the rays contain no marked point. Menger's theorem then ensures that Γ_1 contains j disjoint arms, each connecting S_n to one of the inner active points of Σ — in other words, we can always assume that the j arms land on the circle of radius $2m$ on the inner active points.

The same construction can be made inwards between radii N and $3m$. It is then apparent that there are two ways of connecting the inner active points of Σ to its outer active points pairwise using vertices from Σ, and that these lead to two j-arms configurations between radii n and N with winding angles differing by exactly 2π (note however that it may be the case that the angles of these two configurations are both different from the angle of the initial configuration).

Since the construction above can be performed inside each annulus where there is a j-spiral, and does not modify the arms outside of $S_{m,4m}$, one arrives to the following fact: Whenever there are j arms between radii n and N, and the event $E_j^{(\varepsilon)}(n,N)$ is realized, the set $\bar{I}(n,N)$ contains an interval of length at least $2\pi\varepsilon\log(N/n)$ — and this occurs with conditional probability going to 1 as N/n goes to infinity.

Step 3. Using the BK inequality, it is not hard to see that the winding angle of the arms cannot be larger than $\log(N/n)\,[\log\log(N/n)]^2$. Indeed, assume this is not the case and consider ca. $C\log(N/n)$ overlapping "rectangles" between angles $-\pi/10$ and $\pi/10$, located between radii n and $4n$, $2n$ and $8n$, and so on: One of them has to be crossed (in either direction) at least $\frac{1}{C}\,[\log\log(N/n)]^2$ times, which has a probability at most

$$C\log(N/n)(1-\delta)^{\frac{1}{C}[\log\log(N/n)]^2}$$

by RSW estimates. Hence, the probability that $\bar{I}(n,N)$ is contained in the interval $[\pm\log(N/n)\log\log(N/n)]$ is larger than $1/2$ for N/n large enough. Dividing that interval into sub-intervals of length $\frac{\varepsilon}{2}\log(N/n)$, and using the previous step, we get that for one of them, say $i_\varepsilon(n,N)$,

$$\mathbb{P}\left(i_\varepsilon(n,N)\subseteq\bar{I}(n,N)\mid A_{j,B\ldots BB}(n,N)\right)\geqslant\frac{C'}{[\log\log(N/n)]^2},$$

where $C'>0$ is a universal constant.

Step 4. We are now in a position to conclude. If we take $\alpha_{\min}$ such that

$$\mathbb{P}\left(A_{j,B\ldots BW}(n,N)\cap\{\alpha_{\min}\in\bar{I}(n,N)\}\right)$$

is minimal among $\alpha_{\min}\in i_\varepsilon(n,N)\cap(6\pi\mathbb{Z})$, then

$$\mathbb{P}\left(A_{j,B\ldots BW}(n,N)\cap\{\alpha_{\min}\in\bar{I}(n,N)\}\right)\leqslant\frac{6\pi}{\frac{\varepsilon}{2}\log(N/n)}\,\mathbb{P}\left(A_{j,B\ldots BW}(n,N)\right)$$

since, as we noted earlier, whenever there are arms of different colors, $6\pi\mathbb{Z}$ cannot contain more than one element of $\bar{I}(n,N)$. On the other hand, we know

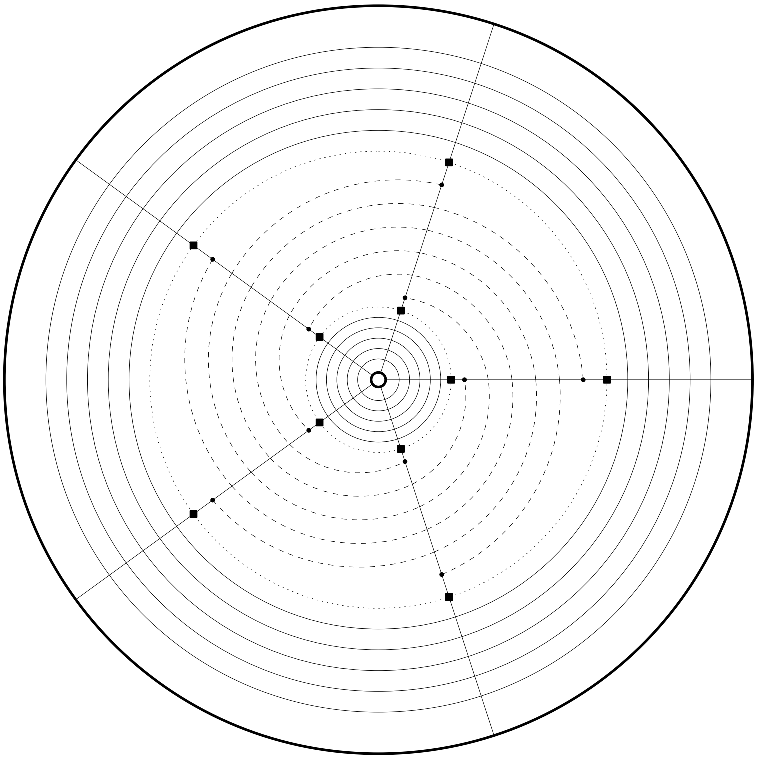

Figure 3: Generalization of Figure 2 in the case of $j \geqslant 3$ arms. The additional circuits are needed to apply Menger's theorem; the circles of radii m and $4m$ (resp. $2m$ and $3m$) are drawn in strong (resp. dotted) lines, the spiraling paths in dashed lines and the active points are marked with a black square.

from the previous step that

$$\begin{aligned}\mathbb{P}\left(A_{j,B\dots BB}(n,N) \cap \{\alpha_{\min} \in \bar{I}(n,N)\}\right) \\ \geqslant \mathbb{P}\left(A_{j,B\dots BB}(n,N) \cap \{i_{\varepsilon}(n,N) \subseteq \bar{I}(n,N)\}\right) \\ \geqslant \frac{C'}{[\log\log(N/n)]^2}\,\mathbb{P}\left(A_{j,B\dots BB}(n,N)\right).\end{aligned}$$

If we apply inequality (2.5) to $A = A_{j-1,B\dots B}(n,N) \cap \{\alpha_{\min} \in \bar{I}_{j-1,B\dots B}(n,N)\}$

and $B = A_{1,B}(n,N)$, we obtain that

$$\begin{aligned}
\frac{C'}{[\log\log(N/n)]^2}\,&\mathbb{P}\left(A_{j,B\dots BB}(n,N)\right)\\
&\leqslant \mathbb{P}\left(A_{j,B\dots BB}(n,N)\cap\{\alpha_{\min}\in \bar{I}_{j-1,B\dots B}(n,N)\}\right)\\
&= \mathbb{P}\left(A\circ B\right)\\
&\leqslant \mathbb{P}\left(A\cap \bar{B}\right)\\
&= \mathbb{P}\left(A_{j,B\dots BW}(n,N)\cap\{\alpha_{\min}\in \bar{I}_{j-1,B\dots B}(n,N)\}\right)\\
&\leqslant \mathbb{P}\left(A_{j,B\dots BW}(n,N)\cap\{\alpha_{\min}\in \bar{I}_{j,B\dots BW}(n,N)\}\right)\\
&\leqslant \frac{6\pi}{\frac{\varepsilon}{2}\log(N/n)}\,\mathbb{P}\left(A_{j,B\dots BW}(n,N)\right),
\end{aligned}$$

which concludes the proof. □

3.2 The density of the set of angles

In this section, we further describe the set of angles $I(n,N)$ — which happened to be a key tool in the previous proof — in the monochromatic case. We prove that (conditionally on the existence of j disjoint black arms) $\bar{I}(n,N)$ is always an interval, as in the polychromatic case. For that, we use the following deterministic statement that $I(n,N)$ does not have large "holes":

Proposition 7. *Let $j \geqslant 1$ and $\sigma = B\dots BB$ of length j. Let $\alpha, \alpha' \in I_{j,\sigma}(n,N)$ with $\alpha < \alpha'$; then there exists a sequence $(\alpha_i)_{0\leqslant i\leqslant r}$ of elements of $I_{j,\sigma}(n,N)$, satisfying the following two properties:*

- *$\alpha = \alpha_0 < \alpha_1 < \dots < \alpha_r = \alpha'$;*
- *for every $i \in \{0,\dots,r-1\}$, $\alpha_{i+1} - \alpha_i < 2\pi$.*

This result directly implies that $\bar{I}(n,N)$ is an interval, and the construction of the previous sub-section, creating extra turns (step 2 of the proof), gives a lower bound on the diameter of $\bar{I}(n,N)$: We hence get that for σ constant, there exists some $\varepsilon > 0$ (depending only on j) such that $\bar{I}(n,N)$ is an interval of length at least $\varepsilon\log(N/n)$ with probability tending to 1 as N/n gets large.

The main step in the proof of the density result is the following topological lemma:

Lemma 8. *Let $j > 0$, and let $\gamma_1, \dots, \gamma_j$ be j disjoint Jordan curves contained in the (closed) annulus $\{n \leqslant |z| \leqslant N\}$, each having its starting point on the circle of radius n and its endpoint on the circle of radius N. For each $k \in \{1,\dots,j\}$, let α_k be the winding angle of γ_k (as defined above) and let δ_k be the ray $[ne^{2i\pi k/j}, Ne^{2i\pi k/j}]$. Assume that, for each pair (k,k'), the intersection of γ_k and $\delta_{k'}$ is finite. Then, provided all the α_k are larger than $2\pi(1+2/j)$, the union of all the paths γ_k and δ_k contains j disjoint paths $\tilde{\delta}_1, \dots, \tilde{\delta}_j$, all having angle $2\pi/j$.*

In other words: starting from two collections of paths, if their angles differ enough, one can "correct" the one with the smaller angle in such a way as to make it turn a little bit more.

Proof. We shall construct the paths $\tilde{\delta}_k$ explicitly. The first step is to reduce the situation to one of lower combinatorial complexity, namely to the case where the starting points of the γ_k are separated by those of the δ_k. For each $k \leqslant j$, let $\tau_k = \inf\{t : \gamma_k(t) \in [ne^{i\pi(2k-1)/j}, Ne^{i\pi(2k-1)/j}]\}$ (which is always finite by our hypotheses), and let

$$\Gamma := \bigcup_{k=1}^{j} \{\gamma_k(t) : 0 \leqslant t \leqslant \tau_k\}.$$

Γ intersects each of the δ_k finitely many times, so each of the $\delta_k \setminus \Gamma$ has finitely many connected components: let Δ be the union of those components that do not intersect the circle of radius N, and let

$$\Omega_0 := \{n \leqslant |z| \leqslant N\} \setminus (\Gamma \cup \Delta).$$

Let Ω be the connected component of Ω_0 having the circle of radius N as a boundary component. Ω is homeomorphic to an annulus, and for each k, the point $\gamma_k(\tau_k)$ is on its boundary; by construction, the $\gamma_k(\tau_k)$ are intertwined with the (remaining portions of the) rays of angles $2\pi k/j$. We will perform our construction of the $\tilde{\delta}_k$ inside Ω; continuing them with the δ_k outside of Ω then produces j disjoint paths satisfying the conditions we need.

Up to homeomorphism, we can now assume without loss of generality that for each k, $\gamma_k(0) = ne^{i\pi(2k-1)/j}$. The only thing we lose in the above reduction is the assumption on the angles of the γ_k; but since it takes at most one turn for each of the γ_k to reach the appropriate argument, we can still assume that the remaining angles are all larger than $4\pi/j$. In particular, each of the γ_k will cross the wedge between angles $2\pi k/j$ and $2\pi(k+1)/j$ in the positive direction before hitting the circle of radius N.

For every $k \leqslant j$, let $\theta_k(t)$ be the continuous determination of the argument of $\gamma_k(t)$ satisfying $\theta_k(0) = (2k-1)\pi/j$, and let

$$\mathcal{T}_k := \left\{ t > 0 : \frac{2\pi k}{j} < \theta_j(t) < \frac{2\pi(k+1)}{j} \right\} \quad \text{and} \quad \tilde{\Gamma}_k = \overline{\{\gamma_k(t) : t \in \mathcal{T}_k\}}.$$

We now describe informally the construction of $\tilde{\delta}_k$. Start from the point $ne^{2i\pi k/j}$, and start following δ_k outwards, until the first intersection of δ_k with $\tilde{\Gamma}_k$. Then, follow the corresponding connected component of $\tilde{\Gamma}_k$, until intersecting either δ_k or δ_{k+1}; follow that one outwards until it intersects either $\tilde{\Gamma}_k$ or the circle of radius N; iterating the construction, one finally obtains a Jordan path joining $ne^{2\pi k/j}$ to $Ne^{2\pi(k+1)/j}$, and contained in the union of δ_k, δ_{k+1} and $\tilde{\Gamma}_k$ (see Figure 4).

All that remains is to prove that the $\tilde{\delta}_k$ are indeed disjoint; by symmetry, it is enough to do so for $\tilde{\delta}_1$ and $\tilde{\delta}_2$. Besides, because the γ_k are themselves disjoint, any intersection point between $\tilde{\delta}_1$ and $\tilde{\delta}_2$ has to occur on δ_2 (at least in the case $j > 2$ — but the case $j = 2$, where they could also intersect along δ_1, again follows by symmetry).

The intersection of $\tilde{\delta}_1$ with δ_2 consists in a finite collection (I_m) of compact intervals; besides, the points of the intersection are visited by $\tilde{\delta}_1$ in order of increasing distance to the origin. Similarly, the intersection of $\tilde{\delta}_2$ with δ_2 consists in a finite collection (J_l) of compact intervals, which are also visited in order of increasing distance to the origin.

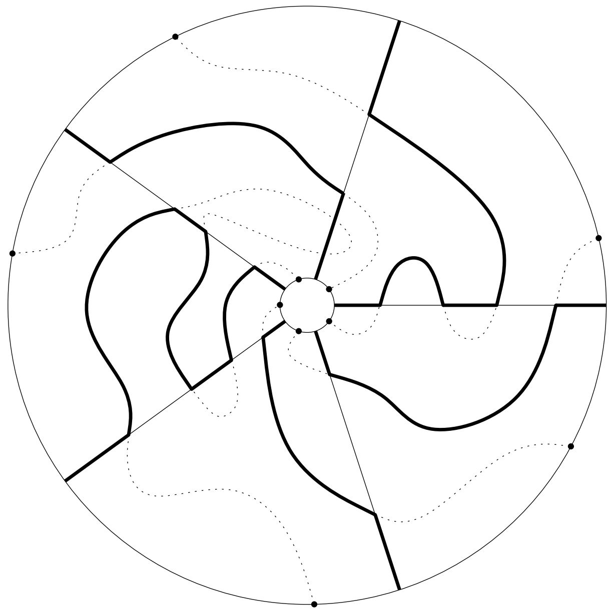

Figure 4: The construction of the $\tilde{\delta}_k$ (in the case $j = 5$). The dotted lines are the paths γ_k, and the strong lines are the $\tilde{\delta}_k$ obtained at the end of the construction.

Suppose that $\bigcup I_p$ and $\bigcup J_p$ have a non-empty intersection; and let z_0 be the intersection point lying closest to the origin. Let p_0 and q_0 be such that $z_0 \in I_{p_0} \cap J_{q_0}$; notice that z_0 is the endpoint closest to the origin of either I_{p_0} or J_{q_0}. According to the order in which γ_1 (resp. γ_2) visits the endpoints of I_{p_0} (resp. J_{q_0}), this gives rise to eight possible configurations; it is straightforward in all cases to apply Jordan's theorem to prove that γ_1 and γ_2 then have to intersect, thus leading to a contradiction. □

For the purpose of the proof of Proposition 7, we will need a slight variation of the lemma, where the hypothesis of finiteness of the intersections between paths is replaced with the assumption that the paths considered are all polygonal lines. The proof is exactly the same though, and does not even require any additional notation: whenever two paths, say γ_k and $\delta_{k'}$, coincide along a line segment, the definition of Γ_k amounts to considering some of the endpoints of this segment as intersections, which in other words is equivalent to shifting γ_k by an infinitesimal amount towards the exterior of the wedge used to define $\mathcal{T}_k$ in order to recover finiteness.

Proof of Proposition 7. The previous Lemma is stated with particular curves on which a surgery can be dons, but it can obviously be applied to more general cases through a homeomorphism of the annulus. The general statement is then the following (roughly speaking): Assuming the existence of two families of j arms with different enough winding angles, it is possible to produce a third family using the same endpoints as the first one but with a slightly larger winding angle.

We are now ready to prove Proposition 7. Consider a configuration in which one can find two families of crossings, say (λ_k) and (λ'_k), in such a way that for every k, the difference between the winding angles of λ_k and λ'_k is at least 2π. Let α_0 be the minimal angle in the first family, and apply the topological lemma with $\delta_k = \lambda_k$ and $\gamma_k = \lambda'_k$: One obtains a new family of pairwise disjoint paths (λ^1_k), which share the same family of endpoints as the (λ_k), the endpoint of λ^1_k being that of λ_{k+1} (with the obvious convention that $j+1=1$).

One can then iterate the procedure, applying the topological lemma with this time $\delta_k = \lambda^1_k$, and still letting $\gamma_k = \lambda'_k$; one gets a new family (λ^2_k) with the endpoints again shifted amongst the paths in the same direction. Continuing as long as the winding angle difference is at least 2π, this construction produces a sequence (λ^i_k) of j-tuples of disjoint paths, the winding angles of which vary by less than 2π at each step. Besides, the construction ends in finitely many steps, for after j steps, each of the winding angles has increased by exactly 2π. This readily implies our claim. □

Remark 9. *Notice that, as early as the second step of the procedure, (λ^n_k) and (λ'_k) will always coincide on a positive fraction of their length, which is why we needed the above extension of the lemma.*

4 Existence of the monochromatic arm exponents

We now prove Theorem 4, stating the existence of the monochromatic exponents α'_j. For that, we use a rather common argument, as presented *e.g.* in [16]: since the quasi-multiplicativity property holds (item 2. above), it is actually enough to check that there exists a function f_j (which will automatically be sub-multiplicative itself – one can take $C_2 = 1$ in the quasi-multiplicativity property) such that, for every $R > 1$,

$$\mathbb{P}\left(A_{j,B\ldots BB}(n, Rn)\right) \to f_j(R) \tag{4.1}$$

as $n \to \infty$ — notice that RSW-type estimates provide both the fact that the left-hand term in bounded above and below by constants for fixed R as $n \to \infty$, and *a priori* estimates on any (potentially subsequential) limit, of the form

$$R^{-\varepsilon_j} \leqslant f_j(R) \leqslant R^{-1/\varepsilon_j}$$

where ε_j depends only on j.

By Menger's theorem (see [5]), the complement of the event $A_{j,B\ldots BB}(n, N)$ can be written as

$$D_j(n,N) = \{\text{There exists a circuit in } S_{n,N} \text{ that surrounds } \partial S_n \text{ and contains at most } j-1 \text{ black sites}\}.$$

This makes it possible to express the event $A_{j,B\ldots BB}(n, N)$ in terms of the collection of all cluster interfaces (or "loops"): It is just the event that there does not exist a "necklace" of at most $(j-1)$ loops, with white vertices on their inner boundary and black ones on their outer boundary, forming a chain around ∂S_n and such that two consecutive loops are separated by only one black site.

Standard arguments show that the probability that two interfaces touch in the scaling limit is exactly the asymptotic probability that they "almost touch" (in the sense that they are separated by exactly one vertex) on discrete lattices — it is *e.g.* a simple consequence of the fact that the polychromatic 6-arm exponent is strictly larger than 2, which in turn is a consequence of RSW-type estimates (the fact that the polychromatic 5-arm exponent is equal to 2 being true on any lattice on which RSW holds – at least for colors $BWBWW$).

What this means, is that to show convergence of the probability in Equation (4.1), it is enough to know the probability of the corresponding continuous event. While we do not know the exact value of the limit, it is nevertheless easy to check that the event itself is measurable with respect to the *full scaling limit* of percolation, as constructed by Camia and Newman in [4], and that is enough for our purpose. Notice that the measurablility of the event in terms of the full scaling limit is ensured by the exploration procedure described in that paper: It is proved there that for every $\varepsilon > 0$, all loops of diameter at least ε are discovered after finitely many steps of the exploration procedure.

References

[1] M. Aizenman, B. Duplantier, A. Aharony, Path crossing exponents and the external perimeter in 2D percolation, Phys. Rev. Lett. **83**, 1359-1362 (1999).

[2] J. Van den Berg, H. Kesten, Inequalities with applications to percolation and reliability, J. Appl. Probab. **22**, 556-569 (1985).

[3] C. Borgs, J.T. Chayes, D. Randall, The van den Berg-Kesten-Reimer inequality: a review, in Perplexing Problems in Probability: Festschrift in Honor of Harry Kesten, Birkhäuser (1999).

[4] F. Camia, C.M. Newman, Two-dimensional critical percolation: the full scaling limit, Comm. Math. Phys. **268**, 1-38 (2006).

[5] R. Diestel, Graph theory, 2nd edition, Springer, New York (2000).

[6] G.R. Grimmett, Percolation, 2nd edition, Springer, New York (1999).

[7] H. Kesten, Scaling relations for 2D-percolation, Comm. Math. Phys. **109**, 109-156 (1987).

[8] G.F. Lawler, O. Schramm, W. Werner, Values of Brownian intersection exponents I: Half-plane exponents, Acta Mathematica **187**, 237-273 (2001).

[9] G.F. Lawler, O. Schramm, W. Werner, Values of Brownian intersection exponents II: Plane exponents, Acta Mathematica **187**, 275-308 (2001).

[10] G.F. Lawler, O. Schramm, W. Werner, One-arm exponent for critical 2D percolation, Elec. J. Probab. **7**, paper no.2 (2002).

[11] P. Nolin, Near-critical percolation in two dimensions, Elec. J. Probab. **13**, paper no.55 (2008).

[12] D. Reimer, Proof of the van den Berg-Kesten conjecture, Combin. Probab. Comput. **9**, 27-32 (2000).

[13] O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees, Israel J. Math. **118**, 221-288 (2000).

[14] O. Schramm, J.E. Steif, Quantitative noise sensitivity and exceptional times for percolation, Ann. Math., to appear.

[15] S. Smirnov, Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits, C. R. Acad. Sci. Paris Sér. I Math. **333**, 239-244 (2001).

[16] S. Smirnov, W. Werner, Critical exponents for two-dimensional percolation, Math. Res. Lett. **8**, 729-744 (2001).

[17] W. Werner, Lectures on two-dimensional critical percolation, IAS-Park City 2007 summer school, PCMI, AMS (2009).