Extending Routing Games to Flows over Time

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Abstract

Routing game presents an interesting framework to analyse the practical problem of source routing in the Internet. It is particularly useful in quantifying the inefficiency of selfish user behavior that results in any transportation network without any central authority. This game assumes that the only user criteria for decision making is path cost. In this work, we take a step further, and model a routing game where user decision is based not only on path but also on time. We show that, under convex cost functions, this new routing game over time can be mapped to the classical routing game, thereby presenting a model that can exploit well-established results in the subject. Using a simple example, we demonstrate the usefulness of the model, and motivate the need for resource coordination to minimize inefficiency or cost.

Keywords: Routing game

Résumé

Routing game presents an interesting framework to analyse the practical problem of source routing in the Internet. It is particularly useful in quantifying the inefficiency of selfish user behavior that results in any transportation network without any central authority. This game assumes that the only user criteria for decision making is path cost. In this work, we take a step further, and model a routing game where user decision is based not only on path but also on time. We show that, under convex cost functions, this new routing game over time can be mapped to the classical routing game, thereby presenting a model that can exploit well-established results in the subject. Using a simple example, we demonstrate the usefulness of the model, and motivate the need for resource coordination to minimize inefficiency or cost.

Mots-clés: Routing game

1 Introduction

Routing games is a game-theoretical subject that sheds light on an important practical problem in the Internet (or for that matter, any transportation network) — the benefits and implications of routing traffic without any central authority. The 'routing' here refers to source routing, where an individual user is able to decide its route, or path, from the source to the destination.

The literature on routing games is wealthy, briefed in Section 2. While the previous works consider path selection as the only decision variable for a user, we explore, along with path selection, another important variable — time preference. We call this game, *routing game over time*, RGoT in short. The users in this game face a cost on the selected path which depends on the time as well as the total amount of traffic sent on this path.

The motivation for including the time criteria for decision making comes from the need to meet time constraints during data transfers. This has lead researchers to explore in-advance reservation, for example [1] and [2]. We propose a model which enables to quantify the inefficiency — price of anarchy — of realizing such a system as a result of selfish decisions, with no centralized coordination. The model for RGoT in presented in Section 3. Therein, we define the concept of equilibrium and optimal allocation. We show that under convex cost function, allocation can be described using step functions. In Section 4 we prove that RGoT can be solved by solving an instance of classical routing game, and map some of the important results known for this kind of game. Finally, in Section 5, using a simple example we illustrate and motivate the need for coordination.

2 Related work

Since the introduction of the concept of equilibrium in transportation network by J. G. Wardrop in 1952, this kind of Nash equilibrium has been widely studied and extended. Among these, we note the introduction of price of anarchy and the results on classes of cost functions by Roughgarden et al. [3]. The dynamics of convergence to equilibrium has been investigated in [4]. The concepts of non-atomic and atomic routing games are detailed in [5, Ch. 18].

On the other side, while concepts of network flows and flows over time was introduced by Ford and Fulkerson, network flows have been extensively studied as summarized in [6, Ch. 26], development on the version considering time is recent [7]. But since it is used in several works on bandwidth and flow allocation, it is interesting and important to extend non-atomic routing games to routing games that take time into consideration. This paper presents this extension.

3 Routing games over time

In this section, we define the model for RGoT, and then extend the concepts of *equilibrium flows* and *optimal allocations* to this model.

3.1 Model

This section describes the RGoT model used in this paper. It is inspired from the *non-atomic routing game* model and from the model of flow over time. This model of routing

v_r	demand (volume)
(s_r, d_r)	source/destination pair
t_r^s	start time (assumed to be a rational)
t_r^d	deadline (assumed to be a rational)
P_r	set of unique labels for $s_r - d_r$ paths of $G = (V, E)$

Table 1: Table of notations.

game assumes infinite number of *users*. Each user is small, and as such doesn't contribute significantly to the *cost*. *Request* as presented in the following definition refers to a kind of request; meaning that the users that form a particular request have same constraints in terms of source, destination, and paths; and total aggregate volume is known. We now proceed to give some important definitions. The notations used are listed in Table 1 for a given request r.

Definition 3.1 (Request)**.** *In the directed graph G, $r \triangleq (s_r, d_r, v_r, t_r^s, t_r^d, P_r)$, is a request between vertices s_r and d_r, dates t_r^s, t_r^d, allowed to transfer over one or more paths from P_r and with a total volume of v_r. Every path in P_r is unique, in the sense that if two requests use the same path (as set of edges) their labels will be different. In addition, we let $\mathcal{P} \triangleq \bigcup_i P_r$.*

These requests induce a traffic on the network which is characterized by the rate traffic going on each path. This is called *allocation vector.*

Definition 3.2 (Allocation vector)**.** *If f_p is the rate for request r over the path $p \in P_r$, the allocation vector f is defined as being the vector $[f_p]_{p\in\mathcal{P}}$ with $f_p : \underset{t \mapsto f_p(t)}{\mathbb{R}_+ \to \mathbb{R}_+}$. In addition, we define* link allocation *as: $f_e : \underset{t \mapsto f_e(t)}{\mathbb{R}_+ \to \mathbb{R}_+}$ where $f_e(t) \triangleq \sum_{p\in\mathcal{P}:e\in p} f_p(t)$*

This allocation vector gives the rates over time with which each request will send on its paths to achieve transfer of the specified volume during the time window. If it does so, it is said to be a feasible allocation. Link allocation gives the total rate seen on a link at a given date.

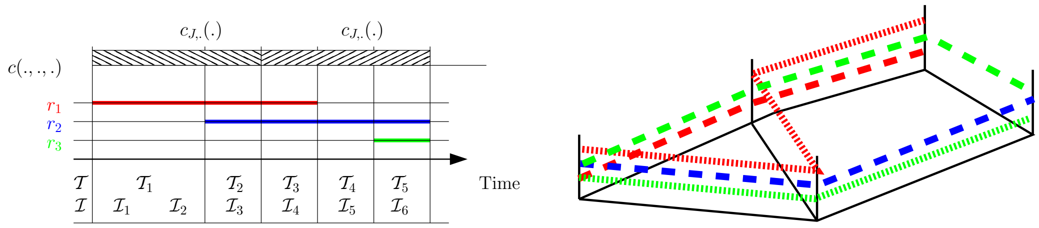
Definition 3.3 (Feasible allocation)**.**

$$\forall r \in R, \int_{t_r^s}^{t_r^d} \sum_{p \in P_r} f_p(t)\mathrm{d}t = v_r$$

Next, we define per-edge cost. It is assumed to be a piecewise constant function with regard to the time.

Definition 3.4 (Per-edge cost function)**.** *For an edge e, $c(e, b, t)$ is the cost function for transferring at rate b at time t over link e. For the cost function, we furthermore assume: given I_c, a finite partition of time ($\mathbb{R}_+$) with rational breaks, c has the following structure: $c(e, b, t) \triangleq \sum_{J\in I_c} \mathbb{1}_{t\in J} c_{J,e}(b)$ and $c_{J,e}(.)$ is assumed to be non-negative, continuous and non-decreasing.*

Using the definitions of allocation vector and cost, allocation f incurs cost $c(e, f_e(t), t)$ at time t on edge e. From the assumption on the form of cost, cost on this interval can be split on intervals I_c as done below. But before doing this, we define RGoT.



(a) Timeline and its discrete versions.

(b) Example of requests with their alternate paths.

Figure 1: Example of requests with their intervals.

Definition 3.5 (RGoT)**.** *(G, R, c), where G is a network, R a set of requests and c a cost function, defines an RGoT.*

In the remaining, we focus on the interval:

$$\mathbb{T} \triangleq \left[\min_{r\in R}\{t_r^s\}, \max_{r\in R}\{t_r^d\}\right]$$

All time breaks — t_r^s, t_r^d, and changes of cost function — are rationals. Hence, we can divide time axis in intervals of same lengths, such that all these breaks come at the boundaries by taking this length as the least common multiplier of denominators of interval lengths. This partitioning of time is used in Def. 3.7 to define the set of time intervals. It is illustrated in Fig. 1(a). This is an artifact, used later, to reduce this game to normal non-atomic routing games.

Definition 3.6 (Time intervals)**.** *For a game (G, R, c) with I_c being the set of time interval used to define cost function c, we define $\mathcal{T}$ as the set of time intervals defined by all the start time t_r^s and deadline t_r^d of request $i \in R$ and start and end of intervals I_c included in $\mathbb{T}$. $\mathcal{T}$ is a partition of $\mathbb{T}$.*

Definition 3.7 (Same length time intervals)**.** *We define $\mathcal{I}$ as the refined set of intervals of $\mathcal{T}$ such that all intervals of $\mathcal{I}$ have the same length (called $|J|$ in the remaining). This set is obtained by subdividing $\mathcal{I}$.*

Definition 3.8 (Intervals of a request)**.**

$$\mathcal{I}_r \triangleq \{J \in \mathcal{I} | J \subset [t_r^s, t_r^d]\}$$

In the remainder of this work, $c_{J,e}$ has been extended to J in $\mathcal{I}$ by using the function $c_{I,e}$ of the interval $I \in I_c$ that contains J.

Definition 3.9 (Per-path cost on an interval)**.** *For any time interval J,*

$$c_{J,p}(f) \triangleq \int_{t\in J} \sum_{e\in p} c(e, f_e(t), t)\mathrm{d}t$$

Definition 3.10 (Per-path cost)**.**

$$
\begin{aligned}
c_p(f) &\triangleq c_{\mathbb{T},p}(f) \\
&= \int_{t\in\mathbb{T}} \sum_{e\in p} c(e, f_e(t), t)\mathrm{d}t \text{ (Def. 3.9)} \\
&= \int_{t\in\mathbb{T}} \sum_{e\in p} \sum_{J\in I_c} \mathbf{1}_{t\in J} c_{J,e}(f_e(t))\mathrm{d}t \text{ (def. of } c()) \\
c_p(f) &= \sum_{J\in\mathcal{I}} \sum_{e\in p} \int_{t\in J} c_{J,e}(f_e(t))\mathrm{d}t \qquad (1)
\end{aligned}
$$

Having defined the model, next sections define some specific feasible flows which arise from selfish user behaviors, equilibrium flows, and from social interest, optimal allocations.

3.2 Equilibrium Flows

As said before, users behave selfishly on their infinitely small fraction of volume belonging to one request. This leads to allocations which form a subset of feasible allocations, called equilibrium flows. We show that they can be defined using step functions with steps on $\mathcal{I}$ when cost is convex. The definition of equilibrium that follows, is similar to (characterization of) Wardrop equilibrium.

Definition 3.11 (Equilibrium)**.** *f is an equilibrium flow in (G, R, c) if:*

1. *f is a feasible allocation vector, and;*
2. *for every $r \in R$, for every interval J and $\tilde{J}$ included in $[t_r^s, t_r^d]$ such that $|J| = |\tilde{J}|$ and any p, $\tilde{p} \in P_r$ where $\int_{t\in J} f_p \mathrm{d}t > 0$, $c_{J,p}(f) \le c_{\tilde{J},\tilde{p}}(f)$.*

If f is an equilibrium flow, for every $r \in R$ and any $p \in P_r$, $c_p(f) \le c_p(\tilde{f})$ with $\tilde{f}$ such that, f_p has been replaced by $\tilde{f}_p$, and $\int_{t_r^s}^{t_r^d} f_p(t)\mathrm{d}t = \int_{t_r^s}^{t_r^d} \tilde{f}_p(t)\mathrm{d}t$, as it is a feasible flow and following second point of Def. 3.11 with $I = J = \mathcal{T}$ (f_p is supposed to be null out of $[t_r^s, t_r^d]$ as it doesn't contribute to feasibility).

We proceed to show that step functions are suitable for equilibrium flows, when the cost is convex.

Proposition 3.12. *If cost function $c_{J,e}(b)$ is convex, for any link allocation f_e, there is one step function constant on each $J \in \mathcal{I}$ that transfers the same volume on each time interval for a cost at least as good.*

Proof. We proceed in two steps.

(1) $f_e(t)$ is not better than constant function on J:
Let J be an interval without arrivals/departures/cost-breaks, we suppose $f_e(t)$ is integrable, $\int_J f_e(t)\mathrm{d}t = f_{J,e}|J|$ and $c_{J,e}(b)$ be a convex function. Let $\mu(.)$ be the Lebesgue measure, $\mu(J) = |J|$ and $\eta(.) \triangleq \mu(.)/\mu(J)$. Obviously $\eta(J) = 1$ and:

$$\int_J c_{J,e}(f_e(t))\mu(\mathrm{d}t) = \mu(J) \int_J c_{J,e}(f_e(\frac{t}{\mu(J)}))\eta(\mathrm{d}t)$$

As $\eta(J) = 1$, using Jensen's inequality:

$$\mu(J)c_{J,e}(\int_J f_e(\frac{t}{\mu(J)})\eta(\mathrm{d}t)) \leq \mu(J)\int_J c_{J,e}(f_e(\frac{t}{\mu(J)}))\eta(\mathrm{d}t)$$

(from definition of $\eta(.)$ and above equation)

$$\Rightarrow \mu(J)c_{J,e}(\int_J \frac{f_e(t)}{\mu(J)}\mu(\mathrm{d}t)) \leq \int_J c_{J,e}(f_e(t))\mu(\mathrm{d}t)$$

(from definition of $f_{J,e}$)

$$\Rightarrow \mu(J)c_{J,e}(\frac{f_{J,e}|J|}{\mu(J)}) \leq \int_J c_{J,e}(f_e(t))\mu(\mathrm{d}t)$$

$$\Rightarrow |J|c_{J,e}(f_{J,e}) \leq \int_J c_{J,e}(f_e(t))\mu(\mathrm{d}t) \tag{2}$$

(2) says that the cost of a function constant on J and equal to $f_{J,e}$ has a cost as good as any other function which transfer the same volume on J.

(2) For any link allocation $f_e(.)$, there is one step function as "good":
First part of this proof can be applied on each intervals and $f_e(.)$ be replaced by a step function with a cost at least as good.

□

Proposition 3.13. *If cost function $c_{J,e}(b)$ is convex, for any feasible allocation f there is $\tilde{f}$ made of step functions which: (1) for all $p \in \mathcal{P}$ has a cost $c_p(\tilde{f})$ not worse than $c_p(f)$; (2) has steps on $\mathcal{I}$.*

Proof. We start with one f. For each $p \in \mathcal{P}$, we define $\tilde{f}_p$ as the step function which transfers the same volume as f_p on each intervals of $\mathcal{I}$ but using a constant rate. This allocation is also feasible as it transfers the same volumes during the same intervals and thus volume constraints are satisfied. Furthermore, its steps are on $\mathcal{I}$. $\tilde{f}_e$ are step functions, as sum of step functions $\tilde{f}_p$. Using Eq. (1) and Prop. 3.12, $c_p(\tilde{f}) \leq c_p(f)$. □

Next we show that for a convex cost function, for any equilibrium flow, there is a corresponding equilibrium flow made of step function which is as good in terms of cost. If we come back to infinitely small users making requests, this basically means, they have no incentive of using anything other than a step function constant on intervals of $\mathcal{T}$ or $\mathcal{I}$.

Proposition 3.14. *Equilibrium flows can be taken as step functions.*

Proof. Equilibrium flows are feasible allocation and can thus be taken as step function for a cost not higher, as proved by Prop. 3.13. □

It can be seen that with cost functions that are not non-negative, convex or non-decreasing in f_e, the link allocation, this proposition does not apply. This is because, we can't exploit the structure of time intervals to define the step functions since it might be cheaper to reduce the duration of transfer while increasing the utilized rate, eventually without limits. As an example, this kind of allocation (equilibrium flow and later optimal allocation) as step functions of time do not apply for sub-linear cost as it is always better to group the utilization of the resources.

3.3 Social Cost and Optimal Allocation

Having defined equilibrium allocation that will result from selfish decisions of users realizing requests of R, we now define optimal allocation that minimizes the total cost charged to serve the requests. The total cost is referred as social cost. By dividing it by $\int_t \sum_p f_p$, average cost can be obtained.

Definition 3.15 (Social cost)**.**

$$\begin{aligned} C(f) &\triangleq \int_t \sum_e f_e(t) c(e, f_e(t), t) \mathrm{d}t \\ &= \int_t \sum_e \sum_{J \in I_c} \mathbf{1}_{t \in J} f_e(t) c_{J,e}(f_e(t)) \mathrm{d}t \ \textit{(def. of c())} \\ &= \sum_{J \in I_c} \int_{t \in J} \sum_e f_e(t) c_{J,e}(f_e(t)) \mathrm{d}t \\ C(f) &= \sum_{J \in \mathcal{I}} \sum_e \int_{t \in J} f_e(t) c_{J,e}(f_e(t)) \mathrm{d}t \end{aligned} \tag{3}$$

Optimal allocation — *in the social sense* — is:

Definition 3.16 (Optimal allocation)**.** *A feasible flow of (G, R, c) is optimal if it minimizes $C(f)$ over other feasible flows.*

As in equilibrium flow, if the social cost is convex, we can take optimal allocation as step function as stated in Prop. 3.20. To begin with, we observe that convexity of cost function implies convexity of social cost.

Remark 3.17. *$x \mapsto x.c_{J,e}(x)$ is convex on $\mathbb{R}_+$. Since this is a product of two convex, non-decreasing and positive functions on $\mathbb{R}_+$.*

Proposition 3.18. *If cost function $c_{J,e}(b)$ is convex, positive and non-decreasing, for any link allocation f_e, there is one link allocation as step function, constant on each $J \in \mathcal{I}$, that transfers the same volume on each time interval for a social cost at least as good.*

Proof. As stated in Remark 3.17, $x \mapsto x.c_{J,e}(x)$ is convex on $\mathbb{R}_+$. Allocation are positive functions. Then, using Jensen's inequality (or Hermite-Hadamard inequality) we show that there is a step function with steps in $\mathcal{I}$ which is as good as any feasible allocation and transfer the same volume on each time interval. □

Proposition 3.19. *If cost function $c_{J,e}(b)$ is convex, positive and non-decreasing, for any feasible allocation f, there is one $\tilde{f}$ made of step functions which: (1) has a costs $C(\tilde{f})$ not worse than $C(f)$; and (2) has steps on $\mathcal{I}$.*

Proof. We start with one f. For each $p \in \mathcal{P}$, we define $\tilde{f}_p$ as the step function which transfers the same volume as f_p on each interval of $\mathcal{I}$, but using a constant rate. This allocation is also feasible as it transfers the same volumes during the same intervals and thus volume constraints are satisfied, steps are on $\mathcal{I}$. $\tilde{f}_e$ are step functions as sum of step functions $\tilde{f}_p$. Using Eq. (3) and Prop. 3.18, $C(\tilde{f}) \leq C(f)$. □

We conclude this section showing that there is an optimal allocation among the set of feasible allocations made of step functions.

Proposition 3.20. *Optimal allocation can be taken as step function with steps on $\mathcal{I}$.*

Proof. Optimal allocations are feasible allocation and can thus be taken as step function with step on $\mathcal{I}$ for a cost not higher as proved by Prop. 3.19. □

In the next section, we exploit the structure of step function and revisit previously introduced definitions to establish equivalence between step allocations and generic allocations under convex costs.

3.4 Discrete Allocations — Reduction to Step Function

As users and regulator try to minimize their costs (equilibrium flow) or social cost (optimal allocation), all the feasible allocations of interest are step functions and their steps are on $\mathcal{I}$ (Prop. 3.13 and 3.19).

Hence, interesting feasible allocations are step functions with steps on $\mathcal{I}$, they can be described by discrete value giving their value on the steps.

Definition 3.21 ($f_{J,p}$)**.** *For a feasible allocation f, we define the $|\mathcal{I}|\times|\mathcal{P}|$ matrix $[f_{J,p}]_{(J,p)\in\mathcal{I}\times\mathcal{P}}$, where $f_{J,p} \triangleq \int_{t\in J} \frac{f_p(t)}{|J|}\mathrm{d}t$ is the constant rate of a step function which transfers as much as f on interval J.*

Remark 3.22. *It can be noted that in Def. 3.21, $f_{J,p}$ for J not used by a request r using path p is supposed to be null. More formally:*

$$\forall r \in R, \forall p \in P_r, \forall J \in \mathcal{I}\backslash\mathcal{I}_r, f_{J,p} = 0$$

This can be justified, as it would not contribute to the total volume, but will add to the cost; hence rendering useless from a social cost and individual cost point of view.

Remark 3.23. *As f is supposed to be a step function on $\mathcal{I}$, it can be reconstructed from the matrix $[f_{J,p}]$ by:*

$$\forall p \in \mathcal{P}, f_p(t) = \sum_{J\in\mathcal{I}} \mathbb{1}_J(t) f_{J,p} \tag{4}$$

Similarly we define discrete link allocation and show that continuous-time link allocation can be reconstructed from the discrete version.

Definition 3.24 ($f_{J,e}$)**.**

$$f_{J,e} \triangleq \sum_{p\in\mathcal{P}:e\in p} f_{J,p} \tag{5}$$

Proposition 3.25.

$$f_e(t) = \sum_{J\in\mathcal{I}} \mathbb{1}_J(t) f_{J,e} \tag{6}$$

Proof.

$$\begin{aligned} f_e(t) &= \sum_{p\in\mathcal{P}:e\in p} f_p(t) \text{ (from Def. 3.2)} \\ &= \sum_{p\in\mathcal{P}:e\in p} \sum_{J\in\mathcal{I}} \mathbb{1}_J(t) f_{J,p} \text{ (from Eq. (4))} \\ f_e(t) &= \sum_{J\in\mathcal{I}} \mathbb{1}_J(t) f_{J,e} \text{ (from Def. 3.24)} \end{aligned}$$

□

From above definition and definition of feasible flow, we get:

Proposition 3.26. *f is feasible iff:*

$$\forall r \in R, \sum_{J\in\mathcal{I}_r} \sum_{p\in P_r} |J| f_{J,p} = v_r \tag{7}$$

Proof. From Def. 3.3, f is feasible iff:

$$\begin{aligned} & \forall r, \int_{t_r^s}^{t_r^d} \sum_{p\in P_r} f_p(t)\mathrm{d}t = v_r \\ \Longleftrightarrow\ & \forall r, \int_{t_r^s}^{t_r^d} \sum_{p\in P_r} \sum_{J\in\mathcal{I}} \mathbb{1}_J(t) f_{J,p}\mathrm{d}t = v_r \text{ (from Remark 3.23)} \\ \Longleftrightarrow\ & \forall r, \sum_{p\in P_r} \sum_{J\in\mathcal{I}} \int_{t_r^s}^{t_r^d} \mathbb{1}_J(t) f_{J,p}\mathrm{d}t = v_r \\ \Longleftrightarrow\ & \forall r, \sum_{p\in P_r} \sum_{J\in\mathcal{I}_r} |J| f_{J,p} = v_r \text{ (from Remark 3.22)} \end{aligned}$$

□

3.5 Reduction of Cost Function, Equilibrium and Optimal

In this section, we prove that, under convex cost function, both continuous and discrete formulations are equivalent.

Proposition 3.27. *If cost function $c_{J,e}(b)$ is convex, for any f (supposed to be a vector of step functions constant on each J), for all $p \in \mathcal{P}$, and for any J in $\mathcal{I}$,*

$$c_{J,p}(f) = \sum_{e\in p} |J| c_{J,e}(f_{J,e}) \tag{8}$$

Proof. This comes from Def. (3.9) applied to step function f and interval J of $\mathcal{I}$ and $[f_{J,e}]$ defined using equations (6) and (5). □

Proposition 3.28. *If cost function $c_{J,e}(b)$ is convex, for any f (supposed to be a vector of step functions constant on each J), for all $p \in \mathcal{P}$,*

$$c_p(f) = \sum_{J\in\mathcal{I}} \sum_{e\in p} |J| c_{J,e}(f_{J,e}) \tag{9}$$

Proof. This is obtained by applying Eq. (1) to the step function f and $[f_{J,e}]$ defined using $[f_{J,p}]$ as given by Eq. (6) and (5). □

Proposition 3.29. *If cost function $c_{J,e}(b)$ is convex, for any f (supposed to be a vector of step functions constant on each J),*

$$C(f) = \sum_{J\in\mathcal{I}}\sum_{e}|J|f_{J,e}c_{J,e}(f_{J,e}) \tag{10}$$

Proof. Same as previous proof but using Eq. (3). □

Proposition 3.30. *f is an equilibrium flow in (G, R, c) iff:*

1. *f is a feasible allocation vector, and;*
2. *for every $r \in R$, for every $J, \tilde{J} \in \mathcal{I}_r$, $c_{J,p}(f) \le c_{\tilde{J},\tilde{p}}(f)$ with p, $\tilde{p}$ in P_r where $\sum_J f_{J,p} > 0$.*

Proof. We prove the equivalence in two parts:

- ⇒ f is supposed to be made of step functions with steps on $\mathcal{I}$. First point is straightforward. By replacing (7) in second point of Def. 3.11 and applying it on J and $\tilde{J}$ in $\mathcal{I}_r$ we obtain above mentioned conditions.
- ⇐ First point is again straightforward. Regarding second point, condition 2) in Def. 3.11 is different is the sense that it is for any pair of sub-interval of $[t_r^s, t_r^d]$. For any such sub-interval and by linearity of the integration over time, cost can be expressed as a sum of cost $c_{J,p}$ with J in $\mathcal{I}_r$ or part of a such interval. From this, we get two partitions of J and $\tilde{J}$. They can be different but we can obtain a one to one mapping of sub-intervals of same length from J to $\tilde{J}$. On each of these pairs of sub-intervals, costs are independent of time and f_P is constant. Applying condition 2) of Prop. 3.30 on the underlying intervals in $\mathcal{I}_r$ and scaling the results with respect to the length of the sub-interval, once summed, proves condition 2) of Def. 3.11.

□

We summarize this section by the following proposition:

Proposition 3.31. *f is an optimal allocation iff $[f_{J,p}]$ is feasible and $C(f)$ as given in (10) is minimized.*

Proof. Trivial. □

With this, we have shown the equivalence of the continuous and discrete versions of the routing game under convex cost functions. Next, we move on to the time-expanded network.

4 Reduction to Non-atomic Routing Game

Since we have defined the RGoT and exhibited the step function structure under convex costs, we can now reduce the game to non-atomic routing game, and then exploit the results already known on this class of games.

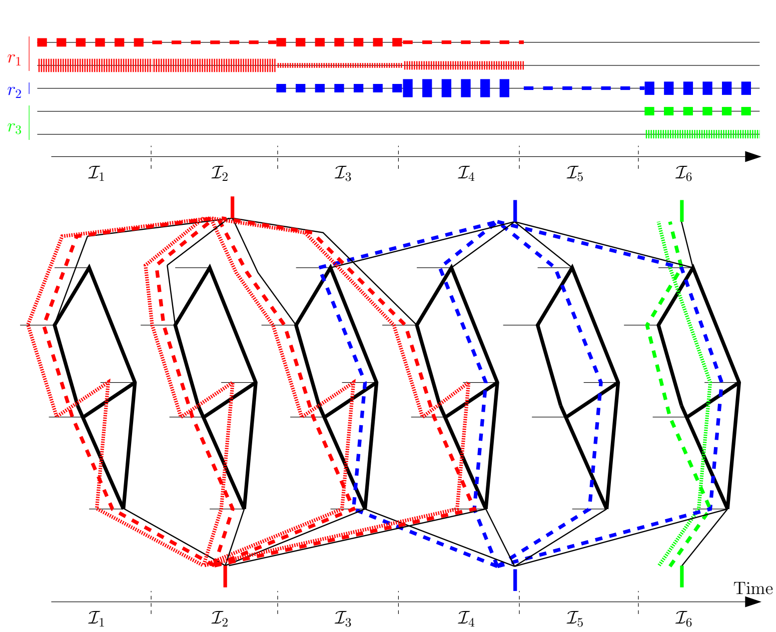


Figure 2: Time-expanded network.

4.1 Time-expanded Routing Game

To use the results from the classical non-atomic routing game, which we refer as *Routing Game in Time-Expanded Network*, we now present that the two games can be mapped, and we show how to do so.

Definition 4.1 (Time-expanded (TE) network)**.** *For a RGoT* (G, R, c)*,* $\bar{G}$ *is the time-expanded graph obtained from* G *by duplicating it one for each interval of* $\mathcal{I}$ *and adding one pair of virtual source and virtual sink for each request* r *in* R*. These extra sources/sinks are connected to the actual source/sink of each duplicate of* G *in time intervals of* $\mathcal{I}_r$*. We note* $\bar{e} = (J, e)$*.*

All edges of $\bar{G}$, except those connecting the actual and virtual sources/sinks, are of the form $\bar{e} = (J, e)$. $\bar{G}$ has $|V||\mathcal{I}| + 2|R|$ vertices and $|E||\mathcal{I}| + 2\sum_{r\in R}|\mathcal{I}_r|$ edges.

Definition 4.2 (Path in TE network)**.** *For every request* r *in* R*, we define* $\bar{P}_r$ *as the set of paths in* $\bar{G}$*: for every path* p *in* P_r *and every interval* J *in* $\mathcal{I}_r$*,* $\bar{P}_r$ *contains a path* $\bar{p} = (J, p)$ *which contains the same edges in* $\bar{G}$ *as* p *in* G *in addition to the two extra edges that connect the virtual source and virtual sink of* r *to the nodes in* $\bar{G}$*, which represents, for each interval their actual source/destination in* G*. We let,* $\bar{\mathcal{P}} = \bigcup_{r\in R}\bar{P}_r$

Observe that $\bar{P}_r$ has $|P_r||\mathcal{I}_r|$ paths.

To illustrate, the time-expanded graph of requests presented in Fig. 1 is as in Fig. 2. The network of Fig. 1(b) can be seen replicated in Fig. 2 for the intervals of a request. If two requests (e.g., r_1 and r_2) have overlapping interval ($\mathcal{I}_3$), the graph is shared by paths of the same requests ($\bar{P}_1$ and $\bar{P}_2$). Virtual source/sink nodes connect paths of one request.

Proposition 4.3. *$\bar{e} = (J_e, e) \in \bar{p} = (J_p, p)$ iff*

$$\begin{cases} e \in p \\ J_e = J_p \end{cases}$$

Proof. By construction of $\bar{G}$ in Def. 4.1. □

The edges of $\bar{G}$ that don't have their counterparts in G are only used by one request and have a null cost in the time-expanded routing games. Hence, they won't contribute to the social cost.

We now define the requests that are used in the new graph: $\bar{R}$, the allocations and feasible allocations in these games.

Definition 4.4 (Requests in the TE network)**.** *For every request r in R, we define: $\bar{r} \triangleq (\bar{s}_r, \bar{d}_r, \bar{P}_r, \bar{v}_r \triangleq \frac{v_r}{|J|})$ and the set $\bar{R}$ as the set of requests $\bar{r}$.*

Definition 4.5 (Allocations in the TE network)**.** *Allocations in the TE network: $\bar{f} \triangleq [f_{\bar{p}}] \triangleq [f_{J,p}]$*

Link allocation: $f_{\bar{e}} \triangleq \sum\limits_{\bar{p} \in \bar{\mathcal{P}} : \bar{e} \in \bar{p}} f_{\bar{p}}$

Remark 4.6. *For each edge of $\bar{G}$ which doesn't have a $\bar{e} = (J, e)$ form, there is no need to define $f_{\bar{e}}$. In the remaining, we use the notation $\bar{e} = (J, e)$ to refer to edges that contribute to the costs and define the counterpart in G and time intervals.*

Proposition 4.7. *For any $\bar{e} = (J_e, e)$, $f_{\bar{e}} = f_{J_e, e}$.*

Proof. By Def. 4.5, $f_{\bar{e}} = \sum\limits_{\bar{p} \in \bar{\mathcal{P}} : \bar{e} \in \bar{p}} f_{\bar{p}}$ and by Def. 4.5, $f_{\bar{p}} = f_{J,p}$. Using Prop. 4.3 and definition of $\bar{p}$ and $\bar{e}$, we get: for any $\bar{e} = (J_e, e)$,

$$\begin{aligned} f_{\bar{e}} &= \sum_{(J,p) \in \mathcal{I} \times \mathcal{P} : e \in p \wedge J_e = J} f_{J,p} \\ &= \sum_{p \in \mathcal{P} : e \in p} f_{J_e, p} \\ f_{\bar{e}} &= f_{J_e, e} \end{aligned}$$

where $f_{J_e, e}$ is the one defined in Def. 3.24. □

Remark 4.8. *Using Prop. 4.7 and Prop. 3.25, we get:*

$$f_e(t) = \sum_{\bar{e} = (J,e) \in \mathcal{I} \times \{e\}} \nVdash_J(t) f_{\bar{e}}$$

Using Remark 3.23 and Def. 4.5, we also have:

$$f_p(t) = \sum_{\bar{p} = (J,p) \in \mathcal{I} \times \{p\}} \nVdash_J(t) f_{\bar{p}}$$

Flow allocation f can thus be reconstructed.

Definition 4.9 (Feasible allocation)**.** *$\bar{f}$ is a feasible allocation for $\bar{R}$ iff:*

$$\forall r \in \bar{R}, \sum_{\bar{p} \in \bar{P}_r} f_{\bar{p}} = \bar{v}_r$$

This is the definition of feasibility in normal routing game.

Proposition 4.10. *f is feasible iff $\bar{f}$ is feasible.*

Proof.

$$\begin{aligned}
& f \text{ is feasible} \\
\iff & \forall r \in R, \sum_{J \in \mathcal{I}_r} \sum_{p \in P_r} |J| f_{J,p} = v_r \text{ (from Prop. 3.26)} \\
\iff & \forall r \in R, \sum_{(J,p) \in \mathcal{I}_r \times P_r} f_{J,p} = \frac{v_r}{|J|} \text{ (as } |J| \text{ is uniform)} \\
\iff & \forall r \in \bar{R}, \sum_{\bar{p} \in \bar{P}_r} f_{\bar{p}} = \bar{v}_r \text{ (Def. of } \bar{G}, \bar{R} \text{ and } f_{\bar{p}}) \\
\iff & \bar{f} \text{ is feasible}
\end{aligned}$$

□

In the following two sections, we map the cost functions, equilibrium and optimal allocation from RGoT to routing games on the TE graph defined in this section.

4.2 Cost Function in TE Network

Here we show the mapping of cost between RGoT and time-expanded routing games.

Definition 4.11 (Costs)**.** *For all $\bar{e} = (J, e)$, $c_{\bar{e}}(.) \triangleq |J| c_{J,e}(.)$. We call $\bar{c}$ the function that associates $c_{\bar{e}}(.)$ to $\bar{e}$. For any $\bar{f}$ and for all $\bar{p}$, $c_{\bar{p}}(\bar{f}) \triangleq \sum_{\bar{e} \in \bar{p}} c_{\bar{e}}(f_{\bar{e}})$.*

Proposition 4.12. *For all p in $\mathcal{P}$,*

$$c_p(f) = \sum_{\bar{p} \in \mathcal{I} \times \{p\}} c_{\bar{p}}(\bar{f})$$

$f_p(t)$ is still supposed to be null out of I_r for each $p \in P_r$.

Proof.

$$\begin{aligned}
c_p(f) &= \sum_{J \in \mathcal{I}} \sum_{e \in p} |J| c_{J,e}(f_{J,e}) \text{ (from Prop. 3.28)} \\
&= \sum_{J \in \mathcal{I}} \sum_{\bar{e} \in \bar{p}} c_{\bar{e}}(f_{\bar{e}}) \text{ (from Def. 4.5, 4.11 and Prop. 4.3)} \\
c_p(f) &= \sum_{\bar{p} \in \mathcal{I} \times \{p\}} c_{\bar{p}}(\bar{f}) \text{ (from Def. 4.11)}
\end{aligned}$$

□

Proposition 4.13. *For all p in $\mathcal{P}$ and J in $\mathcal{I}$, $c_{\bar{p}}(\bar{f}) = c_{J,p}(f)$ with $\bar{p} = (J, p)$.*

Proof.

$$\begin{aligned} c_{\bar{p}}(\bar{f}) &= \sum_{\bar{e} \in \bar{p}} c_{\bar{e}}(f_{\bar{e}}) \text{ (from Def. 4.11)} \\ &= \sum_{e \in p} |J| c_{J,e}(f_{J,e}) \text{ (from Def. 4.5, 4.11 and Prop. 4.3)} \\ c_{\bar{p}}(\bar{f}) &= c_{J,p}(f) \text{ (from Prop. 3.27)} \end{aligned}$$

□

Definition 4.14 (Social cost)**.** $\bar{C}(\bar{f}) = \sum_{\bar{e}} f_{\bar{e}} c_{\bar{e}}(f_{\bar{e}})$

Proposition 4.15. $\bar{C}(\bar{f}) = C(f)$

Proof.

$$\begin{aligned} C(f) &= \sum_{J \in \mathcal{I}} \sum_{e} |J| f_{J,e} c_{J,e}(f_{J,e}) \text{ (by Prop. 3.29)} \\ &= \sum_{\bar{e}} f_{\bar{e}} c_{\bar{e}}(f_{\bar{e}}) \text{ (by Prop. 4.7 and Def. 4.11)} \\ C(f) &= \bar{C}(\bar{f}) \text{ (by Def. 4.14)} \end{aligned}$$

Transition between first and second line is obtained by recalling that per-edge cost over edges that are not of the form $\bar{e} = (J, e)$ is null. □

4.3 Equilibrium and Optimal Allocations

The two definitions given here are from non-atomic routing games. The two propositions basically state that our RGoT can be *solved* by recasting it to this well-known game and using established results, as will be shown in next section.

Definition 4.16. *$\bar{f}$ is an equilibrium flow in $(\bar{G}, \bar{R}, \bar{c})$ if:*

1. *$\bar{f}$ is a feasible allocation vector, and;*
2. *for every $r \in \bar{R}$, $c_{\bar{p}}(\bar{f}) \leq c_{\tilde{\bar{p}}}(\bar{f})$ with $\bar{p}$, $\tilde{\bar{p}}$ in $\bar{P}_r$ where $f_{\bar{p}} > 0$.*

Proposition 4.17. *$\bar{f}$ is an equilibrium flow in $(\bar{G}, \bar{R}, \bar{c})$ iff f is an equilibrium flow in (G, R, c).*

Proof. First point comes from Prop. 4.10. Second point is obtained by replacing $c_{J,p}$ in Prop. 3.30 as established in Prop. 4.13. □

Definition 4.18. *A feasible flow of $(\bar{G}, \bar{R}, \bar{c})$ is optimal if it minimizes $\bar{C}(\bar{f})$ over other feasible flows.*

Proposition 4.19. *A feasible flow $\bar{f}$ of $(\bar{G}, \bar{R}, \bar{c})$ is optimal iff f is optimal in (G, R, c).*

Proof. The proof comes from the equivalence established in propositions 4.10 and 4.15. □

4.4 Results — Existence and PoA

Next theorem on the existence of equilibrium flows for non-atomic routing games comes from the literature [5].

Theorem 4.20 ([5, Theorem 18.8])**.** *Under the assumption of existence of feasible allocation(s), for a non-atomic instance $(\bar{G}, \bar{R}, \bar{c})$ there exists at least one equilibrium, and for any two equilibria $\bar{f}$ and $\tilde{\bar{f}}$, $\bar{c}_{\bar{e}}(\bar{f}_e) = \bar{c}_{\bar{e}}(\tilde{\bar{f}}_e)$.*

From Theorem 4.20, it follows that, $\bar{C}(\bar{f}) = \bar{C}(\tilde{\bar{f}})$; i.e., all equilibrium have the same social cost. We also derive from the theorem that our routing game over time have equilibria since the time-expansion of the problem always exists and this always has at least one equilibrium.

Corollary 4.21. *From Theorem 4.20 and reduction of the game made in previous sections (game, costs, equilibrium and optimal), it follows that (G, R, c) has at least one equilibrium, and they all have the same social cost.*

Existence of optimal allocation is trivial, as there is always some (possibly one) feasible allocation.

Definition 4.22 (PoA)**.** *Price of Anarchy (PoA) is the ratio of the cost of worst equilibrium to the cost of optimal allocation.*

Since all equilibria in routing game have the same cost, PoA is the ratio of the cost of any equilibrium to the optimal social cost. Furthermore, since cost and social cost are same for RGoT and its TE counterpart, PoA of $(\bar{G}, \bar{R}, \bar{c})$ is equal to the PoA of the corresponding (G, R, c).

5 Application to Scheduling

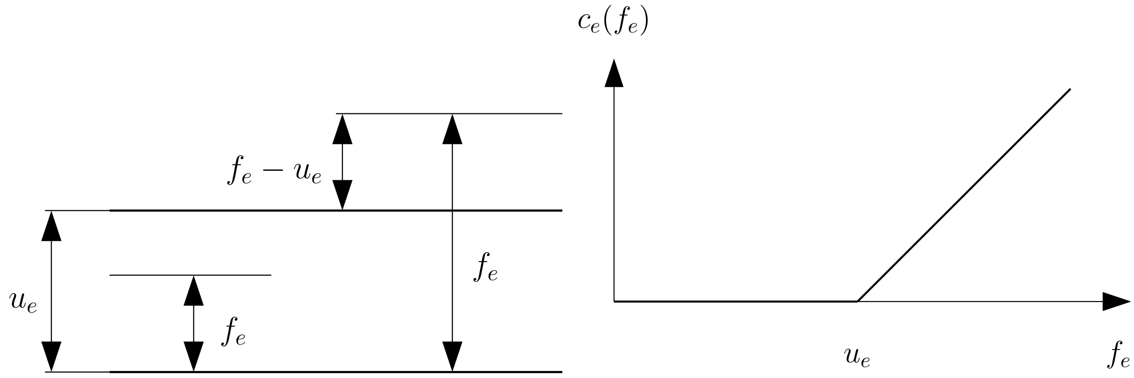
In this section, we illustrate the mapping of PoA between the two forms of game, and its implications, for example, on the benefit of doing centralized scheduling to transfer data. We consider a simple example of a single link network with time-slot preference and congestion. First, we describe the cost functions associated with this model, and then discuss equilibrium flows, optimal allocations and PoA.

5.1 Lateness and Time-slot Preference as Cost

In this model, f_p (and thus f_e) is the expected rate that players plan to have. As long as the link is not oversubscribed, they won't be late with respect to their deadlines. If it is congested, lack of bandwidth increases lateness — cost — as a linear function of this over-subscription. Figures 3(b) and 3(a) shows the cost model for congestion. Another component of this cost represents users' preference of one time-slot over the other.

We illustrate this cost on the following example. This RGoT uses one single link and one request of total volume v, but the time interval $[0, T]$, where $0 < T$, is divided in two equal parts and is assigned user preferences: d_1 and d_2 with $0 \leq d_1 < d_2$. The cost function adopted for this link of capacity u is then:

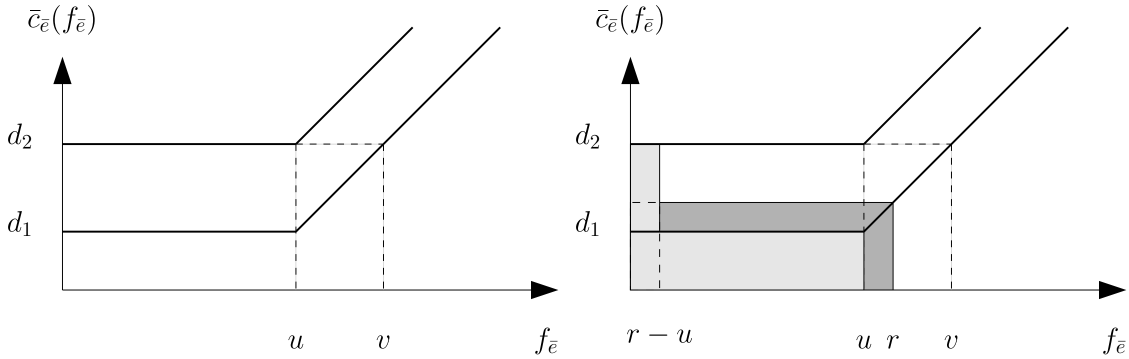
$$c(e, f_e, t) = \begin{cases} d_1 + c_e(f_e) & \text{if } t < T/2 \\ d_2 + c_e(f_e) & \text{else} \end{cases}$$



(a) Cost model for congestion on a link of capacity u_e.

(b) Cost of congestion $c_e(f_e)$ for edge e.

Figure 3: Cost model for congestion.



(a) Cost functions.

(b) Two feasible flows.

Figure 4: Time preference and congestion aversion.

with:

$$c_e(f_e) = \begin{cases} 0 & \text{if } f_e \in [0,u] \\ \frac{f_e - u}{u_e.\alpha(e,R)} & \text{else} \end{cases}$$

Let $s = \frac{1}{u_e.\alpha(e,R)}$, $s > 0$. For each time interval, the cost is convex, positive and non-decreasing in link allocation.

Using time-expansion as defined earlier, we get the routing game with two links (1 and 2) sharing same source and destination; first of these links has cost $\bar{c}_1$, and second $\bar{c}_2$ as defined presently:

$$\bar{c}_{\bar{e}}(f_{\bar{e}}) = \begin{cases} d_{\bar{e}} & \text{if } f_{\bar{e}} \in [0,u] \\ d_{\bar{e}} + s(f_{\bar{e}} - u) & \text{else} \end{cases}$$

with $f_e(t) = \mathbb{1}_{t\in[0,T/2)} f_{\bar{1}} + \mathbb{1}_{t\in[T/2,T]} f_{\bar{2}}$ in RGoT. Fig. 4(a) show the cost functions on the time-expanded network. It includes time preference and congestion aversion for the two links of $\bar{G}$. Note that, the slope s equals $\frac{d_2-d_1}{v-u}$.

Let $r = v/T$. If the cheapest link is not congested, i.e., $r \leq u$, as its cost is less than that of the other link, both equilibrium flow and optimal allocation exclusively use this path/link. This results in a PoA of 1 as social cost of both are the same. But when the cheaper link starts to get congested, it might remain less expensive than the second link up to a point. Beyond this point, it is less expensive from a social cost point of view to route the flow on the two links. But the equilibrium flow still uses only one link as its cost $c_p(.)$ remains lesser than the other. Fig. 4(b) illustrates this situation with two alternative feasible flow and their social cost as shaded. The sum of the areas of the two light gray rectangle are less than the one of the dark rectangle (which is the social cost of equilibrium). More formally:

Proposition 5.1. *For* $\max\{u, v-u\} < r$ *and* $r < \min\{v, 2u\}$*, equilibrium flow uses exclusively link 1 and PoA* > 1.

Proof. Provided r is between $\max\{u, v-u\}$ and $\min\{v, 2u\}$, equilibrium flows only use link 1 and the social cost of such an allocation is:

$$C_{eq} = (d_1 + (r-u)s)r$$

If we consider a different feasible allocation which allocates only u on first link and the remaining on second link, its social cost is:

$$C = u\ d_1 + (r-u)d_2$$

We now demonstrate that this allocation has a social cost strictly less than the equilibrium flow:

$$\begin{aligned}
&C < C_{eq} \\
&\iff u\ d_1 + (r-u)\ d_2 < (d_1 + (r-u)s)r \\
&\iff (r-u)\ d_2 < (r-u)\ d_1 + (r-u)s\ r \\
&\iff \frac{d_2 - d_1}{s} < r \\
&\iff v - u < r
\end{aligned}$$

which holds by assumption.

As the cost of this feasible allocation is strictly less than the one of equilibrium flow, PoA is strictly more than 1. □

5.2 Equilibrium and Optimal Allocation

Define $x_1(r)$ and $x_2(r)$ as the part of the traffic r sent on links 1 and 2 for equilibrium flow. Similarly, $x_1^*(r)$ and $x_2^*(r)$ are the fraction for optimal allocation. We have $x_1(r) + x_2(r) = r$ and $x_1^*(r) + x_2^*(r) = r$. $C_{eq}(r)$ denotes the social cost of equilibrium under demand r and $C^*(r)$ the social cost of optimal allocation.

Optimal allocations Since costs are defined in pieces, social cost are also piecewise. In the following, x denotes x_1 and x_2 is $r - x$.

It follows that the social cost of a feasible allocation $C(x, r) = x\ c_1(x) + (r-x)c_2(r-x)$ has the following expression in the different regions (a),(b),(c) and (d) of Fig. 5:

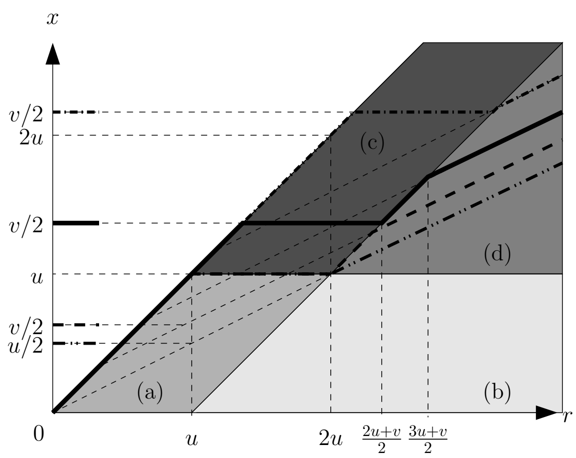


Figure 5: Domain of feasible allocations $x(r)$ and optimal allocations for different values of $v/2$.

(a) $\underline{x < u \text{ and } r - x < u}$: $C(x,r) = x\ d_1 + (r-x)d_2$ and $\frac{\partial C}{\partial x} = d_1 - d_2 < 0$

(b) $\underline{x < u \text{ and } r - x > u}$: $C(x,r) = x\ d_1 + (r-x)(d_2 + s(r-x-u))$ and $\frac{\partial C}{\partial x} = d_1 - d_2 - (2r+u-x)s < 0$

(c) $\underline{x > u \text{ and } r - x < u}$: $C(x,r) = x(d_1+s(x-u))+(r-x)d_2$ and $\frac{\partial C}{\partial x} = d_1-d_2+(2x-u)s$, which is strictly positive when $x > v/2$, and strictly negative when $x < v/2$.

(d) $\underline{x > u \text{ and } r - x > u}$: $C(x,r) = x(d_1 + s(x-u)) + (r-x)(d_2 + s(r-x-u))$ and $\frac{\partial C}{\partial x} = d_1 - d_2 + 2(2x-r)s$, which is strictly negative below $x = (r + (v-u)/2))/2$ and strictly positive above.

As cost functions are continuous, so are optimal and equilibrium allocations as functions of r.

We conclude from previous list that optimal allocations are:

- $\underline{r \leq u}$: $x_1^*(r) = r$ and $x_2^*(r) = 0$
- $\underline{u \leq r \leq 2u}$: $x_1^*(r) = \max\{\min\{r, v/2\}, u\}$ and $x_2^*(r) = r - \max\{\min\{r, v/2\}, u\}$
- $\underline{2u \leq r}$: $x_1^*(r) = \max\{\min\{r,(r+(v-u)/2))/2\}, u\}$ and $x_2^*(r) = r - \max\{\min\{r,(r+(v-u)/2))/2\}\}$

The optimal allocations as a function of r are shown on Fig. 5 by the emphasized lines (plain and dashed) for different values of $v/2$.

Equilibrium flows Regarding equilibrium flows:

- $\underline{r < v}$: Since $c_1(r) < c_2(r)$, $x_1(r) = r$ and $x_2(r) = 0$.
- $\underline{v \leq r < u + v}$: $x_1(r) = v$ and $x_2(r) = r - v$ is the equilibrium flow.
- $\underline{u + v \leq r}$: $x_1(r) = (v-u)/2 + r/2$ and $x_2(r) = (u-v)/2 + r/2$ is the equilibrium flow.

PoA For the specific case where $u < v/2 < 2u$, we have:

- $r \leq v/2$: since $x_1^*(r) = r$, $PoA = 1$.
- $v/2 \leq r \leq u + v/2$: in this region, $x_1^*(r) = v/2$ and $C^* = (d_1 - d_2 + (v/2 - u)s)\ v/2 + r\ d_2$ while $x_1(r) = r$ and $C_{eq} = r\ (d_1 + (r-u)s)$.

$$\begin{aligned} PoA = \frac{C_{eq}}{C^*} &= \frac{r\ (d_1 + (r-u)s)}{(d_1 - d_2 + (v/2 - u)s)v/2 + r\ d_2} \\ &= \frac{r\ (d_1 + (r-u)s)}{r\ d_2 - (v/2)^2 s} \end{aligned}$$

PoA is increasing as long as $s < (2r+v)d/v^2$ since r is less than v if s is less than $2(u+v)d/v^2$ maximum PoA is reached for $r = u + v/2$ and is:

$$PoA = \frac{(2u+v)(-2d + s\ v - 2s\ u)}{-4d\ u - 2d\ v + s\ v^2}$$

In the other case, PoA will first increase and then decrease before leaving the interval. Maximum PoA is reached for $r = \frac{s\ v^2/d - v}{2}$ and its value is:

$$PoA = \frac{(s\ v - d)^2}{d^2}$$

- $u + v/2 < r$: Since $v/2 > u$, $r > 2u$ and then the system is clearly overloaded (both links congested) which is not really interesting in this scenario.

Conclusion In this example, we see that as long as none of the links are congested, PoA remains 1. This basically means that selfish behavior leads to optimal allocation. Therefore, under the presented model no centralized control is needed to achieve optimal allocation. This model, following Wardrop equilibrium concept, assumes that users have complete and up-to-date information to decide which path to use based on the cost. The cost depends on the link allocation reached so far. Thus, it assumes that users know the current link allocation. This information is not needed for the time-slot preference part of the cost, but for the congestion aversion part. Hence, implementing this scheduling mechanism without coordinator mandates the congestion information be available to users, that too, in advance. In both cases coordination is essential: either to take the decision, or to publish the information.

In the case where the system is near congestion, as studied above, PoA is greater than 1. Here, coordination can improve the social cost by sacrificing some of the flow. This can not be achieved by selfish users unwilling to spontaneously take a more costly path.

From [8], we know that using affine cost functions it is possible to construct instance of the non-atomic routing game with PoA up to 4/3. This is value is attained in two-link settings. When the class of function quadratic functions, this bound becomes $3\sqrt{3}/(3\sqrt{3}-2)$. Actually, the more non-linear the cost functions are, the higher is the worst PoA. Using polynomial cost functions of degree p, PoA tends to infinity as p does. In the next section, we see how this inefficiency can be mitigated through coordination or modification of the cost function.

5.3 Reducing Social Cost

Since PoA can be large, it is worth using optimal allocation to minimize social cost. Minimization of social cost can be achieved by a centralized system using optimal allocation or the game can be modified so that equilibrium of the new game is an optimal allocation for the original game. This corresponds to a reduction of the PoA in the new instance.

Cost Sharing and Resource Coordination Consider a resources manager (RM) who proposes some commodities for a cost which depends on the total demand. Users know only about their own individual decisions. The cost function is likely to be convex, positive and non-decreasing as it ultimately results from the cost of congestion. A resource coordinator (RC) aggregating the requests and demanding the optimal allocation for the aggregate can do better than selfish user behaviors. It follows that RC can improve the social cost, i.e., total cost. RC can then share the benefit among all entities: RC and clients. This makes this configuration profitable for both clients and RC.

Pigouvian Tax Another alternative is to modify the game so that cost perceived by users deter them from ultimately reaching an allocation other than the optimal of original game. For this purpose the related theorem is:

Theorem 5.2 (Theorem 18.27 of [5]). *Let $\bar{f}^*$ be an optimal flow for $(\bar{G}, \bar{R}, \bar{c})$ and let $\tau_{\bar{e}} = \bar{f}^*_{\bar{e}}\ \bar{c}'_{\bar{e}}(\bar{f}^*_{\bar{e}})$ denote the marginal cost tax for edge $\bar{e}$ with respect to $\bar{f}^*$. Then $\bar{f}^*$ is an equilibrium flow for $(\bar{G}, \bar{R}, \bar{c} + \tau)$*

It basically says that, equilibrium of the modified game is optimal allocation of the original game. From the realization point of view, this solution still has the problem of sharing the information, and also in addition making the users value the modified cost.

6 Concluding Remarks

This paper presented a new model for RGoT. It shows that under convex cost this can be solved using the results from non-atomic routing games. Based on an example, we have seen, selfish user behavior increases the social cost. Just because they prefer one time interval to the other, users selfishly raise the cost of cheapest links instead of sacrificing to keep it and social cost low. This motivates the need to coordinate. Possible extension of this work is to use atomic or splittable flow model instead of non-atomic model in order to consider the perception users have of their own impact.

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