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# Sparse time-frequency distributions of chirps from a compressed sensing perspective

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## Abstract

Considering that multicomponent chirp signals are sparse in the time-frequency domain, it is possible to attach to them highly localized distributions thanks to a compressed sensing approach based on very few measurements in the ambiguity plane. The principle of the technique is described, with emphasis on the choice of the measurement subset for which an optimality criterion is proposed.

## 1 Introduction

In the case of a multicomponent chirp signal of the form

$$x(t) = \sum_{k=1}^K a_k(t) e^{i\varphi_k(t)}, \quad (1)$$

an idealized representation amounts to distributing its energy along trajectories of the time-frequency (TF) plane according to

$$\rho(t, f) = \sum_{k=1}^K a_k^2(t) \delta(f - \dot{\varphi}_k(t)/2\pi). \quad (2)$$

If the number of components of  $x(t)$  happens to be small, i.e., if — in a discrete-time setting —  $K \ll M$ , where  $M$  is the number of frequency bins, this corresponds to a representation that is intrinsically sparse since, for a signal of length  $N$ , the TF matrix of total cardinality  $|\rho| = NM$  has at most  $KN \ll |\rho|$  non-zero entries.

Some recent work [1] took advantage of this observation and proposed a new way of getting a sparse and sharply localized TF distribution of chirps on the basis of “compressed sensing” ideas [2]. The purpose of the present contribution is to go beyond the preliminary results reported in [1] and, in particular, to better address key questions related to a meaningful selection of the measurement subset on which the approach relies.

## 2 Rationale

It is well-known [3] that classical (quadratic) TF analysis offers a possibility of perfect localization on specific types of monocomponent chirps, while facing a trade-off between such a localization and the creation of spurious cross-terms in the case of multicomponent signals. Thorough analyses of this problem have been conducted (see, e.g., [4, 5, 6]), whose main result is that the signature of auto-terms of any TF distribution is mainly concentrated in the vicinity of the origin of the ambiguity plane, suggesting to build localized distributions on such contributions only so as to reduce the importance of cross-terms. From a standard perspective (i.e., Fourier inversion with no constraint), this however leads to a severe trade-off, the localization in the TF domain becoming all the more poor as the selected domain is shrunk in the ambiguity plane.

Assuming for the target TF distribution a model such as in eq. (2), a way out is however possible. Indeed, as illustrated in Fig. 1, the rationale is to exploit sparsity in the TF plane by making very few (incoherent) measurements in the ambiguity domain so as to get rid of most cross-terms, while

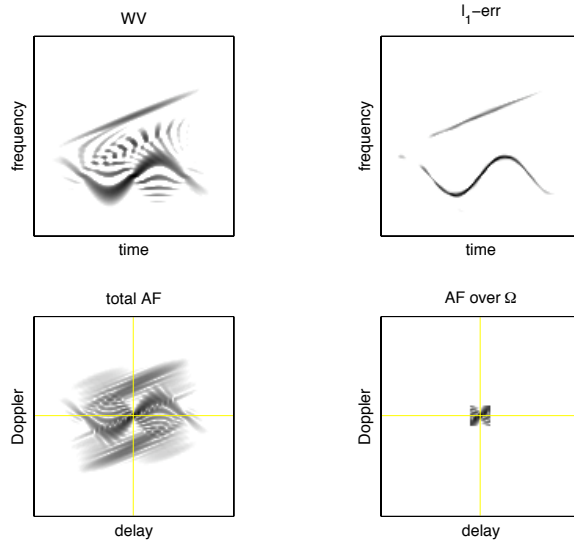


Figure 1: *Example with a 128 points 2-component signal* — Top left: Wigner-Ville Distribution (WVD) of size  $128 \times 128$ . Bottom left: total ambiguity function (AF) of size  $128 \times 128$ , defined as the 2D Fourier transform of the WVD. Bottom right: restriction of the AF to a domain  $\Omega$  consisting of  $13 \times 13$  samples neighbouring the origin of the plane. Top right: minimum  $\ell_1$ -norm TF solution whose 2D Fourier transform coincides with the AF over  $\Omega$ , up to a  $\ell_2$ -norm approximation with  $\epsilon = 0.05 \|x\|_2$ .

constraining the inversion so as to guarantee that only few coefficients are significantly non-zero. From such a “compressed sensing” perspective (see, e.g., [2]), a better TF localization is thus expected to be obtained via the solving of the linear program

$$\left\{ \min_{\rho} \|\rho\|_1 ; \|\mathcal{F}\{\rho\} - A_x\|_2 \leq \epsilon \mid_{(\xi, \tau) \in \Omega} \right\}, \quad (3)$$

where  $\rho$  stands for the target distribution,  $\mathcal{F}$  for the 2D Fourier transform operator,  $A_x$  for the ambiguity function — classically computed as the 2D Fourier transform of the Wigner-Ville Distribution (WVD) [3] —,  $\epsilon$  for a slack variable and  $\Omega$  for some domain neighbouring the origin of the ambiguity plane. As discussed in [1], introducing in eq. (3) an approximate satisfaction of the constraint (in a combined  $\ell_1 - \ell_2$  sense) guarantees a smoother solution as compared to the oversparse, discontinuous solution that would be obtained with a strict equality ( $\epsilon = 0$ ).

### 3 Choosing the measurement subset

Fig. 1 evidences a dramatic improvement when switching from the WVD to the  $\ell_1$ -norm solution of the program (3), in both terms of localization and cross-terms reduction<sup>1</sup>. However, such a result relies heavily on the choice of the measurement subset  $\Omega$ , raising the issue of how to choose it in an appropriate way.

#### 3.1 Oracle

A qualitative appreciation can first be gained from a comparison with an “oracle” based on the assumed knowledge of the ideal distribution. In this case, the linear program (3) is run exactly as in Fig. 1, while replacing *mutatis mutandis* the WVD by the TF model (2). This is illustrated in Fig. 2, where an optimum cardinality  $|\Omega|_*$  is shown to exist, at least qualitatively, for trading off auto-terms localization and cross-terms reduction.

At this point, it might be worth recalling that the approach described here differs from a *reconstruction* problem in the sense that there would be no point in exactly recovering the whole WVD from a limited set

<sup>1</sup>All the computations have been made in MATLAB, with the TIME-FREQUENCY TOOLBOX (<http://tftb.nongnu.org>) for the TF computations and the  $\ell_1$ -MAGIC TOOLBOX (<http://www.11-magic.org>) for the optimization.

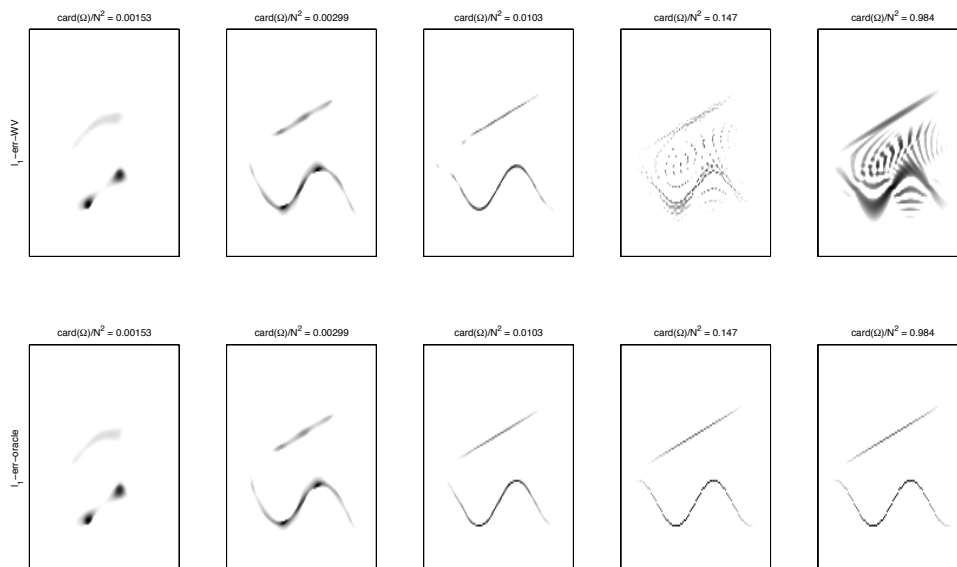


Figure 2: *Qualitative selection of the optimal cardinality  $|\Omega|$*  — Top row: TF solutions obtained from the WVD as in Fig. 1, for increasing values of  $|\Omega|$ . The evolution shows that a too small cardinality does not permit to gain enough information about auto-terms whereas a too large one leads to take into account more and more undesired cross-terms, converging eventually to the actual WVD when the support of  $\Omega$  identifies to the whole AF plane. Bottom row: companion “oracle” solutions obtained the same way, but based on the ideal distribution in place of the WVD. Comparison of both evolutions evidences that an optimum cardinality  $|\Omega|_*$  exists for trading off auto-terms localization and cross-terms reduction (about 1% of the AF support in the present case).

of measurements (in fact, such a perfect reconstruction is of course obtained when the entire ambiguity plane is chosen as the measurement domain, see Fig. 2). The situation is much more that of a *construction* problem in which is created some idealized object which does not exist *per se* prior optimization.

### 3.2 Optimum cardinality

Fig. 2 suggests that some “optimum” cardinality exists for trading-off localization and cross-terms, thus calling for some automatic selection. Interpreting the way the  $\ell_1$ -solutions (WVD and oracle) evolve as a function of  $|\Omega|$ , one can say that essentially two regimes are observed. For small cardinalities, both behaviors happen to be quite similar, focusing primarily on the auto-terms of interest: in this respect, a measure of concentration such as the Shannon entropy can be a good indicator of performance. In a second stage, enlarging  $|\Omega|$  is of little interest for auto-terms while letting cross-terms appear: in this regime, the entropy criterion can then be misleading since it may decrease not because of better localized auto-terms but rather of “spiky” structures due to the oscillatory nature of cross-terms.

In order to circumvent this drawback, it is therefore proposed to combine entropy (aimed at quantifying localization) with Total Variation (aimed at penalizing spiky structures) according to:

$$C = - \iint_{-\infty}^{+\infty} |\rho|_n(t, f) \log_2 |\rho|_n(t, f) dt df + \lambda \iint_{-\infty}^{+\infty} \sqrt{\left| \frac{\partial \rho_n}{\partial t}(t, f) \right|^2 + \left| \frac{\partial \rho_n}{\partial f}(t, f) \right|^2} dt df, \quad (4)$$

where the subscript  $n$  stands for a normalization to unity of the considered TF distribution with respect to its total volume, and  $\lambda$  for a trade-off parameter between entropy and total variation. The result is reported in Fig. 3 with  $\lambda = 1$ , evidencing that the “optimum” cardinality  $|\Omega|_* = \arg \min_{|\Omega|} C$  obtained this way is in good agreement with the one suggested by the oracle approach in Fig. 2.

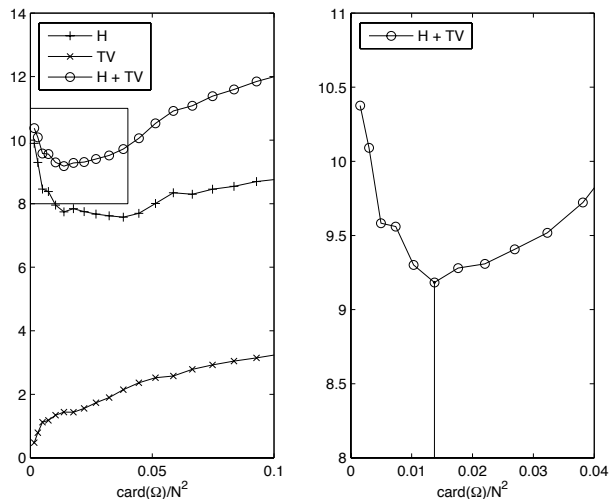


Figure 3: *Quantitative selection of the optimal cardinality  $|\Omega|$*  — Whereas the Shannon entropy (H) can measure TF localization, the Total Variation (TV) is proposed to be used as a penalty function for spiky solutions. A combination of both criteria (left panel, with an enlargement of the box in the right panel) turns out to attain its minimum value for a cardinality of the AF support that is in good agreement with the “optimum” cardinality  $|\Omega|_*$  suggested by the oracle approach in Fig. 2.

### 3.3 On geometric adaptivity

It has been stressed that the selection of the Fourier samples in the AF domain deserved special attention in terms of area, but the question of its shape could be addressed as well. At first sight, it might be expected that the use of adapted kernels (as proposed, e.g., in [7]) would prove useful. This however seems not to be the case, which does not necessarily come as a surprise. Indeed, besides an assumed *sparsity* of the solution, the other ingredient for a successful CS-based approach is that of its *incoherence* with the measurements on which it is based. Operating in the Fourier domain clearly goes this way, but *adapting* the selected subset to the signal structure goes the opposite.

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