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Abstract

We describe a method to derive, from first principles, the long-distance asymptotic behavior of correlation functions of integrable models in the framework of the algebraic Bethe ansatz. We apply this approach to the longitudinal spin-spin correlation function of the XXZ Heisenberg spin-1/2 chain (with magnetic field) in the disordered regime as well as to the density-density correlation function of the interacting one-dimensional Bose gas. At leading order, the results confirm the Luttinger liquid and conformal field theory predictions.

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Contents

1 Introduction

The aim of this article is to derive the long-distance asymptotic behavior of correlation functions of integrable models "from first principles". To reach this goal, we will use the framework of the Bethe ansatz [7, 75, 80, 86, 87] in its algebraic version [28, 79]. For simplicity, we shall explain the main features of our approach using the example of the XXZ spin-$\frac{1}{2}$ Heisenberg model [36, 7]. However, it will become apparent that the method we propose here is quite general and applies, for example, even to continuous integrable models like the interacting one-dimensional Bose gas [62, 61].

The starting point of our approach is an exact expression, that we called the master equation representation, for the generating function of the two-point correlation functions of these models that we obtained in our previous works [48, 50, 51, 45]. In the case of the XXZ spin-$\frac{1}{2}$ Heisenberg model (at non zero magnetic field and in the disordered regime), this means that we will begin our investigation from the exact expressions for the correlation functions on a finite lattice, and then take successively the thermodynamic limit and the large distance limit. The long-distance asymptotic behavior that we obtain confirms, at the leading order, the predictions given from bosonization [67, 33, 34, 35] and conformal field theories [17, 1, 18, 8].

1.1 Historical context: a brief survey

Bethe ansatz and especially its algebraic versions [7, 75, 80, 86, 87, 28, 79, 6, 29, 63, 59] provide a powerful framework for the construction and the resolution of a wide class of quantum integrable models in low dimension. The central object of this approach is the R-matrix satisfying the (cubic) Yang-Baxter equation and providing the structure constants of the associated (quadratic) Yang-Baxter algebra for the operator entries of the quantum monodromy matrix $T(\lambda)$. The representation theory for this algebra leads to the construction of integrable models operator algebras including in a natural way the Hamiltonian, its associated commuting conserved charges generated by the transfer matrix $\mathcal{T}(\lambda) = \operatorname{tr} T(\lambda)$ together with the creation-annihilation operators determining their common spectrum.

The archetype of such models is provided by the XXZ Heisenberg spin-$\frac{1}{2}$ chain in a magnetic field with periodic boundary conditions [36]:

$$H = \sum_{k=1}^{M} \big(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta(\sigma_k^z \sigma_{k+1}^z - 1)\big) - hS_z, \tag{1.1}$$

where

$$S_z = \frac{1}{2}\sum_{k=1}^{M} \sigma_k^z, \qquad [H, S_z] = 0. \tag{1.2}$$

Here Δ is the anisotropy parameter, h an external classical magnetic field, and the length M of the chain is assumed even. The quantum space of states is $\mathcal{H} = \otimes_{k=1}^{M} \mathcal{H}_k$, where $\mathcal{H}_k \sim \mathbb{C}^2$ is called the local quantum space at site k. The operators $\sigma_k^{x,y,z}$ act as the corresponding Pauli matrices in the space $\mathcal{H}_k$ and as the identity operator elsewhere.

The description of the full dynamical properties of such a model amounts to the determination of its correlation functions and associated structure factors. This was first achieved in the case equivalent to free fermions, namely at $\Delta = 0$ for the XXZ chain. We stress that these free fermion results already required tremendous efforts to get satisfactory answers, in particular for the spin-spin correlation functions [73, 69, 4, 70, 74, 85, 72, 76, 77, 44, 71, 22, 23, 21]. Going beyond this free fermion point turned out to be very involved. The first attempts to compute the correlation functions at arbitrary anisotropy Δ came from the algebraic Bethe ansatz [39] (see also [38] for the one-dimensional Bose gas). They led however to rather implicit representations. This was due to the complicated combinatorics involved in the description of the Bethe states, in the action of the local operators on them and in the computation of their scalar products. The notion of dual fields (which are quantum operators) was introduced in [58] to try to overcome such combinatorial difficulties. In this approach (see [59] and references therein), the correlation functions are obtained in terms of expectation values (with respect to the dual fields vacuum) of Fredholm determinants depending on several such dual fields. Unfortunately, these auxiliary quantum fields can hardly be eliminated from the final answers, making the use of such results quite problematic. More explicit representations were obtained later on in terms of multiple integrals for the elementary blocks of the correlation functions; it uses a completely different approach, working directly with infinite chains (hence with several hypothesis), based on the representation theory of quantum affine algebras and their associated q-vertex operators [41, 43, 42]. More algebraic representations of these elementary blocks have been obtained recently along similar lines together with a link to fermionic operator expressions of the corresponding correlation functions [14, 13, 12, 15]. Extensions of this scheme to non zero temperature have been considered in [31, 32].

The resolution of the so-called quantum inverse scattering problem for this model (namely the reconstruction of the local spin operators in terms of the operator entries of the monodromy matrix) opened the way to the actual computation of the form factors (the matrix elements of the local spin operators in the eigenstates basis) and correlation functions [52, 53, 68] in the framework of the algebraic Bethe ansatz. With the help of previous results (like determinant representations for the partition function with domain wall boundary conditions, and for the norms and scalar products of Bethe states [30, 57, 38, 37, 78]), this approach led to explicit determinant representations of the form factors of the Heisenberg spin chains in a magnetic field as well as to multiple integral expressions for the elementary blocks (or reduced density matrix) for their correlation functions in the thermodynamic limit. Remarkably, these integral representations coincide (in the zero magnetic field limit) with those obtained previously directly for the infinite chains using q-vertex operator methods and the representation theory of the quantum affine algebras [41, 43, 42].

These advances immediately raised the next challenge, namely the question of obtaining manageable expressions for physical correlation functions such as the spin-spin correlation functions. This is a central problem in the field of integrable models, on the one hand from the viewpoint of effective applications of these models to the realm of condensed matter physics (the associated dynamical structure factors are measurable quantities), and on the other hand concerning the more fundamental question of the long distance asymptotic behavior of the correlation functions. While the practical determination of the dynamical structure factors can effectively be addressed using the above form factors expressions in terms of determinants combined with powerful numerical techniques to sum up their corresponding series (see [20, 19]), answering the second question from purely analytical techniques looks like a fantastic theoretical problem. It amounts to understanding how the microscopic nearest neighbors interactions, for example in Heisenberg spin chains, integrate to produce an effective long distance behavior of the spin correlations. In particular, for models having no gap in the spectrum, the spin-spin correlation functions are believed to have, besides their possible trivial constant value, a power law decay with the distance. For generic integrable models, in particular not equivalent to free fermions, extracting such an asymptotic behavior "from first principles" has been a challenge for many years.

In the case of fully interacting gapless models (like the XXZ Heisenberg spin-$\frac{1}{2}$ chain for anisotropy $-1 < \Delta < 1$), the first predictions for this power law behavior came from a conjectured correspondence between such integrable lattice models with nearest neighbors interactions and a continuum theory having long-range interactions, the Luttinger model. Using this hypothesis, Luther and Peschel [67] succeeded to predict the exact value for the XXZ critical exponents at zero magnetic field as functions of the anisotropy parameter Δ. The above correspondence was further enlighten by Haldane's development of the Luttinger liquid concept, leading in turn to predictions for the critical exponents of the XXZ model in a magnetic field [33, 34, 35]. The Bethe ansatz techniques were used there to compute the parameters describing the low energy excitation spectrum of the XXZ model and to show that they indeed satisfy relations that are characteristic of the Luttinger liquid universality class. This conjecture was further studied in a series of papers [9, 10, 40] where, in particular, the long-distance behavior of the longitudinal spin-spin correlation function was computed in perturbation to the second order in Δ around the free fermion point $\Delta = 0$, with results in agreement with the above predictions.

Another approach to this problem stemmed from the hypothesis that critical statistical systems with short range interactions should be described by a conformal field theory. In this framework, the mapping of a conformal field theory defined on the plane into one on a strip of finite width l led to the determination of the critical exponents in terms of the eigenvalues of the transfer matrix along the strip [17, 1, 18, 8]. In particular, for large width l, the central charge and the scaling dimensions of scaling operators of the theory are given by the l^{-1} behavior of respectively the ground state and the different excited states energy levels. The possibility to compute the finite size corrections to the spectrum of integrable models in the framework of Bethe ansatz methods gave rise to a prediction for the corresponding critical exponents, in full agreement with the above bosonization approach [26, 82, 83, 24, 84, 54, 55, 56, 25].

Further works in these directions (based however on several conjectures) sharpened this picture by providing, in addition to the critical exponents, predictions (at zero magnetic field) for the amplitudes of the spin-spin correlation functions asymptotic behavior, which are not directly accessible from the bosonization or conformal field theory approaches [64, 65, 66].

1.2 Overview of the method

As already mentioned, the purpose of the present article is to describe an effective method allowing us to derive the long-distance asymptotic behavior of the correlation functions "from first principles". We will use the algebraic Bethe ansatz framework and our previous results [48, 50, 45] giving an exact (master equation) formula for the spin-spin correlation functions of a finite XXZ chain. Starting from this formula, we will explain how to take the thermodynamic limit and extract the large distance asymptotic behavior of the correlation functions step by step from their lattice expressions. As the master equation representation also applies to other integrable models [45], we will briefly describe how to implement the XXZ related analysis to the computation of the asymptotic behavior of the density-density correlation functions in the interacting Bose gas in one dimension. Let us now briefly explain the main features of the *master equation representation* (we refer the reader to the original articles [50, 45] for more details).

In the case of physical correlation functions, the dependence on the distance m appears in the form of the m^{th} power of the elementary shift operator. This operator in its turn, is some function of the transfer matrix of the integrable model at hand. The first step towards the master equation is to consider an *integrable deformation (twist) of the transfer matrix* (and hence of the shift operator) depending on some continuous parameter κ. This parameter is chosen in such a way that one recovers the usual correlation function at say $\kappa = 1$, and such that the spectrum of the twisted transfer matrix for κ around some κ_0 is simple and disjoint from the one for $\kappa = 1$. These properties allow us to sum up exactly the form factor type expansion of the κ-deformed correlation function with respect to the eigenstates of this κ-twisted transfer matrix (for any κ in some neighborhood of κ_0) in terms of a single multiple contour integral that we call the master equation representation. The original correlation function one started with (corresponding to the point $\kappa = 1$) is then reconstructed at the end of the computation from the knowledge of its values in an open neighborhood of κ_0.

Originally, such a representation was first derived in [50] for the *generating function of the longitudinal spin-spin correlation functions* of the finite XXZ chain. This generating function is given by the ground state average value of the operator $e^{\beta \mathcal{Q}_m}$ [39], where β is an arbitrary complex number and

$$\mathcal{Q}_m = \frac{1}{2} \sum_{n=1}^{m} (1 - \sigma_n^z) , \tag{1.3}$$

so that we have

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = 2D_m^2 \, \frac{\partial^2}{\partial \beta^2} \langle e^{\beta \mathcal{Q}_m} \rangle \Big|_{\beta=0} + 2\langle \sigma^z \rangle - 1 \, , \tag{1.4}$$

where the symbol D_m^2 stands for the second lattice derivative :

$$D_m^2 \, f(m) = f(m+1) + f(m-1) - 2f(m).$$

Setting $\kappa = e^\beta$, the operator $e^{\beta \mathcal{Q}_m}$ can be rewritten as

$$e^{\beta \mathcal{Q}_m} = \prod_{n=1}^{m} \left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2} \, \sigma_n^z \right), \tag{1.5}$$

and, thanks to the solution of the quantum inverse problem, it is shown to be equal to the m^{th}-power of the κ-twisted transfer matrix multiplied by the m^{th}-power of the inverse of the usual transfer matrix. Hence it is the simplest example for which the master equation representation can be obtained.

Analogous representations were obtained for other correlation functions of the XXZ chain in [51]. Moreover, the time dependent case can even be considered along the same lines [49]. Later on, we also derived similar representations for correlation functions of a wide class of integrable systems (including continuous theories) possessing the R-matrix of the six-vertex model [45].

One of the first problem to solve in this approach is that it is *a priori* not easy to take the thermodynamic limit directly in the master equation, for example for $\langle e^{\beta \mathcal{Q}_m} \rangle$. This is due to the fact that, in this limit, the number of integration variables involved into the master equation tends to infinity with the size of the lattice. However, starting from this representation, one can obtain various series expansions for the above master equation representation for the generating function. In these series representations the thermodynamic limit can be taken rather easily [50, 45].

In this article we give a *new expansion of the master equation*, appropriate for the study of the long-distance asymptotic behavior of the generating function $\langle e^{\beta \mathcal{Q}_m} \rangle$. The key idea is to consider the master equation written as a multiple contour integral such that the only poles of the integrand inside the contour are at the Bethe parameters characterizing the ground state. It turns out that the algebraic structure of these poles is given by the square of a Cauchy type determinant, in complete analogy with the free fermion case. In fact this feature is quite general and will appear for other correlation functions and other models as well, although with various minor modifications.

Then, the next step of the asymptotic analysis also takes its roots in the free fermion case. There, the obtained series is the expansion of a Fredholm determinant of an integral operator $I + V_0$ (see (2.43)),

$$\langle e^{\beta \mathcal{Q}_m} \rangle_{\Delta=0} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^n \lambda \; \det_n V_0(\lambda_j, \lambda_k) \, , \tag{1.6}$$

its kernel V_0 depending on the distance m. It is known from the analysis of such kernel that the main contribution to the $m \to \infty$ asymptotic behavior is generated by the traces of powers of the kernel V_0, $\mathrm{tr}(V_0^k)$. Independently of the power k involved, every trace behaves as $O(m)$. Therefore the natural idea would be to reorder the series (1.6) in such a way as to obtain an expansion with respect to $\mathrm{tr}(V_0^k)$. This can be easily done by presenting the determinants in (1.6) as a sums over permutations provided the latter are ordered with respect to their *cycles decomposition* (see for example [11]).

In the case of a general anisotropy parameter Δ, the series of multiple integrals for the generating function cannot be reduced to the form (1.6). Nevertheless, it can still be reordered in a way similar to the case of free fermions. Such a reordering leads to the appearance of multiple integrals of a special type that we call *cycle integrals* (see Section 3). These cycle integrals play the role of analogs of $\mathrm{tr}(V_0^k)$. Each of them has a computable long-distance asymptotic behavior that we recently obtained using Riemann-Hilbert techniques applied to the Fredholm determinant of an integral operator with generalized sine kernel [46]. Our strategy is then rather simple: *using the asymptotic behavior for each cycle integral, we can sum up asymptotically the multiple series* corresponding to the generating function $\langle e^{\beta \mathcal{Q}_m} \rangle$. It so happens that such asymptotic series exponentiates in a natural way that mimic a Fredholm determinant expansion. However, in our case, all integrals are coupled in a non trivial way, and the correlation function itself is not a Fredholm determinant (except in the free fermion point). We believe that this strategy can be applied to quite general cases although the details should be adjusted accordingly.

As a result of this procedure, we are able to find the desired long-distance asymptotic behavior for $\langle e^{\beta \mathcal{Q}_m} \rangle$:

$$\langle e^{\beta \mathcal{Q}_m} \rangle = \sum_{\sigma=0,\pm} \widehat{G}^{(0)}(\beta + 2\pi i\sigma, m)\big[1 + \mathrm{o}(1)\big] \, , \tag{1.7}$$

where we have defined the function

$$\widehat{G}^{(0)}(\beta, m) = \mathcal{C}(\beta) \cdot e^{\beta m D} m^{\frac{\beta^2 \mathcal{Z}^2}{2\pi^2}} \, . \tag{1.8}$$

Here $\mathcal{C}(\beta)$, D and $\mathcal{Z}$ are constants depending on the thermodynamic quantities that can be computed in terms of the low energy spectrum properties of the XXZ chain. This spectrum can be characterized by the solutions of the system of Bethe equations (2.2) [7, 75]. In the thermodynamic limit, the ground state of the model appears to be the Dirac sea in the momentum space: the spectral parameters of the particles occupy the interval $[-q, q]$ with a density $\rho(\lambda)$. The Fermi boundary q of the interval depends on the anisotropy parameter Δ and on the magnetic field h. Note that q remains finite for a non zero magnetic field, while it tends to infinity in the vanishing magnetic field limit. In the present article, q *will be kept finite in all computations.* However, the final expressions that we obtain are well defined in the zero magnetic field limit as well. In the Bethe ansatz framework, q and $\rho(\lambda)$ can be obtained in the following way. Let $\epsilon(\lambda)$ be the dressed energy, that is to say the solution of the integral equation

$$\epsilon(\lambda) + \frac{1}{2\pi} \int_{-q}^{q} K(\lambda - \mu)\, \epsilon(\mu)\, \mathrm{d}\mu = h - 2p_0'(\lambda) \sin\zeta, \tag{1.9}$$

where

$$K(\lambda) = \frac{\sin 2\zeta}{\sinh(\lambda + i\zeta)\sinh(\lambda - i\zeta)}, \qquad \cos\zeta = \Delta, \qquad 0 < \zeta < \pi, \tag{1.10}$$

and $p_0(\lambda)$ is the bare momentum of the particle (see (2.6)). Then q is such that $\epsilon(q) = 0$. Similarly, the ground state density $\rho(\lambda)$ [60, 62] satisfies a linear integral equation

$$\rho(\lambda) + \frac{1}{2\pi} \int_{-q}^{q} K(\lambda - \mu)\, \rho(\mu)\, \mathrm{d}\mu = \frac{1}{2\pi} p_0'(\lambda), \tag{1.11}$$

The dressed momentum is closely related to the density as

$$p(\lambda) = 2\pi \int_0^{\lambda} \rho(\mu)\, \mathrm{d}\mu. \tag{1.12}$$

The Fermi momentum, denoted as p_F, is given by its value at the Fermi boundary, $p_F = p(q)$. The average density D is related to the ground state magnetization $\langle \sigma^z \rangle$ and is expressed in terms of the Fermi momentum as

$$\langle \sigma^z \rangle = 1 - 2D, \qquad D = \int_{-q}^{q} \rho(\mu)\, \mathrm{d}\mu = \frac{p_F}{\pi}. \tag{1.13}$$

Note that $D = \frac{1}{2}$ in the case of a zero magnetic field hence implying a vanishing magnetization.

Another constant appearing in the asymptotic behavior (1.8) is the quantity $\mathcal{Z}$. It can be identified with the value of the dressed charge $Z(\lambda)$ at the Fermi boundary, $\mathcal{Z} = Z(\pm q)$. We recall that $Z(\lambda)$ satisfies an integral equation similar to (1.11)

$$Z(\lambda) + \frac{1}{2\pi} \int_{-q}^{q} K(\lambda - \mu)\, Z(\mu)\, \mathrm{d}\mu = 1, \tag{1.14}$$

and can be interpreted in the XXZ model as the intrinsic magnetic moment of the elementary excitations [59].

Finally, the constant $\mathcal{C}(\beta)$ in (1.8) can be computed in terms of a ratio of four Fredholm determinants. The associated integral operators have compact support on contours surrounding the interval $[-q, q]$. The explicit formulas are given in Section 5.1. At this stage, we would like to make two important remarks.

The first one concerns the fact that all the Fredholm determinants involved in our computations, for example those appearing in the constant $\mathcal{C}(\beta)$, are associated to *bounded integral operators acting on some compact contours surrounding the finite interval* $[-q, q]$. Due to the finiteness of q, all these determinants take finite values, which is a direct consequence of working at a non zero value of the magnetic field. When $q \to \infty$ the contours are no more compact and the corresponding Fredholm determinants diverge. It can nevertheless be checked that, in the zero magnetic field limit, the overall constant $\mathcal{C}(\beta)$ appearing in the final answer (1.8) remains finite although each of its individual parts (i.e. the four above mentioned Fredholm determinants) diverges in this limit.

The second remark is the following: since the operator $e^{\beta \mathcal{Q}_m}$ (1.5) is a polynomial in $\kappa = e^{\beta}$, the generating function is a $2\pi i$-periodic function of β. Of course, the leading term of the long-distance asymptotic behavior is itself not necessarily a periodical function of β. Nonetheless, our analysis shows that *this a priori broken periodicity is at least partly restored by the first oscillating correction to the leading term*, as it can be seen from the equation (1.7). More precisely, the result of the direct computation of the first oscillating correction to the asymptotics (1.7) can be reproduced by a simple shift $\beta \to \beta \pm 2\pi i$ from the non-oscillating term. This property, that we will prove here, has been recently used in [16] to get predictions for the asymptotic behavior of $\langle e^{\beta \mathcal{Q}_m} \rangle$ at zero magnetic field.

The long-distance asymptotic behavior of the longitudinal spin-spin correlation function then can be extracted from the one of $\langle e^{\beta \mathcal{Q}_m} \rangle$ using (1.4). At the first leading orders, it reads,

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle_{\text{leading}} = \langle \sigma^z \rangle^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + 2|F_\sigma|^2 \cdot \frac{\cos(2m p_{_F})}{m^{2\mathcal{Z}^2}}, \tag{1.15}$$

where F_σ is related to the properly normalized form factor of the operator σ^z between the ground state and an excited state containing one particle and one hole sited at the two different boundaries of the Fermi sphere, namely at q and $-q$; hence, it corresponds to an umklapp process [27, 35, 34]. The factor 2 in front of $|F_\sigma|^2$ just corresponds to the two possible states of this kind. This result agrees with bosonization and conformal field theory analysis. Note however that, in the thermodynamic limit, all form factors scale to zero as some power of the size of the lattice. The precise coefficient and proper scaling behavior of the form factor which results from our computations is therefore rather non-trivial. We have nevertheless checked that, at zero magnetic field, this constant F_σ goes to a finite value, and that it agrees to second order in Δ around the point $\Delta = 0$ with the predictions given in [64, 65, 66].

This article is organized as follows.

In Section 2, we explain how to obtain, in the case of the XXZ spin chain, a new series expansion for the master equation representation of $\langle e^{\beta \mathcal{Q}_m} \rangle$, that is suitable first for taking the thermodynamic limit and then for extracting the asymptotic behavior of this correlation function.

This expansion is written in Section 3 in terms of cycle integrals. Then, we recall the results of [46] and apply them to get an asymptotic expansion of the cycle integrals in a form adapted for their asymptotic summation.

The asymptotic summation of these cycle integrals is described in Section 4. It is divided into several steps according to the different nature of the terms to sum up. In particular one of the steps uses a generalization of the Lagrange series (the details are given in an appendix).

This procedure enables us, in Section 5, to describe the asymptotic behavior of the correlation function $\langle e^{\beta \mathcal{Q}_m} \rangle$ at large distances m. We show the existence of both oscillating and non-oscillating terms that we compute at the leading orders. The asymptotic behavior of the longitudinal spin-spin correlation function follows and is also given in the same section. It confirms at the leading order the predictions from bosonization and conformal field theory.

In Section 6, the whole procedure is applied to another model, the one-dimensional interacting Bose gas, for which we compute the asymptotic behavior of the density-density correlation function.

Finally, various technicalities are gathered in the appendices. We would like in particular to draw the reader's attention to Appendix C which contains rather essential ingredients for the summation described in Section 4.

2 Master equation and its thermodynamic limit

As already mentionned in the Introduction, the main object of study of this article (at least up to Section 5) is the ground state expectation value of the operator $e^{\beta \mathcal{Q}_m}$ (1.3) for the XXZ model (1.1) in an external magnetic field. In this section, we recall the master equation representation obtained in [50, 45] for this operator in the finite chain, and show how such a master equation can be expanded into some series suitable both for the thermodynamic limit and for the asymptotic analysis performed in the following sections.

2.1 Master equation for $\langle e^{\beta \mathcal{Q}_m} \rangle$

The master equation gives a multiple integral representation for the expectation value of the operator $e^{\beta \mathcal{Q}_m}$ with respect to an N-particles Bethe eigenstate $|\psi(\{\lambda\})\rangle$ of the Hamiltonian. It reads,

$$\begin{aligned} \langle e^{\beta \mathcal{Q}_m} \rangle &= \frac{\langle \psi(\{\lambda\}) | e^{\beta \mathcal{Q}_m} | \psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\}) | \psi(\{\lambda\}) \rangle} \\ &= \frac{(-1)^N}{N!} \oint\limits_{\Gamma(\{\lambda\})} \prod_{j=1}^{N} \left(\frac{\mathrm{d}z_j}{2\pi i} \, \frac{l(z_j)\, d(z_j)}{l(\lambda_j)\, d(\lambda_j)} \right) \frac{\Big[\det_N \Omega_\kappa(\{z\}, \{\lambda\} | \{z\}) \Big]^2}{\prod\limits_{j=1}^{N} \mathcal{Y}_\kappa(z_j | \{z\}) \cdot \det_N \frac{\partial \mathcal{Y}(\lambda_j | \{\lambda\})}{\partial \lambda_k}} . \end{aligned} \tag{2.1}$$

The various quantities entering this representation are all rather universal objects in the context of the Bethe ansatz. Let us explain their meaning.

The parameters $\lambda_1, \ldots, \lambda_N$ describe the specific eigenstate $|\psi(\{\lambda\})\rangle$ in which the average value is computed. From now on, this state will be the ground state of the XXZ chain (1.1) in a given magnetic field h and for some anisotropy $\Delta = \cos\zeta$, $0 < \zeta < \pi$. In this case, these parameters are real numbers [86] which satisfy the system of Bethe equations

$$\mathcal{Y}(\lambda_j|\{\lambda\}) = 0, \qquad j = 1, \ldots, N, \tag{2.2}$$

with

$$\mathcal{Y}(\mu|\{\lambda\}) = a(\mu) \prod_{k=1}^{N} \sinh(\lambda_k - \mu - i\zeta) + d(\mu) \prod_{k=1}^{N} \sinh(\lambda_k - \mu + i\zeta)\,. \tag{2.3}$$

Here $a(\mu) = a_0^M(\mu)$ and $d(\mu) = d_0^M(\mu)$ are the vacuum eigenvalues of the transfer matrix, with

$$a_0(\mu) = \sinh(\mu - \tfrac{i\zeta}{2}), \qquad d_0(\mu) = \sinh(\mu + \tfrac{i\zeta}{2}). \tag{2.4}$$

The whole explicit dependence of (2.1) on the distance m is contained in the function $l(z)$, which can be expressed in terms of a_0 and d_0 as

$$l(\mu) = \left(\frac{a_0(\mu)}{d_0(\mu)}\right)^m = \left(\frac{\sinh(\mu - \frac{i\zeta}{2})}{\sinh(\mu + \frac{i\zeta}{2})}\right)^m \equiv e^{im(p_0(\mu)+\pi)}, \tag{2.5}$$

where

$$p_0(\mu) = -i \log\left(\frac{\sinh(\frac{i\zeta}{2} - \mu)}{\sinh(\frac{i\zeta}{2} + \mu)}\right), \qquad p_0(0) = 0, \tag{2.6}$$

plays the role of the bare momentum of the pseudo particles (see also (1.10), (1.11)).

Besides the function $\mathcal{Y}(\lambda_j|\{\lambda\})$, the master equation (2.1) contains also a function $\mathcal{Y}_\kappa(z_j|\{z\})$, $\kappa = e^\beta$, which naturally appears in the definition of the κ-twisted Bethe equations associated to the spectrum of the κ-twisted monodromy matrix (see [50]). It is given as

$$\mathcal{Y}_\kappa(\mu|\{z\}) = a(\mu) \prod_{k=1}^{N} \sinh(z_k - \mu - i\zeta) + \kappa\, d(\mu) \prod_{k=1}^{N} \sinh(z_k - \mu + i\zeta)\,. \tag{2.7}$$

One more object entering (2.1) is the determinant of the matrix Ω_κ, with matrix elements

$$\begin{aligned}(\Omega_\kappa)_{jk}(\{z\},\{\lambda\}|\{z\}) = a(\lambda_j)\, t(z_k, \lambda_j) \prod_{a=1}^{N} \sinh(z_a - \lambda_j - i\zeta) \\ - \kappa\, d(\lambda_j)\, t(\lambda_j, z_k) \prod_{a=1}^{N} \sinh(z_a - \lambda_j + i\zeta)\,,\end{aligned} \tag{2.8}$$

where

$$t(z, \lambda) = \frac{-i \sin\zeta}{\sinh(z - \lambda) \sinh(z - \lambda - i\zeta)}. \tag{2.9}$$

The determinant of this matrix describes the scalar product between any eigenstate of the twisted transfer-matrix and some arbitrary state (see [50, 45]).

It remains to describe the integration contour $\Gamma(\{\lambda\})$ appearing in equation (2.1). For every variable z_j the integral is taken over a closed contour surrounding the points $\lambda_1, \ldots, \lambda_N$ and such that any other singularities of the integrand (i.e. the roots of the system $\mathcal{Y}_\kappa(z_j|\{z\}) = 0$) are located outside this contour. Since here the parameters $\lambda_1, \ldots, \lambda_N$ decribe the ground state of the XXZ chain in a non-zero magnetic field h, they are real numbers and, in the thermodynamic limit, they fill the interval $[-q, q]$ (with q finite, see (1.9)) with a density $\rho(\lambda)$ satisfying the linear integral equation (1.11). Therefore, the integration contour will be chosen such as to surround the interval $[-q, q]$.

It is worth mentioning that the integral representation (2.1) only holds for $|\kappa|$ small enough. This restriction does not cause any problems: the expectation value of $e^{\beta \mathcal{Q}_m}$ is a polynomial in κ; thus it is enough to know this polynomial in a vicinity of $\kappa = 0$ and then consider its analytic continuation. One should however remember that it is not always possible to analytically continue directly the integrand of (2.1). For instance, at $\kappa \to 1$, one of the solutions of the system $\mathcal{Y}_\kappa(z_j|\{z\}) = 0$ goes to $\{\lambda\}$, and an integration contour $\Gamma(\{\lambda\})$ satisfying the above desired properties does not exist anymore.

2.2 Transformation of the determinants

The integrand of (2.1) has poles at $z_k = \lambda_j$ and, formally, the multiple contour integral in (2.1) can be computed by residues at these points. These poles are contained in the determinants $\det \Omega_\kappa$. More precisely, each of these determinants has simple poles at $z_k = \lambda_j$, which means that the integrand has double poles at these points.

In order to be able to analyze the contribution of these poles, it is convenient to extract them explicitly from $\det \Omega_\kappa$. This extraction can be done in several ways. The method we present here consists in extracting a Cauchy determinant from each $\det \Omega_\kappa$ using the fact that the parameters λ are solution of the Bethe equations. It allows us in particular to show that $\det \Omega_\kappa$ is proportional to $\kappa - 1$.

Proposition 2.1. *If the parameters $\lambda_1, \ldots, \lambda_N$ satisfy the system of Bethe equations, then*

$$\det_N \Omega_\kappa(\{z\}, \{\lambda\}|\{z\}) = \det_N \left(\frac{1}{\sinh(\lambda_j - z_k)} \right) \cdot \prod_{j=1}^{N} \left\{ a(\lambda_j) \left[\kappa \frac{V_+(\lambda_j)}{V_-(\lambda_j)} - 1 \right] \right\}$$
$$\times \frac{1-\kappa}{V_+^{-1}(\theta) - \kappa V_-^{-1}(\theta)} \cdot \prod_{a,b=1}^{N} \sinh(z_a - \lambda_b - i\zeta) \cdot \det_N \left(\delta_{jk} + U_{jk}^{(\lambda)}(\theta) \right), \quad (2.10)$$

or

$$\det_N \Omega_\kappa(\{z\},\{\lambda\}|\{z\}) = \det_N\left(\frac{1}{\sinh(z_k-\lambda_j)}\right)\cdot\prod_{j=1}^{N}\left\{d(\lambda_j)\left[\frac{V_-(z_j)}{V_+(z_j)}-\kappa\right]\right\}$$
$$\times\frac{1-\kappa}{V_-(\theta)-\kappa V_+(\theta)}\cdot\prod_{a,b=1}^{N}\sinh(\lambda_a-z_b-i\zeta)\cdot\det_N\left(\delta_{jk}+U^{(z)}_{jk}(\theta)\right). \quad (2.11)$$

In these expressions, θ is an arbitrary complex number, and

$$V_\pm(\mu)\equiv V_\pm\left(\mu\,\middle|\,\begin{matrix}\{\lambda\}\\ \{z\}\end{matrix}\right)=\prod_{a=1}^{N}\frac{\sinh(\mu-\lambda_a\pm i\zeta)}{\sinh(\mu-z_a\pm i\zeta)}. \quad (2.12)$$

The entries of the matrices $U^{(\lambda)}(\theta)$ and $U^{(z)}(\theta)$ are given by

$$U^{(\lambda)}_{jk}(\theta)=\frac{\prod\limits_{a=1}^{N}\sinh(z_a-\lambda_j)}{\prod\limits_{\substack{a=1\\a\neq j}}^{N}\sinh(\lambda_a-\lambda_j)}\cdot\frac{K_\kappa(\lambda_j-\lambda_k)-K_\kappa(\theta-\lambda_k)}{V_+^{-1}(\lambda_j)-\kappa V_-^{-1}(\lambda_j)}\,, \quad (2.13)$$

$$U^{(z)}_{jk}(\theta)=\frac{\prod\limits_{a=1}^{N}\sinh(z_k-\lambda_a)}{\prod\limits_{\substack{a=1\\a\neq k}}^{N}\sinh(z_k-z_a)}\cdot\frac{K_\kappa(z_j-z_k)-K_\kappa(z_j-\theta)}{V_-(z_k)-\kappa V_+(z_k)}\,, \quad (2.14)$$

with

$$K_\kappa(\lambda)=\coth(\lambda+i\zeta)-\kappa\coth(\lambda-i\zeta)\,. \quad (2.15)$$

The proof of this proposition is given in Appendix A.

Remark 2.1. One can observe that $\det\Omega_\kappa$ is proportional to $1-\kappa$. However, the second line of (2.10) or of (2.11) does not necessarily vanish at $\kappa=1$ ($\beta=0$). Indeed, if $z_j=\lambda_j$ (modulo the permutation group) for all $j=1,\dots,N$, then $V_\pm(\theta)=1$ and the factor $1-\kappa$ cancels. Such situation can arise in the process of evaluating the multiple integral (2.1).

In the representations (2.10), (2.11), the poles at $z_k=\lambda_j$ are gathered explicitly in the two Cauchy determinants. The entries of the matrices $U^{(\lambda,z)}$ contain singularities at $\lambda_j=\lambda_k$ (respectively $z_j=z_k$), but the corresponding determinants are obviously not singular, since the original $\det\Omega_\kappa$ vanishes at $\lambda_j=\lambda_k$ or $z_j=z_k$. To make this fact explicit, we present the determinants of the finite size matrices $\delta_{jk}+U^{(\lambda,z)}_{jk}$ as Fredholm determinants of integral operators acting on a contour Γ surrounding the points $\{\lambda\}$ and $\{z\}$ (see Fig. 1):

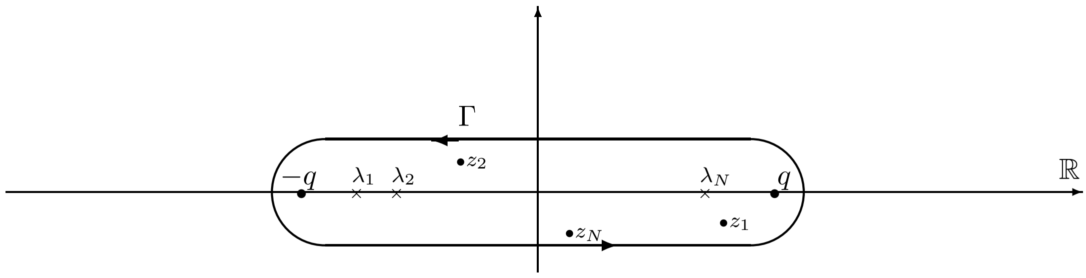

Figure 1: The contour Γ.

$$\det_N\left[\delta_{jk}+U^{(\lambda,z)}_{jk}(\theta)\right]=\det\left[I+\frac{1}{2\pi i}\hat{U}^{(\lambda,z)}_{\theta}(w,w')\right], \tag{2.16}$$

where

$$\hat{U}^{(\lambda)}_{\theta}(w,w')=-\prod_{a=1}^{N}\frac{\sinh(w-z_a)}{\sinh(w-\lambda_a)}\cdot\frac{K_\kappa(w-w')-K_\kappa(\theta-w')}{V_+^{-1}(w)-\kappa V_-^{-1}(w)}, \tag{2.17}$$

and

$$\hat{U}^{(z)}_{\theta}(w,w')=\prod_{a=1}^{N}\frac{\sinh(w'-\lambda_a)}{\sinh(w'-z_a)}\cdot\frac{K_\kappa(w-w')-K_\kappa(w-\theta)}{V_-(w')-\kappa V_+(w')}. \tag{2.18}$$

The equivalence of the two representations (2.16) is proven in Appendix A.2.

Using these results, we can rewrite the master equation (2.1) in the following form:

$$\langle e^{\beta\mathcal{Q}_m}\rangle=\frac{1}{N!}\oint\limits_{\Gamma(\{\lambda\})}\prod_{j=1}^{N}\left\{\frac{\mathrm{d}z_j}{2\pi i}\,e^{im(p_0(z_j)-p_0(\lambda_j))}\,\frac{\left(\kappa\frac{V_+(\lambda_j)}{V_-(\lambda_j)}-1\right)\left(\kappa-\frac{V_-(z_j)}{V_+(z_j)}\right)}{\kappa+(-1)^N\frac{a(z_j)}{d(z_j)}\prod\limits_{a=1}^{N}\frac{\sinh(z_a-z_j-i\zeta)}{\sinh(z_j-z_a-i\zeta)}}\right\}$$
$$\times\frac{1}{\det_N\Theta_{jk}}\cdot\widetilde{W}_N\begin{pmatrix}\lambda_1,\ldots,\lambda_N\\ z_1,\ldots,z_N\end{pmatrix}\cdot\left(\det_N\frac{1}{\sinh(z_k-\lambda_j)}\right)^2, \tag{2.19}$$

where

$$\widetilde{W}_N\begin{pmatrix}\lambda_1,\,\ldots,\,\lambda_N\\ z_1,\,\ldots,\,z_N\end{pmatrix}\equiv\widetilde{W}_N\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix}=\prod_{a,b=1}^{N}\frac{\sinh(z_a-\lambda_b-i\zeta)\sinh(\lambda_b-z_a-i\zeta)}{\sinh(z_a-z_b-i\zeta)\sinh(\lambda_a-\lambda_b-i\zeta)}$$
$$\times\frac{(\kappa-1)^2\det\left(I+\frac{1}{2\pi i}\hat{U}^{(\lambda)}_{\theta_1}(w,w')\right)\det\left(I+\frac{1}{2\pi i}\hat{U}^{(z)}_{\theta_2}(w,w')\right)}{(V_+^{-1}(\theta_1)-\kappa V_-^{-1}(\theta_1))(V_-(\theta_2)-\kappa V_+(\theta_2))}, \tag{2.20}$$

and

$$\Theta_{jk} = \left(a(\lambda_j) \prod_{a=1}^{N} \sinh(\lambda_a - \lambda_j - i\zeta) \right)^{-1} \cdot \frac{\partial \mathcal{Y}(\lambda_j|\{\lambda\})}{\partial \lambda_k}. \tag{2.21}$$

It is important to note that the function $\widetilde{W}$ (2.20) is a symmetric function of the variables $\{z\}$ and also of the variables $\{\lambda\}$. Moreover, it possesses the recursive reduction property

$$\widetilde{W}_N \begin{pmatrix} \lambda_1, \ldots, \lambda_N \\ z_1, \ldots, z_N \end{pmatrix} \Bigg|_{z_N=\lambda_N} = \widetilde{W}_{N-1} \begin{pmatrix} \lambda_1, \ldots, \lambda_{N-1} \\ z_1, \ldots, z_{N-1} \end{pmatrix}. \tag{2.22}$$

The entries of the matrix Θ contain the Lieb kernel $K(\lambda)$ defined in (1.10),

$$\Theta_{jk} = \delta_{jk} \left(\log' \frac{a(\lambda_j)}{d(\lambda_j)} - i \sum_{a=1}^{N} K(\lambda_j - \lambda_a) \right) + iK(\lambda_j - \lambda_k) \ . \tag{2.23}$$

2.3 Expansion of the master equation

We now express the master equation as series appropriate for taking the thermodynamic limit. Replacing one of the Cauchy determinant by the product of its diagonal elements we re-write equation (2.19) as

$$\langle e^{\beta \mathcal{Q}_m} \rangle = \frac{1}{\det_N \Theta} \oint\limits_{\Gamma(\{\lambda\})} \prod_{j=1}^{N} \left\{ \frac{\mathrm{d}z_j}{2\pi i} \cdot e^{im(p_0(z_j)-p_0(\lambda_j))} \cdot \mathcal{V}_N \left(\lambda_j \mid \begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right) \right\} \cdot \widetilde{W}_N \left(\begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right)$$
$$\times \prod_{k=1}^{N} \frac{1}{\sinh(z_k - \lambda_k)} \cdot \det_N \left[\frac{h_N(z_k|\{z\}|\{\lambda\}) - 1}{\sinh(z_k - \lambda_j)} + \frac{1}{\sinh(z_k - \lambda_j)} \right], \tag{2.24}$$

where

$$\mathcal{V}_N \left(\mu \mid \begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right) = \kappa \, V_+ \left(\mu \mid \begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right) V_-^{-1} \left(\mu \mid \begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right) - 1, \tag{2.25}$$

and

$$h_N(z_k|\{z\}|\{\lambda\}) = \frac{\kappa - V_- \left(z_k \mid \begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right) V_+^{-1} \left(z_k \mid \begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right)}{\kappa + (-1)^N \frac{a(z_k)}{d(z_k)} \prod\limits_{a=1}^{N} \frac{\sinh(z_a - z_k - i\zeta)}{\sinh(z_k - z_a - i\zeta)}}. \tag{2.26}$$

Note that the function $\mathcal{V}$ defined by (2.25) satisfy similar properties to $\widetilde{W}$: it is also a symmetric function of the variables $\{\lambda\}$ and of the variables $\{z\}$ separately, and possesses the same reduction property:

$$\mathcal{V}_N \left(\mu \mid \begin{matrix} \lambda_1, \ldots, \lambda_N \\ z_1, \ldots, z_N \end{matrix} \right) \Bigg|_{z_N=\lambda_N} = \mathcal{V}_{N-1} \left(\mu \mid \begin{matrix} \lambda_1, \ldots, \lambda_{N-1} \\ z_1, \ldots, z_{N-1} \end{matrix} \right). \tag{2.27}$$

As for h_N (2.26), it possesses also interesting reduction properties. Indeed, the Bethe equations (2.2) for $\{\lambda\}$ ensure that, for any k and s in $1,\dots,N$,

$$h_N(z_k|\{z\}|\{\lambda\})|_{z_k=\lambda_s} = 1\,, \tag{2.28}$$

and

$$\frac{\partial}{\partial z_k} h_N(z_k|\{z\}|\{\lambda\})\bigg|_{z_k=\lambda_s} = \mathcal{V}_{N-1}^{-1}\left(\lambda_s \mid \begin{matrix}\{\lambda\}\setminus\lambda_s\\ \{z\}\setminus z_k\end{matrix}\right)\left(\log'\frac{a(\lambda_s)}{d(\lambda_s)} - i\sum_{a=1}^{N} K(\lambda_s-\lambda_a)\right). \tag{2.29}$$

It follows in particular from (2.28) that the first part of the determinant in (2.24) is holomorphic at $z_k = \lambda_j$.

Let us now expand this determinant via the Laplace formula. We obtain

$$\langle e^{\beta\mathcal{Q}_m}\rangle = \frac{1}{\det_N\Theta}\oint\limits_{\Gamma(\{\lambda\})}\prod_{j=1}^{N}\left\{\frac{\mathrm{d}z_j}{2\pi i}\cdot\frac{e^{im(p_0(z_j)-p_0(\lambda_j))}}{\sinh(z_j-\lambda_j)}\cdot\mathcal{V}_N\left(\lambda_j \mid \begin{matrix}\{\lambda\}\\ \{z\}\end{matrix}\right)\right\}\cdot\widetilde{W}_N\left(\begin{matrix}\{\lambda\}\\ \{z\}\end{matrix}\right)$$
$$\times\sum_{n=0}^{N}\sum_{\substack{\{z\}=\{z\}_\gamma\cup\{z\}_{\bar\gamma}\\ \{\lambda\}=\{\lambda\}_\alpha\cup\{\lambda\}_{\bar\alpha}\\ \#\bar\gamma=\#\bar\alpha=n}}(-1)^{[P(\alpha)]+[P(\gamma)]}\det_{\substack{k\in\gamma\\ j\in\alpha}}\left[\frac{h_N(z_k|\{z\}|\{\lambda\})-1}{\sinh(z_k-\lambda_j)}\right]\det_{\substack{k\in\bar\gamma\\ j\in\bar\alpha}}\left[\frac{1}{\sinh(z_k-\lambda_j)}\right]. \tag{2.30}$$

The above sum runs through all possible partitions $\alpha\cup\bar\alpha$ and $\gamma\cup\bar\gamma$ of $1,\dots,N$ such that $\#\bar\gamma=\#\bar\alpha=n$. The sets of parameters $\{\lambda\}$ and $\{z\}$ are divided accordingly into disjoint subsets $\{\lambda\}=\{\lambda\}_\alpha\cup\{\lambda\}_{\bar\alpha}$, and similarly $\{z\}=\{z\}_\gamma\cup\{z\}_{\bar\gamma}$. Moreover, we specify that the parameters in each subset are ordered in the canonical way, namely $\{\lambda\}_\alpha=\{\lambda_{\alpha_1},\lambda_{\alpha_2},\dots\}$ with $\alpha_1<\alpha_2<\dots$ etc. Finally, $[P(\alpha)]$ (respectively $[P(\gamma)]$) stands for the signature the permutation P such that $P(\alpha,\bar\alpha)=1,\dots,N$ (respectively $P(\gamma,\bar\gamma)=1,\dots,N$).

Lemma 2.1. *In the multiple integral (2.30),*

1. *the only non-zero contributions come from partitions such that* $\gamma=\alpha$*;*
2. *the first determinant in (2.30) contributes only through the product of its diagonal elements.*

Proof — Let us consider a partition such that $\gamma\neq\alpha$. Then there exists $\ell\in\gamma$ such that $\ell\notin\alpha$. Let us consider the integrand of (2.30) as a function of z_ℓ. At first sight it has a simple pole at $z_\ell=\lambda_\ell$ and also simple poles at the points $z_\ell=\lambda_j$ for all $j\in\alpha$. Yet, due to the property (2.28), all the corresponding residues are equal to zero. Hence, the integral over z_ℓ vanishes. Thus we conclude that only partitions such that $\gamma=\alpha$ yield non-vanishing contributions to the integral.

It remains to prove the second part of the lemma. Let us consider an off-diagonal contribution coming from the first determinant. Hence there is a variable $z_\ell \in \{z\}_\gamma$ such that the integrand, considered as a function of this z_ℓ, has at most simple poles. However, due to the presence of the factor $h_N(z_\ell|\{z\}|\{\lambda\}) - 1$, their residues vanish. □

Thus,

$$\langle e^{\beta \mathcal{Q}_m} \rangle = \frac{1}{\det_N \Theta} \oint_{\Gamma(\{\lambda\})} \prod_{j=1}^{N} \left\{ \frac{\mathrm{d}z_j}{2\pi i} \cdot e^{im(p_0(z_j)-p_0(\lambda_j))} \cdot \mathcal{V}_N \left(\lambda_j \mid \begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right) \right\} \cdot \widetilde{W}_N \left(\begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right)$$

$$\times \sum_{n=0}^{N} \sum_{\substack{\alpha \cup \bar{\alpha} \\ \#\bar{\alpha}=n}} \prod_{k\in\bar{\alpha}} \frac{1}{\sinh(z_k-\lambda_k)} \prod_{k\in\alpha} \left[\frac{h_N(z_k|\{z\}|\{\lambda\})-1}{\sinh^2(z_k-\lambda_k)} \right] \det_{j,k\in\bar{\alpha}} \frac{1}{\sinh(z_k-\lambda_j)}. \quad (2.31)$$

The integrals over $\{z\}_\alpha$ can now be easily computed using (2.29). We obtain

$$\langle e^{\beta \mathcal{Q}_m} \rangle = \frac{1}{\det_N \Theta} \sum_{n=0}^{N} \sum_{\substack{\alpha \cup \bar{\alpha} \\ \#\bar{\alpha}=n}} \oint_{\Gamma(\{\lambda\})} \prod_{j\in\bar{\alpha}} \left\{ \frac{\mathrm{d}z_j}{2\pi i} \cdot e^{im(p_0(z_j)-p_0(\lambda_j))} \cdot \mathcal{V}_n \left(\lambda_j \mid \begin{matrix} \{\lambda\}_{\bar{\alpha}} \\ \{z\}_{\bar{\alpha}} \end{matrix} \right) \right\}$$

$$\times \widetilde{W}_n \left(\begin{matrix} \{\lambda\}_{\bar{\alpha}} \\ \{z\}_{\bar{\alpha}} \end{matrix} \right) \cdot \prod_{j\in\alpha} (2\pi i M \tilde{\rho}(\lambda_j)) \cdot \prod_{k\in\bar{\alpha}} \frac{1}{\sinh(z_k-\lambda_k)} \cdot \det_{j,k\in\bar{\alpha}} \frac{1}{\sinh(z_k-\lambda_j)}, \quad (2.32)$$

where

$$2\pi i M \tilde{\rho}(\lambda_j) = \log' \frac{a(\lambda_j)}{d(\lambda_j)} - i \sum_{a=1}^{N} K(\lambda_j - \lambda_a). \quad (2.33)$$

The notation $\tilde{\rho}(\lambda)$ is motivated by the fact that $\tilde{\rho}(\lambda)$ goes to the ground state density $\rho(\lambda)$ (1.11) in the thermodynamic limit.

It remains to replace the summation over partitions by a sum over individual λ's. Let us denote by Λ the original set of spectral parameters $\{\lambda_1, \ldots, \lambda_N\}$ describing the ground state. We then have

$$\langle e^{\beta \mathcal{Q}_m} \rangle = \frac{\prod_{j=1}^{N} (2\pi i M \tilde{\rho}(\lambda_j))}{\det_N \Theta} \sum_{n=0}^{N} \frac{1}{n!} \sum_{\lambda_1,\ldots,\lambda_n \in \Lambda} \prod_{j=1}^{n} \left(\frac{1}{2\pi i M \tilde{\rho}(\lambda_j)} \right)$$

$$\times \oint_{\Gamma(\{\lambda\})} \prod_{j=1}^{n} \frac{\mathrm{d}z_j}{2\pi i} \cdot \prod_{j=1}^{n} \frac{e^{im(p_0(z_j)-p_0(\lambda_j))}}{\sinh(z_j-\lambda_j)} \cdot \mathcal{F}_n \left(\begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right) \cdot \det_n \frac{1}{\sinh(z_k-\lambda_j)}. \quad (2.34)$$

with

$$\mathcal{F}_n \left(\begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right) = \widetilde{W}_n \left(\begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right) \prod_{j=1}^{n} \mathcal{V}_n \left(\lambda_j \mid \begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right). \quad (2.35)$$

2.4 Thermodynamic limit

Recall that the thermodynamic limit corresponds to $N, M \to \infty$ in such a way that the average density $D = N/M$ remains fixed. In this limit, discrete sums over the ground state Bethe parameters λ_k turn into integrals over $[-q, q]$ weighted by a density function $\rho(\lambda)$,

$$\frac{1}{M} \sum_{\lambda_j \in \Lambda} f(\lambda_j) \to \int_{-q}^{q} f(\lambda)\rho(\lambda)\, d\lambda. \tag{2.36}$$

It is easy to compute the limit of the pre-factor in (2.34). Using the notation (2.33), we can express the determinant of the matrix Θ (2.23) as

$$\det_N \Theta = \prod_{j=1}^{N} (2\pi i M \tilde{\rho}(\lambda_j)) \cdot \det_N \left[\delta_{jk} + \frac{K(\lambda_j - \lambda_k)}{2\pi M \tilde{\rho}(\lambda_j)} \right]. \tag{2.37}$$

The equation (2.33) for $\tilde{\rho}(\lambda)$ can be written in the form

$$2\pi \tilde{\rho}(\lambda_j) = p_0'(\lambda_j) - \frac{1}{M} \sum_{a=1}^{N} K(\lambda_j - \lambda_a)\,. \tag{2.38}$$

Replacing the sum by the integral and using the integral equation for the ground state density (1.11), we find that in the thermodynamic limit $\tilde{\rho}(\lambda) \to \rho(\lambda)$. Therefore the determinant in the r.h.s. of (2.37) goes to the Fredholm determinant of the integral operator $I + \frac{1}{2\pi}K$ acting on $[-q, q]$, namely,

$$\det_N \Theta \cdot \prod_{j=1}^{N} (2\pi i M \tilde{\rho}(\lambda_j))^{-1} \to \det \left[I + \frac{1}{2\pi} K \right], \qquad N, M \to \infty, \tag{2.39}$$

where the integral operator acts on the interval $[-q, q]$.

Hence, in the thermodynamic limit, the generating function $\langle e^{\beta \mathcal{Q}_m} \rangle$ is given by the following series of multiple integrals:

$$\langle e^{\beta \mathcal{Q}_m} \rangle = \frac{1}{\det\left(I + \frac{1}{2\pi}K\right)} \sum_{n=0}^{\infty} \frac{1}{n!} \oint_{\Gamma([-q,q])} \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \int_{-q}^{q} \prod_{j=1}^{n} \frac{d\lambda_j}{2\pi i}$$
$$\times \mathcal{F}_n \begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} \cdot \prod_{j=1}^{n} \frac{e^{im(p_0(z_j) - p_0(\lambda_j))}}{\sinh(z_j - \lambda_j)} \cdot \det_n \frac{1}{\sinh(z_k - \lambda_j)}\,. \tag{2.40}$$

The question of convergence of this series is quite complicated. To understand this topic a little better, let us consider the free fermions point $\Delta = 0$ ($\zeta = \frac{\pi}{2}$).

In this point the terms of the series can be significantly simplified. Indeed, it is obvious from (2.12) that $\mathcal{V}_n \equiv \kappa - 1$ for $\zeta = \frac{\pi}{2}$. It is less trivial, but true (see Appendix B), that $\widetilde{W}_n \equiv 1$. Therefore $\mathcal{F}_n \equiv (\kappa-1)^n$, and we have

$$\langle e^{\beta\mathcal{Q}_m}\rangle_{\zeta=\frac{\pi}{2}} = \sum_{n=0}^{\infty}\frac{(\kappa-1)^n}{n!}\oint\limits_{\Gamma([-q,q])}\prod_{j=1}^{n}\frac{\mathrm{d}z_j}{2\pi i}\int\limits_{-q}^{q}\prod_{j=1}^{n}\frac{\mathrm{d}\lambda_j}{2\pi i} \times\prod_{j=1}^{n}\frac{e^{im(p_0(z_j)-p_0(\lambda_j))}}{\sinh(z_j-\lambda_j)}\cdot\det{}_n\frac{1}{\sinh(z_k-\lambda_j)}. \tag{2.41}$$

The integrals over z_j are now factorized and can be taken explicitly. We obtain

$$\langle e^{\beta\mathcal{Q}_m}\rangle_{\zeta=\frac{\pi}{2}} = \sum_{n=0}^{\infty}\frac{(\kappa-1)^n}{n!}\int\limits_{-q}^{q}\prod_{j=1}^{n}\mathrm{d}\lambda_j\cdot\det{}_n\left[\frac{\sin\left(\frac{m}{2}(p_0(\lambda_j)-p_0(\lambda_k))\right)}{\pi\sinh(\lambda_j-\lambda_k)}\right]. \tag{2.42}$$

The series (2.42) is an expansion of the Fredholm determinant of the integral operator $I+V_0$ with kernel

$$V_0(\lambda,\mu) = (\kappa-1)\frac{\sin\left\{\frac{m}{2}[p_0(\lambda)-p_0(\mu)]\right\}}{\pi\sinh(\lambda-\mu)}. \tag{2.43}$$

The general theory of Fredholm determinants ensures that the series (2.42) is absolutely convergent and define an entire function of κ. It means that the expansion coefficients decay faster than exponentially.

In the case of general Δ, the series (2.40) cannot be reduced to a simple form as in (2.42). As mentioned above, the analysis of its convergence is therefore rather complicated. Nevertheless, taking into account that the XXZ chain with a general anisotropy parameter can be considered as a smooth deformation of the $\Delta = 0$ case, we shall assume in the following that the expansion (2.40) also is absolutely convergent.

As a conclusion to this section, we would like to stress the role of the non-zero magnetic field. As we already mentioned, the support of the ground state density at non-zero magnetic field is a finite interval $[-q,q]$. Therefore, all the multiple integrals in the series (2.40) are convergent. The pre-factor $\det[I+\frac{1}{2\pi}K]^{-1}$ is also finite. However, at zero magnetic field, $q\to\infty$, and it is easy to see that each separate integral in the series (2.40) becomes divergent. This divergence should be of course compensated by the divergence of the Fredholm determinant $\det[I+\frac{1}{2\pi}K]$. The latter becomes ill-defined at $q\to\infty$, because the kernel of the integral operator depends on the difference of the parameters, i.e. $K = K(\lambda-\mu)$. It is not clear, however, how one can extract the finite part from the representation (2.40) directly at $h=0$. Therefore one can say that the non-zero magnetic field plays the role of a regularization in the framework of our approach. One can check however that, in the end of the computation, the obtained results have smooth limits at $h\to 0$, although the corresponding proof is highly non-trivial.

3 Cycle integrals

Even in the simplest case of free fermions where the multiple integrals are decoupled, the calculation of the large m behavior of $\langle e^{\beta \mathcal{Q}_m} \rangle$ remains non-trivial. Indeed, one has to analyze the large m behavior of the Fredholm determinant of the integral operator $I + V_0$ whose kernel is given in (2.43). In the general case, the situation is even worse as the multiple integrals are highly coupled and hence the corresponding series cannot be identified with the Fredholm series for the determinant of some integral operator.

However, the series (2.40) can be rearranged in a specific way, making possible the asymptotic analysis of its separate terms. The key observation towards such a rearrangement comes from the free fermion point. There the leading asymptotic effect is produced by traces of powers of V_0 : $\mathrm{tr}(V_0^s)$ behaves as $\mathrm{O}(m)$ when $m \to \infty$ for $s = 1, 2, \dots$. Analogs of such traces of powers of V_0 will be given, in the general case, by the following type of integrals:

$$\mathcal{I}_s\left[\mathcal{G}_s\right] = \oint\limits_{\Gamma([-q,q])} \frac{\mathrm{d}^s z}{(2\pi i)^s} \int\limits_{-q}^{q} \frac{\mathrm{d}^s \lambda}{(2\pi i)^s}\, \mathcal{G}_s \begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} \prod_{j=1}^{s} \frac{e^{im(p_0(z_j)-p_0(\lambda_j))}}{\sinh\left(z_j - \lambda_j\right) \sinh\left(z_j - \lambda_{j+1}\right)}\,, \tag{3.1}$$

in which the $2s$ integrals are coupled via some holomorphic function $\mathcal{G}_s$ of $2s$ variables $\lambda_1, \dots, \lambda_s$, $z_1, \dots, z_s$, symmetric separately in the s variables λ and in the s variables z. Note that, in (3.1), we identify λ_{s+1} with λ_1.

In this section, we explain how the representation (2.40) can be decomposed into such integrals, and how the asymptotic behavior of the latter can be analyzed in the large m limit.

3.1 The cycle expansion

The decomposition of the series (2.40) into multiple integrals of the form (3.1), which plays a key role in its asymptotic analysis, comes directly from the *cycle expansion* of the Cauchy determinant. In this article we use the following definition for a *cycle*:

Definition 3.1. *Let $A = (a_{i,j})$ be an arbitrary $n \times n$ matrix. A cycle of length ℓ is any product of entries of A of the form*

$$a_{j_1,j_2}\, a_{j_2,j_3} \dots a_{j_{\ell-1},j_\ell}\, a_{j_\ell,j_1}\,.$$

With this terminology, integrals of the form (3.1) *will be called* cycle integrals.

It is well known (see e.g. [11]) that the determinant of any matrix A can be presented as a sum of products of cycles of different lengths. We call such a representation of a determinant its *cycle expansion*. In our case, the determinant we need to decompose into cycles (namely, the Cauchy determinant in (2.40)) is part, together with some symmetric function, of the integrand of a n-fold integral. For such kind of representations we have the following result:

Proposition 3.1. *Let $\mathscr{C}_\lambda$ and $\mathscr{C}_z$ be two curves in $\mathbb{C}$, g a continuous function in $(z,\lambda) \in \mathscr{C}_z \times \mathscr{C}_\lambda$, and $\mathcal{G}_n$ a continuous function in n variables λ and n variables z on $\mathscr{C}_\lambda^n \times \mathscr{C}_z^n$. Assume moreover that, for any permutation $\sigma \in \mathfrak{S}_n$, $\mathcal{G}_n$ is invariant under replacements of pairs $(\lambda_i, z_i) \mapsto (\lambda_{\sigma(i)}, z_{\sigma(i)})$. Then,*

$$\frac{1}{n!}\int_{\mathscr{C}_\lambda} \mathrm{d}^n\lambda \int_{\mathscr{C}_z} \mathrm{d}^n z \; \mathcal{G}_n\begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} \det_n\left[g(z_i,\lambda_j)\right]$$
$$= \sum_{\substack{\ell_1,\dots,\ell_n=0 \\ \sum_{k=1}^n k\ell_k=n}}^{n} C(n|\{\ell\}) \int_{\mathscr{C}_\lambda} \mathrm{d}^n\lambda \int_{\mathscr{C}_z} \mathrm{d}^n z \; \mathcal{G}_n\begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} \prod_{t\in J_{\{\ell\}}} g(z_t,\lambda_{t+1}), \quad (3.2)$$

where

$$C(n|\{\ell\}) = \prod_{s=1}^{n} \frac{1}{\ell_s!}\left(\frac{(-1)^{s+1}}{s}\right)^{\ell_s}. \quad (3.3)$$

Here $J_{\{\ell\}}$ denotes the following set of triplets, associated to the configuration given by $\ell_1,\dots,\ell_n$,

$$J_{\{\ell\}} = \{(s,p,j) : 1 \le s \le n,\, 1 \le p \le \ell_s,\, 1 \le j \le s\}, \quad (3.4)$$

and, for $t=(s,p,j)$, $t+1$ denotes the triplet $(s,p,j+1)$, with $(s,p,s+1) \equiv (s,p,1)$.

Remark 3.1. The notations introduced above should be understood as follows: s labels the length of a cycle, ℓ_s stands for the number of cycles of length s and j marks the position of the variable in the corresponding cycle. Hence, for $t=(s,p,j)$, $\mu_t \equiv \mu_{s,p,j}$ stands for the j^{th} variable of the p^{th} cycle of length s. In this notation we assume that $\mu_{s,p,s+1} \equiv \mu_{s,p,1}$.

Proof — Let g be a $n \times n$ matrix. Then,

$$\begin{aligned}
\det_n[g] &= \frac{1}{n!}\,\frac{\partial^n}{\partial\gamma^n}\det_n[I+\gamma g]\Big|_{\gamma=0} = \frac{1}{n!}\,\frac{\partial^n}{\partial\gamma^n}\exp\bigl(\operatorname{tr}\log[I+\gamma g]\bigr)\Big|_{\gamma=0} \\
&= \frac{1}{n!}\,\frac{\partial^n}{\partial\gamma^n}\prod_{s=1}^{n}\exp\left\{\frac{(-1)^{s+1}\gamma^s}{s}\operatorname{tr}(g^s)\right\}\Bigg|_{\gamma=0} \\
&= \frac{1}{n!}\,\frac{\partial^n}{\partial\gamma^n}\sum_{\ell_1,\dots,\ell_n=0}^{\infty} C(n|\{\ell\})\prod_{s=1}^{n}\bigl[\gamma^s\operatorname{tr}(g^s)\bigr]^{\ell_s}\Bigg|_{\gamma=0} \\
&= \sum_{\substack{\ell_1,\dots,\ell_n=0 \\ \Sigma k\ell_k=n}}^{n} C(n|\{\ell\})\prod_{s=1}^{n}\bigl[\operatorname{tr}(g^s)\bigr]^{\ell_s}.
\end{aligned} \quad (3.5)$$

Substituting here $g_{ij} = g(z_i, \lambda_j)$ and writing explicitly all $\left[\mathrm{tr}(g^s)\right]^{\ell_s}$, we obtain

$$\det_n \left[g(z_i, \lambda_j)\right] = \sum_{\substack{\ell_1,\dots,\ell_n=0 \\ \Sigma k\ell_k = n}}^{n} C(n|\{\ell\}) \prod_{s=1}^{n} \prod_{p=1}^{\ell_s} \sum_{i_{(s,p,1)},\dots,i_{(s,p,s)}=1}^{n} \prod_{j=1}^{s} g\big(z_{i_{(s,p,j)}}, \lambda_{i_{(s,p,j+1)}}\big) , \quad (3.6)$$

It remains to observe that, in the product of the functions $g\big(z_{i_{(s,p,j)}}, \lambda_{i_{(s,p,j+1)}}\big)$, all $i_{(s,p,j)}$ should be different, otherwise the corresponding term does not contribute to the determinant (all other contributions eventually cancel out). Hence, the multiple sum over $i_{(s,p,j)}$ is in fact the sum over permutations:

$$\det_n \left[g(z_i, \lambda_j)\right] = \sum_{\substack{\ell_1,\dots,\ell_n=0 \\ \Sigma k\ell_k = n}}^{n} C(n|\{\ell\}) \sum_{\sigma \in \mathfrak{S}_n} \prod_{t \in J_{\{\ell\}}} g\big(z_{\sigma(t)}, \lambda_{\sigma(t)+1}\big). \quad (3.7)$$

Due to the symmetry properties of the function $\mathcal{G}_n$, all the terms of the sum over permutations give the same contribution to the integral. □

Let us apply this result to the Cauchy determinant under the multiple integrals in the series (2.40) for $\langle e^{\beta \mathcal{Q}_m} \rangle$. Denoting

$$\langle e^{\beta \mathcal{Q}_m} \rangle = \frac{\langle\!\langle e^{\beta \mathcal{Q}_m} \rangle\!\rangle}{\det \left[I + \frac{1}{2\pi} K\right]} , \quad (3.8)$$

we obtain

$$\langle\!\langle e^{\beta \mathcal{Q}_m} \rangle\!\rangle = \sum_{n=0}^{\infty} \sum_{\substack{\ell_1,\dots,\ell_n=0 \\ \Sigma k\ell_k = n}}^{\infty} C(n|\{\ell\}) \oint\limits_{\Gamma([-q,q])} \frac{\mathrm{d}^n z}{(2\pi i)^n} \int\limits_{-q}^{q} \frac{\mathrm{d}^n \lambda}{(2\pi i)^n} \, \mathcal{F}_n \begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix}$$

$$\times \prod_{t \in J_{\{l\}}} \frac{e^{im(p_0(z_t) - p_0(\lambda_t))}}{\sinh(z_t - \lambda_t) \sinh(z_t - \lambda_{t+1})} . \quad (3.9)$$

It is actually convenient to remove the constraint $\Sigma k\ell_k = n$ on the summation variables ℓ_k in the sum (3.9). This can be done by introducing, in the n^{th} term of the series, an n^{th}-derivative over some auxiliary parameter γ, similarly as in the proof of Proposition 3.1:

$$\langle\!\langle e^{\beta \mathcal{Q}_m} \rangle\!\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \gamma^n} \sum_{\ell_1,\dots,\ell_n=0}^{\infty} \prod_{k=1}^{n} \frac{1}{\ell_k!} \left(\frac{(-1)^{k+1} \gamma^k}{k} \right)^{\ell_k} \oint\limits_{\Gamma([-q,q])} \prod_{t \in J_{\{\ell\}}} \frac{\mathrm{d}z_t}{2\pi i} \int\limits_{-q}^{q} \prod_{t \in J_{\{\ell\}}} \frac{\mathrm{d}\lambda_t}{2\pi i}$$

$$\times \mathcal{F}_{|J_{\{\ell\}}|} \begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} \prod_{t \in J_{\{\ell\}}} \frac{e^{im(p_0(z_t) - p_0(\lambda_t))}}{\sinh(z_t - \lambda_t) \sinh(z_t - \lambda_{t+1})} \Bigg|_{\gamma=0} , \quad (3.10)$$

where $|J_{\{\ell\}}| = \Sigma k\ell_k$ denotes the cardinality of the set $J_{\{\ell\}}$ (this cardinality is no longer equal to n since we have removed the constraint).

The n^{th} term in the series for $\langle\!\langle e^{\beta \mathcal{Q}_m} \rangle\!\rangle$ is thus expressed as some sum of multiple cycle integrals of the form (3.1). Note however that these cycle integrals are coupled through the function $\mathcal{F}_{|J_{\{\ell\}}|}$ which involves the total number $2|J_{\{\ell\}}|$ of variables, and not only the $2s$ variables associated to a given cycle. In order to formalize this fact, we need to introduce some notations.

Let us first recall the standard order on $\mathbb{N}^3$:

$$(t_1, t_2, t_3) < (u_1, u_2, u_3) \Leftrightarrow \{t_1 < u_1 \text{ or } \{t_1 = u_1 \text{ and } \{t_2 < u_2 \text{ or } \{t_2 = u_2 \text{ and } t_3 < u_3\}\}\}\},$$

and let $\mathfrak{id}$ be the identity operator on functions of one variable λ and one variable z:

$$\mathfrak{id}\left[\mathcal{G}\begin{pmatrix} \lambda \\ z \end{pmatrix}\right] = \mathcal{G}\begin{pmatrix} \lambda \\ z \end{pmatrix}. \tag{3.11}$$

For some cycle configuration $\{\ell\}$, we define $\mathcal{I}_{(s,p)}$ as the operator acting on functions $\mathcal{G}_{|J_{\{\ell\}}|}$ of $2|J_{\{\ell\}}|$ variables λ_t and z_t, $t \in J_{\{\ell\}}$, as the cycle integral $\mathcal{I}_s$ (3.1) over the variables $\lambda_{s,p,j}$, $z_{s,p,j}$, $1 \le j \le s$, i.e.

$$\mathcal{I}_{(s,p)} = \bigotimes_{t<(s,m,1)} \underbrace{\mathfrak{id}_t}_{\substack{\text{variables} \\ \lambda_t,\, z_t}} \bigotimes \mathcal{I}_s \bigotimes_{t>(s,m,s)} \underbrace{\mathfrak{id}_t}_{\substack{\text{variables} \\ \lambda_t,\, z_t}}. \tag{3.12}$$

Note that the action on $\mathcal{G}_{|J_{\{\ell\}}|}$ of a single $\mathcal{I}_{(s,p)}$ (associated to a given cycle (s,p)) still produces a function (with respect to the other variables). However, the action on $\mathcal{G}_{|J_{\{\ell\}}|}$ (denoted here with the symbol *) of the product of such operators $\mathcal{I}_{(s,p)}$ with respect to all cycles,

$$\left(\prod_{s=1}^{n}\prod_{p=1}^{\ell_s}\mathcal{I}_{(s,p)}\right) * \mathcal{G}_{|J_{\{\ell\}}|} = \prod_{s=1}^{n}\prod_{p=1}^{\ell_s}\prod_{j=1}^{s}\left(\oint\frac{dz_{s,p,j}}{2\pi i}\int_{-q}^{q}\frac{d\lambda_{s,p,j}}{2\pi i}\right)\cdot\mathcal{G}_{|J_{\{\ell\}}|}\begin{pmatrix}\{\lambda_{s,p,j}\} \\ \{z_{s,p,j}\}\end{pmatrix}$$
$$\times\prod_{s=1}^{n}\prod_{p=1}^{\ell_s}\prod_{j=1}^{s}\frac{e^{im(p_0(z_{s,p,j})-p_0(\lambda_{s,p,j}))}}{\sinh(z_{s,p,j}-\lambda_{s,p,j})\sinh(z_{s,p,j}-\lambda_{s,p,j+1})}, \tag{3.13}$$

produces a number, since all variables λ and z are integrated.

This notation enables us to recast the generating function in a quite compact form:

$$\langle\!\langle e^{\beta\mathcal{Q}_m}\rangle\!\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\gamma^n}G_{1\ldots n}(\gamma)\bigg|_{\gamma=0}, \tag{3.14}$$

where

$$G_{1\ldots n}(\gamma) = \sum_{\ell_1,\ldots,\ell_n=0}^{\infty}\prod_{k=1}^{n}\frac{1}{\ell_k!}\left(\frac{(-1)^{k+1}\gamma^k}{k}\right)^{\ell_k}\cdot\left(\prod_{s=1}^{n}\prod_{p=1}^{\ell_s}\mathcal{I}_{(s,p)}\right) * \mathcal{F}_{|J_{\{\ell\}}|}. \tag{3.15}$$

Therefore, the whole problem of the asymptotic analysis of the series (2.40) for $\langle e^{\beta \mathcal{Q}_m}\rangle$ reduces to the derivation of an asymptotic formula for $G_{1\dots n}(\gamma)$. The advantage of such a representation is the following: in spite of the fact that all cycle integrals are coupled with each others through the function $\mathcal{F}_{|J_{\{\ell\}}|}$, the asymptotic behavior for large m of every set of cycle integrals can nevertheless be computed separately, independently of the other integrals; then the asymptotic formula for multiple cycle integrals can be obtained by applying consecutively the results of the analysis of each pure cycle integral $\mathcal{I}_{(s,p)}$ to the corresponding group of cycle variables $\{\lambda_{s,p,j}\}_{1\le j\le s}$ and $\{z_{s,p,j}\}_{1\le j\le s}$, while considering the remaining variables fixed. Passing from the exact formula to the asymptotic one produces a very powerful effect: due to peculiar properties of the function $\mathcal{F}_{|J_{\{\ell\}}|}$, the cycles become "quasi-decoupled" in the $m\to+\infty$ limit. Such a quasi-decoupling is enough to perform the sum over all the possible numbers ℓ_k of cycles of length k.

The summation procedure of the asymptotics of multi-cycle integrals will be presented in Section 4. However, first we need to remind the results of [46] concerning the asymptotic analysis of pure cycle integrals.

3.2 Asymptotic analysis of cycle integrals

The asymptotic study of cycle integrals of the form (3.1) was performed in [46] from the one of the Fredholm determinant of a generalized sine kernel. It was shown there that such integrals admit, up to any arbitrary order n_0, an asymptotic expansion of the form

$$\mathcal{I}_s[\mathcal{G}_s] = I_s^{(0)}[\mathcal{G}_s] + \sum_{n=1}^{n_0} \frac{1}{m^n} I_s^{(n;\,\mathrm{nosc})}[\mathcal{G}_s] + \sum_{\substack{r\in\mathbb{Z}^* \\ |r|\le \frac{n_0}{2}}} e^{irm[p_0(q)-p_0(-q)]} \sum_{n=2|r|}^{n_0} \frac{1}{m^n} I_s^{(n;\,r)}[\mathcal{G}_s] + \mathrm{O}\Big(\frac{\log^s m}{m^{n_0+1}}\Big). \tag{3.16}$$

The leading behavior (up to o(1) corrections) is given by $I_s^{(0)}[\mathcal{G}_s]$, which contains a linear term in m, a term proportional to $\log m$, and a constant term. Its explicit form, which was computed in [46], will be specified later. This leading term admits two kinds of corrections: *oscillating* (containing an oscillating factor $e^{irm[p_0(q)-p_0(-q)]}$), and *non-oscillating* ones[1]. Note that the leading term $I_s^{(0)}[\mathcal{G}_s]$ itself is *non-oscillating*.

In order to derive the asymptotic behavior of $\langle e^{\beta \mathcal{Q}_m}\rangle$, one will have to sum up (3.16) according to (3.15). If we suppose that subleading corrections in (3.16) remain subleading through this process of summation (which is not completely true, as it will be explained later), then it is in principle enough, so as to obtain the leading asymptotic behavior of $\langle e^{\beta \mathcal{Q}_m}\rangle$, to consider only the leading term $I_s^{(0)}[\mathcal{G}_s]$ in (3.16). Recall however that our main goal is not $\langle e^{\beta \mathcal{Q}_m}\rangle$ itself, but the spin-spin correlation function $\langle \sigma_1^z\,\sigma_{m+1}^z\rangle$ obtained from $\langle e^{\beta \mathcal{Q}_m}\rangle$ by second derivative with

[1] $I_s^{(n;\,r)}[\mathcal{G}_s]$ and $I_s^{(n;\,\mathrm{nosc})}[\mathcal{G}_s]$ still depend on m, but simply as a polynomial (of degree at most s) of $\log m$.

respect to β and by *second lattice derivative* (see (1.4)). This process of taking the second lattice derivative will decrease the order of the non-oscillating terms, but not of the oscillating ones, as we may merely differentiate the exponential. Therefore, to be sure to get the leading behavior of $\langle \sigma_1^z \, \sigma_{m+1}^z \rangle$, we will have to consider not only the leading (non-oscillating) term $I_s^{(0)}[\mathcal{G}_s]$, but also the *leading oscillating correction*, given by

$$O_s[\mathcal{G}_s] = O_s^+[\mathcal{G}_s] + O_s^-[\mathcal{G}_s], \quad \text{with} \quad O_s^\pm = \frac{1}{m^2} e^{\pm im[p_0(q) - p_0(-q)]} I_s^{(2;\,\pm 1)}. \tag{3.17}$$

It will indeed appear that the non-oscillating term gives rise to an m^{-2} behavior for $\langle \sigma_1^z \, \sigma_{m+1}^z \rangle$, whereas the oscillating one produces an $m^{-\theta}$ decrease, θ being lower or higher than 2 according to the value of Δ.

To specify the functional action of $I_s^{(0)}$ and O_s on the function $\mathcal{G}_s$ and to apply it to our particular case, let us first briefly remind the main idea of the analysis performed in [46].

3.2.1 From generalized sine kernel to cycle integrals

Suppose that the function $\mathcal{G}_s$ appearing in (3.1) is realized as a product of functions of one variable, i.e.

$$\mathcal{G}_s \left(\begin{array}{c} \{\lambda\} \\ \{z\} \end{array} \right) = \prod_{i=1}^{s} \varphi\left(\lambda_i\right) \, e^{g(z_i)}, \tag{3.18}$$

with g and φ holomorphic in some vicinity of $[-q, q]$. Then the cycle integral $\mathcal{I}_s[\mathcal{G}_s]$ (3.1) can be explicitly computed in terms of the s^{th} γ-derivative of the Fredholm determinant of the operator $I + \gamma V$ on $[-q, q]$, with kernel

$$V(\lambda, \mu) = F(\lambda) \frac{\sin\left\{\frac{m}{2}\left[p_0(\lambda) - p_0(\mu)\right] - \frac{i}{2}\left[g(\lambda) - g(\mu)\right]\right\}}{\pi \sinh(\lambda - \mu)}, \qquad F\left(\lambda\right) = e^{g(\lambda)} \varphi(\lambda). \tag{3.19}$$

Indeed, from the series representation of $\log\det\left[I + V\right]$, one has,

$$\frac{(-1)^{s-1}}{(s-1)!} \partial_\gamma^s \log\det\left[I + \gamma V\right]\Big|_{\gamma=0} = \int_{-q}^{q} \mathrm{d}^s\lambda \prod_{k=1}^{s} V\left(\lambda_k, \lambda_{k+1}\right), \quad \text{with} \quad \lambda_{s+1} \equiv \lambda_1, \tag{3.20}$$

which is exactly the cycle integral $\mathcal{I}_s\left[\mathcal{G}_s\right]$ once the residues in z have all been computed.

In order to obtain the asymptotic expansion for cycle integrals involving more general holomorphic functions $\mathcal{G}_s$ that are symmetric with respect to the variables $\lambda_1, \ldots, \lambda_s$ and $z_1, \ldots, z_s$ separately, one uses the fact that such functions admit an expansion of the form

$$\mathcal{G}_s \left(\begin{array}{c} \{\lambda\} \\ \{z\} \end{array} \right) = \sum_{k=1}^{\infty} \prod_{j=1}^{s} \varphi_k(\lambda_j) \, \phi_k(z_j), \tag{3.21}$$

where $\varphi_k(\lambda)$ and $\phi_k(z)$ are holomorphic in a neighborhood of the interval $[-q, q]$. Setting then

$$F_k(\lambda) = \varphi_k(\lambda)\phi_k(\lambda) \qquad e^{g_k(\lambda)} = \phi_k(\lambda), \tag{3.22}$$

and denoting the corresponding kernel (3.19) as V_k, we obtain

$$\mathcal{I}_s\left[\mathcal{G}_s\right] = \sum_{k=1}^{\infty} \frac{(-1)^{s+1}}{(s-1)!}\, \partial_\gamma^s \log\det\left[I + \gamma V_k\right]\big|_{\gamma=0}. \tag{3.23}$$

Hence, the asymptotics of cycle integrals for such $\mathcal{G}_s$ can be inferred from those of the Fredholm determinant of $I + \gamma V$, V being a generalized sine kernel of the form (3.19) (see [46] for a more rigorous proof of all this procedure).

The large m asymptotic behavior of $\log\det[I+\gamma V]$ was obtained in [46] by Riemann-Hilbert approach. There it was proved that, in the $m \to +\infty$ limit,

$$\log\det[I + \gamma V] = \mathcal{W}_0(m) + \mathrm{o}(1), \tag{3.24}$$

and the leading asymptotic term $\mathcal{W}_0$ was computed explicitly in terms of the function

$$\nu(\lambda) = -\frac{1}{2\pi i}\log\bigl(1 + \gamma F(\lambda)\bigr), \qquad \nu_\pm = \nu(\pm q). \tag{3.25}$$

It reads

$$\begin{aligned}\mathcal{W}_0(m,[\nu]) = -\int_{-q}^{q}\bigl[imp_0'(\lambda) + g'(\lambda)\bigr]\nu(\lambda)\,\mathrm{d}\lambda - \sum_{\sigma=\pm}\bigl\{\nu_\sigma^2\log[m\sinh(2q)p_0'(\sigma q)] - \log G(1,\nu_\sigma)\bigr\}\\ + \frac{1}{2}\int_{-q}^{q}\frac{\nu'(\lambda)\nu(\mu) - \nu(\lambda)\nu'(\mu)}{\tanh(\lambda-\mu)}\,\mathrm{d}\lambda\,\mathrm{d}\mu + \sum_{\sigma=\pm}\sigma\nu_\sigma\int_{-q}^{q}\frac{\nu_\sigma - \nu(\lambda)}{\tanh(\sigma q - \lambda)}\,\mathrm{d}\lambda,\end{aligned} \tag{3.26}$$

where $G(1,z) = G(1+z)G(1-z)$, and $G(z)$ is the Barnes function defined as the unique solution of the equation

$$G(1+z) = \Gamma(z)G(z), \quad \text{with} \quad G(1) = 1 \quad \text{and} \quad \frac{\mathrm{d}^3}{\mathrm{d}z^3}\log G(z) \geq 0, \quad z > 0. \tag{3.27}$$

It was proved in [46] that this leading term $\mathcal{W}_0$ gives rise to the leading term $I_s^{(0)}[\mathcal{G}_s]$ of (3.16) through the procedure described above.

In [46] were also computed the first oscillating and non-oscillating corrections to (3.24). Non-oscillating corrections to (3.24) lead to non-oscillating corrections in (3.16), and we *a priori* do not need them for our purpose. We have seen however that, even if the first oscillating correction (3.17) in (3.16) gives rise, through the process of summation that we will see in the next section, to a subleading term in the expansion for $\langle e^{\beta\mathcal{Q}_m}\rangle$, it may become leading once the difference-differential operator (1.4) is applied. Therefore we also recall the explicit expression of the first oscillating correction which was computed in [46]. It reads,

$$\mathcal{W}_1(m,[\nu])\, e^{im[p_0(q)-p_0(-q)]} + \mathcal{W}_{-1}(m,[\nu])\, e^{-im[p_0(q)-p_0(-q)]}, \tag{3.28}$$

with

$$\mathcal{W}_{\pm 1}(m,[\nu]) = \frac{\nu_+\nu_-\, u(q)^{\pm 1}}{\sinh^2(2q)\, p_0'(q)\, p_0'(-q)}\, m^{-2\pm 2(\nu_+ + \nu_-)} \tag{3.29}$$

and

$$u(q) = \prod_{\sigma=\pm} \left\{ e^{\sigma g(\sigma q)} \left[\sinh(2q)\, p_0'(\sigma q)\right]^{2\nu_\sigma} \frac{\Gamma(-\nu_\sigma)}{\Gamma(\nu_\sigma)} \exp\left[-2\sigma \int_{-q}^{q} \frac{\nu_\sigma - \nu(\lambda)}{\tanh(q-\sigma\lambda)} \mathrm{d}\lambda \right] \right\}. \tag{3.30}$$

The asymptotic estimates given above are uniform over γ for γ small enough: for all $\varepsilon > 0$, there exists γ_0 such that, $\forall \gamma < \gamma_0$, $2|\nu_+ + \nu_-| < \varepsilon$ and the corrections to (3.28) are of order less than $\mathrm{O}(m^{-3+\varepsilon})$. Therefore (3.28) indeed gives rise, through the process described above (see (3.23)), to the first oscillating correction O_s (3.17) of (3.16).

Remark 3.2. We would like to draw the reader's attention to a remarkable relation between the terms $\mathcal{W}_0$ and $\mathcal{W}_{\pm 1}$:

$$\mathcal{W}_{\pm 1}(m,[\nu])\, e^{\pm i m [p_0(q) - p_0(-q)]} = e^{\mathcal{W}_0(m,[\nu \mp 1]) - \mathcal{W}_0(m,[\nu])}\,. \tag{3.31}$$

Hence, the terms $\mathcal{W}_{\pm 1}$ partly restore the original periodicity $\nu \to \nu + n$, $n \in \mathbb{Z}$, of $\det[I + \gamma V]$, that is broken if one considers the leading term $\mathcal{W}_0$ alone.

3.2.2 Leading asymptotic behavior of cycle integrals

We have seen how to generate the leading term $I_s^{(0)}[\mathcal{G}_s]$ and its first oscillating correction $O_s[\mathcal{G}_s]$, in the asymptotic expansion (3.16) of the cycle integral (3.1): $I_s^{(0)}[\mathcal{G}_s]$ is generated through (3.23) by the leading asymptotic part $\mathcal{W}_0$ (3.26), whereas $O_s[\mathcal{G}_s]$ comes from (3.28). It remains, in order to be able to sum up their respective contributions in (3.15), to describe more precisely their action as a functional of $\mathcal{G}_s$.

For the purpose of the summation of the series (3.15), it will be convenient to distinguish the g-dependent part of $\mathcal{W}_0$, presenting the latter as a sum of two terms $\mathcal{W}_0 = \mathcal{W}_0^{(0)} + \mathcal{W}_0^{(g)}$, where

$$\mathcal{W}_0^{(0)}(m,[\nu]) = \mathcal{W}_0(m,[\nu])\Big|_{g=0} \qquad \text{and} \qquad \mathcal{W}_0^{(g)}[\nu] = -\int_{-q}^{q} g'(\lambda)\, \nu(\lambda)\, \mathrm{d}\lambda. \tag{3.32}$$

It will split accordingly the functional $I_s^{(0)}[\mathcal{G}_s]$ into two parts:

$$I_s^{(0)}[\mathcal{G}_s] = (H_s + D_s)\,[\mathcal{G}_s], \tag{3.33}$$

where $H_s[\mathcal{G}_s]$ is the part of $I_s^{(0)}[\mathcal{G}_s]$ issued from $\mathcal{W}_0^{(0)}$, whereas $D_s[\mathcal{G}_s]$ comes from $\mathcal{W}_0^{(g)}$.

To obtain the explicit expression of D_s as a functional of $\mathcal{G}_s$, one should proceed as follows: compute the s^{th} γ-derivative of $\mathcal{W}_0^{(g)}[\nu]$ at $\gamma = 0$,

$$\frac{(-1)^{s-1}}{(s-1)!}\,\partial_\gamma^s\,\mathcal{W}_0^{(g)}[\nu]\Big|_{\gamma=0} = \int_{-q}^{q} \frac{\mathrm{d}\lambda}{2\pi i}\, g'(\lambda)\, F^s(\lambda)\,, \tag{3.34}$$

substitute in this expression $F(\lambda)$ and $g(\lambda)$ by (3.22),

$$\frac{(-1)^{s-1}}{(s-1)!}\,\partial_\gamma^s\,\mathcal{W}_0^{(g)}[\nu]\Big|_{\gamma=0} = \int_{-q}^{q} \frac{\mathrm{d}\lambda}{2\pi i}\, g_k'(\lambda)\, \left[\varphi_k(\lambda)\, e^{g_k(\lambda)}\right]^s, \tag{3.35}$$

and take finally the sum over k as in (3.23). This gives

$$D_s\left[\mathcal{G}_s\right] = \int_{-q}^{q} \frac{\mathrm{d}\lambda}{2\pi i}\, \partial_\epsilon \mathcal{G}_s\begin{pmatrix} \{\lambda\}^s \\ \lambda+\epsilon, \{\lambda\}^{s-1}\end{pmatrix}\Bigg|_{\epsilon=0}. \tag{3.36}$$

The notation $\{\lambda\}^s$ means here that all the original cycle variables $\lambda_1, \ldots, \lambda_s$ are now set equal to the same value λ. The variables $z_2, \ldots, z_s$ are likewise equal to λ. Notwithstanding, one should set here $z_1 = \lambda$ only after taking the derivative with respect to this variable.

One can derive similarly explicit formulas for the action of H_s and O_s. However, for general functions $\mathcal{G}_s$, the corresponding expressions are rather cumbersome (see [46]). Therefore we restrict our analysis to a specific class of functions. Let us suppose from now on that $\mathcal{G}_s$ can be presented as

$$\mathcal{G}_s\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix} = \prod_{j=1}^{s} \Phi_1\left(\lambda_j \mid \begin{matrix}\{\lambda\}\\ \{z\}\end{matrix}\right)\cdot \Phi_2\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix}, \tag{3.37}$$

where the functions

$$\Phi_1\left(\omega \mid \begin{matrix}\{\lambda\}\\ \{z\}\end{matrix}\right), \qquad \Phi_2\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix}, \tag{3.38}$$

are symmetric functions of the s variables $\{\lambda\}$ and of the s variables $\{z\}$ separately, and satisfy the reduction property

$$\Phi_1\left(\omega \mid \begin{matrix}\{\lambda\}\\ \{z\}\end{matrix}\right)\Bigg|_{\lambda_j=z_j} = \Phi_1\left(\omega \mid \begin{matrix}\{\lambda\}\setminus\lambda_j\\ \{z\}\setminus z_j\end{matrix}\right), \quad \Phi_2\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix}\Bigg|_{\lambda_j=z_j} = \Phi_2\begin{pmatrix}\{\lambda\}\setminus\lambda_j\\ \{z\}\setminus z_j\end{pmatrix}. \tag{3.39}$$

Note that, if the set $\{\lambda\}$ coincides with the set $\{z\}$, then the function Φ_2 becomes a constant whereas Φ_1 becomes a one-variable function:

$$\Phi_2\begin{pmatrix}\{\lambda\}\\ \{\lambda\}\end{pmatrix} = \Phi_2\begin{pmatrix}\emptyset\\ \emptyset\end{pmatrix} \equiv \Phi_2 = \text{const}, \quad \Phi_1\left(\omega \mid \begin{matrix}\{\lambda\}\\ \{\lambda\}\end{matrix}\right) = \Phi_1\left(\omega \mid \begin{matrix}\emptyset\\ \emptyset\end{matrix}\right) \equiv \Phi_1(\omega). \tag{3.40}$$

Observe that the functions $\mathcal{V}$ and $\widetilde{W}$ in (2.35) are precisely of this type and, hence, $\mathcal{F}_{|J_{\{\ell\}}|}$ is a particular case of (3.37). Moreover, in the process of consecutive summation of the series (3.15), we will permanently deal with functions of the form (3.37). A remarkable property of such functions is that, regardless of their multi-variable nature, everything happens in the asymptotic regime as if they were a pure product of one variable functions.

Let us give now the explicit form of the action of the functionals H_s and O_s on such functions.

Proposition 3.2. *Let $\mathcal{G}_s$ be a function of the form (3.37). Then*

$$H_s[\mathcal{G}_s] = \frac{(-1)^{s-1}}{(s-1)!}\, \partial_\gamma^s \mathcal{W}_0^{(0)}[\hat{\nu}]\Big|_{\gamma=0} \cdot \Phi_2 \,, \tag{3.41}$$

where

$$\hat{\nu}(\omega) = -\frac{1}{2\pi i} \log\left[1+\gamma\Phi_1(\omega)\right], \tag{3.42}$$

and with $\Phi_1(\omega)$ and Φ_2 defined as in (3.40).

Proof — The explicit form of the functional $\mathcal{W}_0^{(0)}[\nu]$ follows from (3.26). However, to prove Proposition 3.2, it is enough that it can be written in the following quite general form:

$$\mathcal{W}_0^{(0)}[\nu] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-q}^{q} T_n(\xi_1,\dots,\xi_n) \prod_{j=1}^{n} \nu(\xi_j)\, \mathrm{d}^n\xi \,, \tag{3.43}$$

where $\nu(\xi)$ is given by (3.25) and T_n are some functions or distributions. In order to obtain the corresponding part of the asymptotics of the cycle integral of length s, one should first take the s^{th} γ-derivative

$$\partial_\gamma^s \mathcal{W}_0^{(0)}[\nu]\Big|_{\gamma=0} = \sum_{n=0}^{\infty} \sum_{k_1,\dots,k_n=1}^{s}{}' \prod_{j=1}^{n} \frac{\partial_\gamma^{k_j}\nu_0}{k_j!}\Big|_{\gamma=0} \int_{-q}^{q} T_n(\xi_1,\dots,\xi_n) \prod_{j=1}^{n} F^{k_j}(\xi_j)\, \mathrm{d}^n\xi \,, \tag{3.44}$$

where ν_0 is given by

$$\nu_0 = \frac{-1}{2\pi i} \log(1+\gamma)\,, \tag{3.45}$$

and prime means that the summation over $k_1,\dots,k_n$ is taken under the constraint $\sum_{j=1}^{n} k_j = s$. Then, using (3.21) and (3.22), we obtain

$$H_s[\mathcal{G}_s] = \frac{(-1)^{s-1}}{(s-1)!} \sum_{n=0}^{\infty} \sum_{k_1,\dots,k_n=1}^{s}{}' \prod_{j=1}^{n} \frac{\partial_\gamma^{k_j}\nu_0}{k_j!}\Big|_{\gamma=0} \int_{-q}^{q} T_n(\xi_1,\dots,\xi_n)\, \mathcal{G}_s\begin{pmatrix} \{\xi_1\}^{k_1},\dots,\{\xi_n\}^{k_n} \\ \{\xi_1\}^{k_1},\dots,\{\xi_n\}^{k_n} \end{pmatrix} \mathrm{d}^n\xi.$$

It remains to observe that, due to the reduction property (3.39),

$$\mathcal{G}_s\begin{pmatrix} \{\xi_1\}^{k_1},\dots,\{\xi_n\}^{k_n} \\ \{\xi_1\}^{k_1},\dots,\{\xi_n\}^{k_n} \end{pmatrix} = \Phi_2 \prod_{j=1}^{n} \Phi_1^{k_j}(\xi_j)\,. \tag{3.46}$$

Comparing with (3.44) we see that it gives (3.41) with $\hat{\nu}$ defined in (3.42). □

Proposition 3.3. *Let $\mathcal{G}_s$ be as in (3.37). Then*

$$O_s^{\pm}[\mathcal{G}_s] = \frac{(-1)^{s-1}}{(s-1)!}\,\partial_\gamma^s\, e^{\mathcal{W}_0^{(0)}[\hat{\nu}^{(\pm)}\mp 1]-\mathcal{W}_0^{(0)}[\hat{\nu}^{(\pm)}]}\Big|_{\gamma=0}\cdot\,\Phi_2\begin{pmatrix}\mp q\\ \pm q\end{pmatrix}, \tag{3.47}$$

where

$$\hat{\nu}^{(\pm)}(\omega) = -\frac{1}{2\pi i}\log\left[1+\gamma\Phi_1\left(\omega\,\Big|\,\begin{matrix}\mp q\\ \pm q\end{matrix}\right)\right]. \tag{3.48}$$

Proof — The proof is very similar to the one of Proposition 3.2. Consider, for example, the oscillating functional O_s^+. The action of this functional is generated by the oscillating term $\mathcal{W}_{+1}[\nu]\,e^{im[p_0(q)-p_0(-q)]}$, which in its turn is expressed in terms of $\mathcal{W}_0^{(0)}[\nu]$ by (3.31):

$$\mathcal{W}_{+1}[\nu]\;e^{im[p_0(q)-p_0(-q)]} = e^{\mathcal{W}_0^{(0)}[\nu-1]-\mathcal{W}_0^{(0)}[\nu]+g(q)-g(-q)}. \tag{3.49}$$

Once again, we do not need the complete explicit expression of $\mathcal{W}_0^{(0)}$. Let simply

$$e^{\mathcal{W}_0^{(0)}[\nu-1]-\mathcal{W}_0^{(0)}[\nu]} = \nu(q)\,\nu(-q)\;T[\nu]\,, \tag{3.50}$$

in which we have stressed that the oscillating term of the asymptotics of the Fredholm determinant is proportional to $\nu(q)\,\nu(-q)$. For the remaining part $T[\nu]$, we can use a representation similar to (3.43):

$$T[\nu] = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{-q}^{q} T_n(\xi_1,\ldots,\xi_n)\prod_{j=1}^{n}\nu(\xi_j)\;\mathrm{d}^n\xi\,, \tag{3.51}$$

where T_n are some functions or distributions. The s^{th} γ-derivative then reads

$$\begin{aligned}\partial_\gamma^s\Big(\nu(q)\,\nu(-q)\;T[\nu]\Big)\Big|_{\gamma=0} &= \sum_{n=0}^{\infty}\;\sum_{k_+,\,k_-,\,k_i=1}^{s}{}^{'}\;\frac{\partial_\gamma^{k_+}\nu_0}{k_+!}\,\frac{\partial_\gamma^{k_-}\nu_0}{k_-!}\,\prod_{j=1}^{n}\frac{\partial_\gamma^{k_j}\nu_0}{k_j!}\Big|_{\gamma=0}\\ &\quad\times\int_{-q}^{q}T_n(\xi_1,\ldots,\xi_n)\,F^{k_+}(q)\,F^{k_-}(-q)\prod_{j=1}^{n}F^{k_j}(\xi_j)\;\mathrm{d}^n\xi\,,\end{aligned} \tag{3.52}$$

where ν_0 is defined in (3.45), and prime means that the sum over k_+, k_-, $k_1,\ldots,k_n$ is taken under the constraint $k_+ + k_- + \sum_{j=1}^{n}k_j = s$. Then, using (3.21), (3.22), and multiplying (3.52) by $e^{g(q)-g(-q)}$, we obtain

$$\begin{aligned}O_s^+[\mathcal{G}_s] &= \sum_{n=0}^{\infty}\;\sum_{k_+,\,k_-,\,k_i=1}^{s}{}^{'}\;\frac{\partial_\gamma^{k_+}\nu_0}{k_+!}\,\frac{\partial_\gamma^{k_-}\nu_0}{k_-!}\,\prod_{j=1}^{n}\frac{\partial_\gamma^{k_j}\nu_0}{k_j!}\Big|_{\gamma=0}\\ &\quad\times\int_{-q}^{q}T_n(\xi_1,\ldots,\xi_n)\;\mathcal{G}_s\begin{pmatrix}-q,\;\{q\}^{k_+},\{-q\}^{k_--1},\{\xi_1\}^{k_1},\ldots,\{\xi_n\}^{k_n}\\ q,\;\{q\}^{k_+},\{-q\}^{k_--1},\{\xi_1\}^{k_1},\ldots,\{\xi_n\}^{k_n}\end{pmatrix}\mathrm{d}^n\xi\,.\end{aligned} \tag{3.53}$$

It remains to observe that, due to the reduction property (3.39),

$$\mathcal{G}_s\begin{pmatrix} -q,\ \{q\}^{k_+}, \{-q\}^{k_--1}, \{\xi_1\}^{k_1}, \dots, \{\xi_n\}^{k_n} \\ q,\ \{q\}^{k_+}, \{-q\}^{k_--1}, \{\xi_1\}^{k_1}, \dots, \{\xi_n\}^{k_n} \end{pmatrix}$$

$$= \Phi_1^{k_+}\begin{pmatrix} q \mid {-q} \\ \quad q \end{pmatrix} \Phi_1^{k_-}\begin{pmatrix} -q \mid {-q} \\ \quad q \end{pmatrix} \prod_{j=1}^{n} \Phi_1^{k_j}\begin{pmatrix} \xi_j \mid {-q} \\ \quad q \end{pmatrix} \cdot \Phi_2\begin{pmatrix} -q \\ q \end{pmatrix}. \qquad (3.54)$$

Comparing with (3.52) we arrive at the action (3.47) with $\hat{\nu}^{(\pm)}$ defined in (3.48). □

Thus, the action of the functionals D_s, H_s and O_s is explicitly defined at least on the class of functions $\mathcal{G}_s$ admitting the representation (3.37). In that way we have found the leading non-oscillating and oscillating asymptotics of cycle integrals. However, this is not yet enough to obtain a correct estimate of the infinite series of multi-cycle integrals (3.15). The matter is that some a priori sub-leading corrections after their summation may give non-vanishing leading contribution. We will consider this question in more details in Section 4.2, where the mechanism of summation will become clear. Here we merely would like to note that some part of the corrections could easily be included into the action of the functionals considered above. For example, we did not use the explicit form of the functional $\mathcal{W}_0^{(0)}[\nu]$ in the proof of Proposition 3.2. Hence, instead of $\mathcal{W}_0^{(0)}[\nu]$ alone, we could as well have considered all the terms in the asymptotic expansion of $\log\det[I+\gamma V]$ which can be written in the form (3.43), and the statement of this proposition would still be valid.

4 Asymptotic summation of cycle integrals

In this section we sum up the series (3.15) in the asymptotic regime $m \to \infty$. We already labeled each pure cycle integral $\mathcal{I}_{(s,p)}$ with respect to the position of the variables on which it acts. We define similarly $D_{(s,p)}$ (resp. $H_{(s,p)}$ and $O_{(s,p)}$) as the operator that acts on the variables of the p^{th} cycle of length s as D_s (resp. H_s and O_s), and by doing nothing to all the other variables. Then, if we also define R_s to be the functional given by the remaining part of (3.16) (which is in principle of order o(1) for non-oscillating corrections, and of order $\mathrm{o}(\frac{1}{m^2})$ for oscillating ones), and $R_{(s,p)}$ to be its conterpart acting on the cycle (s,p), we can reexpress (3.15) as

$$G_{1\dots n}(\gamma) = \sum_{\ell_1,\dots,\ell_n=0}^{\infty} \prod_{s=1}^{n} \frac{(u_s)^{\ell_s}}{\ell_s!} \prod_{p=1}^{\ell_s} \left[D_{(s,p)} + H_{(s,p)} + O_{(s,p)} + R_{(s,p)}\right] * \mathcal{F}_{|J_{\{\ell\}}|}, \qquad (4.1)$$

where $u_s = \frac{(-1)^{s-1}\gamma^s}{s}$. There, just as in the case of multi-cycle integrals, the order of the different operators is not important due to the symmetry of the function $\mathcal{F}_{|J_{\{\ell\}}|}$ in its two sets of arguments $\{\lambda\}$ and $\{z\}$.

Before starting to sum up explicitely the series (4.1), let us make a usefull remark: since we only need the n^{th} γ-derivative of $G_{1\dots n}$ at $\gamma = 0$ (see (3.14)), it is enough for our purpose to obtain a γ-*equivalent* form of the result.

Definition 4.1. *Two functions* $\varphi_1(\gamma)$ *and* $\varphi_2(\gamma)$ *are said to be* γ-equivalent of order n *if*

$$\partial_\gamma^k \varphi_1(\gamma)\Big|_{\gamma=0} = \partial_\gamma^k \varphi_2(\gamma)\Big|_{\gamma=0}, \qquad k = 0, 1, \ldots, n\,. \tag{4.2}$$

Therefore, at any stage of the computation, we may replace $G_{1\ldots n}$ by some γ-equivalent function of order n which, by abuse of notations, will still be called $G_{1\ldots n}$.

4.1 Summation of the action of $H_{(s,p)}$

Let us start by summing up the action of the operators $H_{(s,p)}$. We first expand the products of the operators in (4.1) according to the binomial formula:

$$\frac{1}{\ell_s!}\prod_{p=1}^{\ell_s}\left[D_{(s,p)} + H_{(s,p)} + O_{(s,p)} + R_{(s,p)}\right]$$
$$= \sum_{r_s=0}^{\ell_s} \frac{1}{r_s!\,(\ell_s - r_s)!}\prod_{p=1}^{r_s}\left[D_{(s,p)} + O_{(s,p)} + R_{(s,p)}\right]\prod_{p=r_s+1}^{\ell_s} H_{(s,p)}\,. \tag{4.3}$$

As we already mentionned, such an operation is possible due to the symmetry property of the function $\mathcal{F}_{|J_{\{\ell\}}|}$ on which the operators in (4.3) act. Substituting this into (4.1) and changing the order of summation, we arrive at

$$G_{1\ldots n}(\gamma) = \sum_{r_1,\ldots,r_n=0}^{\infty}\prod_{s=1}^{n}\frac{(u_s)^{r_s}}{r_s!}\prod_{s=1}^{n}\prod_{p=1}^{r_s}\left[D_{(s,p)} + O_{(s,p)} + R_{(s,p)}\right]$$
$$\times \sum_{\ell_1,\ldots,\ell_n=0}^{\infty}\prod_{s=1}^{n}\frac{(u_s)^{\ell_s}}{\ell_s!}\prod_{s=1}^{n}\prod_{p=1}^{\ell_s} H_{(s,p)} * \mathcal{F}_{|J_{\{r\}}|+|J_{\{\ell\}}|}\begin{pmatrix}\{\lambda\}_{J_{\{r\}}}\cup\{\mu\}_{J_{\{\ell\}}}\\ \{z\}_{J_{\{r\}}}\cup\{y\}_{J_{\{\ell\}}}\end{pmatrix}. \tag{4.4}$$

Here, for convenience, we have divided the set of arguments of $\mathcal{F}_{|J_{\{\ell\}}|+|J_{\{r\}}|}$ into two subsets: the operators $D_{(s,p)}$, $O_{(s,p)}$ and $R_{(s,p)}$ act on the variables $\{\lambda\}_{J_{\{r\}}}$ and $\{z\}_{J_{\{r\}}}$, while the operators $H_{(s,p)}$ act on $\{\mu\}_{J_{\{\ell\}}}$ and $\{y\}_{J_{\{\ell\}}}$.

Recall that each individual $H_{(s,p)}$ acts non trivially (as the functional H_s) only on variables $\mu_{s,p,j}$ and $y_{s,p,j}$, with $j = 1, \ldots, s$. Therefore one can write

$$H_{(s,p)} * \mathcal{F}_{|J_{\{r\}}|+|J_{\{\ell\}}|} = H_s\left[\mathcal{F}_{|J_{\{r\}}|+|J_{\{\ell\}}|}\begin{pmatrix}\{\mu_{s,p,j}\}_{1\le j\le s}\cup\{\lambda\}_{J_{\{r\}}}\cup\{\mu\}_{J_{\{\ell\}}^{\widehat{s,p}}}\\ \{y_{s,p,j}\}_{1\le j\le s}\cup\{z\}_{J_{\{r\}}}\cup\{y\}_{J_{\{\ell\}}^{\widehat{s,p}}}\end{pmatrix}\right], \tag{4.5}$$

where H_s acts on the first sets of variables $\{\mu_{s,p,j}\}_{1\le j\le s}$ and $\{y_{s,p,j}\}_{1\le j\le s}$ according to (3.41), while all other variables can be considered as auxiliary parameters. The set $J_{\{\ell\}}^{\widehat{s,p}}$ is defined as

$$J_{\{\ell\}}^{\widehat{s,p}} = J_{\{\ell\}} \setminus \{(s,p,j)\ :\ 1 \le j \le s\}\,. \tag{4.6}$$

It is clear that $\mathcal{F}_{|J_{\{r\}}|+|J_{\{\ell\}}|}$ has the form (3.37), with

$$\Phi_1(\omega) = \mathcal{V}_{|J_{\{r\}}|+|J_{\{\ell\}}|}\left(\omega \mid \begin{array}{c} \{\mu_{s,p,j}\}_{1\le j\le s} \\ \{y_{s,p,j}\}_{1\le j\le s}\end{array}\right), \tag{4.7}$$

$$\Phi_2 = \prod_{\omega\in\{\lambda\}_{J_{\{r\}}}\cup\{\mu\}_{J^{\widehat{s,p}}_{\{\ell\}}}} \mathcal{V}_{|J_{\{r\}}|+|J_{\{\ell\}}|}\left(\omega \mid \begin{array}{c} \{\mu_{s,p,j}\}_{1\le j\le s} \\ \{y_{s,p,j}\}_{1\le j\le s}\end{array}\right)\cdot \widetilde{W}_{|J_{\{r\}}|+|J_{\{\ell\}}|}\left(\begin{array}{c} \{\mu_{s,p,j}\}_{1\le j\le s} \\ \{y_{s,p,j}\}_{1\le j\le s}\end{array}\right), \tag{4.8}$$

where, for simplicity, we have omitted the additional sets $\{\lambda\}_{J_{\{r\}}}\cup\{\mu\}_{J^{\widehat{s,p}}_{\{\ell\}}}$ and $\{z\}_{J_{\{r\}}}\cup\{y\}_{J^{\widehat{s,p}}_{\{\ell\}}}$ in all the arguments in (4.7), (4.8). Applying Proposition 3.2, we obtain

$$H_{(s,p)} * \mathcal{F}_{|J_{\{r\}}|+|J_{\{\ell\}}|} = \frac{(-1)^{s-1}}{(s-1)!}\,\partial_\gamma^s \mathcal{W}_0^{(0)}[\nu^{\widehat{s,p}}]\Big|_{\gamma=0}\cdot \mathcal{F}_{|J_{\{r\}}|+|J^{\widehat{s,p}}_{\{\ell\}}|}, \tag{4.9}$$

where

$$\nu^{\widehat{s,p}}(\omega) = \frac{-1}{2\pi i}\log\left[1+\gamma\mathcal{V}_{|J_{\{r\}}|+|J^{\widehat{s,p}}_{\{\ell\}}|}\left(\omega \mid \begin{array}{c} \{\lambda\}_{J_{\{r\}}}\cup\{\mu\}_{J^{\widehat{s,p}}_{\{\ell\}}} \\ \{z\}_{J_{\{r\}}}\cup\{y\}_{J^{\widehat{s,p}}_{\{\ell\}}}\end{array}\right)\right]. \tag{4.10}$$

The obtained result still admitting the representation (3.37), we can act with all operators $H_{(s,p)}$ consecutively. It gives

$$\prod_{s=1}^{n}\prod_{p=1}^{\ell_s} H_{(s,p)} * \mathcal{F}_{|J_{\{r\}}|+|J_{\{\ell\}}|} = \prod_{s=1}^{n}\left(\frac{(-1)^{s-1}}{(s-1)!}\,\partial_\gamma^s \mathcal{W}_0^{(0)}[\hat\nu]\Big|_{\gamma=0}\right)^{\ell_s}\mathcal{F}_{|J_{\{r\}}|}\left(\begin{array}{c}\{\lambda\}_{J_{\{r\}}}\\ \{z\}_{J_{\{r\}}}\end{array}\right), \tag{4.11}$$

where

$$\hat\nu(\omega) = \frac{-1}{2\pi i}\log\left[1+\gamma\mathcal{V}_{|J_{\{r\}}|}\left(\omega \mid \begin{array}{c}\{\lambda\}_{J_{\{r\}}}\\ \{z\}_{J_{\{r\}}}\end{array}\right)\right]. \tag{4.12}$$

Substituting this into (4.4) and summing up over all ℓ_s we arrive at

$$G_{1\dots n}(\gamma) = \sum_{r_1,\dots,r_n=0}^{\infty}\prod_{s=1}^{n}\frac{(u_s)^{r_s}}{r_s!}\prod_{s=1}^{n}\prod_{p=1}^{r_s}\left[D_{(s,p)}+O_{(s,p)}+R_{(s,p)}\right]*\mathcal{F}^{(H)}_{|J_{\{r\}}|}\left(\begin{array}{c}\{\lambda\}_{J_{\{r\}}}\\ \{z\}_{J_{\{r\}}}\end{array}\right), \tag{4.13}$$

where

$$\mathcal{F}^{(H)}_{|J_{\{r\}}|}\left(\begin{array}{c}\{\lambda\}_{J_{\{r\}}}\\ \{z\}_{J_{\{r\}}}\end{array}\right) = \mathcal{F}_{|J_{\{r\}}|}\left(\begin{array}{c}\{\lambda\}_{J_{\{r\}}}\\ \{z\}_{J_{\{r\}}}\end{array}\right)\cdot\exp\left\{\sum_{s=1}^{n}\frac{\gamma^s}{s!}\,\partial_\gamma^s\mathcal{W}_0^{(0)}(m,[\hat\nu])\Big|_{\gamma=0}\right\}. \tag{4.14}$$

Since it will turn out to be rather important for the summation of the remaing terms, we have here explicitly indicated that the functional $\mathcal{W}_0^{(0)}(m,[\hat\nu])$ depends on the distance m.

We can now perform the summation over s in (4.14) by extending it up to infinity, which means that we replace $\mathcal{F}^{(H)}$ by the following γ-equivalent function, still denoted by $\mathcal{F}^{(H)}$:

$$\mathcal{F}^{(H)}_{|J_{\{r\}}|}\begin{pmatrix}\{\lambda\}_{J_{\{r\}}}\\ \{z\}_{J_{\{r\}}}\end{pmatrix} = \mathcal{F}_{|J_{\{r\}}|}\begin{pmatrix}\{\lambda\}_{J_{\{r\}}}\\ \{z\}_{J_{\{r\}}}\end{pmatrix}\cdot \exp\left\{\mathcal{W}_0^{(0)}(m,[\hat{\nu}])\right\}. \tag{4.15}$$

Thus, the successive action of operators $H_{(s,p)}$ produces a complete decoupling of parts of the variables, and eventually restores some of the Fredholm determinant asymptotics (3.26).

Remark 4.1. Note that the summation performed here is *exact*, and that we did not use the explicit expression of $\mathcal{W}_0^{(0)}(m,[\hat{\nu}])$. We could as well have considered the action of some operator $\widetilde{H}_s$ including not only the leading contribution H_s, but also subleading (non-oscillating) ones, provided they originate from the same kind of terms (3.43). In fact, the very same process will enable us, in the next subsection, to sum up some part of R_s as well.

4.2 Contributions from $R_{(s,p)}$

Unlike the original function $\mathcal{F}_{|J_{\{r\}}|}$ (2.35), the obtained function $\mathcal{F}^{(H)}_{|J_{\{r\}}|}$ depends on the distance m. Let us specify this dependence. We have

$$\mathcal{W}_0^{(0)}(m,[\hat{\nu}]) = -im\int_{-q}^{q} p_0'(\lambda)\,\hat{\nu}(\lambda)\,\mathrm{d}\lambda - \sum_{\sigma=\pm}\hat{\nu}_\sigma^2\,\log\left[m\,\sinh(2q)\,p_0'(\sigma q)\right] + \tilde{C}[\hat{\nu}], \tag{4.16}$$

with

$$\tilde{C}[\hat{\nu}] = \frac{1}{2}\int_{-q}^{q}\frac{\hat{\nu}'(\lambda)\hat{\nu}(\mu)-\hat{\nu}(\lambda)\hat{\nu}'(\mu)}{\tanh(\lambda-\mu)}\mathrm{d}\lambda\mathrm{d}\mu + \sum_{\sigma=\pm}\left[\sigma\hat{\nu}_\sigma\int_{-q}^{q}\frac{\hat{\nu}_\sigma-\hat{\nu}(\lambda)}{\tanh(\sigma q-\lambda)}\mathrm{d}\lambda + \log G(1,\hat{\nu}_\sigma)\right], \tag{4.17}$$

in which $\hat{\nu}(\omega)$ is given by (4.12). Then the function $\mathcal{F}^{(H)}_{|J_{\{r\}}|}$ reads

$$\mathcal{F}^{(H)}_{|J_{\{r\}}|} = \exp\left\{-im\int_{-q}^{q}p_0'(\omega)\,\hat{\nu}(\omega)\,\mathrm{d}\omega\right\}\cdot\prod_{\sigma=\pm}\left[m\,\sinh(2q)\,p_0'(\sigma q)\right]^{-\hat{\nu}^2(\sigma q)}\cdot e^{\tilde{C}[\hat{\nu}]}$$
$$\times\,\widetilde{W}_{|J_{\{r\}}|}\begin{pmatrix}\{\lambda\}_{J_{\{r\}}}\\ \{z\}_{J_{\{r\}}}\end{pmatrix}\prod_{t\in J_{\{r\}}}\mathcal{V}_{|J_{\{r\}}|}\left(\lambda_t\;\middle|\;\begin{matrix}\{\lambda\}_{J_{\{r\}}}\\ \{z\}_{J_{\{r\}}}\end{matrix}\right). \tag{4.18}$$

In order to complete the calculations, we should in principle sum up the action of the operators $D_{(s,p)}$ and $O_{(s,p)}$ on $\mathcal{F}^{(H)}_{|J_{\{r\}}|}$, *provided that $R_{(s,p)}$ produces only subleading corrections.* This was the case when acting on m-independent functions of the form (3.37) but, due to the fact that $\mathcal{F}^{(H)}_{|J_{\{r\}}|}$ now depends on m, this is no longer true. Let us explain why.

The new function $\mathcal{F}^{(H)}_{|J_{\{r\}}|}$ is still of the form (3.37), but now Φ_2 depends on m as

$$\Phi_2\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix} = \exp\left\{m\,\Phi\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix} + \log m\ \widetilde{\Phi}\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix}\right\}\cdot\widetilde{\Phi}_2\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix}, \tag{4.19}$$

with Φ, $\widetilde{\Phi}$ and $\widetilde{\Phi}_2$ satisfying the reduction property (3.39). We see that the differential operator in $D_{(s,p)}$, when acting on such m-dependent functions, may produce terms that are linear over the distance m: hence, although this operator originates from a term of order O(1) in (3.16), it behaves as a term of order m when acting on (4.18). The same phenomenon may happen for the action of $R_{(s,p)}$: if the latter contains some derivative operators, then these operators may produce non vanishing contributions to the final answer although they are effectively subleading when acting on each cycle; for instance, a differential operator $\frac{1}{m}\partial$ gives a term of order $\mathrm{O}(\frac{1}{m})$ when acting on the original function $\mathcal{F}_{|J_{\{r\}}|}$, but it gives a contribution of order O(1) when acting on the new function $\mathcal{F}^{(H)}_{|J_{\{r\}}|}$ which is the result of summation over all cycles of the action of $H_{(s,p)}$ on $\mathcal{F}_{|J_{\{r\}}|}$. Therefore, we need more informations on the structure of the subleading terms in (3.16) in order to be able to extract from R_s the part of the corrections that eventually contributes to the leading order.

The structure of the series (3.16) for $\mathcal{I}_s[\mathcal{G}_s]$ was studied from the asymptotic expansion of the Fredholm determinant (3.19) and related Riemann–Hilbert problem in [46]. This analysis being extremely cumbersome, we do not have the explicit expression of the series (3.16). However, we obtained in [46] some information on the generic form of the corrections: they are given in terms of the function $\mathcal{G}_s$ or of its derivatives, the latter being evaluated at the endpoints $\pm q$ or integrated, together with some weight function, on the interval $[-q, q]$. Therefore the functional R_s indeed contains differential operators. However, it was proved in [46] that the non-oscillating correction $I_s^{(n;\,\mathrm{nosc})}[\mathcal{G}_s]$ of order n in (3.16) contains *at most derivatives of order* n of $\mathcal{G}_s$, whereas the oscillating correction $I_s^{(n;\,\mathrm{osc})}[\mathcal{G}_s]$ of order n contains *at most derivatives of order* $n-2$. They correspond respectively to the contribution of some differential operator $\frac{1}{m^n}\mathfrak{D}_s^{(n;\,\mathrm{nosc})}$ and $\frac{1}{m^n}\mathfrak{D}_s^{(n;\,\mathrm{osc})}$ in R_s, $\mathfrak{D}_s^{(n;\,\mathrm{nosc})}$ being of order at most n and $\mathfrak{D}_s^{(n;\,\mathrm{osc})}$ of order at most $n-2$. Therefore, only the part of *maximal order* of these operators may produce, when acting on functions of the form (3.37), (4.19) (and more precisely on the exponent), contributions on the same order as D_s or O_s. These are the contributions that we need to determine.

For convenience, in order to distinguish the effect of non-oscillating and oscillating corrections, we define two functionals R_s^{nosc} and R_s^{osc}: R_s^{nosc} contains all the non-oscillating corrections with respect to $I_s^{(0)}$, and R_s^{osc} contains all the oscillating corrections with respect to O_s, in the sense of the asymptotic series (3.16)[2].

[2]It means that we choose some arbitrary integer n_0 and consider the asymptotic development for R_s^{nosc} and R_s^{osc} up to $\mathrm{O}(\log^s m/m^{n_0+1})$.

4.2.1 Non-oscillating contribution

Let us first discuss the effect of the non-oscillating part R_s^{nosc} of the corrections. Clearly, as we already noticed, at each order n, only the differential operators $\frac{1}{m^n}\mathfrak{D}_s^{(n;\,\text{nosc})}$, and more precisely *the part* $\frac{1}{m^n}\widetilde{\mathfrak{D}}_s^{(n;\,\text{nosc})}$ *of* $\frac{1}{m^n}\mathfrak{D}_s^{(n;\,\text{nosc})}$ *which results in the maximal order n of derivatives*, may eventually produce, when acting on m-dependent functions (4.19), contributions of order $\mathrm{O}(1)$. For $n \geq 1$, the explicit expression of the differential operator $\widetilde{\mathfrak{D}}_s^{(n;\,\text{nosc})}$ of maximal order n is not known, but it can be shown from [46] that it has the form

$$\widetilde{\mathfrak{D}}_s^{(n;\,\text{nosc})} = \sum_{\sigma=\pm} \sum_{\{k\},\{\ell\}} C_\sigma(\{k\},\{\ell\}) \prod_{j=1}^{s} \partial_{z_j}^{k_j} \partial_{\lambda_j}^{\ell_j} \bigg|_{\lambda_j = z_j = \sigma q}, \qquad \sum_{j=1}^{s} (k_j + \ell_j) = n, \tag{4.20}$$

where $C_\sigma(\{k\},\{\ell\})$ are some computable coefficients. Here the action of the derivatives (on some function $\mathcal{G}_s$) is followed by the evaluation of all the variables z and λ at the same point σq, $\sigma = \pm$.

Obviously, when acting on some m-dependent function $\mathcal{G}_s$ of the type (3.37), (4.19), this operator can produce $\mathrm{O}(1)$ contribution only if it acts on the part of the exponent which is linear in m, i.e. on

$$\exp\left\{ m\, \Phi \begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} \right\}. \tag{4.21}$$

Using the reduction property (3.39) for Φ_1 and Φ_2 and the fact that the contributions of order $\mathrm{O}(1)$ can be obtained only in case each derivative hits exactly once the exponent (4.21), we obtain

$$\frac{1}{m^n} \widetilde{\mathfrak{D}}_s^{(n;\,\text{nosc})}[\mathcal{G}_s] = \sum_{\sigma=\pm} \Phi_1^s(\sigma q) \cdot \Phi_2 \\ \times \left\{ {\sum_{\{k\},\{\ell\}}}' C_\sigma(\{k\},\{\ell\}) \prod_{j=1}^{s} \left[\tilde{g}'(z_j)\right]^{k_j} \left[-\tilde{g}'(\lambda_j)\right]^{\ell_j} \right\} \bigg|_{\lambda_j = z_j = \sigma q} \cdot \left[1 + \mathrm{O}\left(\frac{\log m}{m} \right) \right], \tag{4.22}$$

where the prime means that the sum is taken under the constraint $\sum (k_j + \ell_j) = n$. Here we have used the conventions (3.40) for the reduced functions Φ_1 and Φ_2, and we have defined the one-variable function $\tilde{g}'$ to be given in terms of the reduced function Φ as

$$\tilde{g}'(\lambda) = \partial_\epsilon \Phi \begin{pmatrix} \lambda \\ \lambda + \epsilon \end{pmatrix} \bigg|_{\epsilon=0}. \tag{4.23}$$

Hence, everything happens (at least as far as the leading term is concerned) as if we would be acting on functions $\widetilde{\mathcal{G}}_s$ of the type

$$\widetilde{\mathcal{G}}_s \begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} = \Phi_2 \cdot \prod_{j=1}^{s} \Phi_1(\lambda_j) \cdot \exp\left\{ m \sum_{j=1}^{s} \left[\widetilde{g}(z_j) - \widetilde{g}(\lambda_j) \right] \right\}. \tag{4.24}$$

Thus, the problem of finding the structure of the O(1) contributions coming from the action of $R_s^{\rm nosc}$ on some m-dependent function $\mathcal{G}_s$ of the form (4.19) reduces to the computation of the leading behavior of the functional action of $\mathcal{I}_s$ on $\widetilde{\mathcal{G}}_s$ (4.24). Strictly speaking, the functional argument of the cycle integral $\mathcal{I}_s[\widetilde{\mathcal{G}}_s]$ should not depend on the distance m but, in the case (4.24), we can simply re-define $p_0 \to \hat{p} = p_0 - i\tilde{g}$ and $\widetilde{\mathcal{G}}_s \to \hat{\mathcal{G}}_s = \prod_{j=1}^{s} \Phi_1(\lambda_j)$ (we recall that Φ_2 is just a constant that will be in factor of the result). This is then a special case of (3.18), and therefore the leading contribution of the corresponding cycle integral can be obtained from the $s^{\rm th}$ γ-derivative at $\gamma = 0$ of

$$\mathcal{W}_0^{(0)}(m,[\hat{\nu}])\Big|_{p_0\to\hat{p}} = -im\int_{-q}^{q} \hat{p}'(\lambda)\,\hat{\nu}(\lambda)\,\mathrm{d}\lambda - \sum_{\sigma=\pm}\left\{\hat{\nu}_\sigma^2 \log\left[m\sinh(2q)\,\hat{p}'(\sigma q)\right]\right\} + C[\hat{\nu}]\,, \quad (4.25)$$

where $\hat{\nu}$ is given in terms of Φ_1 as in (3.42). The fisrt term of the r.h.s. of (4.25), which is linear in m, corresponds to the action on (4.24) of the operator D_s that we already know. It is the other (constant in m) $\hat{p}'$-dependent part that eventually reproduces the contribution we want to compute, namely the O(1) contribution coming from the action of the whole series $\sum_{n\geq 1}\frac{1}{m^n}\widetilde{\mathfrak{D}}_s^{(n;\,\rm nosc)}$ on m-dependent functions of the form (4.24). It reads

$$\frac{(-1)^{s-1}}{(s-1)!}\,\partial_\gamma^s \sum_{\sigma=\pm}\left\{-\hat{\nu}^2(\sigma q)\,\log\frac{\hat{p}'(\sigma q)}{p_0'(\sigma q)}\right\}\Bigg|_{\gamma=0}\cdot\,\Phi_2\,, \quad (4.26)$$

where we recall that $\hat{\nu}$ is given in terms of Φ_1 as in (3.42), and that

$$\hat{p}'(\lambda) = p_0'(\lambda) - i\,\partial_\epsilon \Phi\begin{pmatrix}\lambda\\ \lambda+\epsilon\end{pmatrix}\Bigg|_{\epsilon=0}. \quad (4.27)$$

We have thus determined the explicit contribution of order O(1) coming from the action of $R_s^{\rm nosc}$ on m-dependent function $\mathcal{G}_s$ of the type (3.37), (4.19). Let us call this quantity $R_s^{(1;\,\rm nosc)}[\mathcal{G}_s]$: explicitely, $R_s^{(1;\,\rm nosc)}[\mathcal{G}_s]$ is given by the expression (4.26). If moreover we denote by $R_s^{(\rm sub;\,nosc)}[\mathcal{G}_s]$ the remaining part of $R_s^{\rm nosc}[\mathcal{G}_s]$, we have

$$R_s^{\rm nosc}[\mathcal{G}_s] = R_s^{(1;\,\rm nosc)}[\mathcal{G}_s] + R_s^{(\rm sub;\,nosc)}[\mathcal{G}_s]. \quad (4.28)$$

Remark 4.2. What we have done here is to act formally with the asymptotic series (3.16) on m-dependent functions $\mathcal{G}_s$ of the form (3.37), (4.19), and to identify, at each formal order n in m, the part of $\frac{1}{m^n}I_s^{(n;\,\rm nosc)}[\mathcal{G}_s]$ contributing to the order O(1). We know that the summed contribution of these leading parts, that we have called $R_s^{(1;\,\rm nosc)}[\mathcal{G}_s]$, is given by (4.26). We also know that the remaining terms in each $\frac{1}{m^n}I_s^{(n;\,\rm nosc)}[\mathcal{G}_s]$ are of order $\mathrm{O}(\frac{\log m}{m})$. However, it is not obvious to prove rigorously that the sum $R_s^{(\rm sub;\,nosc)}[\mathcal{G}_s]$ of these remaining terms stay subdominant in the final answer. It should follow from the analysis presented in [46] but here we just assume that it is so.

It is now possible to sum up the successive actions of the corresponding operators $R^{(1;\,\mathrm{nosc})}_{(s,p)}$ on the function $\mathcal{F}^{(H)}$ (4.18). The action of $R^{(1;\,\mathrm{nosc})}_s$ being similar to the one of H_s (compare (4.26) to (3.41)), we can apply exactly the same procedure as in the previous subsection, and the action of $R^{(1;\,\mathrm{nosc})}_{(s,p)}$ exponentiates similarly as in (4.14). Therefore, a γ-equivalent form of the series (4.13) is

$$G_{1\dots n}(\gamma) = \sum_{r_1,\dots,r_n=0}^{\infty} \frac{u_s^{r_s}}{r_s!} \prod_{p=1}^{r_s} \Big[D_{(s,p)} + O_{(s,p)} + R^{(\mathrm{sub};\,\mathrm{nosc})}_{(s,p)} + R^{\mathrm{osc}}_{(s,p)} \Big] * \mathcal{F}^{(R)}_{|J_{\{r\}}|} \begin{pmatrix} \{\lambda\}_{J_{\{r\}}} \\ \{z\}_{J_{\{r\}}} \end{pmatrix}, \quad (4.29)$$

where

$$\mathcal{F}^{(R)}_{|J_{\{r\}}|} \begin{pmatrix} \{\lambda\}_{J_{\{r\}}} \\ \{z\}_{J_{\{r\}}} \end{pmatrix} = \mathcal{F}^{(H)}_{|J_{\{r\}}|} \begin{pmatrix} \{\lambda\}_{J_{\{r\}}} \\ \{z\}_{J_{\{r\}}} \end{pmatrix} \cdot \prod_{\sigma=\pm} \left[\frac{\hat{p}'(\sigma q)}{p_0'(\sigma q)} \right]^{-\hat{\nu}^2(\sigma q)} . \quad (4.30)$$

Using the explicit expression (4.18) for $\mathcal{F}^{(H)}$, we obtain for the new function $\mathcal{F}^{(R)}$:

$$\mathcal{F}^{(R)}_{|J_{\{r\}}|} = \exp\left\{ -im \int_{-q}^{q} p_0'(\omega)\, \hat{\nu}(\omega)\, \mathrm{d}\omega \right\} \cdot \prod_{\sigma=\pm} \big[m \,\sinh(2q)\, \hat{p}'(\sigma q) \big]^{-\hat{\nu}^2(\sigma q)} \cdot e^{\tilde{C}[\hat{\nu}]}$$
$$\times \widetilde{W}_{|J_{\{r\}}|} \begin{pmatrix} \{\lambda\}_{J_{\{r\}}} \\ \{z\}_{J_{\{r\}}} \end{pmatrix} \prod_{t \in J_{\{r\}}} \mathcal{V}_{|J_{\{r\}}|} \left(\lambda_t \;\middle|\; \begin{matrix} \{\lambda\}_{J_{\{r\}}} \\ \{z\}_{J_{\{r\}}} \end{matrix} \right), \quad (4.31)$$

in which $\hat{\nu}$ is still given by (4.12) and

$$\hat{p}'(\mu) = p_0'(\mu) - \int_{-q}^{q} p_0'(\omega)\, \partial_\epsilon \hat{\nu} \left(\omega \;\middle|\; \begin{matrix} \mu, & \{\lambda\}_{J_{\{r\}}} \\ \mu+\epsilon, & \{z\}_{J_{\{r\}}} \end{matrix} \right) \bigg|_{\epsilon=0} \mathrm{d}\omega \,. \quad (4.32)$$

4.2.2 Oscillating contribution

Let us also compute the leading action of the oscillating part R^{osc}_s on m-dependent function $\mathcal{G}_s$ of the type (3.37), (4.19). Similar considerations based on the explicit structure of the oscillating corrections in the asymptotic series (3.16) [46] allow us to decompose $R^{\mathrm{osc}}_s[\mathcal{G}_s]$ as

$$R^{\mathrm{osc}}_s[\mathcal{G}_s] = R^{(1;\,\mathrm{osc})}_s[\mathcal{G}_s] + R^{(\mathrm{sub};\,\mathrm{osc})}_s[\mathcal{G}_s]. \quad (4.33)$$

Here $R^{(1;\,\mathrm{osc})}_s[\mathcal{G}_s]$ takes into account the contributions to the leading oscillating order of the action of each subleading oscillating functional in (3.16), and therefore is an oscillating contribution of the same order as $O_s[\mathcal{G}_s]$, whereas the remainder $R^{(\mathrm{sub};\,\mathrm{osc})}_s[\mathcal{G}_s]$ is subleading with respect to $O_s[\mathcal{G}_s]$. Like in the previous case, the modification at the leading order reduces to the extension of the term $\log p_0'$ to $\log \hat{p}' = \log\big(p_0' - \frac{i}{m}\partial_\epsilon\big)$. Namely, let us define $\widetilde{O}_s = O_s + R^{(1;\,\mathrm{osc})}_s \equiv \widetilde{O}^+_s + \widetilde{O}^-_s$.

Then, for $\mathcal{G}_s$ being given by (3.37), (4.19), we have

$$\widetilde{O}_s^{\pm}[\mathcal{G}_s] = \frac{(-1)^{s-1}}{(s-1)!}\,\partial_\gamma^s \exp\Big\{\pm im\big[p_0'(q) - p_0'(-q)\big] + \tilde{C}[\hat{\nu}^{(\pm)} \mp 1] - \tilde{C}[\hat{\nu}^{(\pm)}]\Big\} \times \prod_{\sigma=\pm}\big[m\sinh(2q)\,\hat{p}_\pm'(\sigma q)\big]^{\pm 2\hat{\nu}^{(\pm)}(\sigma q)-1}\Big|_{\gamma=0} \cdot \Phi_2\begin{pmatrix} \mp q \\ \pm q\end{pmatrix}, \quad (4.34)$$

where $\hat{\nu}^{(\pm)}(\omega)$ is still given by (3.48) and

$$\hat{p}_\pm'(\sigma q) = p_0'(\sigma q) \mp i\sigma\,\partial_{\epsilon_\sigma}\Phi\begin{pmatrix} \mp q + \epsilon_\mp \\ \pm q + \epsilon_\pm\end{pmatrix}\Bigg|_{\epsilon=0}. \quad (4.35)$$

The successive action of the oscillating operators $\widetilde{O}_{(s,p)}$ will be summed up perturbatively in Section 4.4.

4.3 The action of $D_{(s,p)}$ as a continuous generalization of multiple Lagrange series

Let us now sum up the successive action of operators $D_{(s,p)}$. Applying again the binomial formula, we obtain

$$G_{1\ldots n}(\gamma) = \sum_{r_1,\ldots,r_n=0}^{\infty}\prod_{s=1}^{n}\frac{(u_s)^{r_s}}{r_s!}\prod_{s=1}^{n}\prod_{p=1}^{r_s}\Big[\widetilde{O}_{(s,p)} + \widetilde{R}_{(s,p)}\Big] \times \sum_{\ell_1,\ldots,\ell_n=0}^{\infty}\prod_{s=1}^{n}\frac{(u_s)^{\ell_s}}{\ell_s!}\prod_{s=1}^{n}\prod_{p=1}^{\ell_s} D_{(s,p)} * \mathcal{F}^{(R)}_{|J_{\{r\}}|+|J_{\{\ell\}}|}\begin{pmatrix} \{\mu\}_{J_{\{r\}}}\cup\{\lambda\}_{J_{\{\ell\}}} \\ \{y\}_{J_{\{r\}}}\cup\{z\}_{J_{\{\ell\}}}\end{pmatrix}, \quad (4.36)$$

in which the operators $D_{(s,p)}$ act on the variables $\{\lambda\}$ and $\{z\}$, whereas $\widetilde{O}_{(s,p)}$ and $\widetilde{R}_{(s,p)} \equiv R^{(\mathrm{sub;\,nosc})}_{(s,p)} + R^{(\mathrm{sub;\,osc})}_{(s,p)}$ act on $\{\mu\}$ and $\{y\}$. We therefore need to compute

$$\mathcal{F}^{(D)}_{|J_{\{r\}}|}\begin{pmatrix} \{\mu\}_{J_{\{r\}}} \\ \{y\}_{J_{\{r\}}}\end{pmatrix} = \sum_{\ell_1,\ldots,\ell_n=0}^{\infty}\prod_{s=1}^{n}\frac{(u_s)^{\ell_s}}{\ell_s!} \times \prod_{s=1}^{n}\prod_{p=1}^{r_s} D_{(s,p)} * \mathcal{F}^{(R)}_{|J_{\{r\}}|+|J_{\{\ell\}}|}\begin{pmatrix} \{\mu\}_{J_{\{r\}}}\cup\{\lambda\}_{J_{\{\ell\}}} \\ \{y\}_{J_{\{r\}}}\cup\{z\}_{J_{\{\ell\}}}\end{pmatrix}. \quad (4.37)$$

This series is actually a *continuous generalization of the multiple Lagrange series.*

Recall that the standard Lagrange series has the form (see e.g. [81])

$$G_0 = \sum_{n=0}^{\infty}\frac{1}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}\big(F(\epsilon)\phi^n(\epsilon)\big)\Big|_{\epsilon=0}, \quad (4.38)$$

where $F(z)$ and $\phi(z)$ are some functions holomorphic in a vicinity of the origin. If the series (4.38) is convergent, then it can be summed up in terms of the solution of the equation

$$z - \phi(z) = 0, \tag{4.39}$$

and the sum is given by

$$G_0 = \frac{F(z)}{1 - \phi'(z)}\,. \tag{4.40}$$

It is possible to generalize the Lagrange series (4.38) to the case of functions ϕ and F depending on several variables, and even to consider the corresponding continuous limit. The sum of such generalized series is then expressed in terms of a *solution of an integral equation.* All the details about these generalizations are given in Appendix C. We just apply here the result of this appendix to our particular case.

Substituting the action (3.36) into (4.37) and setting $k = |J_{\{r\}}|$, we obtain

$$\mathcal{F}_k^{(D)}\begin{pmatrix} \{\mu\} \\ \{y\} \end{pmatrix} = \sum_{\ell_1,\dots,\ell_n=0}^{\infty} \prod_{s=1}^{n} \frac{u_s^{\ell_s}}{\ell_s!} \int_{-q}^{q} \left\{ \prod_{s=1}^{n}\prod_{p=1}^{\ell_s} \frac{\mathrm{d}\lambda_{s,p}}{2\pi i} \cdot \partial_{\epsilon_{s,p}} \right\}\Bigg|_{\epsilon_{s,p}=0}$$
$$\times\, \mathcal{F}^{(R)}_{|J_{\{\ell\}}|+k}\left(\begin{array}{c} \{\mu\}\ ,\ \bigcup_{s,p} \{\lambda_{s,p}\}^{s} \\ \{y\}\ ,\ \bigcup_{s,p} \big(\{\lambda_{s,p} + \epsilon_{s,p}\} \cup \{\lambda_{s,p}\}^{s-1} \big) \end{array} \right). \tag{4.41}$$

Recall that the notation $\{\lambda_{s,p}\}^{s}$ means that the variable $\lambda_{s,p}$ is repeated s times. Moreover, here and in the following, we use simplified notations for

$$\bigcup_{s,p} \equiv \bigcup_{\substack{1\le s\le n \\ 1\le p\le \ell_s}}\,, \qquad \sum_{s,p} \equiv \sum_{s=1}^{n}\sum_{p=1}^{\ell_s}\,. \tag{4.42}$$

Using now the explicit expression (4.31) of $\mathcal{F}^{(R)}_{|J_{\{\ell\}}|}$, we have

$$\mathcal{F}_k^{(D)} = \sum_{\ell_1,\dots,\ell_n=0}^{\infty} \prod_{s=1}^{n} \frac{u_s^{\ell_s}}{\ell_s!} \int_{-q}^{q} \left\{ \prod_{s=1}^{n}\prod_{p=1}^{\ell_s} \frac{\mathrm{d}\lambda_{s,p}}{2\pi i} \cdot \partial_{\epsilon_{s,p}} \right\}\Bigg|_{\epsilon_{s,p}=0} \exp\left\{ -im \int_{-q}^{q} p_0'(\omega)\,\hat{\nu}(\omega)\,\mathrm{d}\omega \right\}$$
$$\times \prod_{\sigma=\pm} \big[m \sinh(2q)\, \hat{p}'(\sigma q) \big]^{-\hat{\nu}^2(\sigma q)} \cdot e^{\tilde{C}[\hat{\nu}]} \cdot \widetilde{W}_{|\ell|+k}\left(\begin{array}{c} \bigcup_{u,v} \{\lambda_{u,v}\} \\ \bigcup_{u,v} \{\lambda_{u,v} + \epsilon_{u,v}\} \end{array} \right)$$
$$\times \prod_{t=1}^{k} \mathcal{V}_{|\ell|+k}\left(\mu_t \,\Big|\, \begin{array}{c} \bigcup_{u,v} \{\lambda_{u,v}\} \\ \bigcup_{u,v} \{\lambda_{u,v} + \epsilon_{u,v}\} \end{array} \right) \prod_{s=1}^{n}\prod_{p=1}^{\ell_s} \mathcal{V}^{s}_{|\ell|+k}\left(\lambda_{s,p} \,\Big|\, \begin{array}{c} \bigcup_{u,v} \{\lambda_{u,v}\} \\ \bigcup_{u,v} \{\lambda_{u,v} + \epsilon_{u,v}\} \end{array} \right), \tag{4.43}$$

where $|\ell| = \sum_{s=1}^{n} \ell_s$. In order to lighten notations, we did not write explicitly the arguments $\{\mu\}$ and $\{y\}$ of the functions $\mathcal{V}$ and $\widetilde{W}$. We also have omitted most arguments of the functions $\hat{\nu}$ and p'. For instance,

$$\hat{\nu}(\omega) \equiv \hat{\nu}\left(\omega \mid \begin{array}{cc} \{\mu\}, & \bigcup_{u,v}\{\lambda_{u,v}\} \\ \{y\}, & \bigcup_{u,v}\{\lambda_{u,v}+\epsilon_{u,v}\} \end{array}\right) = \frac{-1}{2\pi i}\log\left[1+\gamma\mathcal{V}\left(\omega \mid \begin{array}{cc} \{\mu\}, & \bigcup_{u,v}\{\lambda_{u,v}\} \\ \{y\}, & \bigcup_{u,v}\{\lambda_{u,v}+\epsilon_{u,v}\} \end{array}\right)\right]. \tag{4.44}$$

and one should understand $p'(\sigma q)$ in the similar way.

Observe that we have, at most, to compute the first order derivative with respect to each variable $\epsilon_{s,p}$. It is thus enough to linearize the arguments of the functions $\mathcal{V}$ and $\widetilde{W}$ in the vicinities of $\epsilon_{s,p} = 0$. The linearized version of the function $\mathcal{V}$ reads

$$\mathcal{V}_{|\ell|+k}\left(\omega \mid \begin{array}{c} \bigcup_{s,p}\{\lambda_{s,p}\} \\ \bigcup_{s,p}\{\lambda_{s,p}+\epsilon_{s,p}\} \end{array}\right) \to \exp\left\{\beta - i\sum_{s,p}\epsilon_{s,p}\,K\left(\omega-\lambda_{s,p}\right)+\Psi_k(\omega)\right\}-1\,, \tag{4.45}$$

where $K(\omega-\lambda)$ is given by (1.10), and the function $\Psi_k(\omega)$ by

$$\Psi_k(\omega) \equiv \Psi_k\left(\omega \mid \begin{array}{c} \{\mu\} \\ \{y\} \end{array}\right)$$
$$= \sum_{t=1}^{k}\log\frac{\sinh(\omega-\mu_t+i\zeta)\,\sinh(\omega-y_t-i\zeta)}{\sinh(\omega-y_t+i\zeta)\,\sinh(\omega-\mu_t-i\zeta)} = i\sum_{t=1}^{k}\int_{y_t}^{\mu_t}K(\omega-\lambda)\,\mathrm{d}\lambda\,. \tag{4.46}$$

The linearized form of $\hat{\nu}(\omega)$ follows immediately from (4.45):

$$\hat{\nu}(\omega) \to -\frac{1}{2\pi i}\log\left\{1+\gamma\left[\exp\left(\beta-i\sum_{s,p}\epsilon_{s,p}\,K\left(\omega-\lambda_{s,p}\right)+\Psi_k(\omega)\right)-1\right]\right\}. \tag{4.47}$$

As for the function $\widetilde{W}$, its linearized form is rather cumbersome. On the other hand this function will not play a significant role in this section, therefore we give here the linearization of $\widetilde{W}$ in a rather formal way, namely

$$\widetilde{W}_{|\ell|+k}\left(\begin{array}{c} \bigcup_{s,p}\{\lambda_{s,p}\} \\ \bigcup_{s,p}\{\lambda_{s,p}+\epsilon_{s,p}\} \end{array}\right) \to F_k\left(\sum_{s,p}\epsilon_{s,p}\,g^{(1)}(\lambda_{s,p});\sum_{s,p}\sum_{t,r}\epsilon_{s,p}\,\epsilon_{t,r}\,g^{(2)}(\lambda_{s,p},\lambda_{t,r});\ldots\right), \tag{4.48}$$

where $g^{(1)}$, $g^{(2)}, \ldots$ are some functions of $\lambda_{s,p}$. The explicit formulas for them will be given in

Section 5.1. Then we have

$$\mathcal{F}_k^{(D)} = \sum_{\ell_1,\dots,\ell_n=0}^{\infty} \prod_{s=1}^{n} \frac{1}{\ell_s!} \int\limits_{-q}^{q} \left\{ \prod_{s=1}^{n} \prod_{p=1}^{\ell_s} \mathrm{d}\lambda_{s,p} \cdot \partial_{\epsilon_{s,p}} \right\} \Bigg|_{\epsilon_{s,p}=0} \exp\left\{ -im \int\limits_{-q}^{q} p_0'(\omega)\, \hat{\nu}(\omega)\, \mathrm{d}\omega \right\}$$
$$\times \prod_{\sigma=\pm} \left[m \sinh(2q)\, \hat{p}'(\sigma q) \right]^{-\hat{\nu}^2(\sigma q)} \cdot e^{\tilde{C}[\hat{\nu}]} \cdot F_k\left(\{g^{(i)}\} \right)$$
$$\times \prod_{t=1}^{k} f\left(\mu_t \mid -i \sum_{u,v} \epsilon_{u,v}\, K(\mu_t - \lambda_{u,v}) \right) \prod_{s=1}^{n} \prod_{p=1}^{\ell_s} \frac{u_s}{2\pi i} f^s \left(\lambda_{s,p} \mid -i \sum_{u,v} \epsilon_{u,v}\, K(\lambda_{s,p} - \lambda_{u,v}) \right), \tag{4.49}$$

where the notation $F_k\left(\{g^{(i)}\}\right)$ is a compact form of (4.48) and where we have set

$$f\,(\lambda \mid x) = e^{\beta + x + \Psi_k(\lambda)} - 1\,. \tag{4.50}$$

The representation (4.49) is exactly the continuous generalization of the multiple Lagrange series considered in Appendix C (see (C.19)). Thus, we can simply apply the results derived in this appendix to the concrete case of functions f, F_k.

Let us give the final result: a γ-equivalent form of the new function $\mathcal{F}_k^{(D)}$ (4.49) reads

$$\mathcal{F}_k^{(D)} = \exp\left\{ im \int\limits_{-q}^{q} p_0'(\omega)\, z(\omega)\, \mathrm{d}\omega \right\} \prod_{\sigma=\pm} \left[m \sinh(2q)\, p'(\sigma q) \right]^{-z^2(\sigma q)} \cdot e^{\tilde{C}[-z]}$$
$$\times \frac{F_k\left(\int\limits_{-q}^{q} g^{(1)}(\omega)\, z(\omega)\, \mathrm{d}\omega; \quad \dots \right) \prod\limits_{t=1}^{k} f\left(\mu_t \mid -i \int\limits_{-q}^{q} K(\omega - \mu_t)\, z(\omega)\, \mathrm{d}\omega \right)}{\det\left[I + iK(\xi - \lambda)\, f_\Sigma'\left(\xi \mid -i \int\limits_{-q}^{q} K(\omega - \xi)\, z(\omega)\, \mathrm{d}\omega \right) \right]}. \tag{4.51}$$

Here the function $z(\lambda)$ solves a non-linear integral equation,

$$z(\lambda) = f_\Sigma\left(\lambda \mid -i \int\limits_{-q}^{q} K(\omega - \lambda)\, z(\omega)\, \mathrm{d}\omega \right), \tag{4.52}$$

with f_Σ given by

$$f_\Sigma(\lambda \,|\, x) = \frac{1}{2\pi i} \log\left[1 + \gamma \left(e^{\beta + x + \Psi_k(\lambda)} - 1 \right) \right]. \tag{4.53}$$

The symbol f_Σ' means the derivative of $f_\Sigma(\lambda \,|\, x)$ over x. The Fredholm determinant in the denominator of (4.51) is the Jacobian of the equation (4.52) (compare with (4.39) and (4.40)).

The function[1] $p'(\lambda)$, which appears in (4.51) taken in the points $\lambda = \pm q$, satisfies a linear integral equation,

$$p'(\lambda) + \int_{-q}^{q} K_\Sigma(\lambda,\omega)\, p'(\omega)\, \mathrm{d}\omega = p_0'(\lambda)\,, \tag{4.54}$$

where

$$K_\Sigma(\lambda,\omega) = i\, K(\lambda-\omega)\; f_\Sigma'\left(\omega \mid -i\int_{-q}^{q} K(\xi-\omega)\, z(\xi)\, \mathrm{d}\xi\right). \tag{4.55}$$

We would like finally to mention that the functions $z(\lambda)$ and $p'(\lambda)$ depend on the parameters $\{\mu\}$ and $\{y\}$ through Ψ_k (see (4.53)),

$$z(\lambda) = z\left(\lambda \mid \begin{array}{c}\{\mu\}\\ \{y\}\end{array}\right), \qquad p'(\lambda) = p'\left(\lambda \mid \begin{array}{c}\{\mu\}\\ \{y\}\end{array}\right), \tag{4.56}$$

and still satisfy the reduction property (3.39).

Let us now explain how to obtain this result from the results of Appendix C.

According to Appendix C.4, the final answer for $\mathcal{F}_k^{(D)}$ is given in terms of a function $z^{(n)}(\lambda)$ solving an integral equation (see (C.25))

$$z^{(n)}(\lambda) - f_{\Sigma_n}\left(\lambda \mid -i\int_{-q}^{q} K(\omega-\lambda)\, z^{(n)}(\omega)\, \mathrm{d}\omega\right) = 0, \qquad f_{\Sigma_n}(\lambda\,|\,x) = \sum_{s=1}^{n} \frac{u_s}{2\pi i}\, f^s(\lambda\,|\,x). \tag{4.57}$$

Following our usual strategy, we can replace this equation with a γ-equivalent one by sending $n\to\infty$ in f_{Σ_n} as $u_s \propto \gamma^s$. Then the function $z^{(n)}(\mu)$ is replaced by $z(\mu)$ satisfying the integral equation (4.52), since

$$\begin{aligned}\lim_{n\to\infty} f_{\Sigma_n}(\lambda\,|\,x) &= \sum_{s=1}^{\infty} \frac{(-1)^{s-1}\gamma^s}{2\pi i s}\left(e^{\beta+x+\Psi_k(\lambda)}-1\right)^s\\ &= \frac{1}{2\pi i}\log\left[1+\gamma f(\lambda\,|\,x)\right] = f_\Sigma(\lambda\,|\,x)\,.\end{aligned} \tag{4.58}$$

Comparing (4.51) and (4.49) one can observe that the action of the operators $D_{(s,p)}$ leads to the replacement of the functional argument $\hat\nu(\lambda)$ by $-z(\lambda)$. This happens due to the fact that the linearized form of $\hat\nu$ (4.47) coincides with the r.h.s. of the equation (4.52) up to the sign

$$\hat\nu\left(\omega \mid \begin{array}{c}\bigcup_{s,p}\{\lambda_{s,p}\}\\ \bigcup_{s,p}\{\lambda_{s,p}+\epsilon_{s,p}\}\end{array}\right) \to -f_\Sigma\left(\omega \mid -i\sum_{s,p}\epsilon_{s,p}\, K\left(\omega-\lambda_{s,p}\right)\right). \tag{4.59}$$

[1] We use the notation $p'(\lambda)$ for this function, since eventually $p(\lambda)$ will have the sense of the dressed momentum (1.12).

Since the summation of the continuous generalization of the Lagrange series reduces to the replacement of the sum over $\epsilon_{s,p}$ by the integral with $z(\lambda)$ (see (C.24)), we obtain

$$\hat{\nu}(\omega) \to -f_\Sigma\left(\omega \mid -i\int_{-q}^{q} K(\omega-\lambda)\, z(\lambda)\, \mathrm{d}\lambda\right) = -z(\omega). \tag{4.60}$$

Finally, let us explain how the function $p'(\lambda)$ and the equation (4.54) arise. From (4.32) and (4.60), we have[3]

$$\hat{p}'(\lambda) \to p'(\lambda) = p_0'(\lambda) + \int_{-q}^{q} p_0'(\omega)\, \partial_\epsilon z\left(\omega \mid \begin{array}{c}\lambda\\ \lambda+\epsilon\end{array}\right)\Big|_{\epsilon=0}. \tag{4.61}$$

Differentiating the integral equation (4.52) we find

$$\partial_\epsilon z\left(\omega \mid \begin{array}{c}\lambda\\ \lambda+\epsilon\end{array}\right)\Big|_{\epsilon=0} + \int_{-q}^{q} K_\Sigma(\xi,\omega)\, \partial_\epsilon z\left(\xi \mid \begin{array}{c}\lambda\\ \lambda+\epsilon\end{array}\right)\Big|_{\epsilon=0} \mathrm{d}\xi = -K_\Sigma(\lambda,\omega)\,. \tag{4.62}$$

Here $-K_\Sigma(\lambda,\omega)$ in the r.h.s. appears due to the derivative of the function Ψ_k (4.46). Hence the derivative $\partial_\epsilon z$ can be expressed in terms of the resolvent $R_\Sigma(\lambda,\omega)$ of the integral operator K_Σ:

$$\partial_\epsilon z\left(\omega \mid \begin{array}{c}\lambda\\ \lambda+\epsilon\end{array}\right)\Big|_{\epsilon=0} = -R_\Sigma(\lambda,\omega)\,. \tag{4.63}$$

Substituting this into (4.61) we arrive at

$$p'(\lambda) = p_0'(\lambda) - \int_{-q}^{q} R_\Sigma(\lambda,\omega)\, p_0'(\omega)\, \mathrm{d}\omega\,, \tag{4.64}$$

which means that $p'(\lambda)$ solves the equation (4.54).

4.4 Oscillating corrections

The series (4.36) takes now the form

$$G_{1\ldots n}(\gamma) = \sum_{r_1,\ldots,r_n=0}^{\infty} \prod_{s=1}^{n} \frac{(u_s)^{r_s}}{r_s!} \prod_{s=1}^{n}\prod_{p=1}^{r_s} \left[\widetilde{O}_{(s,p)} + \widetilde{R}_{(s,p)}\right] * \mathcal{F}^{(D)}_{|J_{\{r\}}|}\left(\begin{array}{c}\{\mu\}_{J_{\{r\}}}\\ \{y\}_{J_{\{r\}}}\end{array}\right), \tag{4.65}$$

with $\mathcal{F}^{(D)}_{|J_{\{r\}}|}$ given by (4.51). Note that $\mathcal{F}^{(D)}_{|J_{\{r\}}|}$ is still an m-dependent function of the type (3.37), (4.19), and therefore we can use the results of Section 4.2.2.

[3]Here also, we do not specify that z depends on the extra sets of parameters $\{\mu\}$ and $\{y\}$.

At this stage, the functionals which remain to be summed up produce only subleading contributions. Hence, if we were only interested in the leading term of $\langle e^{\beta \mathcal{Q}_m} \rangle$, it would be enough to consider only the leading term of these series, namely the term $r_1 = \cdots = r_n = 0$. Recall however that we need to keep the leading oscillating contribution since, after second lattice derivative, it may produce a term competing with the leading non-oscillating one. We can nevertheless forget completely the remaining terms $\widetilde{R}_{(s,p)}$, since they merely produce corrections to one of these two main contributions. Moreover, since the oscillating operators $\widetilde{O}_{(s,p)}$ are themselves of sub-leading order, it is enough to restrict our analysis to the terms of (4.65) that are linear over $\widetilde{O}_{(s,p)}$, namely the terms for which one r_k takes the value 0 or 1 while all other are equal to 0. We thus obtain

$$G_{1\dots n}(\gamma) = \mathcal{F}_0^{(D)} \begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix} + \sum_{s=1}^{n} u_s \, \widetilde{O}_s \left[\mathcal{F}_s^{(D)} \begin{pmatrix} \{\mu\} \\ \{y\} \end{pmatrix} \right] + \text{corrections}, \tag{4.66}$$

in which the *corrections* are either non-oscillating corrections to the first term, or oscillating corrections to the second term (note that the non-oscillating corrections may be of order higher than this second term, but they are nevertheless not important for the final result). Observe once again that, due to the fact that $u_s \propto \gamma^s$, the summation over s in (4.66) can be extended to infinity. This will be done in the end of our calculations.

To complete the summation of the series, we should now calculate the functionals $\widetilde{O}_s \big[\mathcal{F}_s^{(D)}\big]$. Let us present $G_{1\dots n}(\gamma)$ in (4.66) as the sum of three parts

$$G_{1\dots n}(\gamma) = \sum_{\sigma=0,\pm 1} G_{1\dots n}^{(\sigma)}(\gamma) + \text{corrections}\,, \tag{4.67}$$

where

$$G_{1\dots n}^{(0)}(\gamma) = \mathcal{F}_0^{(D)} \begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix}, \qquad G_{1\dots n}^{(\pm)}(\gamma) = \sum_{k=1}^{n} u_k \, \widetilde{O}_k^{\pm} \left[\mathcal{F}_k^{(D)} \begin{pmatrix} \{\mu\} \\ \{y\} \end{pmatrix} \right]. \tag{4.68}$$

The non-oscillating part $G_{1\dots n}^{(0)}(\gamma)$ has in fact already been computed. It is given by (4.51) in which one should set $\{\mu\} = \{y\} = \emptyset$, which is easy since we almost did not specify this dependence: basically, the parameters $\{\mu\}$ and $\{y\}$ enter the functions $z(\lambda)$ and $p'(\lambda)$ (4.56) through the function Ψ_k (4.46). Thus, in order to obtain $G_{1\dots n}^{(0)}(\gamma)$, it is enough to set $\Psi_k = 0$ in (4.46) and $k = 0$ in (4.51).

Let us now compute the oscillating corrections $G_{1\dots n}^{(\pm)}(\gamma)$. The new function $\mathcal{F}_k^{(D)}$ (4.51) is of the type (3.37), (4.19), with

$$\Phi_1 \left(\xi \mid \begin{array}{c} \{\mu\} \\ \{y\} \end{array} \right) = f \left(\xi \mid -i \int_{-q}^{q} K(\omega - \xi)\, z(\omega)\, \mathrm{d}\omega \right), \tag{4.69}$$

where $f(\lambda|x)$ is given by (4.50) and

$$\Phi\left(\begin{array}{c}\{\mu\}\\ \{y\}\end{array}\right)=i\int_{-q}^{q}p_0'(\omega)\,z(\omega)\,\mathrm{d}\omega\,. \tag{4.70}$$

The functionals $\widetilde{O}_k^{\pm}$ (4.34) send $\mu_1=\mp q$, $y_1=\pm q$ and $\mu_j=y_j$ for $j>1$. This means that the function $z(\lambda)$ turns into $z^{(\pm)}(\lambda)$ satisfying the equation

$$z^{(\pm)}(\lambda)=f_\Sigma^{(\pm)}\left(\lambda\mid -i\int_{-q}^{q}K(\omega-\lambda)\,z^{(\pm)}(\omega)\,\mathrm{d}\omega\right), \tag{4.71}$$

with $f_\Sigma^{(\pm)}$ given by

$$f_\Sigma^{(\pm)}(\lambda\mid x)=\frac{1}{2\pi i}\log\left[1+\gamma\left(e^{\beta+x+\Psi^{(\pm)}(\lambda)}-1\right)\right], \tag{4.72}$$

and

$$\Psi^{(\pm)}(\omega)=\log\frac{\sinh(\omega-\mu+i\zeta)\,\sinh(\omega-y-i\zeta)}{\sinh(\omega-y+i\zeta)\,\sinh(\omega-\mu-i\zeta)}\bigg|_{\substack{\mu=\mp q\\ y=\pm q}}=\mp i\int_{-q}^{q}K(\omega-\lambda)\,\mathrm{d}\lambda\,. \tag{4.73}$$

Similarly the function $p'(\lambda)$ (4.54) turns into $p'_\pm(\lambda)$ satisfying

$$p'_\pm(\lambda)+\int_{-q}^{q}K_\Sigma^{(\pm)}(\lambda,\omega)\,p'_\pm(\omega)\,\mathrm{d}\omega=p_0'(\lambda)\,, \tag{4.74}$$

where

$$K_\Sigma^{(\pm)}(\lambda,\omega)=i\,K(\lambda-\omega)\,\left(f_\Sigma^{(\pm)}\right)'\left(\omega\mid -i\int_{-q}^{q}K(\xi-\omega)\,z^{(\pm)}(\xi)\,\mathrm{d}\xi\right). \tag{4.75}$$

Comparing (4.72) with (3.48) for $\hat{\nu}^{(\pm)}(\lambda)$ we conclude that $\hat{\nu}^{(\pm)}(\lambda)=-z^{(\pm)}(\lambda)$. Thus, we obtain

$$\widetilde{O}_k^{\pm}\big[\mathcal{F}_k^{(D)}\big]=\frac{(-1)^{k-1}}{(k-1)!}\,\partial_\gamma^k\exp\left\{\pm im\big[p_0'(q)-p_0'(-q)\big]+\tilde{C}[-z^{(\pm)}\mp 1]-\tilde{C}[-z^{(\pm)}]\right\}$$
$$\times\prod_{\sigma=\pm}\big[m\sinh(2q)\,\hat{p}'_\pm(\sigma q)\big]^{\mp 2z^{(\pm)}(\sigma q)-1}\Big|_{\gamma=0}\cdot\Phi_2\left(\begin{array}{c}\mp q\\ \pm q\end{array}\right). \tag{4.76}$$

Finally, similar arguments as in Section 4.3 (see (4.62) to (4.64)) enable us to show that the quantity $\hat{p}'_\pm(\sigma q)$ defined by (4.35), (4.70) is equal to $p'_\pm(\sigma q)$, where $p'_\pm(\lambda)$ is the solution of the equation (4.74).

Observe that the k^{th} γ-derivative acts on an expression which is independent on k. Therefore the sum over k in (4.68), once extended to infinity, gives the Taylor series

$$\sum_{k=1}^{\infty} u_k \, \widetilde{O}_k^{\pm}[\mathcal{F}_k^{(D)}] = \exp\Big\{ \pm im\big(p_0'(q) - p_0'(-q)\big) + \tilde{C}[-z^{(\pm)} \mp 1] - \tilde{C}[-z^{(\pm)}] \Big\} \\ \times \prod_{\sigma=\pm} \big[m \sinh(2q)\, p_{\pm}'(\sigma q)\big]^{\mp 2z^{(\pm)}(\sigma q) - 1} \cdot \Phi_2 \begin{pmatrix} \mp q \\ \pm q \end{pmatrix}. \quad (4.77)$$

Combining now all the above results, we obtain

$$G_{1\dots n}^{(\pm)}(\gamma) = \exp\left\{ im \int_{-q}^{q} p_0'(\lambda)\big(z^{(\pm)}(\lambda) \pm 1\big)\,\mathrm{d}\lambda \right\} \prod_{\sigma=\pm} \big[m \sinh(2q)\, p_{\pm}'(\sigma q)\big]^{-\left(z^{(\pm)}(\sigma q) \pm 1\right)^2} \\ \times \frac{F^{(\pm)}\left(\int_{-q}^{q} g_{\pm}^{(1)}(\lambda)\, z^{(\pm)}(\lambda)\,\mathrm{d}\lambda; \quad \dots \right) e^{\tilde{C}[-z^{(\pm)} \mp 1]}}{\det\left[I + iK(\xi - \lambda) \left(f_{\Sigma}^{(\pm)}\right)' \left(\xi \mid -i \int_{-q}^{q} K(\omega - \xi)\, z^{(\pm)}(\omega)\,\mathrm{d}\omega \right) \right]}. \quad (4.78)$$

Here in complete analogy with (4.48) the functions $F^{(\pm)}$ and $\{g_{\pm}^{(i)}\}$ are defined from the linearized form of $\widetilde{W}_{|\ell|+1}$

$$\widetilde{W}_{|\ell|+1} \begin{pmatrix} \mp q \,, \; \bigcup_{s,p} \{\lambda_{s,p}\} \\ \pm q \,, \; \bigcup_{s,p} \{\lambda_{s,p} + \epsilon_{s,p}\} \end{pmatrix} \to F^{(\pm)} \left(\sum_{s,p} \epsilon_{s,p}\, g_{\pm}^{(1)}(\lambda_{s,p}); \; \dots \right). \quad (4.79)$$

5 Leading asymptotic behavior of correlation functions

In the first part of this section, we gather all the previous results for the asymptotic behavior of $\langle e^{\beta \mathcal{Q}_m} \rangle$. This will enable us, in a second part, to obtain the leading asymptotic behavior at large distance of the spin-spin correlation function $\langle \sigma_1^z \, \sigma_{m+1}^z \rangle$.

5.1 Asymptotic expansion for $\langle e^{\beta \mathcal{Q}_m} \rangle$

Let us indicate explicitly that the functions $G_{1\dots n}^{(\sigma)}(\gamma)$ (4.68) depend on the distance m and parameter β: $G_{1\dots n}^{(\sigma)}(\gamma) = G_{1\dots n}^{(\sigma)}(\gamma|\beta, m)$. The remarkable property of the obtained γ-equivalent results is that they do not depend on n: $G_{1\dots n}^{(\sigma)}(\gamma|\beta, m) = G^{(\sigma)}(\gamma|\beta, m)$. Therefore the series (3.14) is the Taylor series of $G^{(\sigma)}(\gamma|\beta, m)$ at $\gamma = 1$, hence leading to :

$$\langle e^{\beta \mathcal{Q}_m} \rangle = \frac{1}{\det[I + \frac{1}{2\pi}K]} \sum_{\sigma=0,\pm} \sum_{n \geq 0} \frac{\partial_\gamma^n G^{(\sigma)}(\gamma|\beta, m)}{n!}\Big|_{\gamma=0} + \text{corrections} \\ = \sum_{\sigma=0,\pm} \widehat{G}^{(\sigma)}(\beta, m)\big[1 + \mathrm{o}(1)\big], \qquad \text{with} \quad \widehat{G}^{(\sigma)}(\beta, m) = \frac{G^{(\sigma)}(1|\beta, m)}{\det[I + \frac{1}{2\pi}K]}. \quad (5.1)$$

The term $\widehat{G}^{(0)}(\beta,m)$ in (5.1) gives the leading non-oscillating asymptotic behavior of $\langle e^{\beta \mathcal{Q}_m}\rangle$, while $\widehat{G}^{(\pm)}(\beta,m)$ describe its leading oscillating correction. Let us consider these two terms separately.

5.1.1 Leading non-oscillating behavior of $\langle e^{\beta \mathcal{Q}_m}\rangle$

It happens that, at $\gamma=1$, *the non-linear integral equation for the function $z(\lambda)$ (4.52) degenerates into a linear one* (taking also into account that, in this case, $\Psi_k=0$):

$$z(\lambda)=\frac{\beta}{2\pi i}-\int\limits_{-q}^{q}\frac{\mathrm{d}\mu}{2\pi}\,K(\lambda-\mu)\,z(\mu)\,. \tag{5.2}$$

Setting $z(\lambda)=\frac{\beta Z(\lambda)}{2\pi i}$, one gets the *integral equation (1.14) for the dressed charge*:

$$Z(\lambda)+\int\limits_{-q}^{q}\frac{\mathrm{d}\mu}{2\pi}\,K(\lambda-\mu)\,Z(\mu)=1\,. \tag{5.3}$$

The linear equation (4.54) also simplifies due to the fact that $f'_\Sigma=1/2\pi i$ at $\gamma=1$,

$$p'(\mu)+\frac{1}{2\pi}\int\limits_{-q}^{q}K(\mu-\lambda)\,p'(\lambda)\,\mathrm{d}\lambda=p'_0(\mu)\,. \tag{5.4}$$

Comparing the last equation with (1.11), we find that $p'(\lambda)=2\pi\rho(\lambda)$ and, hence, *the function $p(\lambda)$ can be identified with the dressed momentum* (see (1.12)).

Due to the same property of f'_Σ, the Fredholm determinant in the denominator of (4.51) turns into $\det[I+\frac{1}{2\pi}K]$. Thus, we have

$$\widehat{G}^{(0)}(\beta,m)=e^{m\beta D}\big[2\pi\rho(q)m\sinh(2q)\big]^{\frac{\beta^2\mathcal{Z}^2}{2\pi^2}}e^{\tilde{C}\left[\frac{\beta Z}{2\pi i}\right]}\,\frac{F_0\left(\frac{\beta}{2\pi i}\int\limits_{-q}^{q}g^{(1)}(\mu)\,Z(\mu)\,\mathrm{d}\mu;\ \ldots\right)}{\det^2\left[I+\frac{1}{2\pi}K\right]}\,, \tag{5.5}$$

where $\mathcal{Z}=Z(\pm q)$, and we have used that $Z(\lambda)$ and $\rho(\lambda)$ are even functions.

It remains to pass from the symbolic form of F_0 to a more specific one. When linearized, the product in the representation (2.20) for the function $\widetilde{W}_{|\ell|}$ turns into

$$\prod_{s,p}\prod_{s',p'}\frac{\sinh(z_{s,p}-\lambda_{s',p'}-i\zeta)\,\sinh(\lambda_{s',p'}-z_{s,p}-i\zeta)}{\sinh(z_{s,p}-z_{s',p'}-i\zeta)\,\sinh(\lambda_{s,p}-\lambda_{s',p'}-i\zeta)}\Bigg|_{\{z_{u,v}\}=\{\lambda_{u,v}+\epsilon_{u,v}\}}$$

$$\rightarrow\quad\exp\left\{\sum_{s,p}\sum_{s',p'}\epsilon_{s,p}\,\epsilon_{s',p'}\,g^{(2)}(\lambda_{s,p},\lambda_{s',p'})\right\}, \tag{5.6}$$

where

$$g^{(2)}(\lambda,\mu) = -\frac{1}{2}\left[\sinh^{-2}(\lambda-\mu+i\zeta)+\sinh^{-2}(\lambda-\mu-i\zeta)\right]. \tag{5.7}$$

After the summation of the continuous Lagrange series, this part becomes

$$\prod_{s,p}\prod_{s',p'}\frac{\sinh(z_{s,p}-\lambda_{s',p'}-i\zeta)\,\sinh(\lambda_{s',p'}-z_{s,p}-i\zeta)}{\sinh(z_{s,p}-z_{s',p'}-i\zeta)\,\sinh(\lambda_{s,p}-\lambda_{s',p'}-i\zeta)}\Bigg|_{\{z_{u,v}\}=\{\lambda_{u,v}+\epsilon_{u,v}\}} \quad\rightarrow\quad e^{\frac{\beta^2}{4\pi^2}C_0}, \tag{5.8}$$

where

$$C_0 = \int\limits_{-q}^{q}\frac{Z(\lambda)\,Z(\mu)}{\sinh^2(\lambda-\mu-i\zeta)}\,\mathrm{d}\lambda\,\mathrm{d}\mu\,. \tag{5.9}$$

The remaining part of $\widetilde{W}$ contains the Fredholm determinants (2.16), whose kernels depend on the following products (see (2.17), (2.18))

$$P_\alpha = \prod_{s,p}\frac{\sinh(w-\lambda_{s,p}+i\alpha\zeta)}{\sinh(w-z_{s,p}+i\alpha\zeta)}\Bigg|_{\{z_{s,p}\}=\{\lambda_{s,p}+\epsilon_{s,p}\}} \quad\rightarrow\quad \exp\left\{\sum_{s,p}\epsilon_{s,p}\,g^{(1,\alpha)}(\lambda_{s,p})\right\},\qquad \alpha=0,\pm1\,, \tag{5.10}$$

where $g^{(1,\alpha)}(\lambda)=\coth(w-\lambda+i\alpha\zeta)$. Let us introduce the $i\pi$-periodic Cauchy transform of the dressed charge Z,

$$\tilde z(\omega) = \frac{1}{2\pi i}\int\limits_{-q}^{q}\coth(\lambda-\omega)\,Z(\lambda)\,\mathrm{d}\lambda\,. \tag{5.11}$$

Then we obtain

$$P_\alpha \rightarrow e^{\beta\tilde z(w+i\alpha\zeta)},\qquad \alpha=0,\pm1\,. \tag{5.12}$$

Observe that the function $\tilde z$ has the following properties

$$\begin{aligned}&\tilde z(w+i\zeta)-\tilde z(w-i\zeta) = 1-Z(w),\\ &\tilde z_+(w)-\tilde z_-(w) = Z(w),\qquad w\in[-q,q],\end{aligned} \tag{5.13}$$

where (1.14) is used in the first equation, and $\tilde z_\pm$ are limiting values of $\tilde z$ from the upper and lower half planes in the second equation.

Therefore, the leading non-oscillating part of the generating function can be written as

$$\widehat{G}^{(0)}(\beta,m) = \mathcal{A}(\beta)\cdot e^{\beta m D}\left[2\pi\,\sinh(2q)\,\rho(q)\,m\right]^{\frac{\beta^2\mathcal{Z}^2}{2\pi^2}} G^2\!\left(1,\frac{\beta\mathcal{Z}}{2\pi i}\right) e^{\frac{\beta^2}{4\pi^2}(C_0-C_1)}\,. \tag{5.14}$$

Here the constant C_0 given by (5.9), $G(1,z)$ is the product of the Barnes functions (see (3.27)), and C_1 reads

$$C_1 = \frac{1}{2}\int\limits_{-q}^{q} \frac{Z'(\lambda)Z(\mu) - Z(\lambda)\, Z'(\mu)}{\tanh(\lambda-\mu)}\, d\lambda\, d\mu + 2\mathcal{Z}\int\limits_{-q}^{q} \frac{\mathcal{Z} - Z(\lambda)}{\tanh(q-\lambda)}\, d\lambda\,. \tag{5.15}$$

The coefficient $\mathcal{A}(\beta)$ is

$$\mathcal{A}(\beta) = \frac{(e^{\beta}-1)^2 \cdot \det\left[I + \frac{1}{2\pi i}\, U^{(\lambda)}_{\theta_1}(w,w')\right] \cdot \det\left[I + \frac{1}{2\pi i}\, U^{(z)}_{\theta_2}(w,w')\right]}{\left(e^{\beta\tilde{z}(\theta_1+i\zeta)} - e^{\beta+\beta\tilde{z}(\theta_1-i\zeta)}\right)\left(e^{-\beta\tilde{z}(\theta_2-i\zeta)} - e^{\beta-\beta\tilde{z}(\theta_2+i\zeta)}\right) \cdot \det^2[I + \frac{1}{2\pi}K]}\,, \tag{5.16}$$

where

$$U^{(\lambda)}_{\theta_1}(w,w') = -\frac{e^{\beta\tilde{z}(w)}[K_\kappa(w-w') - K_\kappa(\theta_1-w')]}{e^{\beta\tilde{z}(w+i\zeta)} - e^{\beta+\beta\tilde{z}(w-i\zeta)}}\,, \tag{5.17}$$

and

$$U^{(z)}_{\theta_2}(w,w') = \frac{e^{-\beta\tilde{z}(w')}[K_\kappa(w-w') - K_\kappa(w-\theta_2)]}{e^{-\beta\tilde{z}(w'-i\zeta)} - e^{\beta-\beta\tilde{z}(w'+i\zeta)}}\,, \tag{5.18}$$

and the kernels $U^{(\lambda)}_{\theta_1}(w,w')$ and $U^{(z)}_{\theta_2}(w,w')$ act on the contour Γ surrounding the interval $[-q,q]$.

5.1.2 Leading oscillating behavior of $\langle e^{\beta\mathcal{Q}_m}\rangle$

Consider for exemple the function $\widehat{G}^{(+)}(\beta,m)$, the case $\widehat{G}^{(-)}(\beta,m)$ being completely analogous. Like in the previous case, the function $f^{(+)}_{\Sigma}$ (4.72) becomes linear at $\gamma=1$, and $(f^{(+)}_{\Sigma})' = 1/2\pi i$. Therefore the Fredholm determinant in (4.78) is equal to $\det[I+\frac{1}{2\pi}K]$ like for the non-oscillating term $\widehat{G}^{(0)}(\beta,m)$ and the integral equation for $p'(\lambda)$ (5.4) remains the same.

As for the linear integral equation for $z^{(+)}(\lambda)$, it reads

$$z^{(+)}(\lambda) = \frac{\beta + \Psi^{(+)}(\lambda)}{2\pi i} - \int\limits_{-q}^{q} \frac{d\mu}{2\pi}\, K(\lambda-\mu)\, z^{(+)}(\mu)\,. \tag{5.19}$$

Using the expression (4.46) of $\Psi(\lambda)$, one can write (5.19) in the form

$$z^{(+)}(\lambda) = \frac{\beta}{2\pi i} - \int\limits_{-q}^{q} \frac{d\mu}{2\pi}\, K(\lambda-\mu)\,(z^{(+)}(\mu)+1)\,, \tag{5.20}$$

and the solution is

$$z^{(+)}(\lambda) = Z(\lambda)\left(\frac{\beta}{2\pi i}+1\right) - 1\,. \tag{5.21}$$

One should now notice that, everywhere but in the argument of $F^{(+)}$, the function $z^{(+)}(\lambda)$ enters in (4.78) only through the combination

$$z^{(+)}(\lambda) + 1 = \frac{(\beta + 2\pi i)\, Z(\lambda)}{2\pi i}, \tag{5.22}$$

which means that this part of the result can merely be obtained from the non-oscillating term via the replacement $\beta \to \beta + 2\pi i$. It is easy to see that it is enough to make the same shift $\beta \to \beta + 2\pi i$ in the function $F^{(+)}$: indeed, the Fredholm determinants in this function contain the following products

$$P_\alpha^{(+)} = \frac{\sinh(w + q + i\alpha\zeta)}{\sinh(w - q + i\alpha\zeta)} \prod_{s,p} \frac{\sinh(w - \lambda_{s,p} + i\alpha\zeta)}{\sinh(w - z_{s,p} + i\alpha\zeta)}, \qquad \alpha = 0, \pm 1\,, \tag{5.23}$$

which, in the linearized limit $\{z_{s,p}\} = \{\lambda_{s,p} + \epsilon_{s,p}\}$ become

$$\begin{aligned} P_\alpha^{(+)} &\to \exp\left\{ \sum_{s,p} \epsilon_{s,p} \coth(w - \lambda_{s,p} + i\alpha\zeta) + \log \frac{\sinh(w + q + i\alpha\zeta)}{\sinh(w - q + i\alpha\zeta)} \right\} \\ &\to \exp\left\{ \int_{-q}^{q} \coth(w - \lambda + i\alpha\zeta)\, (z^{(+)}(\lambda) + 1)\, \mathrm{d}\lambda \right\} = e^{(\beta + 2\pi i)\, \tilde{z}(w + i\alpha\zeta)}, \end{aligned} \tag{5.24}$$

and one can similarly show that the same shift of β has to be done in the equation (5.8).

The term $\widehat{G}^{(-)}(\beta, m)$ can be considered in the same way, and we finally obtain

$$\widehat{G}^{(\pm)}(\beta, m) = \widehat{G}^{(0)}(\beta \pm 2\pi i, m)\,. \tag{5.25}$$

It is worth mentioning at this point that the generating function $\langle e^{\beta \mathcal{Q}_m} \rangle$ is a polynomial of e^β, which means that the exact result should be a $2\pi i$-periodic function of β. In the asymptotic formula this periodicity may of course be broken. We see, however, that *the leading oscillating part of the asymptotics partly restores the original periodicity.* It is therefore very possible that the more rapidly oscillating corrections to this formula can simply be obtained, at their leading order, by the shifts $\beta \to \beta + 2\pi i n$, $n \in \mathbb{Z}^*$, in $\widehat{G}^{(0)}(\beta, m)$.

5.1.3 Final result and comments

We eventually obtain

$$\langle e^{\beta \mathcal{Q}_m} \rangle = \sum_{\sigma = 0, \pm} \widehat{G}^{(0)}(\beta + 2\pi i \sigma, m)\, [1 + \mathrm{o}(1)]\,, \tag{5.26}$$

in which $\widehat{G}^{(0)}(\beta, m)$, given by (5.14), corresponds to the leading non-oscillating term whereas $\widehat{G}^{(0)}(\beta \pm 2\pi i, m)$ are the leading oscillating ones.

To conclude this section, we would like to stress once more that the result (5.26) for the leading asymptotics of the generating function $\langle e^{\beta \mathcal{Q}_m} \rangle$ is formulated in terms of the solution of

(1.14), which is a *linear* integral equation, whereas the integral equations describing the partial sums $G_{1\dots n}(\gamma)$ were *non-linear*. The linearization arises only after taking the last sum over n, in other words, only if we take into account the contributions of cycle integrals of all possible lengths. In the framework of our approach, the necessity to consider cycles of arbitrary lengths is quite natural. Indeed, the asymptotics of all cycle integrals have a rather common form independently on the length of the cycle. In particular it always contains a term which is linear over the distance m, and one might therefore expect that all cycle integrals eventually give a contribution of the same order to the asymptotics of $\langle e^{\beta\mathcal{Q}_m}\rangle$. It is worth mentioning however that, for cycle integrals of length $\ell > 1$, this linear dependence on the distance shows itself only in the asymptotics, while it is explicit for the cycle integral of length $\ell = 1$:

$$\oint \frac{\mathrm{d}z}{2\pi i}\int_{-q}^{q}\frac{\mathrm{d}\lambda}{2\pi i}\,\mathcal{G}_1\begin{pmatrix}\lambda\\ z\end{pmatrix}\frac{e^{im(p_0(z)-p_0(\lambda))}}{\sinh^2(z-\lambda)} = \int_{-q}^{q}\frac{\mathrm{d}\lambda}{2\pi i}\left[imp_0'(\lambda)+\partial_\epsilon\right]\mathcal{G}_1\begin{pmatrix}\lambda\\ \lambda+\epsilon\end{pmatrix}\Big|_{\epsilon=0}. \quad (5.27)$$

Therefore, dealing with the series for $\langle e^{\beta\mathcal{Q}_m}\rangle$ in its initial form (2.40), we see only the explicit linear dependence in m generated by each double pole at $z_j = \lambda_j$ (i.e. by the cycle integral of length $\ell = 1$), but we do not see the hidden linear dependence produced by all other cycle integrals. This may create the fallacious impression that the role of these double poles predominates.

We specify here all these details since they explain the appearance of a non-linear integral dressing equation in the works [39, 38]. Exactly the same non-linear equation can be obtained from the series (2.40) after summing up the contributions of only the double poles. We would like to point out that this way leads to a misleading representation for the generating function $\langle e^{\beta\mathcal{Q}_m}\rangle$. In particular, using such representation, the authors of [39] concluded that the oscillations in the asymptotics of the two-point correlation function $\langle\sigma_1^z\sigma_{m+1}^z\rangle$ are exponentially suppressed at long distance, which is not true.

5.2 Asymptotic expansion of $\langle\sigma_1^z\sigma_{m+1}^z\rangle$

Let us now consider the asymptotic behavior of the two-point correlation function $\langle\sigma_1^z\,\sigma_{m+1}^z\rangle$ which can be obtained via (1.4):

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle = 2D_m^2\,\frac{\partial^2}{\partial\beta^2}\langle e^{\beta\mathcal{Q}_m}\rangle\Big|_{\beta=0} - 4D + 1. \quad (5.28)$$

Similarly as for $\langle e^{\beta\mathcal{Q}_m}\rangle$, we will consider separately the non-oscillating and oscillating parts.

5.2.1 Leading non-oscillating behavior of $\langle\sigma_1^z\sigma_{m+1}^z\rangle$

To obtain the leading non-oscillating term of the asymptotics of $\langle\sigma_1^z\sigma_{m+1}^z\rangle$, we should apply the differential operator $2D_m^2\partial_\beta^2$ to the term $\widehat{G}^{(0)}(\beta, m)$ in (5.1). Evidently, in order to obtain non-vanishing contributions, one has to apply the derivatives over β either to $e^{\beta mD}$ or to the fractional power of m, setting simply $\beta = 0$ in the remaining part. This gives us, for $m \to \infty$,

$$2D_m^2 \frac{\partial^2}{\partial \beta^2} \widehat{G}^{(0)}(\beta, m)\bigg|_{\beta=0} = \left(4D^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2}\right) \cdot \mathcal{A}(\beta = 0). \tag{5.29}$$

Consider now the behavior of the factor $\mathcal{A}$ at $\beta \to 0$. Let us take, for example, the determinant of the operator $U^{(\lambda)}$. Due to the factor $e^{\beta \tilde{z}(w)}$ the kernel $U_{\theta_1}^{(\lambda)}(w, w')$ has a cut on the interval $[-q, q]$, therefore, for β small enough, the action of the integral operator on the closed contour Γ can be reduced to

$$U^{(\lambda)}\Big|_{\Gamma} \to (U_-^{(\lambda)} - U_+^{(\lambda)})\Big|_{[-q,q]}, \tag{5.30}$$

where $U_{\pm}^{(\lambda)}$ are the limiting values of $U^{(\lambda)}$ from the upper (lower) half-planes. Using the equations (5.13) we obtain

$$\det\left[I + \frac{1}{2\pi i} U_{\theta_1}^{(\lambda)}(w, w')\right]\bigg|_{\Gamma} = \det\left[I + \frac{1}{2\pi i} \tilde{U}_{\theta_1}^{(\lambda)}(w, w')\right]\bigg|_{[-q,q]}, \tag{5.31}$$

where the operator in the r.h.s. acts on $[-q, q]$ and its kernel is

$$\tilde{U}_{\theta_1}^{(\lambda)}(w, w') = -e^{\beta \tilde{z}_-(w) - \beta \tilde{z}(w + i\zeta)} [K_\kappa(w - w') - K_\kappa(\theta_1 - w')]. \tag{5.32}$$

Setting now $\beta = 0$ we obtain

$$\det\left[I + \frac{1}{2\pi i} U_{\theta_1}^{(\lambda)}(w, w')\right]\bigg|_{\beta=0} = \det\left[I + \frac{1}{2\pi}[K(w - w') - K(\theta_1 - w')]\right]. \tag{5.33}$$

On the other hand,

$$\begin{aligned} \det\left[I + \frac{1}{2\pi}[K(w - w') - K(\theta_1 - w')]\right] &= \det\left[I + \frac{1}{2\pi} K(w - w')\right] \cdot \det\left[I - R(\theta_1, w')\right] \\ &= \det\left[I + \frac{1}{2\pi} K\right] \cdot \left[1 - \int_{-q}^{q} R(\theta_1, w)\, \mathrm{d}w\right] = Z(\theta_1) \cdot \det\left[I + \frac{1}{2\pi} K\right], \end{aligned} \tag{5.34}$$

where $R(w, w')$ is the resolvent of the operator $I + \frac{1}{2\pi} K$. Thus,

$$\det\left[I + \frac{1}{2\pi i} U_{\theta_1}^{(\lambda)}(w, w')\right]\bigg|_{\beta=0} = Z(\theta_1) \cdot \det\left[I + \frac{1}{2\pi} K\right]. \tag{5.35}$$

One can prove similarly that

$$\det\left[I + \frac{1}{2\pi i} U_{\theta_2}^{(z)}(w, w')\right]\bigg|_{\beta=0} = Z(\theta_2) \cdot \det\left[I + \frac{1}{2\pi} K\right]. \tag{5.36}$$

Substituting these equations into (5.16) and using again (5.13), we find that

$$\lim_{\beta\to 0}\mathcal{A}(\beta)=1. \tag{5.37}$$

Thus, taking into account (5.28) and (5.29), we obtain the following leading non-oscillating asymptotic behavior of the correlation function:

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle_{\rm non-osc}=(2D-1)^2-\frac{2\mathcal{Z}^2}{\pi^2m^2}+{\rm o}\left(\frac{1}{m^2}\right),\qquad m\to\infty. \tag{5.38}$$

5.2.2 Leading oscillating behavior of $\langle\sigma_1^z\sigma_{m+1}^z\rangle$

As we have seen, in order to obtain the leading oscillating term of the asymptotics of the correlation function, it is enough to shift β by $\pm 2\pi i$ in (5.14). Observe that the constant $\mathcal{A}(\beta)$ is proportional to $(e^\beta-1)^2$, therefore $\mathcal{A}(\pm 2\pi i)=\mathcal{A}'(\pm 2\pi i)=0$. Thus, in order to obtain a non-zero contribution, one should differentiate only the factor $(e^\beta-1)^2$ when taking the second derivative with respect to β, setting $\beta=\pm 2\pi i$ in the rest of (5.14).

Let us consider, for example, the case $\beta\to\beta+2\pi i$. It gives

$$\frac{\partial^2}{\partial\beta^2}\widehat{G}^{(0)}(\beta+2\pi i,m)\bigg|_{\beta=0}=\mathcal{A}''(2\pi i)\cdot\frac{e^{2imp_F}\,G^2(1,\mathcal{Z})\,e^{C_1-C_0}}{[2\pi\,\sinh(2q)\,\rho(q)\,m]^{2\mathcal{Z}^2}}. \tag{5.39}$$

Here $p_{_F}$ is the Fermi momentum $p_{_F}=p(q)$ and

$$\mathcal{A}''(2\pi i)=\frac{2\det\left[I+\frac{1}{2\pi i}U_{\theta_1}^{(\lambda)}(w,w')\right]\cdot\det\left[I+\frac{1}{2\pi i}U_{\theta_2}^{(z)}(w,w')\right]\Big|_{\beta=2\pi i}}{\left[e^{2\pi i\tilde z(\theta_1+i\zeta)}-e^{2\pi i\tilde z(\theta_1-i\zeta)}\right]\left[e^{-2\pi i\tilde z(\theta_2-i\zeta)}-e^{-2\pi i\tilde z(\theta_2+i\zeta)}\right]\det^2[I+\frac{1}{2\pi}K]}. \tag{5.40}$$

Recall that $\theta_{1,2}$ in (5.40) are arbitrary complex numbers. Let us set $\theta_1=-q$ and $\theta_2=q$[1]. Then it is easy to see that, at $\beta=2\pi i$,

$$\left(\frac{1}{2\pi i}U_{-q}^{(\lambda)}\right)^\dagger(w,w')=\frac{1}{2\pi i}U_q^{(z)}(-\bar w,-\bar w'), \tag{5.41}$$

where $\dagger$ stays for Hermitian conjugation. Hence, the two determinants in the numerator of (5.40) are complex conjugated. It is also easy to show using (5.13) that

$$\left[e^{2\pi i\tilde z(-q+i\zeta)}-e^{2\pi i\tilde z(-q-i\zeta)}\right]\left[e^{-2\pi i\tilde z(q-i\zeta)}-e^{-2\pi i\tilde z(q+i\zeta)}\right]=-4\sin^2\pi\mathcal{Z}\cdot e^{2\pi i[\tilde z(-q-i\zeta)-\tilde z(q-i\zeta)]}.$$

Combining (5.40) with the Barnes function in (5.39) and defining $\widetilde{\mathcal{A}}\equiv -4\,G^2(1,\mathcal{Z})\,\mathcal{A}''(2\pi i)$, we find

$$\widetilde{\mathcal{A}}=\left|e^{\pi i[\tilde z(q-i\zeta)-\tilde z(-q-i\zeta)]}\,\frac{G(2,\mathcal{Z})\cdot\det\left[I+\frac{1}{2\pi i}U_{-q}^{(\lambda)}(w,w')\right]}{\pi\mathcal{Z}\cdot\det\left[I+\frac{1}{2\pi}K\right]}\right|^2, \tag{5.42}$$

[1] The θ-independence of the expression given in (5.40) is proven in Appendix A.3. This specific choice of θ_j is the most convenient for the calculation of the Fredholm determinants in (5.40) in the vicinity of the free fermion point, see Appendix B.

where $G(2,\mathcal{Z}) = G(2+\mathcal{Z})G(2-\mathcal{Z})$ and where we have used (3.27).

One can transform similarly the expression for $\mathcal{A}''(-2\pi i)$. The result in this case is still given by the same constant $\widetilde{\mathcal{A}}$ (5.42). Then, taking the second lattice derivative, we eventually obtain in the limit $m\to\infty$,

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle_{\text{osc}} = 8\,\widetilde{\mathcal{A}}\,\sin^2 p_{_F}\cdot\frac{\cos(2mp_{_F})\,e^{C_1-C_0}}{[2\pi\,\sinh(2q)\,\rho(q)\,m]^{2\mathcal{Z}^2}} + \mathrm{o}\big(m^{-2\mathcal{Z}^2}\big). \tag{5.43}$$

The obtained amplitude of this leading oscillating term appears to be closely related to a special form factor of the operator σ^z. We will comment this relationship in the conclusion.

5.2.3 Final result and comments

Thus, at $m\to\infty$, the two-point correlation function of the third components of spin in the external magnetic field behaves as

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle_{\text{leading}} = (2D-1)^2 - \frac{2\mathcal{Z}^2}{\pi^2m^2} + 8\,\widetilde{\mathcal{A}}\,e^{C_1-C_0}\,\sin^2 p_{_F}\cdot\frac{\cos(2mp_{_F})}{[2\pi\,\sinh(2q)\,\rho(q)\,m]^{2\mathcal{Z}^2}}. \tag{5.44}$$

Note that, dependending on the value on Δ (and thus on $\mathcal{Z}$), the second term of (5.44) may be dominant compared to the third one or *vice-versa*.

Let us discuss this result from the viewpoint of its comparison with known results and predictions.

As already mentioned in the introduction, the asymptotic equation (5.44) completely confirms the predictions from Luttinger liquid and conformal field theory approaches. In the limit of free fermions $\zeta=\frac{\pi}{2}$ (see Appendix B), this result also agrees with the known answer

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle_{\Delta=0} = (2D-1)^2 - \frac{2}{\pi^2m^2}\big[1-\cos(2mp_{_F})\big]. \tag{5.45}$$

In the general case, there exists no prediction concerning the value of the amplitude of the oscillating term, except in the zero magnetic field limit (see [64, 65, 66]). However, in the framework of our approach, the non-zero magnetic field plays the role of a certain *regularization* which ensures the finiteness of all the constants (C_0, C_1, $\widetilde{\mathcal{A}}$, $\rho(q)\sinh 2q$) entering this amplitude: at $h=0$ all these constants become divergent. It is nevertheless possible to show that the total combination of these constants remains finite. As we have mentioned already the corresponding proof is highly non-trivial, therefore we do not present it here. We would just like to mention that the most complicated part of the calculations is related to the extraction of the divergent part from the Fredholm determinant $\det[I+\frac{1}{2\pi i}U_{-q}^{(\lambda)}(w,w')]$: one can show that it coincides with the divergent part of $\det[I-\frac{1}{2\pi}K]$.

Due to the difficulties that arise when taking this limit $h\to0$ in our result, the complete comparison of (5.44) with the results of [64, 65, 66] still remains an unsolved problem. Note nevertheless that, in the limit of free fermions, the formula (5.44) holds for arbitrary magnetic field including the case $h=0$. We succeeded moreover to compute explicitly the limiting value at $h = 0$ of the amplitude of the oscillating term in the vicinity of free fermions up to the second order in $\epsilon = \frac{\zeta}{\pi} - \frac{1}{2}$. Our result,

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle_{osc} = \frac{2(-1)^m}{\pi^2 m^2 \mathcal{Z}^2} \exp\Big[-4\epsilon(\log 2 + \mathbf{C}) + 8\epsilon^2\big(\tfrac{\pi^2}{6} - 1 - \mathbf{C} - \log 2\big)\Big] + \mathrm{O}(\epsilon^3), \quad (5.46)$$

$\mathbf{C}$ being the Euler constant, coincides at this order in ϵ with the one of [64, 65, 66].

6 Quantum non-linear Schrödinger equation

In this section we briefly explain how to apply the method described above to another model, the quantum one-dimensional Bose gas (or quantum nonlinear Schrödinger equation model).

Recall that the starting point of our derivation is the master equation (2.1). We have stressed already that such integral representation exists not only for the XXZ chain, but also for other models possessing the six-vertex R-matrix [45] and, in particular, for the system of one-dimensional interacting bosons on some finite interval $[0, L]$.

The Hamiltonian of this model is given by

$$H = \int_0^L \Big(\partial_x \Psi^\dagger \partial_x \Psi + c\Psi^\dagger \Psi^\dagger \Psi \Psi - h\Psi^\dagger \Psi\Big)\, \mathrm{d}x\,. \quad (6.1)$$

Here Ψ and $\Psi^\dagger$ are Bose-fields possessing canonical equal-time commutation relations, c is a coupling constant and h a chemical potential. For $c > 0$ and $h > 0$, the ground state of the model goes to a Dirac sea in the thermodynamic limit and can be described by a set of integral equations similar to those of the XXZ chain.

The analog of the $\mathcal{Q}_m$ operator is

$$\mathcal{Q}_x = \int_0^x \Psi^\dagger(z)\, \Psi(z)\, \mathrm{d}z\,, \quad (6.2)$$

and the function $\langle e^{\beta \mathcal{Q}_x} \rangle$ is a generating function for the correlation function of the densities $j(x) = \Psi^\dagger(x)\Psi(x)$ via

$$\langle j(x) j(0) \rangle = \frac{1}{2} \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial \beta^2} \langle e^{\beta \mathcal{Q}_x} \rangle \Big|_{\beta=0}\,. \quad (6.3)$$

The method to compute the asymptotics of (6.3) in the case of the one-dimensional Bose gas literally repeats the steps we have described in details in the case of the XXZ chain. Let us just point a few peculiarities.

The master equation for $\langle e^{\beta \mathcal{Q}_x} \rangle$ has the form (2.1), but the functions $a(\lambda)$, $d(\lambda)$, and $l(\lambda)$ (see (2.4), (2.5)) are now

$$a(\lambda) = e^{-\frac{iL\lambda}{2}}, \qquad d(\lambda) = e^{\frac{iL\lambda}{2}}, \qquad l(\lambda) = e^{-ix\lambda}. \quad (6.4)$$

In the remaining part of the master equation, all hyperbolic functions should be replaced by rational ones, i.e. $\sinh(\lambda - z \pm i\zeta) \to \lambda - z \pm i\zeta$, *etc.* After this operation one should set $\zeta = c$. The transformations of the master equation described in Sections 2.2 and 2.3 are the same for the one-dimensional bosons.

One of the difference between the quantum one-dimensional Bose gas and the XXZ model is due to the choice of notations[1]. In particular, the integral equation for the dressed charge in the Bose gas model coincides with its XXZ analog up to the sign of the integral operator:

$$Z(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda - \mu)\, Z(\mu)\, \mathrm{d}\mu = 1\,, \qquad K(\lambda) = \frac{2c}{\lambda^2 + c^2}\,. \tag{6.5}$$

One should also pay attention to the fact that the function $l(\lambda)$ (6.4) can be analytically continued to the lower half-plane, while its analog (2.5) can be continued to the upper half plane (in the case of $i\pi$-periodic hyperbolic functions, we can of course always consider the domain $|\Im\lambda| \le \frac{\pi}{2}$). This difference leads to some differences of signs in several formulas, but the common strategy of the derivation remains just the same as in the case of the XXZ chain.

The final answer for the leading asymptotic behavior of (6.3) is

$$\langle j(x) j(0)\rangle_{\text{leading}} = D^2 - \frac{\mathcal{Z}^2}{2\pi^2 x^2} + 2\widetilde{\mathcal{A}}\, p_{F}^2 \cdot \frac{\cos(2xp_{F})\, e^{C_1 - C_0}}{[4\pi\, q\, \rho(q)\, x]^{2\mathcal{Z}^2}}\,. \tag{6.6}$$

Here, just as in the previous case, $\mathcal{Z} = Z(\pm q)$, $Z(\lambda)$ satisfies the integral equation (6.5) and q, the value of the spectral parameter at the Fermi boundary, depends on c and h (see e.g. [59]). In the model of the one-dimensional Bose gas the spectral density $\rho(\lambda)$ is related to the dressed charge by $\rho = \frac{Z}{2\pi}$. The Fermi momentum p_{F} and the average density D are given by (1.13).

The expressions for the constants C_0 and C_1 are quite similar to (5.9), (5.15),

$$C_0 = \int_{-q}^{q} \frac{Z(\lambda)\, Z(\mu)}{(\lambda - \mu - ic)^2}\, \mathrm{d}\lambda\, \mathrm{d}\mu\,, \tag{6.7}$$

$$C_1 = \frac{1}{2}\int_{-q}^{q} \frac{Z'(\lambda)\, Z(\mu) - Z(\lambda)\, Z'(\mu)}{\lambda - \mu}\, \mathrm{d}\lambda\, \mathrm{d}\mu + 2\mathcal{Z}\int_{-q}^{q} \frac{\mathcal{Z} - Z(\lambda)}{q - \lambda}\, \mathrm{d}\lambda\,. \tag{6.8}$$

The formula for the constant $\widetilde{\mathcal{A}}$ coincides also basically with (5.42):

$$\widetilde{\mathcal{A}} = \left| e^{\pi i[\tilde{z}(q - ic) - \tilde{z}(-q - ic)]}\, \frac{G(2, \mathcal{Z}) \cdot \det\left[I + \frac{1}{2\pi i} U_{-q}^{(\lambda)}(w, w')\right]}{\pi\mathcal{Z} \cdot \det\left[I - \frac{1}{2\pi} K\right]} \right|^2 . \tag{6.9}$$

[1] One could avoid this difference by replacing $\zeta \to \pi - \zeta$ in the original formulas.

Here

$$\tilde{z}(w) = \frac{1}{2\pi i} \int_{-q}^{q} \frac{Z(\lambda)}{\lambda - w} \, \mathrm{d}\lambda \,, \tag{6.10}$$

and

$$\tilde{U}_{-q}^{(\lambda)}(w, w') = i \frac{e^{2\pi i \tilde{z}(w)}}{e^{2\pi i \tilde{z}(w+ic)} - e^{2\pi i \tilde{z}(w-ic)}} \left[K(w - w') - K(q + w') \right] . \tag{6.11}$$

It is not difficult to check that the result (6.6) has the correct free fermion limit ($c = \infty$). One can also show that it reproduces at first order in c^{-1} the asymptotic behavior obtained in [38].

Conclusion

We have described a method to compute the asymptotic behavior of correlation functions of quantum integrable systems. Our study is based on the multiple integral representation (2.1), called the master equation. Such master equation can be obtained via the algebraic Bethe ansatz for a rather wide class of integrable models and different correlation functions. In the present paper, we have considered only one specific correlation function for the XXZ chain and one-dimensional bosons. To conclude, we would like to discuss possible further developments.

First of all, it should be possible to apply this method to compute the asymptotic behavior of other correlation functions, such as $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$ for the XXZ chain or $\langle \Psi^\dagger(x)\Psi(0) \rangle$ for the Bose gas. It would also be interesting to apply this approach to compute the long-distance correlations in the massive phase $\Delta > 1$ of the XXZ chain. In that case there is no power law corrections in the asymptotic behavior of the cycles. This should lead to an exponential decay of the correlations.

The case of the XXX model is also of special interest. For non-zero magnetic field the result (5.44) apparently remains valid. However, at $h = 0$, one can expect logarithmic contributions to the asymptotic expression [2, 5]. It would be interesting to see how such terms can appear in the framework of our approach. However, the case of zero magnetic field appears to be the most complicated one, even for the XXZ chain. Therefore before studying the XXX model it would be desirable to understand completely what happens for the XXZ chain at $h = 0$, and in particular to compute explicitely the (finite) limiting value for the amplitude of the oscillating term.

We hope that our approach can be used for the asymptotic analysis of the temperature dependent correlation functions. At least in the case of the one-dimensional Bose gas, for which there is no bound states in the spectrum of the Hamiltonian for $c > 0$, the generalization of our method looks quite straightforward. For $T > 0$, the integrals over the spectral parameters should be taken over the whole real axis. This leads to the absence of power law corrections in the asymptotic behavior of the cycle integrals, just as in the massive regime of the XXZ chain. As a result, the asymptotic behavior of the correlation functions should decrease exponentially with the distance.

We would like finally to discuss the nice relation that exists between the amplitude of the leading oscillating term of the asymptotics (5.44) and a special form factor of the operator σ^z, namely the matrix element

$$F_\sigma = \left(\frac{M}{2\pi}\right)^{\mathcal{Z}^2} \cdot \frac{\langle\psi(\{\mu\})|\sigma_k^z|\psi(\{\lambda\})\rangle}{\|\psi(\{\mu\})\| \cdot \|\psi(\{\lambda\})\|}, \tag{6.12}$$

where $|\psi(\{\lambda\})\rangle$ is the N-particle ground state and $\langle\psi(\{\mu\})|$ is an excited state containing one particle and one hole at the different boundaries of the Fermi sphere, for instance, $\lambda_p = q$, $\lambda_h = -q$. We recall that in (6.12) M denotes the length of the chain. Using the determinant representations of form factors obtained in [52], one can show [47] that, in the thermodynamic limit $|F_\sigma|^2$ is equal to,

$$\lim_{M\to\infty} |F_\sigma|^2 = 4\widetilde{\mathcal{A}} \sin^2 p_{_F}\, e^{C_1-C_0} \left[2\pi \sinh(2q)\, \rho(q)\right]^{-2\mathcal{Z}^2}. \tag{6.13}$$

Thus, the amplitude of the leading oscillating term is equal to the square of the norm of the form factor (6.12):

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle_{\text{leading}} = (2D-1)^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + 2|F_\sigma|^2 \cdot \frac{\cos(2mp_{_F})}{m^{2\mathcal{Z}^2}}. \tag{6.14}$$

The coefficient 2 is due to the fact that there are two such form factors. This observation perfectly agrees with the conformal field theory approach. Namely, if we consider the two-point correlation function as a sum of form factors, then the main contribution to the asymptotics comes from the terms corresponding to the excitations in a vicinity of the Fermi boundaries. Hereby, the leading oscillating term in the asymptotics is produced by the excited states having particles and holes on the different sides of the Fermi zone. Thus there is an opportunity to improve drastically the form factor approach. This, in turn, would open a path towards the computation of dynamical correlation functions.

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A Determinant representations

A.1 Proof of Proposition 2.1

If the parameters $\lambda_1, \dots, \lambda_N$ satisfy the system of Bethe equations (2.2), then $d(\lambda_j)$ can be expressed in terms of $a(\lambda_j)$, and the determinant of Ω_κ can be written as

$$\det_N \Omega_\kappa(\{z\}, \{\lambda\}|\{z\}) = \prod_{j=1}^{N} a(\lambda_j) \cdot \prod_{a,b=1}^{N} \sinh(z_a - \lambda_b - i\zeta) \cdot \det_N \tilde{M}_\kappa(\{z\}|\{\lambda\}), \tag{A.1}$$

where

$$\left(\tilde{M}_\kappa\right)_{jk} = t(z_k, \lambda_j) + \kappa\, t(\lambda_j, z_k) \cdot V_+(\lambda_j)\, V_-^{-1}(\lambda_j). \tag{A.2}$$

In order to prove (2.10), we shall prove that

$$\det_N \tilde{M}_\kappa = \det_N \left[\frac{1}{\sinh(\lambda_j - z_k)}\right] \cdot \prod_{j=1}^{N} \left[\kappa \frac{V_+(\lambda_j)}{V_-(\lambda_j)} - 1\right] \\ \times \frac{1-\kappa}{V_+^{-1}(\theta) - \kappa V_-^{-1}(\theta)} \cdot \det_N \left[\delta_{jk} + U_{jk}^{(\lambda)}(\theta)\right]. \tag{A.3}$$

Consider the following transformation of the l.h.s. of (A.3)

$$\det_N \tilde{M}_\kappa = \frac{\det_N(\tilde{M}_\kappa A)}{\det_N A}, \tag{A.4}$$

with

$$A_{jk} = \frac{\prod_{a=1}^{N} \sinh(z_j - \lambda_a)}{\prod_{\substack{a=1\\a\neq j}}^{N} \sinh(z_j - z_a)} \times \begin{cases} \coth(z_j - \lambda_k) & \text{for} \quad k \neq N, \\ 1 & \text{for} \quad k = N. \end{cases} \tag{A.5}$$

The determinant of A can be easily computed. Indeed,

$$\det_N A = \frac{\prod_{a,b=1}^{N} \sinh(z_a - \lambda_b)}{\prod_{\substack{a,b=1\\a\neq b}}^{N} \sinh(z_a - z_b)} \cdot \det_N \left(\begin{array}{c|c} \coth(z_1 - \lambda_k) & 1 \\ \vdots & \vdots \\ \coth(z_N - \lambda_k) & 1 \end{array}\right). \tag{A.6}$$

Subtracting the last line from all others, we reduce the remaining determinant to a Cauchy determinant, which gives

$$\det_N A = \frac{\prod_{a=1}^{N} \sinh(z_a - \lambda_N)}{\prod_{a=1}^{N-1} \sinh(\lambda_a - \lambda_N)} \cdot \prod_{a>b}^{N} \frac{\sinh(\lambda_a - \lambda_b)}{\sinh(z_a - z_b)}. \tag{A.7}$$

The product of matrices $\tilde{M}_\kappa A$ can also be explicitly calculated

$$(\tilde{M}_\kappa A)_{jk} = \delta_{jk} \frac{\prod\limits_{\substack{a=1\\a\neq j}}^{N} \sinh(\lambda_j - \lambda_a)}{\prod\limits_{a=1}^{N} \sinh(\lambda_j - z_a)} \left[\kappa \frac{V_+(\lambda_j)}{V_-(\lambda_j)} - 1\right] + K_\kappa(\lambda_j - \lambda_k)\, V_+(\lambda_j), \qquad k < N, \quad \text{(A.8)}$$

and

$$(\tilde{M}_\kappa A)_{jN} = (1-\kappa)\, V_+(\lambda_j), \qquad k = N. \quad \text{(A.9)}$$

Here the function $K_\kappa(\lambda)$ is given by (2.15). Let us explain how the formulas (A.8), (A.9) were obtained. For instance, in order to obtain (A.8), one has to calculate the sums

$$G^{\pm}_{jk} = \sum_{\ell=1}^{N} \frac{-i \sin\zeta \, \coth(z_\ell - \lambda_k)}{\sinh(z_\ell - \lambda_j)\, \sinh(z_\ell - \lambda_j \pm i\zeta)} \cdot \frac{\prod\limits_{a=1}^{N} \sinh(z_\ell - \lambda_a)}{\prod\limits_{\substack{a=1\\a\neq \ell}}^{N} \sinh(z_\ell - z_a)}. \quad \text{(A.10)}$$

Consider an auxiliary contour integral

$$I_\pm = \frac{1}{2\pi i} \oint \frac{-i \sin\zeta \, \coth(w - \lambda_k)\, \mathrm{d}w}{\sinh(w - \lambda_j)\, \sinh(w - \lambda_j \pm i\zeta)} \cdot \prod_{a=1}^{N} \frac{\sinh(w - \lambda_a)}{\sinh(w - z_a)}, \quad \text{(A.11)}$$

where the integration is taken over the boundary of a horizontal strip of width $i\pi$. On the one hand, the integral vanishes $I_\pm = 0$ as the integrand is an $i\pi$-periodic function that is exponentially decreasing for $\Re(w) \to \pm\infty$. On the other hand, it is equal to the sum of the residues inside of the integration contour. The sum of the residues at the poles $w = z_a$ gives exactly $G^{\pm}_{jk}$. Taking into account the contribution of the additional poles at $w = \lambda_j \mp i\zeta$ and $w = \lambda_j$ (the last one only exists for $j = k$), we arrive at the following identity

$$G^{\pm}_{jk} \pm \delta_{jk} \frac{\prod\limits_{\substack{a=1\\a\neq j}}^{N} \sinh(\lambda_a - \lambda_j)}{\prod\limits_{a=1}^{N} \sinh(z_a - \lambda_j)} \pm \coth(\lambda_j - \lambda_k \mp i\zeta)\, V_\mp(\lambda_j) = 0. \quad \text{(A.12)}$$

We have thus computed $G^{\pm}_{jk}$ and in this way proved the formula (A.8). The result (A.9) can be obtained by a similar method.

The following transformations are trivial. We can extract the factor $V_+(\lambda_j)$ from each line of $(\tilde{M}_\kappa A)_{jk}$, making the elements of the last column equal to $1-\kappa$. After this, subtracting the last line from all the others and extracting the coefficients in front of δ_{jk} we obtain a new representation for $\det \tilde{M}_\kappa$

$$\det_N \tilde{M}_\kappa = \frac{1-\kappa}{V_+^{-1}(\lambda_N) - \kappa V_-^{-1}(\lambda_N)} \prod_{a=1}^{N} \left[\kappa \frac{V_+(\lambda_a)}{V_-(\lambda_a)} - 1\right] \\ \times \det_N \left[\frac{1}{\sinh(\lambda_j - z_k)}\right] \cdot \det_N \left[\delta_{jk} + U_{jk}^{(\lambda)}(\lambda_N)\right]. \quad \text{(A.13)}$$

Thus, we have reproduced the r.h.s. of (A.3) up to replacement $\theta \to \lambda_N$.

Now consider $\det\left[\delta_{jk} + U_{jk}^{(\lambda)}(\theta)\right]$ appearing in the r.h.s. of equation (A.3) (see (2.13) for its explicit form). After a simple similarity transformation we obtain

$$U_{jk}^{(\lambda)}(\theta) \to \tilde{U}_{jk}^{(\lambda)}(\theta) = \frac{\prod\limits_{a=1}^{N} \sinh(z_a - \lambda_k)}{\prod\limits_{\substack{a=1\\a\neq k}}^{N} \sinh(\lambda_a - \lambda_k)} \cdot \frac{K_\kappa(\lambda_j - \lambda_k) - K_\kappa(\theta - \lambda_k)}{V_+^{-1}(\lambda_k) - \kappa V_-^{-1}(\lambda_k)}, \quad \text{(A.14)}$$

Let us multiply the first $N-1$ columns by the coefficients s_k

$$s_k = \frac{V_+^{-1}(\lambda_k) - \kappa V_-^{-1}(\lambda_k)}{V_+^{-1}(\lambda_N) - \kappa V_-^{-1}(\lambda_N)} \quad \text{(A.15)}$$

and add them to the N-th column. Then the last column of the determinant becomes

$$\delta_{jN} + \tilde{U}_{jN}^{(\lambda)}(\theta) + \sum_{k=1}^{N-1} s_k \left(\delta_{jk} + \tilde{U}_{jk}^{(\lambda)}(\theta)\right) = \frac{V_+^{-1}(\theta) - \kappa V_-^{-1}(\theta)}{V_+^{-1}(\lambda_N) - \kappa V_-^{-1}(\lambda_N)}, \quad \text{(A.16)}$$

(the method of calculation is quite similar to one described in the formulas (A.10)–(A.12)). Now it is enough to subtract the last line of the obtained matrix from all others, and we arrive at the following identity

$$\det_N \left[\delta_{jk} + \tilde{U}_{jk}^{(\lambda)}(\theta)\right] = \frac{V_+^{-1}(\theta) - \kappa V_-^{-1}(\theta)}{V_+^{-1}(\lambda_N) - \kappa V_-^{-1}(\lambda_N)} \cdot \det_N \left[\delta_{jk} + \tilde{U}_{jk}^{(\lambda)}(\lambda_N)\right]. \quad \text{(A.17)}$$

In other words, it means that the combination

$$\frac{\det_N \left[\delta_{jk} + U_{jk}^{(\lambda)}(\theta)\right]}{V_+^{-1}(\theta) - \kappa V_-^{-1}(\theta)} \quad \text{(A.18)}$$

does not depend on θ. Thus, substituting (A.17) into (A.13), we obtain the r.h.s. of (A.3).

The representation (2.11) for Ω_κ can be proved by using the identity

$$\det_N \tilde{M}_\kappa = \det_N \hat{M}_\kappa, \quad \text{where} \quad \left(\hat{M}_\kappa\right)_{jk} = t(z_k, \lambda_j) + \kappa\, t(\lambda_j, z_k) \cdot V_+(z_k)\, V_-^{-1}(z_k), \quad \text{(A.19)}$$

established in [45] (see Appendix B of this paper). It follows from (A.19) that

$$\det_N \tilde{M}_\kappa(\{z\},\{\lambda\}) = \kappa^N \prod_{a,b=1}^{N} \frac{\sinh(\lambda_a - z_b - i\zeta)}{\sinh(z_b - \lambda_a - i\zeta)} \cdot \det_N \tilde{M}_{\kappa^{-1}}(\{\lambda\},\{z\}). \tag{A.20}$$

Then the representation (2.11) follows from (2.10) after the replacements $z \leftrightarrow \lambda$ and $\kappa \to \kappa^{-1}$.

A.2 Fredholm determinant representation

Consider the Fredholm determinants $\det\left[I + \frac{1}{2\pi i}\hat{U}_\theta^{(\lambda,z)}(w,w')\right]$ (see (2.16)). We have

$$\log\det\left[I + \frac{1}{2\pi i}\hat{U}_\theta^{(\lambda,z)}\right] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \oint_\Gamma \frac{\mathrm{d}^k w}{(2\pi i)^k}\, \hat{U}_\theta^{(\lambda,z)}(w_1,w_2)\cdots \hat{U}_\theta^{(\lambda,z)}(w_k,w_1). \tag{A.21}$$

Computing the multiple integrals by the residues at $w_j = \lambda_\ell$ (respectively at $w_k = z_\ell$), where $\ell = 1,\dots,N$, we obtain

$$\log\det\left[I + \frac{1}{2\pi i}\hat{U}_\theta^{(\lambda,z)}\right] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sum_{\ell_1,\dots,\ell_k=1}^{N} U_{\ell_1\ell_2}^{(\lambda,z)}(\theta)\cdots U_{\ell_k\ell_1}^{(\lambda,z)}(\theta), \tag{A.22}$$

what is exactly the expansion for $\log\det_N\left[\delta_{jk} + U_{jk}^{(\lambda,z)}(\theta)\right]$.

A.3 Some identities for Fredholm determinants

Here we prove θ-independence of Fredholm determinants considered in Sections 2 and 5.

Consider an integral operator $I+V$ acting on some contour Γ.

Proposition A.1. *Let*

$$g(w) = h(w) + \int_\Gamma V(w,w')\, h(w')\, \mathrm{d}w'. \tag{A.23}$$

Then, for arbitrary $w_0 \in \mathcal{C}$, the following identity holds

$$\det\left[I + V(w,w')\right] = \frac{g(w_0)}{h(w_0)} \cdot \det\left[I + V(w,w') - \frac{g(w)}{g(w_0)} V(w_0,w')\right]. \tag{A.24}$$

Proof — Suppose that the resolvent $R(w,w')$ of the operator $I+V$ exists (if not, then we can consider some regularization $V_\epsilon(w,w')$ of the original kernel, such that the corresponding resolvent R_ϵ exists). Then

$$\frac{g(w_0)}{h(w_0)}\cdot\det\left[I+V(w,w')-\frac{g(w)}{g(w_0)}V(w_0,w')\right]$$
$$=\frac{g(w_0)}{h(w_0)}\cdot\det\left[I+V(w,w')\right]\cdot\det\left[I-\frac{g(w)}{g(w_0)}R(w_0,w')\right]$$
$$=\frac{1}{h(w_0)}\cdot\det\left[I+V(w,w')\right]\left(g(w_0)-\int_\Gamma R(w_0,w')g(w')\,dw'\right)$$
$$=\det\left[I+V(w,w')\right], \quad \text{(A.25)}$$

which ends the proof. □

If the kernel $V(w,w')$ is analytic within some domain $\mathcal{D}$ containing Γ, then the identity (A.24) holds for $w_0\in\mathcal{D}$.

Let now Γ be the contour shown on Fig. 1 and $h(w)$ an $i\pi$-periodic function that is holomorphic outside of Γ and bounded at $w\to\pm\infty$. Then

$$\oint_\Gamma \frac{\mathrm{d}w'}{2\pi i}\big[K_\kappa(w-w')-K_\kappa(\theta-w')\big]h(w')=\kappa h(\theta-i\zeta)-h(\theta+i\zeta)-\kappa h(w-i\zeta)+h(w+i\zeta), \quad \text{(A.26)}$$

where the integral has been computed by the residues lying outside of the contour Γ. Compose a kernel $U_\theta^{(h)}(w,w')$ as

$$U_\theta^{(h)}(w,w')=\frac{h(w)}{2\pi i\left[\kappa h(w-i\zeta)-h(w+i\zeta)\right]}\cdot\big[K_\kappa(w-w')-K_\kappa(\theta-w')\big]. \quad \text{(A.27)}$$

Let us apply (A.24) to the operator $I+U_\theta^{(h)}(w,w')$ with $\theta\in\Gamma$. We have

$$g(w)=h(w)+\oint_\Gamma U_\theta^{(h)}(w,w')\,h(w')\,\mathrm{d}w'=h(\omega)\,\frac{\kappa h(\theta-i\zeta)-h(\theta+i\zeta)}{\kappa h(w-i\zeta)-h(w+i\zeta)}. \quad \text{(A.28)}$$

Substituting this into (A.24) we immediately arrive at

$$\frac{\det\left[I+\frac{1}{2\pi i}U_\theta^{(\lambda)}(w,w')\right]}{\kappa h(\theta-i\zeta)-h(\theta+i\zeta)}=\frac{\det\left[I+\frac{1}{2\pi i}U_{w_0}^{(\lambda)}(w,w')\right]}{\kappa h(w_0-i\zeta)-h(w_0+i\zeta)}, \quad \text{(A.29)}$$

for arbitrary w_0 such that $|\Im(w_0-w)|<\zeta$, $w\in\Gamma$.

In particular one can take

$$h(w)=\prod_{a=1}^N\frac{\sinh(w-z_a)}{\sinh(w-\lambda_a)},\qquad \text{or}\qquad h(w)=e^{\beta\tilde{z}(w)}. \quad \text{(A.30)}$$

In the first case we obtain the identity for the operator $\hat{U}^{(\lambda)}$, in the second for $U^{(\lambda)}$. Similarly one can prove θ-independence of the kernels $\hat{U}^{(z)}$ and $U^{(z)}$.

B Free fermions

Consider the determinant of the matrix $\delta_{jk}+U^{(\lambda)}_{jk}(\theta)$ in the limit of free fermions $\zeta=\frac{\pi}{2}$. Then $V_+(\mu)=V_-(\mu)$ and the entries $U^{(\lambda)}_{jk}(\theta)$ are equal to

$$U^{(\lambda)}_{jk}(\theta)=\frac{\prod\limits_{a=1}^{N}\tanh(z_a-\lambda_j)}{\prod\limits_{\substack{a=1\\a\neq j}}^{N}\tanh(\lambda_a-\lambda_j)}\cdot\big[\tanh(\lambda_j-\lambda_k)-\tanh(\theta-\lambda_k)\big],\qquad \zeta=\frac{\pi}{2}. \tag{B.1}$$

The straightforward calculation of the corresponding determinant causes serious difficulties. However, there exists a way to avoid these problems. Due to (A.3) the determinant of $\delta_{jk}+U^{(\lambda)}_{jk}(\theta)$ can be expressed in terms of $\det\tilde{M}_\kappa$. One has, for the limit of free fermions,

$$\det{}_N\left[\delta_{jk}+U^{(\lambda)}_{jk}(\theta)\right]=V_+^{-1}(\theta)\cdot\frac{\det_N\tilde{M}_\kappa}{\det_N\left[\frac{1-\kappa}{\sinh(z_k-\lambda_j)}\right]}. \tag{B.2}$$

On the other hand, the matrix $\tilde{M}_\kappa$ (A.2) becomes the Cauchy matrix at $\zeta=\frac{\pi}{2}$,

$$\big(\tilde{M}_\kappa\big)_{jk}=\frac{2(1-\kappa)}{\sinh 2(z_k-\lambda_j)},\qquad \text{for}\quad \zeta=\frac{\pi}{2}. \tag{B.3}$$

Thus, both determinants in the r.h.s. of (B.2) are explicitly computable and we arrive at

$$\det{}_N\left[\delta_{jk}+U^{(\lambda)}_{jk}(\theta)\right]=\prod_{a=1}^{N}\frac{\cosh(\theta-z_a)}{\cosh(\theta-\lambda_a)}\cdot\frac{\prod\limits_{a>b}^{N}\cosh(z_a-z_b)\cosh(\lambda_a-\lambda_b)}{\prod\limits_{a,b=1}^{N}\cosh(z_a-\lambda_b)},\quad \zeta=\frac{\pi}{2}. \tag{B.4}$$

Similarly one has

$$\det{}_N\left[\delta_{jk}+U^{(z)}_{jk}(\theta)\right]=\prod_{a=1}^{N}\frac{\cosh(\theta-\lambda_a)}{\cosh(\theta-z_a)}\cdot\frac{\prod\limits_{a>b}^{N}\cosh(z_a-z_b)\cosh(\lambda_a-\lambda_b)}{\prod\limits_{a,b=1}^{N}\cosh(z_a-\lambda_b)},\quad \zeta=\frac{\pi}{2}. \tag{B.5}$$

It is easy to see from the obtained result and the definition (2.20) of $\widetilde{W}_N$ that, in the free fermion limit,

$$\widetilde{W}_N\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix}=1. \tag{B.6}$$

Let us now compute the constant $\widetilde{\mathcal{A}}$ in the limit of free fermions. For $\zeta = \frac{\pi}{2}$ we have $Z(\lambda) \equiv 1$, hence,

$$U_{-q}^{(\lambda)}(w,w') = -\tanh(w-q)\,\coth(w+q)\,\big[\tanh(w-w') + \tanh(q+w')\big]. \tag{B.7}$$

The kernel (B.7) is holomorphic in a vicinity of the interval $[-q,q]$. We draw the reader's attention on the fact that this property only holds due to the special choice of $\theta_1 = -q$. Otherwise the kernel $U_{\theta_1}^{(\lambda)}(w,w')$ would have a simple pole at $w = -q$.

Since the integral operator $U_{-q}^{(\lambda)}(w,w')$ acts on the closed contour surrounding the interval $[-q,q]$ we obtain that

$$\det\left[I + \frac{1}{2\pi i}U_{-q}^{(\lambda)}(w,w')\right] = 1, \tag{B.8}$$

hence,

$$\widetilde{\mathcal{A}} = \frac{1}{\pi^2}e^{2\pi i\left[\tilde{z}(q-i\frac{\pi}{2})-\tilde{z}(-q-i\frac{\pi}{2})\right]}. \tag{B.9}$$

It is also easy to see that for free fermions $C_1 = 0$ (see (5.15)), and

$$2\pi i\left[\tilde{z}(q-\tfrac{i\pi}{2}) - \tilde{z}(-q-\tfrac{i\pi}{2})\right] - C_0 = 0. \tag{B.10}$$

Thus, taking into account that $\rho(\lambda) = \frac{1}{\pi\cosh(2\lambda)}$ for $\zeta = \frac{\pi}{2}$, we obtain

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle_{osc} = \left(\frac{\sin p_F}{\tanh(2q)}\right)^2 \cdot \frac{2\cos(2mp_F)}{\pi^2 m^2}. \tag{B.11}$$

It remains to observe that

$$p_F = \pi D = \int_{-q}^{q}\frac{d\lambda}{\cosh 2\lambda} = \arctan\left(e^{2q}\right) - \arctan\left(e^{-2q}\right). \tag{B.12}$$

From this we find that $\sin p_F = \tanh(2q)$, and hence

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle_{osc} = \frac{2\cos(2mp_F)}{\pi^2 m^2}. \tag{B.13}$$

C The Lagrange series and its generalizations

In this appendix we consider several generalizations of the Lagrange series (see e.g. [81]) used in Section 4.3. We give the detailed proof in the standard (scalar) case. The generalizations then become quite evident.

C.1 Scalar case

Let us consider the series

$$G_0 = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n} \big(\phi^n(\epsilon) F(\epsilon)\big)\Big|_{\epsilon=0}, \tag{C.1}$$

where $F(\epsilon)$ and $\phi(\epsilon)$ are holomorphic for $|\epsilon| < r_0$.

Proposition C.1. *If there exists $r < r_0$ such that $|\phi(\epsilon)| < r$ for $|\epsilon| = r$, then the series (C.1) is absolutely convergent and its sum is given by*

$$G_0 = \frac{F(z)}{1 - \phi'(z)}, \tag{C.2}$$

where z is the root of the equation

$$z - \phi(z) = 0 \tag{C.3}$$

such that $|z| < r$.

Proof — Replacing the n^{th} derivative by a Cauchy integral, we obtain

$$G_0 = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{|\omega|=r} \frac{F(\omega)\,\phi^n(\omega)}{\omega^{n+1}}\,\mathrm{d}\omega. \tag{C.4}$$

Since $|\phi(\omega)| < r = |\omega|$ the obtained series is absolutely convergent, and we arrive at

$$G_0 = \frac{1}{2\pi i} \oint_{|\omega|=r} \frac{F(\omega)}{\omega - \phi(\omega)}\, dz. \tag{C.5}$$

Due to Rouché's Theorem, the equation $\omega - \phi(\omega) = 0$ has exactly one simple zero $\omega = z$ within the circle $|\omega| < r$. Taking the residue in this point we obtain the statement of the proposition. □

C.2 Matrix case

The result obtained for the series (C.1) can easily be generalized to the case of several variables. Consider a multiple series of the form

$$G_N = \sum_{s_1,\ldots,s_N=0}^{\infty} \frac{1}{s_1! \cdots s_N!} \prod_{j=1}^{N} \frac{\partial^{s_j}}{\partial \epsilon_j^{s_j}} \prod_{j=1}^{N} \phi_j^{s_j}(\{\epsilon\}) \cdot F(\{\epsilon\})\,\Big|_{\epsilon_j=0}, \tag{C.6}$$

where $F(\{\epsilon\})$ and $\phi_j(\{\epsilon\})$ are holomorphic for $|\epsilon_k| < r_k^{(0)}$, $k = 1, \ldots, N$.

Proposition C.2. *If there exist $r_j < r_j^{(0)}$ such that $|\phi_j(\{z\})| < r_j$ for $|z_j| = r_j$, then the series (C.6) is absolutely convergent and its sum is given by*

$$G_N = \frac{F(\{z_j\})}{\det_N S_{jk}}, \tag{C.7}$$

where z_j are the roots of the system

$$z_j - \phi_j\left(\{z\}\right) = 0, \tag{C.8}$$

and $\det_N S_{jk}$ is the Jacobian of the system (C.8):

$$S_{jk} = \delta_{jk} - \tfrac{\partial}{\partial z_k}\phi_j\left(\{z\}\right). \tag{C.9}$$

The proof is completely analogous to the one of Proposition C.1. It uses the analog of Rouché's Theorem for several complex variables (see e.g. [3]).

We consider now special cases of the series (C.6). Namely, let $\mu_1, \dots, \mu_N$ be a set of complex parameters. Let

$$\phi_j(\{\epsilon\}) = f\left(\sum_{a=1}^{N} \epsilon_a\, \theta(\mu_a, \mu_j)\right), \tag{C.10}$$

where $\theta(\lambda, \mu)$ is some smooth function. Let also

$$F(\{\epsilon\}) = F\Bigg(\sum_{a=1}^{N} g^{(1)}(\mu_a)\, \epsilon_a;\ \sum_{\substack{a,b=1\\ a\neq b}}^{N} g^{(2)}(\mu_a, \mu_b)\, \epsilon_a\, \epsilon_b;\ \dots\Bigg), \tag{C.11}$$

where $g^{(1)}, g^{(2)} \dots$ are smooth functions. Then the obtained result takes the form

$$G_N = \frac{F\Big(\sum\limits_{a=1}^{N} g^{(1)}(\mu_a)\, z_a;\ \sum\limits_{a,b=1}^{N} g^{(2)}(\mu_a, \mu_b)\, z_a\, z_b;\ \dots\Big)}{\det_N\left[\delta_{jk} - \theta(\mu_k, \mu_j)\, f'\Big(\sum\limits_{a=1}^{N} z_a\, \theta(\mu_a, \mu_j)\Big)\right]}, \tag{C.12}$$

where

$$z_j - f\left(\sum_{a=1}^{N} z_a\, \theta(\mu_a, \mu_j)\right) = 0. \tag{C.13}$$

C.3 Continuous case

Let us now consider a series of multiple integrals

$$\hat{G} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-q}^{q} \mathrm{d}^n\lambda \prod_{j=1}^{n} \frac{\partial}{\partial \varepsilon_j} \prod_{j=1}^{n} f\left(\sum_{a=1}^{n} \varepsilon_a\, \theta(\lambda_a, \lambda_j)\right) \cdot F\left(\sum_{a=1}^{n} g^{(1)}(\lambda_a)\, \varepsilon_a;\ \dots,\right)\Bigg|_{\varepsilon_j=0}, \tag{C.14}$$

where for brevity we have written explicitly only the first argument of the function F. Consider a discreet analog of these multiple integrals,

$$\hat{G}(\Delta) = \sum_{n=0}^{\infty} \frac{\Delta^n}{n!} \sum_{\lambda_1,\dots,\lambda_n\in\Lambda} \prod_{j=1}^{n} \frac{\partial}{\partial\varepsilon_j} \prod_{j=1}^{n} f\left(\sum_{a=1}^{n} \varepsilon_a\, \theta(\lambda_a,\lambda_j)\right) \cdot F\left(\sum_{a=1}^{n} g^{(1)}(\lambda_a)\varepsilon_a;\ \dots\right)\Bigg|_{\varepsilon_j=0}.$$

Here each λ_j independently runs through a finite set of values $\Lambda = \{\mu_1,\dots,\mu_N\}$, where $\mu_{k+1} - \mu_k = \Delta$. Obviously $\hat{G}(\Delta) \to \hat{G}$ at $\Delta \to 0$.

We now reorganize the inner sums over the lattice into sums over the number of λ's equal to some point of the lattice. Namely, we sum up with respect to all the possible numbers s_j such that there are s_j λ's equal to the lattice point μ_j. Obviously there exist $\frac{n!}{s_1!\dots s_N!}$ ways of realizing such a configuration. It is also evident that, for a configuration such that, for each j, there are s_j λ's equal to μ_j, then for any function Φ, one has

$$\prod_{j=1}^{n} \frac{\partial}{\partial\varepsilon_j} \cdot \Phi\left(\sum_{a=1}^{n} g^{(1)}(\lambda_a)\, \varepsilon_a;\ \dots\right)\Bigg|_{\varepsilon_j=0} = \prod_{j=1}^{N} \frac{\partial^{s_j}}{\partial\epsilon_j^{s_j}} \cdot \Phi\left(\sum_{a=1}^{N} g^{(1)}(\mu_a)\, \epsilon_a;\ \dots\right)\Bigg|_{\epsilon_j=0}. \quad \text{(C.15)}$$

Hence,

$$\hat{G}(\Delta) = \sum_{n=0}^{\infty} \sum_{\substack{s_i\ge 0\\ \Sigma s_i=n}} \prod_{j=1}^{N} \frac{\Delta^{s_j}}{s_j!} \frac{\partial^{s_j}}{\partial\epsilon_j^{s_j}}\Bigg|_{\epsilon_j=0} \prod_{j=1}^{N} f^{s_j}\left(\sum_{a=1}^{N} \epsilon_a\theta(\mu_a,\mu_j)\right) F\left(\sum_{a=1}^{N} g^{(1)}(\mu_a)\epsilon_a;\dots\right). \quad \text{(C.16)}$$

Changing the order of summation and re-scaling $\epsilon_j \to \Delta\epsilon_j$, we obtain the series (C.6) with ϕ_j and F in the form (C.10), (C.11), in which the functions θ and $g^{(i)}$ should be replaced by $\Delta\theta(\mu_a,\mu_j)$ and $\Delta g^{(i)}(\mu_a)$. Then the continuous limit $\Delta \to 0$ is trivial, and we finally obtain

$$\hat{G} = \frac{F\left(\int_{-q}^{q} g^{(1)}(\mu)\, z(\mu)\, \mathrm{d}\mu;\ \dots\right)}{\det\left[\delta(\lambda-\mu) - \theta(\mu,\lambda)\, f'\left(\int_{-q}^{q} \theta(\nu,\lambda)\, z(\nu)\, \mathrm{d}\nu\right)\right]}, \quad \text{(C.17)}$$

where the function $z(\mu)$ satisfies the integral equation

$$z(\mu) = f\left(\int_{-q}^{q} \theta(\lambda,\mu)\, z(\lambda)\, \mathrm{d}\lambda\right). \quad \text{(C.18)}$$

C.4 Multiple series of multiple integrals

Consider now a multiple series of the form

$$\hat{G}_{1\ldots n} = \sum_{\ell_1,\ldots,\ell_n=0}^{\infty} \prod_{s=1}^{n} \frac{1}{\ell_s!} \int_{-q}^{q} \prod_{s=1}^{n} \prod_{p=1}^{\ell_s} \left[\mathrm{d}\lambda_{s,p} \, \frac{\partial}{\partial \epsilon_{s,p}} \right]$$
$$\times \prod_{s=1}^{n} \prod_{p=1}^{\ell_s} f_s \left(\sum_{t=1}^{n} \sum_{a=1}^{\ell_t} \epsilon_{t,a} \, \theta(\lambda_{t,a}, \lambda_{s,p}) \right) \cdot F \left(\sum_{t=1}^{n} \sum_{a=1}^{\ell_t} g^{(1)}(\lambda_{t,a}) \, \epsilon_{t,a}; \; \ldots \right) \Bigg|_{\epsilon_{s,p}=0} . \quad \text{(C.19)}$$

Just like in the previous example we can take the lattice approximation of the integrals. Then it is easy to see that

$$\hat{G}_{1\ldots n}(\Delta) = \frac{1}{\det_{nN} S_{jk}} \, F \left(\Delta \sum_{s=1}^{n} \sum_{a=1}^{N} g^{(1)}(\mu_a) \, z_{s,a}; \; \ldots \right) , \quad \text{(C.20)}$$

where

$$z_{s,a} - f_s \left(\Delta \sum_{s=1}^{n} \sum_{b=1}^{N} z_{s,b} \, \theta(\mu_b, \mu_a) \right) = 0, \quad \text{(C.21)}$$

and $\det_{nN} S_{jk}$ is the Jacobian of the system (C.21). Taking the sum over s in (C.21), we obtain

$$z_a - f_{\Sigma_n} \left(\Delta \sum_{b=1}^{N} z_b \, \theta(\mu_b, \mu_a) \right) = 0, \qquad \text{where} \quad z_a = \sum_{s=1}^{n} z_{s,a}, \quad f_{\Sigma_n} = \sum_{s=1}^{n} f_s. \quad \text{(C.22)}$$

It is also clear that $\det_{mN} S_{jk} = \det_N \tilde{S}_{jk}$, where

$$\tilde{S}_{jk} = \delta_{jk} - \Delta \, \theta(\mu_k, \mu_j) \, f'_{\Sigma_n} \left(\Delta \sum_{b=1}^{N} z_b \, \theta(\mu_b, \mu_j) \right) \quad \text{(C.23)}$$

is the Jacobian of the system (C.22). Thus, in the continuous limit, we have

$$\hat{G}_{1\ldots n} = \frac{F\left(\int_{-q}^{q} g^{(1)}(\mu) \, z^{(n)}(\mu) \, \mathrm{d}\mu; \; \ldots \right)}{\det \left[\delta(\lambda - \mu) - \theta(\mu, \lambda) \, f'_{\Sigma_n} \left(\int_{-q}^{q} \theta(\nu, \lambda) \, z^{(n)}(\nu) \, \mathrm{d}\nu \right) \right]}, \quad \text{(C.24)}$$

where $f_{\Sigma_n} = \sum_{s=1}^{n} f_k$, and

$$z^{(n)}(\mu) = f_{\Sigma_n} \left(\int_{-q}^{q} \theta(\lambda, \mu) \, z^{(n)}(\lambda) \, \mathrm{d}\lambda \right) . \quad \text{(C.25)}$$

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