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# GAUGED SUPERGRAVITIES, TENSOR HIERARCHIES, AND M-THEORY

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**Abstract**

Deformations of maximal supergravity theories induced by gauging non-abelian subgroups of the duality group reveal the presence of charged M-theory degrees of freedom that are not necessarily contained in supergravity. The relation with M-theory degrees of freedom is confirmed by the representation assignments under the duality group of the gauge charges and the ensuing vector and tensor gauge fields. The underlying hierarchy of these gauge fields is required for consistency of general gaugings. As an example gauged maximal supergravity in three space-time dimensions is presented in a version where all possible tensor fields appear.

# 1 Introduction

In recent years a wealth of information has become available about general gaugings of supergravity. In particular, it has become clear that these theories may play an essential role in probing and exploring M-theory beyond the supergravity approximations considered so far. The key ingredient in these developments is the so-called *embedding tensor*, which defines the embedding of the gauge group generators (up to possible central extensions) in the rigid symmetry group, which for the maximal supergravities is the duality group that arises upon dimensional reduction of eleven-dimensional or ten-dimensional (IIB) supergravity. With the embedding tensor at hand, all gauged supergravities in various space-time dimensions can now be classified.

The first maximal gauged supergravity, $N = 8$ supergravity in four space-time dimensions with compact gauge group SO(8), was constructed in [1], soon followed by similar gaugings in maximal supergravity in $D = 5$ [2] and $D = 7$ [3] dimensions. Also $D = 4$ gaugings with non-compact versions of SO(8) and contractions thereof were found to exist [4]. Although these results eluded a more systematic understanding for a long time, there were hints of a deeper group-theoretical structure underlying these constructions, and linking the existence of gauged supergravities to certain higher-dimensional representations of the duality groups $E_{n(n)}$: it was known already in 1984 that the so-called $T$-tensor of $N = 8$ supergravity (essentially a 'dressed' version of the embedding tensor) belongs to the **912** representation of $\mathrm{E}_{7(7)}$ [5]. The latter group is the invariance group of (ungauged) maximal supergravity in $D = 4$ dimensions [6].

The more recent developments allowing for a much more systematic exploration of gauged supergravities go back to the discovery of maximal gauged supergravities in *three* space-time dimensions [7, 8, 9], and it was in this context that the notion of embedding tensor was first introduced. The case of three space-time dimensions is special because all higher-rank tensor fields present in higher dimensions have been dualized away in the dimensional reduction of $D = 11$ supergravity [10] to three dimensions, such that all propagating degrees of freedom can be described by scalar fields. An immediate puzzle then (and the reason why these theories had not been found earlier) was the question how to gauge a theory that apparently does not have any vector fields left from the dimensional reduction. This puzzle was finally resolved in [7, 8] by introducing a set of 248 'redundant' vector fields transforming in the adjoint representation of $\mathrm{E}_{8(8)}$; rather than through the standard Yang-Mills kinetic term, these gauge fields appear with a Chern-Simons term in the Lagrangian, ensuring that the number of physical degrees of freedom in the theory remains the same as before.

The systematic investigation of gauged supergravities in dimensions $D \geq 4$ by means of the embedding tensor was initiated in refs. [11, 12], following the discovery of a new maximal gauged supergravity in [13] based on Scherk–Schwarz compactification [14] of $D = 5$ supergravity. This systematic analysis has meanwhile lead to a complete classification of gauged maximal supergravities in $D = 5$ [15], $D = 7$ [16], and, finally, $D = 4$ [17] and $D = 6$ [18] (the situation in an even number of dimensions is more complicated because the

duality group is only a symmetry of the equations of motion, but not of the Lagrangian). In particular, it can be shown that the known examples of gauged supergravities (including more recent constructions such as [19, 20, 21, 22, 23, 24, 25, 26, 27, 28]) can all be accommodated within the systematic approach based on the embedding tensor. Most recently, gaugings of maximal supergravity in $D = 2$ were constructed in [29] — this case being more exotic because the relevant duality group $E_9$ is infinite dimensional.

The appearance of 'redundant' vector fields in $D = 3$ gauged supergravities and the (long known) fact that the consistent gauging of maximal supergravity in $D = 5$ [2] requires the simultaneous use of vector fields and 2-form potentials, has led to the conclusion that a systematic understanding of gauged supergravities makes the consideration of higher-rank tensor fields unavoidable [30]. As pointed out there, and as will be analyzed in further detail in the present paper, gauged supergravities can be consistently and systematically formulated by introducing *a hierarchy of anti-symmetric tensor fields*. The analysis at this point is independent of the number of space-time dimensions, and the hierarchy contains in principle an infinite number of anti-symmetric tensors of any rank. Of course, once the space-time dimension is fixed to some integer $D$, the maximal rank is also fixed to $D$. Maintaining the correct number of propagating degrees of freedom in the presence of these extra fields requires a subtle interplay of ordinary gauge invariance and higher-rank tensor gauge transformations. For non-zero gauge coupling the physical degrees of freedom reside in a finite number of the tensor fields and it is the embedding tensor that determines how these degrees of freedom are distributed over the various tensor fields. Here it is important to note that, in the presence of gauge interactions, the possibility for converting rank-$p$ to rank-$(D-p-2)$ tensors fields is severely restricted. When the gauge coupling constant vanishes the hierarchy can in general be truncated.

In an important and independent development [31, 32, 33] it has been shown that the relevant representations of all higher-rank tensors fields can also be obtained via a level decomposition of the indefinite Kac–Moody algebra $\mathrm{E}_{11}$ (if one omits the $D$-forms, these representations can be equivalently derived from the hyperbolic Kac–Moody algebra $\mathrm{E}_{10}$). In order to arrive at this decomposition, one first selects a 'disabled' node in the Dynkin diagram, and then decomposes the algebra in representations of the remaining finite dimensional subgroups of $\mathrm{E}_{11}$, all of which are direct products $\mathrm{SL}(D) \times \mathrm{E}_{11-D}$ where $D \geq 3$ denotes the number of uncompactified space-time coordinates. Remarkably, it turns out that the low-lying representations in that analysis coincide with the representations found here by a completely different route. However, one should keep in mind that 'higher up' in the level decompositions of $\mathrm{E}_{10}$ and $\mathrm{E}_{11}$ there opens up a *terra incognita* of an exponentially growing spectrum of representations of ever increasing size and complexity, whose ultimate role and significance remain to be understood.

What is the physical significance of these results? As we will argue here, the existence of these gauged supergravities constitutes direct evidence for new M-theoretic degrees of freedom beyond the known maximal supergravities in space-time dimensions $D \leq 11$ (and possibly also beyond string theory as presently understood). This feature is most evident

for $D = 3$ gauged supergravities with semi-simple gauge groups: none of these theories can be obtained from higher-dimensional supergravity by conventional (Kaluza–Klein or Scherk–Schwarz) compactification. Our claim is supported by the fact that several of the 'exotic' representations of the duality groups exhibited here have also been found to occur in toroidally compactified matrix theory [34, 35], as well as in the context of del Pezzo surfaces and compactified M-theory [36]. The process of gauging a given maximal supergravity can thus be interpreted as the process of 'switching on' such new degrees of freedom, which are here encoded into the embedding tensor. A special role is played by the $(D-1)$- and $D$-forms: we will set up a Lagrangian formulation of three-dimensional maximal supergravity containing all higher-rank antisymmetric tensor fields with an initially space-time dependent embedding tensor $\Theta(x)$, in such a way that the $(D-1)$- and $D$-forms, respectively, impose the constancy of $\Theta$, and the closure of the corresponding gauge group. Alternatively, one can eliminate the field $\Theta$ which appears at most quadratically in the Lagrangian by means of its equations of motion, thereby arriving at a Lagrangian that contains the higher-rank tensor fields in a non-polynomial fashion. Gauging would then be realized as a kind of spontaneous symmetry breaking,[1] and equivalent to the process of certain $D$-form field strengths acquiring vacuum expectation values. In this way, the different maximal gauged supergravities can be interpreted as different 'phases' of one and the same Lagrangian theory.

Finally, we should stress that we consider the deformations mainly from the point of view of setting up a consistent gauging. On the other hand, additional deformations are sometimes possible, generated by singlet components in the 'descendants' of the embedding tensor (which, presumably, could induce additional non-singlet terms higher up in the hierarchy). The embedding tensor by definition specifies how the gauge group is embedded in the duality group, but it also encodes many of the interactions of the tensor fields. At the level of these tensor interactions the embedding tensor may be able to accomodate additional components which will still fit into the hierarchy. A well-known example of this phenomenon is the Romans massive deformation of ten-dimensional IIA supergravity [37], which is induced by a nine-form potential. We will comment on this in due course.

This paper is organized as follows. In section 2 we discuss the hierarchy of tensor gauge fields in a general context. In section 3 we discuss the relation with M-theory degrees of freedom. In section 4 we determine the duality representations of the tensor fields in three space-time dimensions. The corresponding supersymmetry algebra is discussed in section 5 and the general Lagrangian for gauged three-dimensional maximal supergravity in section 6. Results of the present investigation have already been announced and discussed by us in several talks [2].

[1] This terminology clearly differs from the usual one, and should thus be understood *cum grano salis*.

[2] See, for instance: http://ggi-www.fi.infn.it/activities/workshops/stringM/talks/dewit.pdf; http://maths.dur.ac.uk/events/Meetings/LMS/2007/TSAS/Talks/dewit.pdf

## 2 A hierarchy of vector and tensor gauge fields

Maximal supergravities in various space-time dimensions can be constructed by dimensional reduction on a torus of supergravity in eleven and/or ten space-time dimensions. In general these theories contain abelian vector fields and antisymmetric tensor fields of various ranks. Their field content is not unique as $p$-rank tensor gauge fields can be dualized to tensor fields of rank $D-p-2$, where $D$ denotes the dimension of space-time of the reduced theory. However, there always exists an optimal choice of the field configuration that most clearly exhibits the invariance under a duality group G. This group is listed for space-time dimensions $D=3,\ldots,7$ in the second column of table 1. The symmetry under the G-transformations is realized non-linearly in view of the fact that the scalar fields parametrize a G/H coset space, where H is the R-symmetry group of the corrresponding supersymmetry algebra. This group equals the maximal compact subgroup of G and it is also listed in table 1. In general the vector and antisymmetric gauge fields transform in specific representations of G.[3] The vector fields, which we denote by $A_\mu{}^{\mathcal{M}}$, transform in the fundamental or in a spinor representation of G. These representations are (implicitly) listed in table 1, as we will explain below. The generators in these representations are denoted by $(t_\alpha)_{\mathcal{M}}{}^{\mathcal{N}}$, so that $\delta A_\mu{}^{\mathcal{M}} = -\Lambda^\alpha (t_\alpha)_{\mathcal{N}}{}^{\mathcal{M}}\, A_\mu{}^{\mathcal{N}}$. Structure constants $f_{\alpha\beta}{}^\gamma$ of the duality group are defined according to $[t_\alpha, t_\beta] = f_{\alpha\beta}{}^\gamma\, t_\gamma$.

Deformations of these maximal supergravities can be constructed by introducing a non-abelian gauge group, which must be a subgroup of the duality group. The dimension of this gauge group is obviously restricted by the number of vector fields in the theory. The discussion in this section will remain rather general and will neither depend on the actual duality group nor on the space-time dimension (we recall, however, that there may be subtleties in even space-time dimensions related to selfduality of vector or tensor gauge fields). We refer to [15, 16, 17] where a number of results were described for maximal supergravity in various dimensions.

The gauge group generators $X_{\mathcal{M}}$, which will couple to the gauge fields $A_\mu{}^{\mathcal{M}}$ in the usual fashion, are obviously decomposed in terms of the independent G generators $t_\alpha$, *i.e.*,

$$X_{\mathcal{M}} = \Theta_{\mathcal{M}}{}^\alpha\, t_\alpha\ . \tag{2.1}$$

The gauging is thus encoded in a real *embedding tensor* $\Theta_{\mathcal{M}}{}^\alpha$ belonging to the product of the representation conjugate to the representation in which the gauge fields transform and the adjoint representation of G. This product representation is reducible and decomposes into a number of irreducible representations as is indicated for the cases of interest in the last column of table 1. However, as is also shown in the table, supersymmetry requires most of these irreducible representations to be absent: only the underlined representations in the table are compatible with local supersymmetry. Actually, for non-supersymmetric theories one may have to impose similar constraints (see, *e.g.* [38]). This constraint on the embedding

[3] In even space-time dimensions this assignment may fail and complete G representations may require the presence of magnetic duals. For four space-time dimensions, this has been demonstrated in [38].

| $D$ | G | H | $\Theta$ |
|---|---|---|---|
| 7 | SL(5) | USp(4) | $\mathbf{10}\times\mathbf{24}=\mathbf{10}+\underline{\mathbf{15}}+\overline{\underline{\mathbf{40}}}+\mathbf{175}$ |
| 6 | SO(5,5) | USp(4) × USp(4) | $\mathbf{16}\times\mathbf{45}=\mathbf{16}+\underline{\mathbf{144}}+\mathbf{560}$ |
| 5 | $E_{6(6)}$ | USp(8) | $\mathbf{27}\times\mathbf{78}=\mathbf{27}+\underline{\mathbf{351}}+\overline{\mathbf{1728}}$ |
| 4 | $E_{7(7)}$ | SU(8) | $\mathbf{56}\times\mathbf{133}=\mathbf{56}+\underline{\mathbf{912}}+\mathbf{6480}$ |
| 3 | $E_{8(8)}$ | SO(16) | $\mathbf{248}\times\mathbf{248}=\underline{\mathbf{1}}+\mathbf{248}+\underline{\mathbf{3875}}+\mathbf{27000}+\mathbf{30380}$ |

Table 1: Decomposition of the embedding tensor $\Theta$ for maximal supergravities in various space-time dimensions in terms of irreducible G representations [11, 30]. Only the underlined representations are allowed by supersymmetry. The R-symmetry group H is the maximal compact subgroup of G.

tensor is known as the *representation constraint*. Here we treat the embedding tensor as a spurionic object, which we allow to transform under the duality group so that the Lagrangian and transformation rules remain formally invariant under G. At the end we will freeze the embedding tensor to a constant, so that the duality invariance will be broken. Later in this paper we see that this last step can also be described in terms of a new action in which the freezing of $\Theta_{\mathcal{M}}{}^{\alpha}$ will be the result of a more dynamical process.

The embedding tensor must satisfy a second constraint, the so-called *closure constraint*, which is quadratic in $\Theta_{\mathcal{M}}{}^{\alpha}$ and more generic. This constraint ensures that the gauge transformations form a group so that the generators (2.1) will close under commutation. Any embedding tensor that satisfies the closure constraint, together with the representation constraint mentioned earlier, defines a consistent gauged supergravity theory that is both supersymmetric and gauge invariant. To spell out the closure constraint in more detail let us write out (2.1) once more, but now with representation indices in the G-representation pertaining to the gauge fields written out explicitly, *viz.*

$$X_{\mathcal{MN}}{}^{\mathcal{P}} \equiv \Theta_{\mathcal{M}}{}^{\alpha}\,(t_\alpha)_{\mathcal{N}}{}^{\mathcal{P}} = X_{[\mathcal{MN}]}{}^{\mathcal{P}} + Z^{\mathcal{P}}{}_{\mathcal{MN}} \tag{2.2}$$

where we will use the notation

$$Z^{\mathcal{P}}{}_{\mathcal{MN}} \equiv X_{(\mathcal{MN})}{}^{\mathcal{P}}\,, \tag{2.3}$$

for the symmetric part throughout this paper. The closure constraint is a consequence of the invariance of the embedding tensor under the gauge group it generates, that is

$$\delta_{\mathcal{P}}\Theta_{\mathcal{M}}{}^{\alpha} = \Theta_{\mathcal{P}}{}^{\beta} t_{\beta\mathcal{M}}{}^{\mathcal{N}}\Theta_{\mathcal{N}}{}^{\alpha} + \Theta_{\mathcal{P}}{}^{\beta} f_{\beta\gamma}{}^{\alpha}\Theta_{\mathcal{M}}{}^{\gamma} = 0\,. \tag{2.4}$$

Contracting this result with $t_\alpha$ we obtain

$$[X_{\mathcal{M}}, X_{\mathcal{N}}] = -X_{\mathcal{MN}}{}^{\mathcal{P}}\, X_{\mathcal{P}} = -X_{[\mathcal{MN}]}{}^{\mathcal{P}}\, X_{\mathcal{P}}\,, \tag{2.5}$$

Hence, the gauge invariance of the embedding tensor is equivalent to the closure of the gauge algebra. It is noteworthy here that the generator $X_{\mathcal{MN}}{}^{\mathcal{P}}$ and the structure constants of the gauge group are thus related, but do not have to be identical. In particular $X_{\mathcal{MN}}{}^{\mathcal{P}}$ is in general not antisymmetric in $[\mathcal{MN}]$, as is evident from (2.2). The embedding tensor acts

as a projector, and only in the projected subspace the matrix $X_{\mathcal{MN}}{}^{\mathcal{P}}$ is antisymmetric in $[\mathcal{MN}]$ and the Jacobi identity will be satisfied. Therefore (2.5) implies in particular that $X_{(\mathcal{MN})}{}^{\mathcal{P}}$ must vanish when contracted with the embedding tensor. In terms of the notation introduced above, this condition reads

$$\Theta_{\mathcal{P}}{}^{\alpha}\, Z^{\mathcal{P}}{}_{\mathcal{MN}} = 0\,. \tag{2.6}$$

The gauge invariant tensor $Z^{\mathcal{P}}{}_{\mathcal{MN}}$ transforms in the same representation as $\Theta_{\mathcal{M}}{}^{\alpha}$, except when the embedding tensor transforms reducibly so that $Z^{\mathcal{P}}{}_{\mathcal{MN}}$ may depend on a smaller representation. As may be expected the tensor $Z^{\mathcal{P}}{}_{\mathcal{MN}}$ characterizes the lack of closure of the generators $X_{\mathcal{M}}$. This can be seen, for instance, by calculating the direct analogue of the Jacobi identity,

$$X_{[\mathcal{NP}}{}^{\mathcal{R}}\, X_{\mathcal{Q}]\mathcal{R}}{}^{\mathcal{M}} = \tfrac{2}{3} Z^{\mathcal{M}}{}_{\mathcal{R}[\mathcal{N}}\, X_{\mathcal{PQ}]}{}^{\mathcal{R}}\,. \tag{2.7}$$

We emphasize that seemingly strange features, such as the appearance of a symmetric contribution in $X_{\mathcal{MN}}{}^{\mathcal{P}}$, or the apparent violation of the Jacobi identity in (2.7), are entirely due to the redundancy in the description: although the actual gauge group is usually smaller than G, we nevertheless continue to label all matrices by G indices $\mathcal{M}$, such that the number of matrices $X_{\mathcal{M}}$ in general will exceed the dimension of the gauge group. The main advantage of this parametrization (and nomenclature) is its universality, which allows us to treat all gaugings (and gauge groups) on the same footing.

Now we return to the field theoretic description. The gauging requires the replacement of ordinary space-time derivatives by covariant ones for all fields except the gauge fields,

$$\partial_\mu \to D_\mu = \partial_\mu - g\, A_\mu{}^{\mathcal{M}}\, X_{\mathcal{M}}\,, \tag{2.8}$$

where the generator $X_{\mathcal{M}}$ must be taken in the appropriate representation. To write down invariant kinetic terms for the gauge fields one needs a suitable covariant field strength tensor. This is an issue because the Jacobi identity is not satisfied. The standard field strength, which follows from the Ricci identity, $[D_\mu, D_\nu] = -g\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\, X_{\mathcal{M}}$, reads,

$$\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} = \partial_\mu A_\nu{}^{\mathcal{M}} - \partial_\nu A_\mu{}^{\mathcal{M}} + g\, X_{[\mathcal{NP}]}{}^{\mathcal{M}}\, A_\mu{}^{\mathcal{N}} A_\nu{}^{\mathcal{P}}\,, \tag{2.9}$$

and is not fully covariant. The lack of covariance can be readily checked by observing that $\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}$ does *not* satisfy the Palatini identity [4]; rather, we have

$$\delta\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} = 2\, D_{[\mu}\delta A_{\nu]}{}^{\mathcal{M}} - 2g\, X^{\mathcal{M}}{}_{(\mathcal{PQ})}\, A_{[\mu}{}^{\mathcal{P}}\, \delta A_{\nu]}{}^{\mathcal{Q}}\,, \tag{2.10}$$

under arbitrary variations $\delta A_\mu{}^{\mathcal{M}}$. Assuming the standard gauge transformation,

$$\delta A_\mu{}^{\mathcal{M}} = D_\mu \Lambda^{\mathcal{M}} = \partial_\mu \Lambda^{\mathcal{M}} + g A_\mu{}^{\mathcal{N}} X_{\mathcal{NP}}{}^{\mathcal{M}} \Lambda^{\mathcal{P}}\,, \tag{2.11}$$

it follows that $\mathcal{F}_{\mu\nu}{}^{M}$ transforms under gauge transformations as

$$\delta\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} = g\, \Lambda^{\mathcal{P}} X_{\mathcal{NP}}{}^{\mathcal{M}}\, \mathcal{F}_{\mu\nu}{}^{\mathcal{N}} - 2g\, Z^{\mathcal{M}}{}_{\mathcal{PQ}}\, A_{[\mu}{}^{\mathcal{P}}\, \delta A_{\nu]}{}^{\mathcal{Q}}\,, \tag{2.12}$$

[4] That is, the standard relation $\delta\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} = 2\, D_{[\mu}\delta A_{\nu]}{}^{\mathcal{M}}$.

which is *not* covariant — not only because of the presence of the second term on the right-hand side, but also because the lack of antisymmetry of the structure constants $X_{\mathcal{NP}}{}^{\mathcal{M}}$ prevents us from getting the correct result (cf. (2.20) below) by simply inverting the order of indices $\mathcal{NP}$ in the first term on the right-hand side

In order to remedy this lack of covariance we now follow the strategy of [15, 30]. Since we know that closure is ensured on the subspace projected by the embedding tensor, we introduce additional gauge transformations in the orthogonal complement so that all difficulties associated with the lack of closure can be compensated for by performing these new transformations. For the gauge fields, this leads to the following transformation rule,

$$\delta A_\mu{}^{\mathcal{M}} = D_\mu \Lambda^{\mathcal{M}} - g\, Z^{\mathcal{M}}{}_{\mathcal{NP}}\, \Xi_\mu{}^{\mathcal{NP}}\,, \tag{2.13}$$

where the transformations proportional to $\Xi_\mu{}^{\mathcal{NP}}$ enable one to gauge away those vector fields that are in the sector of the gauge generators $X_{\mathcal{MN}}{}^{\mathcal{P}}$ where the Jacobi identity is not satisfied (this sector is perpendicular to the embedding tensor by (2.6)). Note that the parameter $\Xi_\mu{}^{\mathcal{NP}}$ in (2.13) appears contracted with the constant tensor $Z^{\mathcal{M}}{}_{\mathcal{NP}}$ defined in (2.3) as a linear function of the embedding tensor. It is important, that this tensor generically does not map onto the full symmetric tensor product $(\mathcal{NP})$ in its lower indices but rather only on a restricted subrepresentation. In other words, there is a non-trivial G-invariant projector $\mathbb{P}$ such that

$$Z^{\mathcal{M}}{}_{\mathcal{NP}} = Z^{\mathcal{M}}{}_{\mathcal{RS}}\, \mathbb{P}^{\mathcal{RS}}{}_{\mathcal{NP}}\,, \tag{2.14}$$

for any choice of the embedding tensor. The precise representation content of $\mathbb{P}$ can be determined for any given theory by carefully inspecting (2.3) and we give examples of this in the later sections (see also [30]). In order not to overburden the formulas with explicit projectors, we denote the projection corresponding to (2.14) by the special brackets $\llbracket\mathcal{NP}\rrbracket$, *i.e.* we use the notation

$$A^{\llbracket\mathcal{M}} A^{\mathcal{N}\rrbracket} \equiv \mathbb{P}^{\mathcal{MN}}{}_{\mathcal{RS}}\, A^{\mathcal{R}} A^{\mathcal{S}}\,, \qquad \text{etc.} \tag{2.15}$$

Similar notation will be used for other index combination that we will encounter shortly.

The combined gauge transformations (2.13) generate a group on the vector fields, as follows from the commutation relations,

$$[\delta(\Lambda_1), \delta(\Lambda_2)] \;=\; \delta(\Lambda_3) + \delta(\Xi_3)\,, \tag{2.16}$$

where

$$\begin{aligned} \Lambda_3{}^{\mathcal{M}} &= g\, X_{[\mathcal{NP}]}{}^{\mathcal{M}}\, \Lambda_2^{\mathcal{N}} \Lambda_1^{\mathcal{P}}\,, \\ \Xi_{3\mu}{}^{\mathcal{MN}} &= \Lambda_1^{\llbracket\mathcal{M}} D_\mu \Lambda_2^{\mathcal{N}\rrbracket} - \Lambda_2^{\llbracket\mathcal{M}} D_\mu \Lambda_1^{\mathcal{N}\rrbracket}\,. \end{aligned} \tag{2.17}$$

Here it is crucial that $\delta(\Lambda)$ and $\delta(\Xi)$ commute on the vector fields. However, these commutators are subject to change when more fields will be introduced. We return to this issue in due course.

Under the combined gauge transformations $\mathcal{F}_{\mu\nu}{}^{M}$ changes as follows,

$$\delta\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} = g\,\Lambda^{\mathcal{P}} X_{\mathcal{N}\mathcal{P}}{}^{\mathcal{M}}\,\mathcal{F}_{\mu\nu}{}^{\mathcal{N}} - 2g\,Z^{\mathcal{M}}{}_{\mathcal{P}\mathcal{Q}}\left(D_{[\mu}\Xi_{\nu]}{}^{\mathcal{P}\mathcal{Q}} + A_{[\mu}{}^{\mathcal{P}}\,\delta A_{\nu]}{}^{\mathcal{Q}}\right), \tag{2.18}$$

which is still not covariant. The standard strategy [15, 30] is therefore to define modified field strengths,

$$\mathcal{H}_{\mu\nu}{}^{\mathcal{M}} = \mathcal{F}_{\mu\nu}{}^{\mathcal{M}} + g\,Z^{\mathcal{M}}{}_{\mathcal{N}\mathcal{P}}\,B_{\mu\nu}{}^{\mathcal{N}\mathcal{P}}\,, \tag{2.19}$$

where we introduce tensor fields $B_{\mu\nu}{}^{\mathcal{N}\mathcal{P}}$, transforming under G in the restricted representation (2.14) *i.e.* $B_{\mu\nu}{}^{\mathcal{N}\mathcal{P}} = B_{\mu\nu}{}^{[\![\mathcal{N}\mathcal{P}]\!]}$. Actually the restricted index pair $[\![\mathcal{M}\mathcal{N}]\!]$ will play the role of a new index belonging to a specific representation, and $Z^{\mathcal{M}}{}_{\mathcal{N}\mathcal{P}}$ is an intertwining tensor between the representations of the vectors and the two-forms. The gauge transformation rules of $B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}$ will be chosen such that the field strengths $\mathcal{H}_{\mu\nu}{}^{\mathcal{M}}$ will transform covariantly under gauge transformations, *i.e.*,

$$\delta\mathcal{H}_{\mu\nu}{}^{\mathcal{M}} = -g\,\Lambda^{\mathcal{P}} X_{\mathcal{P}\mathcal{N}}{}^{\mathcal{M}}\,\mathcal{H}_{\mu\nu}{}^{\mathcal{N}}\,. \tag{2.20}$$

To do this in a systematic manner we first define generic covariant variations of the tensor fields,

$$\Delta B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} \equiv \delta B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} - 2\,A_{[\mu}{}^{[\![\mathcal{M}}\delta A_{\nu]}{}^{\mathcal{N}]\!]}\,, \tag{2.21}$$

so that generic variations of $\mathcal{H}_{\mu\nu}{}^{\mathcal{M}}$ take the form

$$\delta\mathcal{H}_{\mu\nu}{}^{\mathcal{M}} = 2\,D_{[\mu}\delta A_{\nu]}{}^{\mathcal{M}} + g\,Z^{\mathcal{M}}{}_{\mathcal{N}\mathcal{P}}\,\Delta B_{\mu\nu}{}^{\mathcal{N}\mathcal{P}}\,. \tag{2.22}$$

For a combined gauge transformation we choose for $\Delta B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}$,

$$\Delta B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}\Big|_{\text{gauge}} = 2\,D_{[\mu}\Xi_{\nu]}{}^{\mathcal{M}\mathcal{N}} - 2\,\Lambda^{[\![\mathcal{M}}\mathcal{H}_{\mu\nu}{}^{\mathcal{N}]\!]} + \cdots\,, \tag{2.23}$$

where the unspecified contributions vanish when $\Delta B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}$ is contracted with $Z^{\mathcal{P}}{}_{\mathcal{M}\mathcal{N}}$, so that they remain as yet undetermined. Substituting this expression and (2.13) into (2.22) leads indeed to the required result (2.20).[5]

Here it is worth pointing out that the expected gauge transformation on $B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}$ equal to

$$\delta B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} = -g\Lambda^{\mathcal{P}} X_{\mathcal{P}[\![\mathcal{R}\mathcal{S}]\!]}{}^{[\![\mathcal{M}\mathcal{N}]\!]} B_{\mu\nu}{}^{\mathcal{R}\mathcal{S}}\,, \tag{2.24}$$

where the generator $X_{\mathcal{P}[\![\mathcal{R}\mathcal{S}]\!]}{}^{[\![\mathcal{M}\mathcal{N}]\!]} = (X_{\mathcal{P}})_{[\![\mathcal{R}\mathcal{S}]\!]}{}^{[\![\mathcal{M}\mathcal{N}]\!]}$ acts in the restricted representation to which $\delta B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}$ belongs, is already contained in the second term in (2.23), up to an additional gauge transformation associated with a three-rank tensor field, that we will introduce shortly.

The above strategy forms the starting point for the construction of a hierarchy of anti-symmetric tensor gauge fields [30]. To see how one proceeds, let us turn to the construction of the covariant field strength for the tensor fields $B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}$,[6]

$$\mathcal{F}_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N}} = 3\,D_{[\mu}B_{\nu\rho]}{}^{\mathcal{M}\mathcal{N}} + 6\,A_{[\mu}{}^{[\![\mathcal{M}}\left(\partial_{\nu}A_{\rho]}{}^{\mathcal{N}]\!]} + \tfrac{1}{3}gX_{[\mathcal{P}\mathcal{Q}]}{}^{\mathcal{N}]\!]}A_{\nu}{}^{\mathcal{P}}A_{\rho]}{}^{\mathcal{Q}}\right), \tag{2.25}$$

[5] Here we note that the present formulae cannot be compared directly to the ones in [30], as those are derived in a different basis, but they can be compared to later work along the same lines, starting with [16].

[6] We use the same letters $\mathcal{F}$ for the field strengths of vectors and higher $p$-forms. From the number of space-time indices it is always clear to which forms the $\mathcal{F}$ belong.

where the first two coefficients follow from (2.23) and the terms cubic in the vector gauge fields are such that generic variations of $\mathcal{F}_{\mu\nu\rho}{}^{\mathcal{MN}}$ read as follows,

$$\begin{aligned}\delta\mathcal{F}_{\mu\nu\rho}{}^{\mathcal{MN}} &= 3\,D_{[\mu}\,\Delta B_{\nu\rho]}{}^{\mathcal{MN}} + 6\,\mathcal{H}_{[\mu\nu}{}^{\lceil\mathcal{M}}\,\delta A_{\rho]}{}^{\mathcal{N}\rfloor} \\ &\quad - g\,Y^{\mathcal{MN}}{}_{\mathcal{P}\lceil\mathcal{RS}\rfloor}\,(3\,B_{[\mu\nu}{}^{\mathcal{RS}}\,\delta A_{\rho]}{}^{\mathcal{P}} + 2\,A_{[\mu}{}^{\mathcal{P}}A_{\nu}{}^{\mathcal{R}}\delta A_{\rho]}{}^{\mathcal{S}})\,,\end{aligned} \tag{2.26}$$

where

$$Y^{\mathcal{MN}}{}_{\mathcal{P}\lceil\mathcal{RS}\rfloor} = 2\,\delta_{\mathcal{P}}{}^{\lceil\mathcal{M}}\,Z^{\mathcal{N}\rfloor}{}_{\mathcal{RS}} - X_{\mathcal{P}\lceil\mathcal{RS}\rfloor}{}^{\lceil\mathcal{MN}\rfloor}\,. \tag{2.27}$$

Note that this definition can be rewritten as

$$Y^{\mathcal{MN}}{}_{\mathcal{P}\lceil\mathcal{RS}\rfloor} = 2\left(\delta_{\mathcal{P}}{}^{\lceil\mathcal{M}}\,Z^{\mathcal{N}\rfloor}{}_{\mathcal{RS}} - X_{\mathcal{P}\lceil\mathcal{R}}{}^{\lceil\mathcal{M}}\delta_{\mathcal{S}\rfloor}{}^{\mathcal{N}\rfloor}\right)\,. \tag{2.28}$$

Just as before we introduce an extra gauge invariance to eventually deal with the non-covariant variations in the last term of (2.26), which will then provide the missing variations in (2.23),

$$\Delta B_{\mu\nu}{}^{\mathcal{MN}}\Big|_{\text{gauge}} = 2\,D_{[\mu}\Xi_{\nu]}{}^{\mathcal{MN}} - 2\,\Lambda^{\lceil\mathcal{M}}\mathcal{H}_{\mu\nu}{}^{\mathcal{N}\rfloor} - g\,Y^{\mathcal{MN}}{}_{\mathcal{P}\lceil\mathcal{RS}\rfloor}\Phi_{\mu\nu}{}^{\mathcal{P}\lceil\mathcal{RS}\rfloor}\,, \tag{2.29}$$

where $\Phi_{\mu\nu}{}^{\mathcal{P}\lceil\mathcal{RS}\rfloor}$ is the new gauge parameter. Secondly we introduce a corresponding three-form gauge field $C_{\mu\nu\rho}{}^{\mathcal{P}\lceil\mathcal{RS}\rfloor}$, and define the field strength $\mathcal{H}_{\mu\nu\rho}{}^{\mathcal{MN}}$,

$$\mathcal{H}_{\mu\nu\rho}{}^{\mathcal{MN}} = \mathcal{F}_{\mu\nu\rho}{}^{\mathcal{MN}} + g\,Y^{\mathcal{MN}}{}_{\mathcal{P}\lceil\mathcal{RS}\rfloor}\,C_{\mu\nu\rho}{}^{\mathcal{P}\lceil\mathcal{RS}\rfloor}\,. \tag{2.30}$$

such that it transforms covariantly, *i.e.*

$$\delta\mathcal{H}_{\mu\nu\rho}{}^{\mathcal{MN}} = -g\Lambda^{\mathcal{P}}X_{\mathcal{P}\lceil\mathcal{RS}\rfloor}{}^{\lceil\mathcal{MN}\rfloor}\,\mathcal{H}_{\mu\nu\rho}{}^{\mathcal{RS}}\,, \tag{2.31}$$

in complete analogy with (2.20). As before, the tensor $Y^{\mathcal{MN}}{}_{\mathcal{P}\lceil\mathcal{RS}\rfloor}$ does not map onto the full tensor product $\mathcal{P}\lceil\mathcal{RS}\rfloor$ in its lower indices but only on a restricted subrepresentation inside, *i.e.*,

$$Y^{\mathcal{MN}}{}_{\mathcal{P}\lceil\mathcal{RS}\rfloor} = Y^{\mathcal{MN}}{}_{\mathcal{Q}\lceil\mathcal{KL}\rfloor}\,\mathbb{P}^{\mathcal{Q}\lceil\mathcal{KL}\rfloor}{}_{\mathcal{P}\lceil\mathcal{RS}\rfloor}\,, \tag{2.32}$$

for a non-trivial projector $\mathbb{P}$ independent of the embedding tensor. In principle, this projector can be worked out from (2.27), but deriving more explicit expressions requires a case-by-case consideration for each duality group G. As in (2.15) we will denote the corresponding projection by special brackets $\lceil\mathcal{P}\lceil\mathcal{RS}\rfloor\rfloor$. The tensor $Y^{\mathcal{MN}}{}_{\mathcal{P}\lceil\mathcal{RS}\rfloor}$ thus represents an intertwining tensor between the two- and the three-forms. It satisfies the properties

$$Z^{\mathcal{Q}}{}_{\mathcal{MN}}\,Y^{\mathcal{MN}}{}_{\mathcal{P}\lceil\mathcal{RS}\rfloor} = 0\,, \tag{2.33}$$

$$Z^{\mathcal{K}}{}_{\mathcal{PQ}}\,Y^{\mathcal{MN}}{}_{\mathcal{K}\lceil\mathcal{RS}\rfloor} = 2\,Z^{\lceil\mathcal{M}}{}_{\mathcal{PQ}}\,Z^{\mathcal{N}\rfloor}{}_{\mathcal{RS}}\,. \tag{2.34}$$

which are both consequences of the quadratic constraint (2.5). The first identity represents the analogue of (2.6). Another identity follows directly from (2.28),

$$Y^{\mathcal{MN}}{}_{\mathcal{P}\lceil\mathcal{RS}\rfloor}\Big|_{(\mathcal{PRS})} = 0\,, \tag{2.35}$$

Generic variations of the covariant field strength (2.6) can be written as

$$\delta\mathcal{H}_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N}} = 3\,D_{[\mu}\,\Delta B_{\nu\rho]}{}^{\mathcal{M}\mathcal{N}} + 6\,\mathcal{H}_{[\mu\nu}{}^{\lceil\mathcal{M}}\,\delta A_{\rho]}{}^{\mathcal{N}\rfloor} + g\,Y^{\lceil\mathcal{M}\mathcal{N}\rfloor}{}_{\mathcal{P}\lceil\mathcal{R}\mathcal{S}\rfloor}\Delta C_{\mu\nu\rho}{}^{\mathcal{P}\lceil\mathcal{R}\mathcal{S}\rfloor}\,, \tag{2.36}$$

where

$$\Delta C_{\mu\nu\rho}{}^{\mathcal{P}\lceil\mathcal{R}\mathcal{S}\rfloor} = \delta C_{\mu\nu\rho}{}^{\mathcal{P}\lceil\mathcal{R}\mathcal{S}\rfloor} - 3\,\delta A_{[\mu}{}^{\lceil\mathcal{P}}\,B_{\nu\rho]}{}^{\mathcal{R}\mathcal{S}\rfloor} - 2\,A_{[\mu}{}^{\lceil\mathcal{P}}A_{\nu}{}^{\lceil\mathcal{R}}\delta A_{\rho]}{}^{\mathcal{S}\rfloor\rfloor}\,. \tag{2.37}$$

Now we consider again a combined gauge transformation. Requiring that $\mathcal{H}_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N}}$ transforms covariantly, it follows that we must choose

$$\Delta C_{\mu\nu\rho}{}^{\mathcal{P}\lceil\mathcal{R}\mathcal{S}\rfloor}\Big|_{\text{gauge}} = 3\,D_{[\mu}\Phi_{\nu\rho]}{}^{\mathcal{P}\lceil\mathcal{R}\mathcal{S}\rfloor} + 3\,\mathcal{H}_{\mu\nu}{}^{\lceil\mathcal{P}}\,\Xi_{\rho}{}^{\mathcal{R}\mathcal{S}\rfloor} + \Lambda^{\lceil\mathcal{P}}\mathcal{H}_{\mu\nu\rho}{}^{\mathcal{R}\mathcal{S}\rfloor} + \cdots\,, \tag{2.38}$$

where the unspecified contributions vanish upon contracting $\Delta C_{\mu\nu\rho}{}^{\mathcal{P}\lceil\mathcal{R}\mathcal{S}\rfloor}$ with $Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\lceil\mathcal{R}\mathcal{S}\rfloor}$, so that they remain as yet undetermined. Here we made use of the Bianchi identity,

$$D_{[\mu}\mathcal{H}_{\nu\rho]}{}^{\mathcal{M}} = \tfrac{1}{3}g\,Z^{\mathcal{M}}{}_{\mathcal{N}\mathcal{P}}\,\mathcal{H}_{\mu\nu\rho}{}^{\mathcal{N}\mathcal{P}}\,. \tag{2.39}$$

Note that the standard Bianchi is obtained upon contraction with the embedding tensor.

At this point we must verify that the algebra of the various gauge transformations defined so far, will close under commutation. Let us first summarize the various transformation rules,

$$\begin{aligned}
\delta A_{\mu}{}^{\mathcal{M}} &= D_{\mu}\Lambda^{\mathcal{M}} - g\,Z^{\mathcal{M}}{}_{\mathcal{N}\mathcal{P}}\,\Xi_{\mu}{}^{\mathcal{N}\mathcal{P}}\,,\\
\delta B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} &= 2\,D_{[\mu}\Xi_{\nu]}{}^{\mathcal{M}\mathcal{N}} - 2\,\Lambda^{\lceil\mathcal{M}}\mathcal{H}_{\mu\nu}{}^{\mathcal{N}\rfloor} + 2\,A_{[\mu}{}^{\lceil\mathcal{M}}\delta A_{\nu]}{}^{\mathcal{N}\rfloor}\\
&\quad - g\,Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\lceil\mathcal{R}\mathcal{S}\rfloor}\,\Phi_{\mu\nu}{}^{\mathcal{P}\lceil\mathcal{R}\mathcal{S}\rfloor}\,,\\
\delta C_{\mu\nu\rho}{}^{\mathcal{P}\lceil\mathcal{R}\mathcal{S}\rfloor} &= 3\,D_{[\mu}\Phi_{\nu\rho]}{}^{\mathcal{P}\lceil\mathcal{R}\mathcal{S}\rfloor} + 3\,\mathcal{H}_{\mu\nu}{}^{\lceil\mathcal{P}}\,\Xi_{\rho}{}^{\mathcal{R}\mathcal{S}\rfloor} + \Lambda^{\lceil\mathcal{P}}\mathcal{H}_{\mu\nu\rho}{}^{\mathcal{R}\mathcal{S}\rfloor} + 3\,\delta A_{[\mu}{}^{\lceil\mathcal{P}}\,B_{\nu\rho]}{}^{\mathcal{R}\mathcal{S}\rfloor}\\
&\quad + 2\,A_{[\mu}{}^{\lceil\mathcal{P}}A_{\nu}{}^{\lceil\mathcal{R}}\delta A_{\rho]}{}^{\mathcal{S}\rfloor\rfloor} + \cdots\,.
\end{aligned} \tag{2.40}$$

These tranformations indeed yield a closed algebra,

$$\begin{aligned}
[\delta(\Lambda_1),\delta(\Lambda_2)] &= \delta(\Lambda_3) + \delta(\Xi_3) + \delta(\Phi_3)\,,\\
[\delta(\Lambda),\delta(\Xi)] &= \delta(\Phi_4)\,,\\
[\delta(\Xi_1),\delta(\Xi_2)] &= \delta(\Phi_5)\,,\\
[\delta(\Lambda),\delta(\Phi)] &= \cdots\,,\\
[\delta(\Xi),\delta(\Phi)] &= \cdots\,,\\
[\delta(\Phi_1),\delta(\Phi_2)] &= 0\,,
\end{aligned} \tag{2.41}$$

where we will comment on the two unspecified commutators in a sequal. The transformation parameters appearing on the right-hand side of (2.41) take the following form,

$$\begin{aligned}
\Lambda_3{}^{\mathcal{M}} &= g\,X_{[\mathcal{N}\mathcal{P}]}{}^{\mathcal{M}}\,\Lambda_2^{\mathcal{N}}\Lambda_1^{\mathcal{P}}\,,\\
\Xi_{3\mu}{}^{\mathcal{M}\mathcal{N}} &= \Lambda_1^{\lceil\mathcal{M}}D_{\mu}\Lambda_2^{\mathcal{N}\rfloor} - \Lambda_2^{\lceil\mathcal{M}}D_{\mu}\Lambda_1^{\mathcal{N}\rfloor}\,,\\
\Phi_{3\mu\nu}{}^{\mathcal{P}\lceil\mathcal{M}\mathcal{N}\rfloor} &= \mathcal{H}_{\mu\nu}{}^{\lceil\lceil\mathcal{M}}\left(\Lambda_2{}^{\mathrm{N}\rfloor}\Lambda_1{}^{\mathcal{P}\rfloor} - \Lambda_1{}^{\mathrm{N}\rfloor}\Lambda_2{}^{\mathcal{P}\rfloor}\right)\,,\\
\Phi_{4\mu\nu}{}^{\mathcal{P}\lceil\mathcal{M}\mathcal{N}\rfloor} &= 2\,D_{[\mu}\Lambda^{\lceil\mathcal{P}}\Xi_{\nu]}{}^{\mathcal{M}\mathcal{N}\rfloor}\,,\\
\Phi_{5\mu\nu}{}^{\mathcal{P}\lceil\mathcal{M}\mathcal{N}\rfloor} &= -\,gZ^{\lceil\mathcal{P}}{}_{\mathcal{R}\mathcal{S}}\left(\Xi_{1[\mu}{}^{\mathcal{M}\mathcal{N}\rfloor}\Xi_{2\nu]}{}^{\mathcal{R}\mathcal{S}} - \Xi_{2[\mu}{}^{\mathcal{M}\mathcal{N}\rfloor}\Xi_{1\nu]}{}^{\mathcal{R}\mathcal{S}}\right)\,,
\end{aligned} \tag{2.42}$$

where the first two equations were already given in (2.17).

Continuing this pattern one can derive the full hierarchy of $p$-forms by iteration. For instance, the transformation rule for $C_{\mu\nu\rho}{}^{\mathcal{P}\lceil\mathcal{RS}\rfloor}$ contains the expected gauge transformation

$$\delta C_{\mu\nu\rho}{}^{\mathcal{P}\lceil\mathcal{RS}\rfloor} = -g\Lambda^{\mathcal{Q}} X_{\mathcal{Q}\lceil\mathcal{K}\lceil\mathcal{LM}\rfloor\rfloor}{}^{\lceil\mathcal{P}\lceil\mathcal{RS}\rfloor\rfloor} C_{\mu\nu\rho}{}^{\mathcal{K}\lceil\mathcal{LM}\rfloor}\,, \tag{2.43}$$

(where again, $X_{\mathcal{Q}\lceil\mathcal{K}\lceil\mathcal{LM}\rfloor\rfloor}{}^{\lceil\mathcal{P}\lceil\mathcal{RS}\rfloor\rfloor} = (X_{\mathcal{Q}})_{\lceil\mathcal{K}\lceil\mathcal{LM}\rfloor\rfloor}{}^{\lceil\mathcal{P}\lceil\mathcal{RS}\rfloor\rfloor}$) up to a term

$$\delta C_{\mu\nu\rho}{}^{\mathcal{P}\lceil\mathcal{RS}\rfloor} = -g\, Y^{\mathcal{P}\lceil\mathcal{RS}\rfloor}{}_{\mathcal{Q}\lceil\mathcal{P}\lceil\mathcal{RS}\rfloor\rfloor}\, \Phi_{\mu\nu\rho}{}^{\mathcal{Q}\lceil\mathcal{P}\lceil\mathcal{RS}\rfloor\rfloor}\,, \tag{2.44}$$

which characterizes a new gauge transformation with parameter $\Phi_{\mu\nu\rho}{}^{\mathcal{Q}\lceil\mathcal{P}\lceil\mathcal{RS}\rfloor\rfloor}$, associated with a new four-rank tensor field which will again belong to some restricted subrepresentation. It turns out that the two unspecified commutators in (2.41) are precisely given by these transformations. The tensor $Y^{\mathcal{P}\lceil\mathcal{RS}\rfloor}{}_{\mathcal{Q}\lceil\mathcal{P}\lceil\mathcal{RS}\rfloor\rfloor}$ acts as an intertwiner between the three- and four-rank tensor fields, and can easily be written down explicitly,

$$\begin{aligned} Y^{\mathcal{K}\lceil\mathcal{MN}\rfloor}{}_{\mathcal{P}\lceil\mathcal{Q}\lceil\mathcal{RS}\rfloor\rfloor} &= -\delta^{\lceil\mathcal{K}}_{\mathcal{P}}\, Y^{\mathcal{MN}\rfloor}{}_{\mathcal{Q}\lceil\mathcal{RS}\rfloor} - X_{\mathcal{P}\lceil\mathcal{Q}\lceil\mathcal{RS}\rfloor\rfloor}{}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor} \\ &= -2\left(\delta^{\lceil\mathcal{K}}_{\mathcal{Q}}\delta^{\lceil\mathcal{M}}_{\mathcal{S}}\, X_{\mathcal{PR}}{}^{\mathcal{N}\rfloor\rfloor} + \delta^{\lceil\mathcal{K}}_{\mathcal{P}}\delta^{\lceil\mathcal{M}}_{\mathcal{Q}}\, X_{\mathcal{SR}}{}^{\mathcal{N}\rfloor\rfloor}\right) \\ &\quad + 2\left(\delta^{\lceil\mathcal{K}}_{\mathcal{P}}\delta^{\lceil\mathcal{M}}_{\mathcal{S}}\, X_{\mathcal{QR}}{}^{\mathcal{N}\rfloor\rfloor} + \delta^{\lceil\mathcal{K}}_{\mathcal{R}}\delta^{\lceil\mathcal{M}}_{\mathcal{S}}\, X_{\mathcal{PQ}}{}^{\mathcal{N}\rfloor\rfloor}\right). \end{aligned} \tag{2.45}$$

To derive the second formula we made use of (2.35). Observe that on the r.h.s we must apply the projector (2.32) in order to obtain the restricted representations in the index triples $\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor$ and $\lceil\mathcal{Q}\lceil\mathcal{RS}\rfloor\rfloor$, respectively; the result is then automatically projected onto a restricted representation in the indices $\mathcal{P}\lceil\mathcal{Q}\lceil\mathcal{RS}\rfloor\rfloor$. In other words, our recursive procedure 'knows about' the new restricted representations occurring at the next step.

At this point one recognizes that there exists a whole hierarchy of such tensors.[7] They are defined by ($p \geq 3$)

$$\begin{aligned} Y^{\mathcal{M}_1\lceil\mathcal{M}_2\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor}{}_{\mathcal{N}_0\lceil\mathcal{N}_1\lceil\cdots\mathcal{N}_p\rfloor\cdot\cdot\rfloor} &= -\delta^{\lceil\mathcal{M}_1}_{\mathcal{N}_0}\, Y^{\mathcal{M}_2\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor}{}_{\mathcal{N}_1\lceil\mathcal{N}_2\lceil\cdots\mathcal{N}_p\rfloor\cdot\cdot\rfloor} \\ &\quad - X_{\mathcal{N}_0\lceil\mathcal{N}_1\lceil\mathcal{N}_2\lceil\cdots\mathcal{N}_p\rfloor\cdot\cdot\rfloor}{}^{\lceil\mathcal{M}_1\lceil\mathcal{M}_2\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor}\,, \end{aligned} \tag{2.46}$$

where, as before, we employ the notation,

$$X_{\mathcal{N}_0\lceil\mathcal{N}_1\lceil\mathcal{N}_2\lceil\cdots\mathcal{N}_p\rfloor\cdot\cdot\rfloor}{}^{\lceil\mathcal{M}_1\lceil\mathcal{M}_2\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor} = (X_{\mathcal{N}_0})_{\lceil\mathcal{N}_1\lceil\mathcal{N}_2\lceil\cdots\mathcal{N}_p\rfloor\cdot\cdot\rfloor}{}^{\lceil\mathcal{M}_1\lceil\mathcal{M}_2\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor}\,. \tag{2.47}$$

All these tensors are gauge invariant and they are formed from the embedding tensor multiplied by invariant tensors of the duality group G, so that they all transform in (a subset of) the same representations as the embedding tensor. By induction, one can prove their mutual orthogonality,

$$Y^{\mathcal{K}_2\lceil\mathcal{K}_3\lceil\cdots\mathcal{K}_p\rfloor\cdot\cdot\rfloor}{}_{\mathcal{M}_1\lceil\mathcal{M}_2\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor}\, Y^{\mathcal{M}_1\lceil\mathcal{M}_2\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor}{}_{\mathcal{N}_0\lceil\mathcal{N}_1\lceil\cdots\mathcal{N}_p\rfloor\cdot\cdot\rfloor} = 0\,. \tag{2.48}$$

[7] From this point we denote the intertwining tensors and $p$-forms by $Y$ and $C$, respectively, and the corresponding gauge transformation parameters by $\Phi$. Their rank will be obvious from the index structure.

To see this, one substitutes the expression (2.46) for the second $Y$-tensor and uses the gauge invariance of the first $Y$-tensor to obtain the expression,

$$
\begin{aligned}
(2.48) \quad = \quad & - Y^{\mathcal{K}_2\lceil\mathcal{K}_3\lceil\cdots\mathcal{K}_p\rfloor\cdot\cdot\rfloor}{}_{\mathcal{N}_0\lceil\mathcal{M}_2\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor}\; Y^{\mathcal{M}_2\lceil\mathcal{M}_3\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor}{}_{\mathcal{N}_1\lceil\mathcal{N}_2\lceil\cdots\mathcal{N}_p\rfloor\cdot\cdot\rfloor} \\
& - Y^{\mathcal{M}_2\lceil\mathcal{M}_3\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor}{}_{\mathcal{N}_1\lceil\mathcal{N}_2\lceil\cdots\mathcal{N}_p\rfloor\cdot\cdot\rfloor}\; X_{\mathcal{N}_0\lceil\mathcal{M}_2\lceil\mathcal{M}_3\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor}{}^{\lceil\mathcal{K}_2\lceil\mathcal{K}_3\lceil\cdots\mathcal{K}_p\rfloor\cdot\cdot\rfloor} \,. \qquad (2.49)
\end{aligned}
$$

This result vanishes upon expressing the generator $X$ on the right-hand side in terms of the $Y$-tensors, using the definition (2.46), and subsequently using the orthogonality constraint for a lower value of the rank $p$. The fact that symmetrization over the three last indices of the restricted representation will vanish as a result of (2.35), implies that higher-rank tensors will vanish as well under certain index symmetrizations.

The $Y$-tensors form an (infinite, in principle) hierarchy of intertwiners between successive sets of restricted representations of tensor gauge fields. The restrictions on the representations occurring at the ($p$+1)-th step of the iteration are determined inductively via formula (2.46), where on the r.h.s. the projectors obtained at the previous $p$-th step of the iteration must be applied to the $p$-tuples of indices $\mathcal{M}_1\lceil\mathcal{M}_2\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor$ and $\mathcal{N}_1\lceil\mathcal{N}_2\lceil\cdots\mathcal{N}_p\rfloor\cdot\cdot\rfloor$, respectively. We emphasize that *no other information is needed to determine the hierarchy.* However, as we pointed out already, working out more explicit expressions requires a case-by-case study, as we will exemplify for $D = 3$ supergravity and duality group $\mathrm{G} = \mathrm{E}_{8(8)}$ in section 4 of this paper. Consequently, given the $Y$-tensors, and specifying the duality group G, the above results enable a complete determination of the full hierarchy of the higher-rank $p$ forms required for the consistency of the gauging. In particular, we can exhibit some of the terms in the variations of the $p$-form fields that follow rather directly from the previous discussion,

$$
\begin{aligned}
\delta C_{\mu_1\ldots\mu_p}{}^{\mathcal{M}_1\lceil\mathcal{M}_2\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor} \quad = \quad & p\, D_{[\mu_1}\Phi_{\mu_2\cdots\mu_p]}{}^{\mathcal{M}_1\lceil\mathcal{M}_2\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor} \\
& + \Lambda^{\lceil\mathcal{M}_1}\mathcal{H}_{\mu_1\cdots\mu_p}{}^{\lceil\mathcal{M}_2\cdots\rfloor\cdot\cdot\rfloor\rfloor} + p\,\delta A_{[\mu_1}{}^{\lceil\mathcal{M}_1}\, C_{\mu_2\cdots\mu_p]}{}^{\lceil\mathcal{M}_2\cdots\rfloor\cdot\cdot\rfloor\rfloor} \\
& - g\, Y^{\mathcal{M}_1\lceil\mathcal{M}_2\lceil\cdots\mathcal{M}_p\rfloor\cdot\cdot\rfloor}{}_{\mathcal{N}_0\lceil\mathcal{N}_1\lceil\ldots\mathcal{N}_p\rfloor\cdot\rfloor}\; \Phi_{\mu_1\ldots\mu_p}{}^{\mathcal{N}_0\lceil\mathcal{N}_1\lceil\ldots\mathcal{N}_p\rfloor\cdot\cdot\rfloor} \,, \\
& + \cdots \,. \qquad (2.50)
\end{aligned}
$$

Although the number of space-time dimensions does not enter into this analysis (as we said, the iteration procedure can in principle be continued indefinitely) there is, for the maximal supergravities, a consistent correlation between the rank of the tensor fields and the occurrence of conjugate G-representations that is precisely in accord with tensor-tensor and vector-tensor (Hodge) duality [8] corresponding to the space-time dimension where the maximal supergravity with that particular duality group G lives. In the next section we discuss some of the results of this analysis and their implications for M-theory degrees of freedom.

[8] As well as with the count of physical degrees of freedom.

## 3 M-Theory degrees of freedom

The hierarchy of vector and tensor gauge fields that we presented in the previous section can be considered in the context of the maximal gauged supergravities. In that case the gauge group is embedded in the duality group G, which depends on the space-time dimension in which the supergravity is defined. Once we specify the group G the representations can be determined of the various $p$-form potentials. In principle the hierarchy allows a unique determination of the higher $p$-forms, but in practice this determination tends to be somewhat subtle. To see this, let us first briefly consider the possible representations for the two-forms. For that we need the representations in the symmetric product of two representations belonging to the vector fields (we will deal with the case $D = 3$ separately),

$$\begin{array}{rcrcl} D=7 & : & \overline{\mathbf{10}} \times_{\rm sym} \overline{\mathbf{10}} & = & \mathbf{5} + \overline{\mathbf{50}} \,, \\ D=6 & : & \mathbf{16}_c \times_{\rm sym} \mathbf{16}_c & = & \mathbf{10} + \mathbf{126}_c \,, \\ D=5 & : & \overline{\mathbf{27}} \times_{\rm sym} \overline{\mathbf{27}} & = & \mathbf{27} + \overline{\mathbf{351}}' \,, \\ D=4 & : & \mathbf{56} \times_{\rm sym} \mathbf{56} & = & \mathbf{133} + \mathbf{1463} \,. \end{array} \tag{3.1}$$

Hence it seems that the two-forms can belong to two possible representations of the duality group. To see which representation is allowed, we take its conjugate and consider once more the product with the vector field representation, This product should contain the representation associated with the tensor $Z^{\mathcal{M}}{}_{\mathcal{NP}}$. The latter is simply equal to the representation of the embedding tensor. If this representation is contained in the product, then we are dealing with an acceptable candidate representation. If this is not the case, then we must conclude that $Z^{\mathcal{M}}{}_{\mathcal{NP}}$ cannot act as an intertwiner between the corresponding two-forms and the one-form potentials.

Performing this test[9] on each of the two representations in (3.1), it turns out that only the first representation is allowed, leading to the entries for the two-forms presented in the third column of table 2. For the case of $D = 3$ space-time dimensions the above approach leads only to a partial determination of the representation assignment. Here the symmetric product decomposes into six different representations and in section 4 we will proceed diffently to deduce the correct assignment. The results for the two-forms in $4 \leq D \leq 7$ dimensions were originally derived in [30], where also the representations of the three-forms were determined that are shown in the table.

As we stressed already the hierarchy leads to a unique determination of the representations of the higher-rank tensor fields, but this has only partially been carried out. Already for lower-rank tensors, table 2 shows remarkable features. We recall that the analysis described in section 2 did not depend on the number of space-time dimensions. For instance, it is possible to derive representation assignments for $(D+1)$-rank tensors, although these do not live in a $D$-dimensional space-time. On the other hand, whenever there exists a (Hodge) duality relation between fields of different rank at the appropriate value for $D$, then one finds that their G representations turn out to be related by conjugation. This property is already

[9] We used the Lie package [39] for computing such decompositions.

| | | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 7 | SL(5) | $\overline{\mathbf{10}}$ | $\mathbf{5}$ | $\overline{\mathbf{5}}$ | $\mathbf{10}$ | $\mathbf{24}$ | $\overline{\mathbf{15}}+\mathbf{40}$ |
| 6 | SO(5,5) | $\mathbf{16}_c$ | $\mathbf{10}$ | $\mathbf{16}_s$ | $\mathbf{45}$ | $\mathbf{144}_s$ | $\mathbf{10}+\mathbf{126}_s+\mathbf{320}$ |
| 5 | $E_{6(6)}$ | $\overline{\mathbf{27}}$ | $\mathbf{27}$ | $\mathbf{78}$ | $\mathbf{351}$ | $\mathbf{27}+\mathbf{1728}$ | |
| 4 | $E_{7(7)}$ | $\mathbf{56}$ | $\mathbf{133}$ | $\mathbf{912}$ | $\mathbf{133}+\mathbf{8645}$ | | |
| 3 | $E_{8(8)}$ | $\mathbf{248}$ | $\mathbf{1}+\mathbf{3875}$ | $\mathbf{3875}+\mathbf{147250}$ | | | |

Table 2: Duality representations of the vector and tensor gauge fields for gauged maximal supergravities in space-time dimensions $3 \leq D \leq 7$. The first two columns list the space-time dimension and the corresponding duality group. Note that the singlet two-form in three dimensions is not induced by the hierarchy. Its presence follows from independent considerations, which are discussed in the text.

exhibited at the level of the lower-rank tensors and we have simply extrapolated this pattern to higher-rank fields. Furthermore the diagonals pertaining to the $(D-2)$-, $(D-1)$- and $D$-rank tensor fields refer to the adjoint representation and the representations conjugate to those assigned to the embedding tensor and its quadratic constraint, respectively. While not all of these features show up fully for the lower-rank tensors, the pattern is quite suggestive. The underlying reasons for some of this will become apparent in the later sections, where we establish that the $(D-1)$- and $D$-rank tensors play the special role of imposing the constancy of the embedding tensor and the closure of the corresponding gauge group.

It is an obvious question whether these systematic features have a natural explanation in terms of M-theory. Supergravity may contain some of the fields carrying charges that could induce a gauging. For instance, in the toroidal compactification there are towers of massive Kaluza-Klein states whose charges couple to the corresponding Kaluza-Klein gauge fields emerging from the higher-dimensional metric. This is of direct relevance in the so-called Scherk-Schwarz reductions [14]. However, these Kaluza-Klein states cannot generally be assigned to representations of the duality group and therefore there must be extra degrees of freedom whose origin cannot be understood within the context of a dimensional compactification of supergravity.[10] This phenomenon was discussed some time ago, for instance, in [35, 40].

The general gaugings that have been constructed in recent years obviously extend beyond gaugings whose charges are carried by supergravity degrees of freedom. The embedding tensor can be regarded as a duality covariant tensor that, once it is fixed to some constant value, selects a certain subsector of the available charge configurations carried by degrees of freedom that will cover complete representations of the duality group. If this idea is correct these degrees of freedom must exist in M-theory, and there are indeed indications that this is the case. In this way the gauging acts as a probe of M-theory degrees of freedom.

Independent evidence that this relation with M-theory degrees of freedoms is indeed re-

[10] In view of the fact that the Kaluza-Klein states are 1/2-BPS, also these extra degrees of freedom must correspond to 1/2-BPS states.

alized is provided by the work of [34] (see also, [35] and references quoted therein) where matrix theory [41, 42] is considered in a toroidal compactification. These results are based on the correspondence between $N = 4$ super-Yang-Mills theory on a (rectangular) spatial torus $\tilde{T}^n$ with radii $s_1, \ldots, s_n$, and M-theory in the infinite-momentum frame on the dual torus $T^n$ with radii $R_1, R_2, \ldots, R_n$, where $s_i = l_{\rm p}^3/R_{11}R_i$ and $l_{\rm p}$ denotes the Planck length in eleven dimensions. The conjecture then is that the latter should be invariant under permutations of the radii $R_i$ and under T-duality of type-IIA string theory. The relevant T-duality transformations follow from making two consecutive T-dualities on two different circles. When combined with the permutation symmetry, T-duality can be represented by ($i \neq j \neq k \neq i$)

$$R_i \to \frac{l_{\rm p}^3}{R_j R_k}\,, \quad R_j \to \frac{l_{\rm p}^3}{R_k R_i}\,, \quad R_k \to \frac{l_{\rm p}^3}{R_i R_j}\,, \quad l_{\rm p}^3 \to \frac{l_{\rm p}^6}{R_i R_j R_k}\,, \tag{3.2}$$

The above transformations generate a discrete group which coincides with the Weyl group of $\mathrm{E}_n$; on the Yang-Mills side, the elementary Weyl reflections correspond to permutations of the compactified coordinates (generating the Weyl group of $\mathrm{SL}(n)$) and Montonen-Olive duality $g_{\rm eff} \to 1/g_{\rm eff}$ (corresponding to reflections with respect to the exceptional node of the $\mathrm{E}_n$ Dynkin diagram). This Weyl group, which leaves the rectangular shape of the compactification torus invariant, can be realized as a discrete subgroup of the compact subgroup of $\mathrm{E}_{n(n)}$, and consequently as a subgroup of the conjectured non-perturbative duality group $\mathrm{E}_{n(n)}(\mathbb{Z})$ [43]. Representations of this symmetry can now be generated by mapping out the Weyl orbits starting from certain states. For instance, one may start with Kaluza-Klein states on $T^n$, whose masses are proportional to $M \sim 1/R_i$. The action of the Weyl group then generates new states, such as the ones that can be identified with two-branes wrapped around the torus, whose masses are of order $M \sim R_j R_k/l_{\rm p}^3$, and so on. According to [43], the non-perturbative states should combine into multiplets of $\mathrm{E}_{n(n)}(\mathbb{Z})$; if the representation has weights of different lengths, one needs several different Weyl orbits to recover all states in the representation.

Following this procedure one obtains complete multiplets of the duality group (taking into account that some states belonging to the representation will vanish under the Weyl group and will therefore remain inaccessible by this construction). More specifically, using the relation $n = 11 - D$, it turns out that the first two columns of table 2, respectively, correspond to the so-called flux and momentum multiplets of [34]. However, as already pointed out above, the conjecture of [43] is essential in that one may need extra states from different Weyl orbits in order to get the full representation; for instance, there are only 2160 momentum states for $\mathrm{E}_{8(8)}$, which must be supplemented by 8-brane states to obtain the full **3875** representation of $\mathrm{E}_{8(8)}$.

The representations in the table were also found in [36], where a 'mysterious duality' was exhibited between toroidal compactifications of M-theory and del Pezzo surfaces. Here the M-theory dualities are related to global diffeomorphisms that perserve the canonical class of the del Pezzo surface. Again the representations thus found are in good agreement with the representations in table 2.

For $n \geq 9$, the flux and momentum multiplets of [34] have infinitely many components. Indeed, there are hints that the above considerations concerning new M-theoretic degrees of freedom can be extended to infinite-dimensional duality groups: in particular, a recent analysis of the indefinite Kac–Moody algebra $\mathrm{E}_{11}$ has shown that the decomposition of its adjoint representation at low levels under its finite-dimensional subalgebras $\mathrm{SL}(D) \times \mathrm{E}_{11-D}$ for $D \geq 3$ yields the same representations as in table 2 [31, 32, 33]. However, it is far from clear what these (infinitely many) new degrees of freedom would correspond to, and how they would be concretely realized. Concerning the physical interpretation of the new states, a first step was taken in [44], where an infinite multiplet of BPS states is generated from the M2 brane and M5 brane solutions of $D = 11$ supergravity by the iterated action of certain $A_1^{(1)}$ subgroups of the $\mathrm{E}_9$ Weyl group. In the context of gauged supergravities, the significance of these states may become clearer with the exploration of maximal gauged supergravities in *two* space-time dimensions [29], where the embedding tensor transforms in the so-called basic representation of $\mathrm{E}_9$ (which is infinite dimensional).

# 4 Tensor field representations in three space-time dimensions

Here and in the following two sections we will illustrate the preceding discussion and consider maximal supergravity in three space-time dimensions, where the full tensor hierarchy of $p$-forms is short enough to obtain all relevant information from the explicit results given in section 2. This example will show all the characteristic features that are generic for gauged supergravities. In this section we will determine the representation assignments for the tensor fields. The relevant duality group is equal to $\mathrm{E}_{8(8)}$, which is of dimension 248. Its fundamental representation coincides with the adjoint representation, so that the generators in this representation are given by the $\mathrm{E}_{8(8)}$ structure constants, $(t_{\mathcal{M}})_{\mathcal{N}}{}^{\mathcal{P}} = -f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$. Indices may be raised and lowered by means of the Cartan-Killing form $\eta_{\mathcal{M}\mathcal{N}}$. The vector fields $A_\mu{}^{\mathcal{M}}$ transform in the **248** representation and the embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ is a *symmetric* matrix belonging to the $\mathbf{3875} + \mathbf{1}$ representation [7, 8]. Using these data, we may evaluate the general formulas of section 2 for this particular theory.

The gauge group generators are obtained by contracting $\mathrm{E}_{8(8)}$ generators with the embedding tensor $X_{\mathcal{M}} \equiv \Theta_{\mathcal{M}\mathcal{N}}\, t^{\mathcal{N}}$. In the adjoint representation we thus have

$$X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} = -\Theta_{\mathcal{M}}{}^{\mathcal{Q}}\, f_{\mathcal{Q}\mathcal{N}}{}^{\mathcal{P}} = \Theta_{\mathcal{M}\mathcal{Q}}\, f^{\mathcal{Q}\mathcal{P}}{}_{\mathcal{N}} \,. \tag{4.1}$$

The tensor $Z^{\mathcal{P}}{}_{\mathcal{M}\mathcal{N}}$ defined in (2.3) is then given by

$$Z^{\mathcal{P}}{}_{\mathcal{M}\mathcal{N}} \;=\; \Theta_{\mathcal{Q}(\mathcal{M}}\, f^{\mathcal{Q}\mathcal{P}}{}_{\mathcal{N})} \,. \tag{4.2}$$

Because this tensor is a group invariant contraction of the embedding tensor, its representation must overlap with some of the representations of the embedding tensor. Obviously, the singlet component drops out so that we may conclude that (4.2) must belong to the **3875** representation.

As discussed before (*cf.* (2.14)), the tensor $Z^{\mathcal{K}}{}_{\mathcal{MN}}$ generically does not map onto the full symmetric tensor product $(\mathcal{MN})$, which decomposes according to

$$\mathbf{248} \times_{\rm sym} \mathbf{248} = \mathbf{1} + \mathbf{3875} + \mathbf{27000} \,, \tag{4.3}$$

but only on a restricted representation. Since (4.2) represents an infinitesimal $\mathrm{E}_{8(8)}$ transformation on the embedding tensor $\Theta_{\mathcal{MN}}$ which leaves the representation content invariant, it follows the indices $(\mathcal{MN})$ in (4.2) are restricted to the **3875** representation, so that the relevant projector is precisely $\mathbb{P}^{(3875)}$ acting on the symmetric tensor product. This projector can be written as [46]

$$(\mathbb{P}^{(3875)})^{\mathcal{RS}}{}_{\mathcal{MN}} \;=\; \tfrac{1}{7}\,\delta^{(\mathcal{R}}_{\mathcal{M}}\delta^{\mathcal{S})}_{\mathcal{N}} - \tfrac{1}{56}\,\eta_{\mathcal{MN}}\,\eta^{\mathcal{RS}} - \tfrac{1}{14}\,f^{\mathcal{P}}{}_{\mathcal{M}}{}^{(\mathcal{R}}\,f_{\mathcal{PN}}{}^{\mathcal{S})} \,. \tag{4.4}$$

According to the general discussion, it follows that closure of the vector field gauge algebra requires the introduction of two-forms in the **3875** representation. Hence the two-forms transform in the same representation as the embedding tensor. As noted in the previous section, this is a general pattern in gauged supergravities: the embedding tensor in $D$ dimensions transforms in the representation which is conjugate to the $(D{-}1)$-forms. More precisely, the field strength of the $(D-1)$-forms is dual to the embedding tensor. We will discuss the explicit relation in the next sections. In three dimensions there is a subtlety related to the fact that the embedding tensor is not irreducible but contains an additional singlet **1** besides the **3875**. The associated two-form can be defined but does not yet show up in the tensor hierarchy at this point. In order to keep the discussion as simple as possible, we will in the following restrict to the gaugings induced by an embedding tensor in the irreducible **3875**.

Continuing the tensor hierarchy according to the general pattern discussed above, the next intertwining tensor $Y^{\mathcal{MN}}{}_{\mathcal{K}\lceil\mathcal{PQ}\rfloor}$, defined in (2.28), takes the form

$$Y^{\mathcal{MN}}{}_{\mathcal{K}\lceil\mathcal{PQ}\rfloor} \;\equiv\; 2\left(\delta_{\lceil\mathcal{P}}{}^{\lceil\mathcal{R}}\,f^{\mathcal{S}\rfloor\lceil\mathcal{M}}{}_{\mathcal{Q}\rfloor}\,\delta_{\mathcal{K}}{}^{\mathcal{N}\rfloor} - \delta_{\mathcal{K}}{}^{\lceil\mathcal{R}}\,f^{\mathcal{S}\rfloor\lceil\mathcal{M}}{}_{\lceil\mathcal{P}}\,\delta_{\mathcal{Q}\rfloor}{}^{\mathcal{N}\rfloor}\right)\Theta_{\mathcal{RS}} \,. \tag{4.5}$$

In view of the group-invariant contractions, the tensor $Y^{\mathcal{MN}}{}_{\mathcal{K}\lceil\mathcal{PQ}\rfloor}$ transforms again in the **3875** representation. It controls the appearance of three-forms in the gauge transformations of two-forms and thereby determines the (minimal) field content of three-forms required for consistency of the algebra. Again it does not map onto the full tensor product $\mathcal{K}\lceil\mathcal{PQ}\rfloor$ but only onto a restricted subrepresentation, as in (2.14). To determine this subrepresentation, we observe that the expression in parentheses in (4.5) is symmetric under exchange $\lceil\mathcal{RS}\rfloor \leftrightarrow \lceil\mathcal{MN}\rfloor$, and thus transforms in

$$\mathbf{3875} \times_{\rm sym} \mathbf{3875} \;=\; \mathbf{1} + \mathbf{3875} + \mathbf{27000} + \mathbf{147250} + \mathbf{2450240} + \mathbf{4881384} \,. \tag{4.6}$$

On the other hand, by its index structure, the tensor product $\mathcal{K}\lceil\mathcal{PQ}\rfloor$ is given by

$$\mathbf{248} \times \mathbf{3875} \;=\; \mathbf{248} + \mathbf{3875} + \mathbf{30380} + \mathbf{147250} + \mathbf{779247} \,, \tag{4.7}$$

Comparing (4.6) and (4.7), it follows that the index combination $\mathcal{K}\lceil\mathcal{P}\mathcal{Q}\rfloor$ is indeed restricted to certain irreducible representations so that the three-forms transform in the representation[11]

$$C_{\mu\nu\rho}{}^{\mathcal{K}\lceil\mathcal{P}\mathcal{Q}\rfloor} \quad \sim \quad \mathbf{3875} + \mathbf{147250} \,. \tag{4.8}$$

In principle, the argument so far does not exclude the possibility that the image of $Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}\lceil\mathcal{P}\mathcal{Q}\rfloor}$ is restricted to only one of the two irreducible representations in (4.8). To show that both irreducible parts are present, one may *e.g.* compute and diagonalize the action of the $\mathrm{E}_{8(8)}$ Casimir operator on $Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}\lceil\mathcal{P}\mathcal{Q}\rfloor}$.

At this point, it is instructive to have a closer look at the quadratic constraint. In three dimensions, this constraint implies that the tensor

$$\mathcal{Q}_{\mathcal{M}\lceil\mathcal{P}\mathcal{Q}\rfloor} \;\equiv\; \Theta_{\mathcal{M}\mathcal{N}}\, Z^{\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}} \;=\; -\, X_{\mathcal{M}\lceil\mathcal{P}}{}^{\mathcal{N}}\, \Theta_{\mathcal{Q}\rfloor\mathcal{N}} \,, \tag{4.9}$$

must vanish. Let us determine, in which representation $\mathcal{Q}_{\mathcal{M}\lceil\mathcal{P}\mathcal{Q}\rfloor}$ transforms. As we have seen above, the tensor $Z^{\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}}$ in its indices $\mathcal{P}\mathcal{Q}$ projects onto the **3875** representation. As a consequence, $\mathcal{Q}_{\mathcal{M}\lceil\mathcal{P}\mathcal{Q}\rfloor}$ transforms in the tensor product $\mathbf{248} \times \mathbf{3875}$ given in (4.7). On the other hand, as $\mathcal{Q}_{\mathcal{M}\lceil\mathcal{P}\mathcal{Q}\rfloor}$ is quadratic in $\Theta$ it transforms in the symmetric tensor product $\mathbf{3875} \times_{\mathrm{sym}} \mathbf{3875}$ given in (4.6). Comparing (4.6) and (4.7), it follows that also $\mathcal{Q}_{\mathcal{M}\lceil\mathcal{P}\mathcal{Q}\rfloor}$ transforms in the representation,

$$C_{\mathrm{quad}} \quad = \quad \mathbf{3875} + \mathbf{147250} \,, \tag{4.10}$$

and thus in the very same representation as the three-forms (4.8). This is in accord with the general pattern in gauged supergravities noted in the previous section: the quadratic constraint transform in a (reducible) representation whose conjugate is equal to (or at least contained in) the representation of the $D$-forms. We will propose a natural interpretation for this in the last section, where the $D$-forms act as Lagrange multipliers for the quadratic constraint.

Let us finally continue the tensor hierarchy one last step further, *i.e.*, to the four-forms. Although four-forms cannot live in three dimensions, their tensor gauge freedom shows up in the three-dimensional tensor gauge algebra by the shift transformation of the three-forms (2.44). For a complete picture we thus need to work out also their structure. Again, $Y^{\mathcal{K}\lceil\mathcal{M}\mathcal{N}\rfloor}{}_{\mathcal{P}\lceil\mathcal{Q}\lceil\mathcal{R}\mathcal{S}\rfloor\rfloor}$ does not map onto the full tensor product $\mathcal{P}\lceil\mathcal{Q}\lceil\mathcal{R}\mathcal{S}\rfloor\rfloor$ but only onto a restricted subrepresentation of $\mathbf{248} \times (\mathbf{3875} + \mathbf{147250})$, which we do not explicitly work out here. It is interesting to note, that apart from the standard othogonality relations (2.48) which follow as a consequence of the quadratic constraint (2.5), the tensor $Y^{\mathcal{K}\lceil\mathcal{M}\mathcal{N}\rfloor}{}_{\mathcal{P}\lceil\mathcal{Q}\lceil\mathcal{R}\mathcal{S}\rfloor\rfloor}$ also identically satisfies the relation

$$\mathcal{Q}_{\mathcal{K}\lceil\mathcal{M}\mathcal{N}\rfloor}\, Y^{\mathcal{K}\lceil\mathcal{M}\mathcal{N}\rfloor}{}_{\mathcal{P}\lceil\mathcal{Q}\lceil\mathcal{R}\mathcal{S}\rfloor\rfloor} \quad = \quad 0 \,, \tag{4.11}$$

with $\mathcal{Q}_{\mathcal{K}\lceil\mathcal{M}\mathcal{N}\rfloor}$ from (4.9). This identity will also play an important role in the last section. Its proof is not entirely straightforward, as (4.11) involves expressions cubic in $\Theta$ and quadratic in the $\mathrm{E}_8$ structure constants, and is therefore most easily checked on a computer.

[11] The absence of the **248**, **30380** and **779247** representations is in accord with equation (2.35) because those are contained in the fully symmetrized product of three **248** representations.

To summarize, we have explicitly worked out the tensor hierarchy of gauged three-dimensional supergravity and shown that consistency requires two- and three-forms to transform in the **3875** and **3875** + **147250** representation, respectively. The representation content of the (evanescent) four-forms is implicitly defined by (2.45) as a subrepresentation of **248** × (**3875** + **147250**) and shows up through the shift transformations (2.44) of the three-forms. In principle,the precise representation content of of the index combinations in (2.45) can be worked out further, but these details are not necessary in what follows.

# 5 The supersymmetry algebra in three space-time dimensions

In this section we present the complete determination of the supersymmetry transformations and the corresponding algebra for the $p$-forms in three dimensions. Already in a number of cases supersymmetry variations of $p$-forms that do not appear in the ungauged action, have been determined. This was done by making an ansatz for these variations based on their tensorial structure, which involves some undetermined coefficients. These constants are subsequently fixed by imposing the supersymmetry algebra, after which one proceeds by iteration. Here we go one step further and consider also the supersymmetry variations of those $p$-forms that are not required for writing down the most general gaugings, in order to determine what their possible role could be. In three space-time dimensions this implies that we will now also consider the two-, three-, and four-form potentials. Although four-form potentials do not exist in four dimensions, their symmetries will still play a role as they act on the three-form potentials. We note that a similar investigation of maximal supergravity in five dimensions has recently appeared in [45].

We use spinor and $\mathrm{E}_{8(8)}$ conventions from [7, 8].[12] In particular, the $\mathrm{E}_{8(8)}$ generators $t^{\mathcal{M}}$ split into 120 compact ones $X^{IJ} = X^{[IJ]}$, associated with the group SO(16), and 128 non-compact ones denoted by $Y^A$. Here $I, J, \ldots$ and $A, B, \ldots$, respectively, label the $\mathbf{16}_v$ and $\mathbf{128}_s$ representations of SO(16). Eventually we will also need indices $\dot{A}, \dot{B}, \ldots$ labelling the conjugate spinor representation $\mathbf{128}_c$. Naturally we will also encounter SO(16) gamma matrices $\Gamma^I{}_{A\dot{A}}$ in what follows. We will freely raise and lower SO(16) indices.

The scalar fields parametrize the $\mathrm{E}_{8(8)}/\mathrm{SO}(16)$ coset space in terms of an $\mathrm{E}_{8(8)}$-valued matrix $\mathcal{V}^{\mathcal{M}}{}_{\underline{\mathcal{P}}}$, which transforms as

$$\delta\mathcal{V}(x)^{\mathcal{M}}{}_{\underline{\mathcal{P}}} \;=\; -g_{\mathcal{N}}{}^{\mathcal{M}}\,\mathcal{V}(x)^{\mathcal{N}}{}_{\underline{\mathcal{P}}} + \mathcal{V}(x)^{\mathcal{M}}{}_{\underline{\mathcal{Q}}}\,h(x)_{\underline{\mathcal{P}}}{}^{\underline{\mathcal{Q}}}\,, \tag{5.1}$$

under global $\mathrm{E}_{8(8)}$ and local SO(16), characterized by the matrices $g$ and $h(x)$ which take their values in the Lie algebra of the two groups. Note that underlined $\mathrm{E}_{8(8)}$ indices and indices $[IJ]$, $A$ and $\dot{A}$ are always subject to *local* SO(16). The one-forms associated with the scalars are given by

$$\mathcal{V}^{-1}\,D_\mu\mathcal{V} \;=\; \tfrac{1}{2}\,\mathcal{Q}_\mu{}^{IJ}\,X^{IJ} + \mathcal{P}_\mu{}^{A}\,Y^{A}\,, \tag{5.2}$$

[12] To be precise: the only change in notation with respect to [7, 8] is the sign of the vector fields, *i.e.*, $A_\mu{}^{\mathcal{M}} \to -A_\mu{}^{\mathcal{M}}$. The tangent space metric and gamma matrix conventions are as follows: $\eta_{ab} = \mathrm{diag}(+,-,-)$, $\{\gamma^a, \gamma^b\} = 2\eta^{ab}\,\mathbf{1}$, and $\gamma^{abc} = -\mathrm{i}\varepsilon^{abc}\,\mathbf{1}$.

where the derivative $D_\mu$ on the left-hand side is covariant with respect to the chosen gauge group (*cf.* (2.8)). As is well-known, ${\mathcal{Q}_\mu}^{IJ}$ will play the role of a composite SO(16) gauge connection. Both $\mathcal{P}_\mu$ and $\mathcal{Q}_\mu$ will implicitly depend on the gaugings introduced in section 2, through the defining relation (5.2).

For simplicity of the formulas we use the abbreviating notation,

$$\mathcal{V}^{\mathcal{MN}}{}_{\underline{\mathcal{P}}|\underline{\mathcal{R}}} \;\equiv\; \mathcal{V}^{\mathcal{M}}{}_{\underline{\mathcal{P}}}\,\mathcal{V}^{\mathcal{N}}{}_{\underline{\mathcal{R}}}\,, \quad \mathcal{V}^{\mathcal{MNK}}{}_{\underline{\mathcal{P}}|\underline{\mathcal{R}}|\underline{\mathcal{S}}} \;\equiv\; \mathcal{V}^{\mathcal{M}}{}_{\underline{\mathcal{P}}}\,\mathcal{V}^{\mathcal{N}}{}_{\underline{\mathcal{R}}}\,\mathcal{V}^{\mathcal{K}}{}_{\underline{\mathcal{S}}}\,, \quad \text{etc.} \tag{5.3}$$

for multiple tensor products of these matrices. The fermionic field content is given by 16 gravitinos ${\psi_\mu}^I$ and 128 spin-1/2 fermions $\chi^{\dot{A}}$ transforming under SO(16). In the presence of a gauging their supersymmetry variations are given by

$$\delta\,{\psi_\mu}^I \;=\; D_\mu\epsilon^I + \mathrm{i}g\,{A_1}^{IJ}\,\gamma_\mu\epsilon^J\,, \qquad \delta\,\chi^{\dot{A}} \;=\; \tfrac{1}{2}\mathrm{i}\,\gamma^\mu\epsilon^I\,\Gamma^I_{A\dot{A}}\,\mathcal{P}^A_\mu + g\,{A_2}^{I\dot{A}}\,\epsilon^I\,, \tag{5.4}$$

with the tensors $A_1$, $A_2$ given by

$$A_1^{IJ} \;=\; \tfrac{1}{7}\,\mathcal{V}^{\mathcal{MN}}{}_{IK|JK}\,\Theta_{\mathcal{MN}}\,, \qquad A_2^{I\dot{A}} \;=\; -\,\tfrac{1}{7}\,\Gamma^J_{A\dot{A}}\,\mathcal{V}^{\mathcal{MN}}{}_{IJ|A}\,\Theta_{\mathcal{MN}}\,. \tag{5.5}$$

The bosonic fields on the other hand transform as

$$\begin{aligned} \delta{e_\mu}^\alpha \;&=\; \mathrm{i}\bar{\epsilon}^I\gamma^\alpha\psi_\mu{}^I\,, \qquad \mathcal{V}^{-1}\delta\mathcal{V} \;=\; \Gamma^I_{A\dot{A}}\,\bar{\chi}^{\dot{A}}\epsilon^I\,Y^A\,, \\ \delta{A_\mu}^{\mathcal{M}} \;&=\; 2\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\,\bar{\epsilon}^I\psi_\mu{}^J - \mathrm{i}\Gamma^I{}_{A\dot{A}}\,\mathcal{V}^{\mathcal{M}}{}_A\,\bar{\epsilon}^I\gamma_\mu\chi^{\dot{A}}\,. \end{aligned} \tag{5.6}$$

The supersymmetry transformations are expected to close into the various local symmetries, up to field equations. The supersymmetry comutator takes the form,

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \;=\; \xi^\mu\hat{D}_\mu + \delta_\Lambda + \delta_\Xi + \delta_\Phi + \cdots\,, \tag{5.7}$$

where the unspecified terms denote local Lorentz transformations, local supersymmetry transformations and other symmetries which will be discussed below. By $\xi^\mu\hat{D}_\mu$ we denote a covariant translation: a general coordinate transformation with parameter $\xi^\mu$ accompanied by other field-dependent gauge transformations such that the combined result is fully covariant. In the context of this work we are mostly interested in the field-dependent vector and tensor gauge transformations,

$$\xi^\mu\hat{D}_\mu \;\equiv\; \xi^\mu\partial_\mu + \delta_{\Lambda(\xi)} + \delta_{\Xi(\xi)} + \delta_{\Phi(\xi)} + \cdots\,, \tag{5.8}$$

where the vector and tensor gauge parameters are equal to

$$\begin{aligned} \Lambda(\xi)^{\mathcal{M}} \;&\equiv\; -\xi^\rho\,{A_\rho}^{\mathcal{M}}\,, \\ \Xi(\xi)_\mu{}^{\mathcal{MN}} \;&\equiv\; -\xi^\rho\left(B_{\rho\mu}{}^{\mathcal{M}} + A_\rho{}^{[\mathcal{M}}A_\mu{}^{\mathcal{N}]}\right), \\ \Phi(\xi)_{\mu\nu}{}^{\mathcal{K}[\mathcal{MN}]} \;&\equiv\; -\xi^\rho\left(C_{\rho\mu\nu}{}^{\mathcal{K}[\mathcal{MN}]} - A_\rho{}^{[\mathcal{K}}B_{\mu\nu}{}^{[\mathcal{MN}]]} - \tfrac{2}{3}\,A_{[\mu}{}^{[\mathcal{K}}A_{\nu]}{}^{[\mathcal{M}}A_\rho{}^{\mathcal{N}]]}\right), \end{aligned} \tag{5.9}$$

so that

$$\begin{aligned} \xi^\rho\hat{D}_\rho A_\mu{}^{\mathcal{M}} \;&=\; \xi^\rho\,\mathcal{H}_{\rho\mu}{}^{\mathcal{M}}\,, \\ \xi^\rho\hat{D}_\rho B_{\mu\nu}{}^{\mathcal{MN}} - A_\mu{}^{[\mathcal{M}}\xi^\rho\hat{D}_\rho A_\nu{}^{\mathcal{N}]} + A_\nu{}^{[\mathcal{M}}\xi^\rho\hat{D}_\rho A_\mu{}^{\mathcal{N}]} \;&=\; \xi^\rho\,\mathcal{H}_{\rho\mu\nu}{}^{\mathcal{MN}}\,, \end{aligned} \tag{5.10}$$

take a fully covariant form in terms of the covariant variations and field strengths of section 2. Note that we have suppressed the supercovariantizations in this result, as we restrict attention to the terms of lowest order in the fermion fields. Calculating closure of the supersymmetry algebra on the $p$-form tensor fields will determine the parameters $\xi$, $\Lambda$, $\Xi$, $\Phi$ in (5.7).

Let us start from the supersymmetry commutator on vector fields. A short computation starting from (5.4) and (5.6) yields

$$\begin{aligned}[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\, A_\mu{}^{\mathcal{M}} &= -2D_\mu(\mathcal{V}^{\mathcal{M}}{}_{IJ}\, \bar\epsilon_{[1}{}^I \epsilon_{2]}{}^J) + \mathrm{i}\, \epsilon_{\mu\nu\rho}\, \mathcal{V}^{\mathcal{M}}{}_A\, \mathcal{P}^{\nu\, A}\, \bar\epsilon_{[1}{}^I \gamma^\rho \epsilon_{2]}{}^I \\ &\quad + 2\mathrm{i}g \left(\Gamma^I_{A\dot A}\, A_2{}^{J\dot A}\, \mathcal{V}^{\mathcal{M}}{}_A - 2A_1{}^{JK}\, \mathcal{V}^{\mathcal{M}}{}_{IK}\right) \bar\epsilon_{[1}{}^I \gamma_\mu \epsilon_{2]}{}^J \,. \end{aligned} \tag{5.11}$$

The first term is a gauge transformation, while the last term proves to be the dressed version of the constant tensor $Z^{\mathcal{M}}{}_{\mathcal{PQ}}$ defined in (2.3). Indeed, we note the identity,

$$\Gamma^{(I}_{A\dot A}\, A_2{}^{J)\dot A}\, \mathcal{V}^{\mathcal{M}}{}_A - A_1{}^{JK}\, \mathcal{V}^{\mathcal{M}}{}_{IK} - A_1{}^{IK}\, \mathcal{V}^{\mathcal{M}}{}_{JK} = \tfrac{2}{7}\, \mathcal{V}^{\mathcal{PQ}}{}_{IK|JK}\, Z^{\mathcal{M}}{}_{\mathcal{PQ}} \,. \tag{5.12}$$

Upon contraction with $\Theta_{\mathcal{MN}}$, the right-hand side of this equation vanishes, and we re-obtain the identity (3.18) of [8]. The second term in (5.11) shows up in the duality equation relating vector and scalar fields in three dimensions,

$$\mathcal{X}_{\mu\nu}{}^{\mathcal{M}} \equiv \mathcal{H}_{\mu\nu}{}^{\mathcal{M}} + e\, \epsilon_{\mu\nu\rho}\, \mathcal{V}^{\mathcal{M}}{}_A\, \mathcal{P}^{\rho\, A} \,, \tag{5.13}$$

which, at least in the ungauged theory, vanishes on-shell. Hence, we find that

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\, A_\mu{}^{\mathcal{M}} = \xi^\rho\, \mathcal{H}_{\rho\mu}{}^{\mathcal{M}} + D_\mu\, \Lambda^{\mathcal{M}} - g\, Z^{\mathcal{M}}{}_{\mathcal{PQ}}\, \Xi_\mu{}^{\mathcal{PQ}} - \xi^\rho\, \mathcal{X}_{\rho\mu}{}^{\mathcal{M}} \,, \tag{5.14}$$

with parameters

$$\begin{aligned} \xi^\mu &= -\mathrm{i}\, \bar\epsilon_{[1}{}^I \gamma^\mu \epsilon_{2]}{}^I \,, \\ \Lambda^{\mathcal{M}} &= -2\, \mathcal{V}^{\mathcal{M}}{}_{IJ}\, \bar\epsilon_{[1}{}^I \epsilon_{2]}{}^J \,, \\ \Xi_\mu{}^{\mathcal{MN}} &= -\tfrac{4}{7}\mathrm{i}\, \mathcal{V}^{[\mathcal{MN}]}{}_{IK|JK}\, \bar\epsilon_{[1}{}^I \gamma_\mu \epsilon_{2]}{}^J \,. \end{aligned} \tag{5.15}$$

Except for the last term in (5.14) the supersymmetry algebra closes precisely as expected. Usually, this last term is disregarded as the supersymmetry algebra is expected to close modulo the first-order (duality) equations of motion (that is, $\mathcal{X}_{\mu\nu}{}^{\mathcal{M}} = 0$). Nowever, matters are more subtle here, as only a projection of the duality equation with the embedding tensor is expected to correspond to an equation of motion. For the moment, let us just keep this term: we will interpret it later as an additional local symmetry of the Lagrangian.

Let us continue with the two-forms. The supersymmetry variation of $B_{\mu\nu}{}^{\mathcal{MN}}$ is determined by its tensor structure up to two constants, $\alpha_1$ and $\alpha_2$,

$$\Delta B_{\mu\nu}{}^{\mathcal{MN}} = \mathrm{i}\alpha_1\, \mathcal{V}^{[\mathcal{MN}]}{}_{IK|JK}\, \bar\epsilon^I \gamma_{[\mu}\psi_{\nu]}{}^J - \alpha_2\, \mathcal{V}^{[\mathcal{MN}]}{}_{A|IJ}\, \Gamma^I_{A\dot A}\, \bar\epsilon^J \gamma_{\mu\nu} \chi^{\dot A} \,. \tag{5.16}$$

Requiring that the commutator closes into a gauge transformation with parameter $\Xi_\mu{}^{\mathcal{MN}}$ as given in (5.15), leads to $\alpha_1 = -8/7$, $\alpha_2 = -4/7$. From (5.16), we obtain after some further

computation,

$$
\begin{aligned}
[\delta_{\epsilon_1},\delta_{\epsilon_2}]\, B_{\mu\nu}{}^{\mathcal{MN}} &= 2D_{[\mu}\Xi_{\nu]}{}^{\mathcal{MN}} + \tfrac{4}{7}\,\epsilon_{\mu\nu\rho}\,\mathcal{P}^{\rho\,B}\,\mathcal{V}^{\lceil\mathcal{MN}\rfloor}{}_{IJ|A}\,(\Gamma^I\Gamma^K)_{AB}\,\bar\epsilon_{[1}{}^J\epsilon_{2]}{}^K \\
&\quad -\tfrac{8}{7}\,g\left(\mathcal{V}^{\lceil\mathcal{MN}\rfloor}{}_{IK|LK}\,A_1{}^{JL} + \tfrac{1}{2}\mathcal{V}^{\lceil\mathcal{MN}\rfloor}{}_{IK|A}\,\Gamma^K{}_{A\dot A}\,A_2{}^{J\dot A}\right)\bar\epsilon_{[1}{}^I\gamma_{\mu\nu}\epsilon_{2]}{}^J \\
&\quad +2\,A_{[\mu}{}^{\lceil\mathcal{M}}\,[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,A_{\nu]}{}^{\mathcal{N}\rfloor}\,. \qquad (5.17)
\end{aligned}
$$

The first term denotes the tensor gauge transformation. To understand the second term we need to make explicit use of the projection of $\lceil\mathcal{MN}\rfloor$ onto the **3875**, which induces relations such as [7, 8],

$$
\mathcal{V}^{\lceil\mathcal{MN}\rfloor}{}_{IJ|A} = \tfrac{1}{14}\left((\Gamma^I\Gamma^K)_{AB}\,\mathcal{V}^{\lceil\mathcal{MN}\rfloor}{}_{KJ|B} - (\Gamma^J\Gamma^K)_{AB}\,\mathcal{V}^{\lceil\mathcal{MN}\rfloor}{}_{KI|B}\right). \qquad (5.18)
$$

After some calculation, the second term in (5.17) then reduces to $2\,\Lambda^{\lceil\mathcal{M}}\,(\mathcal{X}_{\mu\nu}{}^{\mathcal{N}\rfloor} - \mathcal{H}_{\mu\nu}{}^{\mathcal{N}\rfloor})$, where we again introduced the expression for the duality relation (5.13). The term proportional to $\mathcal{H}_{\mu\nu}{}^{\mathcal{N}}$ then yields a term belonging to the tensor gauge transformation (2.29). The second line in (5.17) can be simplified in a similar way. Its $(IJ)$ traceless part may be brought into the form

$$
\tfrac{1}{7}\,g\,Y^{\mathcal{MN}}{}_{\mathcal{K}\lceil\mathcal{PQ}\rfloor}\,\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{PQ}\rfloor\rfloor}{}_{IK|KL|LJ}\,\bar\epsilon_{[1}{}^I\gamma_{\mu\nu}\epsilon_{2]}{}^J\,, \qquad (5.19)
$$

and thus constitutes the shift transformation of (2.29) with parameter[13]

$$
\Phi_{\mu\nu}{}^{\mathcal{K}\lceil\mathcal{MN}\rfloor} = -\tfrac{1}{7}\,\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}{}_{IK|KL|LJ}\,\bar\epsilon_{[1}{}^I\gamma_{\mu\nu}\epsilon_{2]}{}^J\,. \qquad (5.20)
$$

It remains to consider the $(IJ)$ trace part of the second line in (5.17) which reduces to

$$
4\,eg\,\varepsilon_{\mu\nu\rho}\,\xi^\rho\,\mathbb{V}^{\lceil\mathcal{MN}\rfloor,\lceil\mathcal{KL}\rfloor}\,\Theta_{\mathcal{KL}}\,, \qquad (5.21)
$$

where $\mathbb{V}^{\lceil\mathcal{MN}\rfloor,\lceil\mathcal{KL}\rfloor}$ equals the symmetric scalar-dependent matrix defined by

$$
\mathbb{V}^{\lceil\mathcal{MN}\rfloor,\lceil\mathcal{KL}\rfloor} = \tfrac{1}{392}\left(7\,\mathcal{V}^{\lceil\mathcal{MN}\rfloor\lceil\mathcal{KL}\rfloor}{}_{IJ|A|IJ|A} - 2\,\mathcal{V}^{\lceil\mathcal{MN}\rfloor\lceil\mathcal{KL}\rfloor}{}_{IK|JK|IL|JL}\right). \qquad (5.22)
$$

Putting everything together, the supersymmetry commutator on two-forms takes the form,

$$
\begin{aligned}
[\delta_{\epsilon_1},\delta_{\epsilon_2}]\, B_{\mu\nu}{}^{\mathcal{MN}} &= 4ge\,\xi^\rho\,\varepsilon_{\mu\nu\rho}\,\mathbb{V}^{\lceil\mathcal{MN}\rfloor,\lceil\mathcal{KL}\rfloor}\,\Theta_{\mathcal{KL}} + 2D_{[\mu}\Xi_{\nu]}{}^{\mathcal{MN}} - 2\,\mathcal{H}_{\mu\nu}{}^{\lceil\mathcal{M}}\,\Lambda^{\mathcal{N}\rfloor} \\
&\quad - gY^{\mathcal{MN}}{}_{\mathcal{K}\lceil\mathcal{PQ}\rfloor}\,\Phi_{\mu\nu}{}^{\mathcal{K}\lceil\mathcal{PQ}\rfloor} + 2\,\mathcal{X}_{\mu\nu}{}^{\lceil\mathcal{M}}\,\Lambda^{\mathcal{N}\rfloor} + 2\,A_{[\mu}{}^{\lceil\mathcal{M}}\,[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,A_{\nu]}{}^{\mathcal{N}\rfloor}\,. \\
&= \left(\xi^\rho\hat D_\rho + \delta_\Lambda + \delta_\Xi + \delta_\Phi\right)B_{\mu\nu}{}^{\mathcal{MN}} - \xi^\rho\mathcal{Y}_{\rho\mu\nu}{}^{\mathcal{MN}} \\
&\quad + 2\,\mathcal{X}_{\mu\nu}{}^{\lceil\mathcal{M}}\,(\Lambda^{\mathcal{N}\rfloor} + \Lambda(\xi)^{\mathcal{N}\rfloor}) - 2\,\xi^\rho\,\mathcal{X}_{\rho[\mu}{}^{\lceil\mathcal{M}}A_{\nu]}{}^{\mathcal{N}\rfloor}\,, \qquad (5.23)
\end{aligned}
$$

where in the second equation we introduced the tensor,

$$
\mathcal{Y}_{\mu\nu\rho}{}^{\mathcal{MN}} \equiv \mathcal{H}_{\mu\nu\rho}{}^{\mathcal{MN}} - 4\,g\,e\,\varepsilon_{\mu\nu\rho}\,\mathbb{V}^{\lceil\mathcal{MN}\rfloor,\lceil\mathcal{KL}\rfloor}\,\Theta_{\mathcal{KL}} - 6\,A_{[\mu}{}^{\lceil\mathcal{M}}\,\mathcal{X}_{\nu\rho]}{}^{\mathcal{N}\rfloor}\,. \qquad (5.24)
$$

[13]Note that not only the coefficient is determined. There exists yet another independent term with the correct tensor structure, $\Gamma^{IK}_{AB}\,\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}{}_{A|B|JK}\,\bar\epsilon_{[1}{}^I\gamma_{\mu\nu}\epsilon_{2]}{}^J$, which turns out to be absent. One may verify by explicit calculation that $\Phi^{\mathcal{K}\lceil\mathcal{MN}\rfloor}$ defined in (5.20) has contributions in both irreducible representations **3875** and **147250**.

This tensor takes the form of a duality relation between the field strength of the two-forms (2.30) and the embedding tensor. The supersymmetry commutator thus closes according to (5.7) modulo terms proportional to the duality relations (5.13) and (5.24). These terms are interpreted as follows. The term proportional to $\mathcal{Y}_{\mu\nu\rho}{}^{\mathcal{MN}}$ corresponds to a new symmetry transformation of the two-form potential. The last term proportional to $\mathcal{X}_{\rho\mu}{}^{\mathcal{M}}$ accompanies the extra transformation in the vector fields represented by the last term in (5.14). Finally the preceding terms proportional to $\mathcal{X}_{\mu\nu}{}^{\mathcal{M}}$ are interpreted as deformations of the vector gauge transformation acting on the two-form potential (*cf.* (2.40)). Hence we change this transformation according to,

$$\begin{aligned}\delta_{\rm mod}(\Lambda)B_{\mu\nu}{}^{\mathcal{MN}} &= -2\,\Lambda^{[\mathcal{M}}\mathcal{H}_{\mu\nu}{}^{\mathcal{N}]} + 2\,\Lambda^{[\mathcal{M}}\mathcal{X}_{\mu\nu}{}^{\mathcal{N}]} + 2\,A_{[\mu}{}^{[\mathcal{M}}\,\delta(\Lambda)A_{\nu]}{}^{\mathcal{N}]} \\ &= 2e\,\varepsilon_{\mu\nu\rho}\,\Lambda^{[\mathcal{M}}\mathcal{V}^{\mathcal{N}]}{}_{A}\,\mathcal{P}^{\rho A} + 2\,A_{[\mu}{}^{[\mathcal{M}}\,\delta(\Lambda)A_{\nu]}{}^{\mathcal{N}]}\,. \end{aligned} \tag{5.25}$$

This deformation is reminiscent of what happens, for instance, in $D = 4$ gauged supergravity [38, 26, 17], where the two-form fields acquire also additional variations once they couple to other fields in the Lagrangian. Of course, it remains to see whether this interpretation is correct, but we will present further evidence of this in section 6.

The duality relation (5.24) is remarkable. On-shell, (*i.e.* for $\mathcal{X}^{\mathcal{M}} = 0 = \mathcal{Y}^{\mathcal{MN}}$) it reads

$$\mathcal{H}_{\mu\nu\rho}{}^{\mathcal{MN}} = 4g\,e\,\varepsilon_{\mu\nu\rho}\,\mathbb{V}^{[\mathcal{MN}],[\mathcal{KL}]}\,\Theta_{\mathcal{KL}}\,, \tag{5.26}$$

and it relates the field strengths of the two-forms to the embedding tensor. The scalar matrix $\mathbb{V}^{[\mathcal{MN}],[\mathcal{KL}]}$ defined in (5.22), which shows up in this equation, is related to the scalar potential of the gauged theory in a simple way. With the explicit expression for the scalar potential $V$ from [7, 8] one finds the expression

$$V \;=\; -\tfrac{1}{8}\Big(A_1^{IJ}A_1^{IJ} - \tfrac{1}{2}\,A_2^{I\dot{A}}A_2^{I\dot{A}}\Big) \;=\; \tfrac{1}{2}\,\mathbb{V}^{[\mathcal{MN}],[\mathcal{KL}]}\,\Theta_{\mathcal{MN}}\,\Theta_{\mathcal{KL}}\,. \tag{5.27}$$

In other words, the matrix $\mathbb{V}^{[\mathcal{MN}],[\mathcal{KL}]}$ precisely encodes the scalar potential of the gauged theory. This appears to be a generic pattern for the ($D$–1)-forms in the gauged supergravities, and we shall see its natural interpretation in the next section. We emphasize that the matrix $\mathbb{V}^{[\mathcal{MN}],[\mathcal{KL}]}$ is *not* positive definite — unlike the scalar matrices that show up in the lower-rank $p$-form dualities. This lack of positivity is in accord with the fact that the potentials of gauged supergravities are generically known to be unbounded from below.

At this point let us briefly comment on a similar result in [45] where the form fields are considered for $D = 5$ gauge maximal supergravity. In that work an equation (4.27) appears which seems the direct analogue of (5.26), but now for the field strength of the four-form potential. Although it has the same structure as (5.26), its right-hand side is not related to the potential in the way we described above. However, a direct comparison is subtle as (5.13) only vanishes on shell upon projection with the embedding tensor, so that (5.26) will not be realized as a field equation.

The duality equation (5.26) in particular provides the $\mathrm{E}_{8(8)}$ covariant field equation for two-forms in the three-dimensional ungauged theory:

$$\partial^{\mu}\Big(\mathbb{V}_{[\mathcal{MN}],[\mathcal{KL}]}\,\mathcal{H}_{\mu\nu\rho}{}^{\mathcal{KL}}\Big) \;+\; \text{fermions} \;=\; 0\,, \tag{5.28}$$

with $\mathbb{V}_{\lceil\mathcal{MN}\rfloor,\lceil\mathcal{KL}\rfloor}$ the inverse matrix to $\mathbb{V}^{\lceil\mathcal{MN}\rfloor,\lceil\mathcal{KL}\rfloor}$.

To close this section, we also compute the commutator of supersymmetry transformations on the three-forms. Equation (5.20) suggests to define the supersymmetry variation of the three-forms as

$$\Delta C_{\mu\nu\rho}{}^{\mathcal{K}\lceil\mathcal{MN}\rfloor} \;=\; \tfrac{3}{7}\,\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}{}_{IK|KL|LJ}\;\bar\epsilon^I\gamma_{[\mu\nu}\psi_{\rho]}{}^J \;+\; \cdots\,, \tag{5.29}$$

where the dots refer to the $\bar\epsilon\,\chi$ variations. Indeed,

$$\begin{aligned} 2\delta_{\epsilon_{[1}}\Big(\tfrac{3}{7}\,\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}{}_{IK|KL|LJ}\;\bar\epsilon^I_{2]}\gamma_{[\mu\nu}\psi^J_{\rho]}\Big) \;&=\; 3\,D_{[\mu}\Phi_{\nu\rho]}{}^{\mathcal{KMN}} \\ &\quad+\tfrac{3}{7}\,D_{[\mu}\Big(\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}{}_{IK|KL|LJ}\Big)\;\bar\epsilon_{[1}{}^I\gamma_{\mu\nu}\epsilon_{2]}{}^J+\cdots\,, \end{aligned} \tag{5.30}$$

thus reproducing the correct $\Phi$ term given in (5.20). Evaluating the derivatives of the second term and using the duality equation (5.13), eventually brings this term into the form (modulo $\mathcal{X}^{[\mathcal{M}]}$),

$$\begin{aligned} &\Big(\tfrac{3}{14}\,\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}{}_{A|KM|JM}-\tfrac{3}{7}\,\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}{}_{JM|KM|A}\Big)\,(\Gamma^K\Gamma^N)_{AB}\,\mathcal{P}^B_{[\mu}\,\bar\epsilon^N_{[1}\gamma_{\nu\rho]}\epsilon^J_{2]} \\ &\qquad+3\,\mathcal{H}^{(2)}_{[\mu\nu}{}^{\lceil\mathcal{K}}\,\Xi^{\mathcal{MN}\rfloor}_{\rho]}\,. \end{aligned} \tag{5.31}$$

In order to arrive at this result, we need to make use of the explicit projection onto the $\mathbf{3875}+\mathbf{147250}$ within the tensor product $\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor$. This gives rise to a number of non-trivial identities, like

$$\begin{aligned} 14\,\Gamma^K_{A\dot A}\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}_{\quad A|KM|JM}+16\,\Gamma^K_{A\dot A}\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}_{\quad JM|KM|A} \\ -(\Gamma^K\Gamma^{MN})_{A\dot A}\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}_{\quad MN|JK|A} \;=\; 0\,, \end{aligned} \tag{5.32}$$

which results from the projection of a triple product of $\mathcal{V}$'s onto the $\mathbf{147250}+\mathbf{3875}$ representation in the same way as (5.18) is obtained by applying (4.4) to a double product of $\mathcal{V}$'s. From (5.31) we can infer the full supersymmetry transformation of the three-forms. While the last term is precisely expected from the tensor gauge transformations (2.29), the rest must be cancelled by $\delta\chi$ variations in $\delta C$. Together, this determines the supersymmetry variation of the three-forms to be given by

$$\begin{aligned} \Delta C_{\mu\nu\rho}{}^{\mathcal{K}\lceil\mathcal{MN}\rfloor} \;&=\; \tfrac{3}{7}\,\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}{}_{IK|KL|LJ}\;\bar\epsilon^I\gamma_{[\mu\nu}\psi^J_{\rho]} \\ &\quad-\tfrac{1}{14}\mathrm{i}\,\Big(\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}{}_{A|KM|JM}-2\mathcal{V}^{\lceil\mathcal{K}\lceil\mathcal{MN}\rfloor\rfloor}{}_{JM|KM|A}\Big)\,\Gamma^K{}_{A\dot A}\,\bar\epsilon^J\gamma_{\mu\nu\rho}\chi^{\dot A}\,. \end{aligned} \tag{5.33}$$

To summarize, we have determined the supersymmetry variations of all $p$-forms in three dimensions by closure of the supersymmetry algebra. The full algebra is given by

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}] \;=\; \xi^\mu\hat D_\mu+\delta_\Lambda+\delta_\Xi+\delta_\Phi+\delta_\mathcal{X}+\delta_\mathcal{Y}\,, \tag{5.34}$$

up to supersymmetry and local Lorentz symmetry transformations. The last two terms correspond to additional local symmetries proportional to $\mathcal{X}_{\mu\nu}$ and $\mathcal{Y}_{\mu\nu\rho}$, that have appeared in (5.14) and (5.23) for the one- and two-forms, respectively. Furthermore, we recall that we have made a modification in the vector gauge transformation rule for the two-forms.

Of course, we have to justify both the presence of this deformation and the fact that the two new variations can indeed be regarded as symmetries of a specific Lagrangian. In this respect it is important to recall that $\mathcal{X}_{\mu\nu}$ and $\mathcal{Y}_{\mu\nu\rho}$ take the form of first-order duality equations between $p$-forms in three dimensions and, as it turns out, there are indeed field equations are proportional to $\mathcal{X}_{\mu\nu}$ and $\mathcal{Y}_{\mu\nu\rho}$. This feature plays an important role in realizing the invariance. To understand this issue further we turn to the construction of the Lagrangian in the next section.

# 6 The Lagrangian with all $p$-forms in three dimensions

Finally, we give a Lagrangian which contains all $p$-forms in three dimensions. To this end we start from the gauged Lagrangian of [7, 8],

$$\begin{aligned}
\mathcal{L}_{\text{gauged}} &= -\tfrac{1}{4}eR + \tfrac{1}{4}e\mathcal{P}^{\mu A}\mathcal{P}_{\mu}^{A} + \tfrac{1}{2}\,\varepsilon^{\mu\nu\rho}\bar{\psi}_{\mu}^{I}D_{\nu}\psi_{\rho}^{I} - \tfrac{1}{2}\mathrm{i}e\bar{\chi}^{\dot{A}}\gamma^{\mu}D_{\mu}\chi^{\dot{A}} \\
&\quad - \tfrac{1}{4}\,g\,\varepsilon^{\mu\nu\rho}\,A_{\mu}{}^{\mathcal{M}}\,\Theta_{\mathcal{M}\mathcal{N}}\,(\partial_{\nu}A_{\rho}{}^{\mathcal{N}} + \tfrac{1}{3}\,gX_{[\mathcal{R}\mathcal{S}]}{}^{\mathcal{N}}\,A_{\nu}{}^{\mathcal{R}}A_{\rho}{}^{\mathcal{S}}) \\
&\quad - \tfrac{1}{2}e\,\bar{\chi}^{\dot{A}}\gamma^{\mu}\gamma^{\nu}\psi_{\mu}^{I}\,\Gamma^{I}{}_{A\dot{A}}\mathcal{P}_{\nu}^{A} + \tfrac{1}{2}eg\,A_{1}{}^{IJ}\,\bar{\psi}_{\mu}^{I}\,\gamma^{\mu\nu}\,\psi_{\nu}^{J} + \mathrm{i}eg\,A_{2}{}^{I\dot{A}}\,\bar{\chi}^{\dot{A}}\gamma^{\mu}\,\psi_{\mu}^{I} \\
&\quad + \tfrac{1}{2}eg\,A_{3}{}^{\dot{A}\dot{B}}\,\bar{\chi}^{\dot{A}}\,\chi^{\dot{B}} - \tfrac{1}{2}eg^{2}\,\mathbb{V}^{[\mathcal{M}\mathcal{N}],[\mathcal{K}\mathcal{L}]}\,\Theta_{\mathcal{M}\mathcal{N}}\,\Theta_{\mathcal{K}\mathcal{L}} \;+\; \mathcal{L}_{4-\text{fermi}}\,, \qquad (6.1)
\end{aligned}$$

where,

$$A_{3}{}^{\dot{A}\dot{B}} = \tfrac{1}{48}\,(\Gamma^{IJKL})_{\dot{A}\dot{B}}\,\mathcal{V}^{\mathcal{M}\mathcal{N}}{}_{IJ|KL}\,\Theta_{\mathcal{M}\mathcal{N}}\,. \qquad (6.2)$$

This is the Lagrangian that describes all consistent gaugings with a constant, symmetric, embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ that belongs to the $\mathbf{3875}+\mathbf{1}$ representation and is subject to the quadratic constraint $\mathcal{Q}_{\mathcal{K}[\mathcal{M}\mathcal{N}]} = 0$.

Now consider $\Theta_{\mathcal{M}\mathcal{N}}$ not as a constant tensor but as an $x$-dependent field $\Theta_{\mathcal{M}\mathcal{N}}(x)$ satisfying the representation constraint (*i.e.* living in the **3875**; for convenience we suppress the singlet representation in what follows), but *not* the quadratic constraint on $\Theta_{\mathcal{M}\mathcal{N}}$. To the Lagrangian (6.1) we add a new Lagrangian describing the couplings to two-forms $B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}$ and three-forms $C_{\mu\nu\rho}{}^{\mathcal{K}[\mathcal{M}\mathcal{N}]}$,

$$\mathcal{L}_{BC} \;=\; -\tfrac{1}{8}g\,\varepsilon^{\mu\nu\rho}B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}\,D_{\rho}\Theta_{\mathcal{M}\mathcal{N}} + \tfrac{1}{12}g^{2}\,\varepsilon^{\mu\nu\rho}\,C_{\mu\nu\rho}{}^{\mathcal{K}[\mathcal{M}\mathcal{N}]}\,\mathcal{Q}_{\mathcal{K}[\mathcal{M}\mathcal{N}]}\,, \qquad (6.3)$$

The two- and three-form potentials thus act as Lagrange multipliers to ensure that $\Theta_{\mathcal{M}\mathcal{N}}$ is constant and satisfies the quadratic constraint. As $\Theta_{\mathcal{M}\mathcal{N}}$ is a field now, the quadratic constraint can no longer be imposed by hand but must be implemented in this way.

Since the Lagrangian (6.1) is supersymmetric and gauge invariant for a constant tensor $\Theta_{\mathcal{M}\mathcal{N}}$ satisfying the quadratic constraint, the new Lagrangian $\mathcal{L}_{\text{gauged}} + \mathcal{L}_{BC}$ with $x$-dependent $\Theta_{\mathcal{M}\mathcal{N}}$ can be made supersymmetric and gauge invariant by introducing the proper local transformation laws for the potentials $B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}$ and $C_{\mu\nu\rho}{}^{\mathcal{K}[\mathcal{M}\mathcal{N}]}$, while keeping $\delta\Theta_{\mathcal{M}\mathcal{N}} = 0$. This construction thus shows that the supersymmetry algebra can be extended to two- and three-forms transforming in **3875** and the $\mathbf{3875}+\mathbf{147250}$, respectively. The same construction can be applied in higher dimensions and gives a natural explanation of why in

general the $(D-1)$-forms and the $D$-forms transform in the conjugate representations of the embedding tensor and the quadratic constraint, respectively.

As a first exercise, we can compute the new field equation obtained by varying the full Lagrangian with respect to $\Theta_{\mathcal{MN}}$. Neglecting fermions, we find,

$$\begin{aligned}\delta\mathcal{L}_{\text{gauged}} &= -eg\left(\tfrac{1}{2}\mathcal{V}^{\mathcal{M}}{}_{A}\,\mathcal{P}^{\mu\,A}A_{\mu}{}^{\mathcal{N}}+g\,\mathbb{V}^{[\mathcal{MN}],[\mathcal{KL}]}\,\Theta_{\mathcal{KL}}\right)\delta\Theta_{\mathcal{MN}}\\ &\quad-\tfrac{1}{4}g\,\varepsilon^{\mu\nu\rho}\,A_{\mu}{}^{\mathcal{M}}\left(\partial_{\nu}A_{\rho}{}^{\mathcal{N}}+\tfrac{2}{3}\,gX_{[\mathcal{RS}]}{}^{\mathcal{N}}\,A_{\nu}{}^{\mathcal{R}}A_{\rho}{}^{\mathcal{S}}\right)\delta\Theta_{\mathcal{MN}}\,,\end{aligned}\tag{6.4}$$

and (modulo a total derivative),

$$\begin{aligned}\delta\mathcal{L}_{BC} &= \tfrac{1}{24}g\,\epsilon^{\mu\nu\rho}\Big(3D_{\rho}B_{\mu\nu}{}^{\mathcal{MN}}-6gZ^{\mathcal{M}}{}_{\mathcal{PQ}}A_{\rho}{}^{\mathcal{N}}B_{\mu\nu}{}^{\mathcal{PQ}}\\ &\qquad+gY^{\mathcal{MN}}{}_{\mathcal{K}[\mathcal{PQ}]}\,C_{\mu\nu\rho}{}^{\mathcal{K}[\mathcal{PQ}]}\Big)\,\delta\Theta_{\mathcal{MN}}\,,\end{aligned}\tag{6.5}$$

where we used the identity $\delta\mathcal{Q}_{\mathcal{K}[\mathcal{MN}]}=\frac{1}{2}\delta\Theta_{\mathcal{PQ}}\,Y^{\mathcal{PQ}}{}_{\mathcal{K}[\mathcal{MN}]}$. Therefore the variation of the full Lagrangian $\mathcal{L}=\mathcal{L}_{\text{gauged}}+\mathcal{L}_{BC}$ takes the form,

$$\delta\mathcal{L} = \tfrac{1}{24}g\,\varepsilon^{\mu\nu\rho}\,\mathcal{Y}_{\mu\nu\rho}{}^{\mathcal{MN}}\,\delta\Theta_{\mathcal{MN}}\,,\tag{6.6}$$

so that we obtain precisely the duality relation $\mathcal{Y}_{\mu\nu\rho}{}^{\mathcal{MN}}$ defined in (5.24). In particular, this shows why the scalar matrix that relates the field strength of the $(D-1)$-forms to the embedding tensor according to (5.26) is precisely the (non-positive definite) matrix $\mathbb{V}^{[\mathcal{MN}],[KL]}$ of the scalar potential. Clearly the analogue of this relation will hold in any dimension.

Under general variations of vector and tensor fields, the full Lagrangian varies as (again neglecting fermions),

$$\begin{aligned}\delta\mathcal{L} &= -\tfrac{1}{4}g\,\varepsilon^{\mu\nu\rho}\,\Theta_{\mathcal{MN}}\,\delta A_{\mu}{}^{\mathcal{M}}\,\mathcal{X}_{\nu\rho}{}^{\mathcal{N}}-\tfrac{1}{8}g\,\varepsilon^{\mu\nu\rho}\left(\delta B_{\mu\nu}{}^{\mathcal{MN}}+2A_{[\mu}{}^{\mathcal{M}}\,\delta A_{\nu]}{}^{\mathcal{N}}\right)D_{\rho}\Theta_{\mathcal{MN}}\\ &\quad+\tfrac{1}{12}g^{2}\,\varepsilon^{\mu\nu\rho}\left(\delta C_{\mu\nu\rho}{}^{\mathcal{K}[\mathcal{MN}]}+2\,A_{\mu}{}^{\mathcal{K}}A_{\nu}{}^{\mathcal{M}}\delta A_{\rho}{}^{\mathcal{N}}\right)\mathcal{Q}_{\mathcal{K}[\mathcal{MN}]}\,.\end{aligned}\tag{6.7}$$

Thus, varying the Lagrangian with respect to all $p$-form tensor fields and $\Theta_{\mathcal{MN}}$, one obtains the set of first order and algebraic field equations

$$\Theta_{\mathcal{MN}}\,\mathcal{X}_{\mu\nu}{}^{\mathcal{N}}\;=\;0\,,\quad\mathcal{Y}_{\mu\nu\rho}{}^{\mathcal{MN}}\;=\;0\,,\quad\partial_{\mu}\Theta_{\mathcal{MN}}\;=\;0\,,\quad\mathcal{Q}_{\mathcal{K}[\mathcal{MN}]}\;=\;0\,,\tag{6.8}$$

and we recover the duality relations $\mathcal{X}^{\mathcal{M}}$ and $\mathcal{Y}^{\mathcal{MN}}$ that appeared in the computation of the supersymmetry algebra (5.13) and (5.24), respectively.

Let us further remark that the full Lagrangian is invariant under the additional symmetry

$$\begin{aligned}&\delta_{\mathcal{X}}A_{\mu}{}^{\mathcal{M}}=\xi^{\nu}_{\mathcal{X}}\mathcal{X}_{\nu\mu}{}^{\mathcal{M}}\,,\qquad\delta_{\mathcal{X}}B_{\mu\nu}{}^{\mathcal{MN}}=-2A_{[\mu}{}^{[\mathcal{M}}\,\delta_{\mathcal{X}}A_{\nu]}{}^{\mathcal{N}]}\,,\\ &\delta_{\mathcal{X}}C_{\mu\nu\rho}{}^{\mathcal{K}[\mathcal{MN}]}=-8\,A_{[\mu}{}^{[\mathcal{K}}A_{\nu}{}^{[\mathcal{M}}\,\delta_{\mathcal{X}}A_{\rho]}{}^{\mathcal{N}]]}\,,\end{aligned}\tag{6.9}$$

with an arbitrary vector field $\xi^{\nu}_{\mathcal{X}}$. This follows directly from (6.7):

$$\delta_{\mathcal{X}}\mathcal{L}\;\propto\;\varepsilon^{\mu\nu\rho}\,\Theta_{\mathcal{MN}}\,\mathcal{X}_{\mu\nu}{}^{\mathcal{M}}\,\mathcal{X}_{\rho\sigma}{}^{\mathcal{N}}\,\xi^{\sigma}_{\mathcal{X}}\;=\;0\,.\tag{6.10}$$

Likewise, the Lagrangian is invariant under the additional symmetry

$$\delta_{\mathcal{Y}} B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} = \xi^{\rho}_{\mathcal{Y}}\, \mathcal{Y}_{\rho\mu\nu}{}^{\mathcal{M}\mathcal{N}} \,, \qquad \delta_{\mathcal{Y}} \Theta_{\mathcal{M}\mathcal{N}} = \xi^{\rho}_{\mathcal{Y}}\, D_\rho \Theta_{\mathcal{M}\mathcal{N}} \,, \tag{6.11}$$

with another arbitrary vector field $\xi^{\mu}_{\mathcal{Y}}$. The extra symmetries (6.9) and (6.11) are those which have shown up already in the supersymmetry algebra and correspond to the last two terms in (5.34). The second one is a standard equations-of-motion symmetry', whereas the first one is a little more subtle as its corresponding field variations do not vanish completely upon imposing the equations of motion.

Note that although there are of course no four-forms present in the three-dimensional Lagrangian, their tensor gauge freedom shows up as a shift transformation on the the three-forms (2.43). Since these are the only fields transforming under this symmetry, the Lagrangian must be invariant under the mere shift of three-forms according to (2.43). Fortunately, this invariance is precisely ensured by the additional orthogonality (4.11), showing that the combination $C_{\mu\nu\rho}{}^{\mathcal{K}[\mathcal{M}\mathcal{N}}\, \mathcal{Q}_{\mathcal{K}[\mathcal{M}\mathcal{N}]}$ entering the Lagrangian is invariant under these shifts.

A rather lengthy but straightforward calculation now shows that the full Lagrangian $\mathcal{L} = \mathcal{L}_{\text{gauged}} + \mathcal{L}_{BC}$ is invariant under supersymmetry provided the fields transform as (5.4), (5.6), (5.16), and (5.33). Here no supersymmetry variation is assigned to the field $\Theta_{MN}$, which can still satisfy the supersymmetry variations by virtue of the existence of the new symmetry (6.11). Furthermore we precisely recover the new transformation rules for the higher $p$-forms that we have derived in section 5. A somewhat similar construction has been carried out in [47] to describe Roman's massive deformation of ten-dimensional IIA supergravity [37] in terms of a nine-form potential and an $x$-dependent parameter $m(x)$ rather than a constant deformation parameter $m$. What is new here is the non-trivial representation structure of the deformation parameters and the need to simultaneously implement on them the quadratic constraint, hence the need for $D$-forms acting as the corresponding Lagrange multipliers.

We now return to the possible interpretation of our results, and especially the ones of the present section, in the framework of infinite-dimensional duality symmetries. Let us recall that the representations found in the level decompositions of $\mathrm{E}_{11}$ [31, 32, 33] are in one-to-one correspondence with the various $p$-form fields identified in course of our analysis and displayed in table 2. By contrast, the embedding tensor itself does not show up in this level decomposition, but must be added as an 'extraneous' quantity, even though it is to be treated as a 'field' in the present analysis (otherwise there would be no need for extra $p$-form fields in the Lagrangian (6.3)). In order to better understand the link with infinite-dimensional dualities, it would therefore be desirable to re-formulate the theory entirely in terms of only the fields appearing in the group theoretical analysis, and thus *without* $\Theta$.

At least in principle, it is possible to pass from the total Lagrangian $\mathcal{L} \equiv \mathcal{L}_{\text{gauged}} + \mathcal{L}_{BC}$ to another Lagrangian which does not depend on $\Theta$, by noting that $\mathcal{L}$ depends on $\Theta$ at most quadratically. Accordingly, we now regard the field equation $\mathcal{Y}_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N}} = 0$ as an algebraic equation for the (auxiliary) field $\Theta_{\mathcal{M}\mathcal{N}}$,

$$4\, g\, e\, \varepsilon_{\mu\nu\rho}\, \mathbb{V}^{[\mathcal{M}\mathcal{N}],[\mathcal{K}\mathcal{L}]}\, \Theta_{\mathcal{K}\mathcal{L}} \;=\; 3\, D_{[\mu} B_{\nu\rho]}{}^{\mathcal{M}\mathcal{N}} + 6\, A_{[\mu}{}^{[\mathcal{M}} \left( \partial_\nu A_{\rho]}{}^{\mathcal{N}]} + \tfrac{1}{3} g X_{[\mathcal{P}\mathcal{Q}]}{}^{\mathcal{N}]} A_\nu{}^{\mathcal{P}} A_{\rho]}{}^{\mathcal{Q}} \right)$$

$$+\, g\, Y^{\mathcal{MN}}{}_{\mathcal{P}[\mathcal{RS}]}\, C_{\mu\nu\rho}{}^{\mathcal{P}[\mathcal{RS}]} - 6\, A_{[\mu}{}^{[\mathcal{M}}\, \mathcal{X}_{\nu\rho]}{}^{\mathcal{N}]}\,, \tag{6.12}$$

and use it to eliminate $\Theta_{\mathcal{MN}}$ from the Lagrangian. Although this equation is linear in $\Theta_{\mathcal{MN}}$, its solution is rather complicated due to the hidden $\Theta$ dependence of the tensors $X_{\mathcal{PQ}}{}^{\mathcal{N}}$, $Y^{\mathcal{MN}}{}_{\mathcal{PRS}}$ and $\mathcal{X}_{\mu\nu}{}^{\mathcal{M}}$ on the right-hand side. Consequently, the solution cannot be written in closed form, but only given as an *infinite series* in the $p$-forms and their derivatives.[14] We therefore exhibit only the lowest-order term of the solution which reads

$$\Theta_{\mathcal{MN}} \;=\; \tfrac{3}{4}\, e^{-1} \varepsilon^{\mu\nu\rho}\, \mathbb{V}_{[\mathcal{MN}],[\mathcal{KL}]}\, \partial_\mu B_{\nu\rho}{}^{\mathcal{KL}} + \cdots\,. \tag{6.13}$$

Plugging (6.13) back into (6.1) and (6.3) we derive the bosonic kinetic term for the two-form fields in lowest order, with the result

$$\mathcal{L}_{\rm kin} \;=\; e\, \partial_{[\mu} B_{\nu\rho]}{}^{\mathcal{MN}}\, \partial^{[\mu} B^{\nu\rho]\mathcal{KL}}\, \mathbb{V}_{[\mathcal{MN}],[\mathcal{KL}]} + \ldots\,, \tag{6.14}$$

We thus see that the inverse scalar potential matrix $\mathbb{V}_{[\mathcal{MN}],[\mathcal{KL}]}$ shows up as the kinetic matrix of the $(D-1)$-forms, as would have been expected from (5.28). As we already pointed out above (after (5.27)) this matrix is not positive definite, unlike the kinetic matrices of the lower $p$-forms. Fortunately, we need to require positive definite kinetic terms only for those fields which carry propagating degrees of freedom, whence the non-positivity of the kinetic term for the 2-form fields in the above formula is entirely harmless.

In conclusion it is possible to re-formulate the theory in terms of a Lagrangian that contains only the scalars and $p$-forms, but no embedding tensor. The price we have to pay is that the resulting structure is rather complicated, with non-polynomial interactions and gauge transformations. Nevertheless, the Lagrangian obtained by elimination of $\Theta$ is 'universal' in the sense that it would incorporate *all* gaugings, in such a way that any specific gauging would correspond to the 3-form field strength $\partial_{[\mu} B_{\nu\rho]}{}^{\mathcal{MN}}$ acquiring a vacuum expectation value according to (6.13). One may view this as a kind of 'spontaneous symmetry breaking', but of a novel kind: rather than simply breaking the rigid G invariance of the original theory to some subgroup, this mechanism generates non-abelian gaugings from a theory with purely abelian $p$-forms and interactions!

By construction, the constraints on the embedding tensor exhibited and studied in the foregoing sections must also be consistently encoded into this new Lagrangian. Unfortunately, due to the the non-polynomiality of the latter, it appears difficult to extract this information directly and without explicit use of $\Theta$. For this reason, it would be desirable to go beyond the mere kinematics of level decompositions, and to 'test' this non-polynomial Lagrangian (or at least some of its pieces, and in particular the dependence of (6.14) on the scalars via the kinetic matrix) directly either against the $\mathrm{E}_{11}$ proposal of [48], or alternatively, against the $\mathrm{E}_{10}$ proposal of [49, 50]. Because the latter admits a Lagrangian formulation (but without $D$-forms as these do not appear in the decomposition of $\mathrm{E}_{10}$), such tests are possible in principle. Although this will require much more work, we are confident that the present

[14] Observe that the matrix $\mathbb{V}^{[\mathcal{MN}],[\mathcal{KL}]}$ will have zero eigenvalues at certain points of the scalar field configuration space.

results can serve as useful probes of M theory, or, more succinctly, of the specific proposals made in [48] and [49, 50], respectively, and thereby shed new light on the unresolved issues with them.

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