The motion of the freely falling chain tip. Force measurements.

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Abstract

A chain is firmly attached at one end while the other falls freely in the gravitational field. We report and discuss careful time-resolved measurements of the horizontal and vertical components of the force applied by the chain to the mount. Our results interestingly complement previous laboratory measurements and numerical simulations of the free-end dynamics.

I. INTRODUCTION

The dynamics of a chain falling in a gravitational field is a very interesting problem to be considered in intermediate courses on classical mechanics for teaching how to handle variable mass systems. Among the various possible experimental configurations, we focus here in the following one: The chain is attached to a rigid mount by its two ends, which, separated by the horizontal distance x_0, have initially the same altitude. Then, as one of the ends is released, the chain begins to fall. The main features of the chain dynamics are that the acceleration of the chain tip is greater than g, the acceleration due to gravity, and exhibits a large maximum almost when the chain reaches its maximum extension[1]. The problem has already been the purpose of several experimental, numerical and theoretical studies and we suggest the introduction by Wong and Yasui for a review[2].

Here we report measurements of the force acting on the mount which supplement the existing experimental results and, especially, previous force measurements that were limited to the vertical component of the force and to small initial spacings of the two ends[1]. We extend the measurements to the horizontal and vertical components of the force and to the entire accessible range of the initial spacing. We remind here that the force exerted to the mount, $\vec{F}$, is related to the acceleration $\vec{a}_G$ of the center of mass G of the chain as $\vec{F} = M\vec{g} - M\vec{a}_G$, where M stands for the mass of the chain. We point out that $\vec{F}$ can be inferred from the motion of the center of mass, which could be determined either from experimental images of the chain or from the numerical simulations[3]. However, the analysis of the images or the numerical simulations are not readily accessible to the audience of a laboratory course and measuring the force $\vec{F}$ is an alternative and easier way to point out some striking features of the chain motion. Our aim is not to give an exhaustive description of the chain dynamics but to describe the experimental method and discuss, at least qualitatively, some of the observations.

II. EXPERIMENTAL SETUP

The experimental setup is very similar to the one already described in reference 1 but we designed, in addition, specific parts making possible to measure independently the vertical and horizontal components of the force applied by the chain to the mount at the end which remains at rest.

We report experiments performed with a ball chain consisting of stainless-steel identical segments that are made from rods [length $l = (4.46 \pm 0.01)\, 10^{-3}$ m] and spheres [diameter $\phi = (3.26 \pm 0.01)\, 10^{-3}$ m] attached to each other. The total length of the chain is $L = 1.022$ m, which corresponds to $n = 229$ segments for a total mass of $M = (2.08 \pm 0.01)\, 10^{-2}$ kg.

The experimental configuration is as follows (Fig. 1). The chain is tightly attached at one end in O, where the force shall be measured. At the other edge, the chain ends with a rod to which we attach a thin nylon cord (fishing line, diameter 10^{-4} m). We then extend the nylon cord between two nails and a thin metallic wire (nickel, diameter 10^{-4} m) as sketched in the figure 1b. In order to release the extremity of the chain, we shall suddenly cut the nylon wire by injecting a large electric current I (about 1 A) through the metallic wire. The time at which the end is released (the origin of time, $t = 0$) is obtained by means of an accelerometer (Dytran, 3035BG) attached to the mount holding the wire. In O, the horizontal, F_x, and vertical, F_y, components of the force, $\vec{F}$, applied by the chain to the mount are measured by means of two identical force-sensors (Testwell, KD40S, maximum force 10 N) making a 90-degrees angle between them (Fig. 1a). On the one hand, the chain is firmly attached to the vertical sensor by means of a thin nickel wire (diameter 100 μm) which prevents any vertical motion without exerting any significant horizontal force. On the other hand, the first ball is in contact with the inner surface of a metallic ring attached to the horizontal sensor, which prevents any horizontal motion of the first link as long as the horizontal force is positive ($F_x > 0$). The first link is thus not strickly trapped so as to insure that the experimental setup does not apply any significant vertical force to the chain, the only contribution reducing to a negligible friction force due to the contact between the first ball and the ring. The whole system is adjusted in order to ensure that both ends of the chain are initially at the same level, $y_0 = 0$, and the setup can be displaced horizontally to vary the initial horizontal separation, x_0, between the two ends of the chain. Finally, the signals from the three sensors (the accelerometer and the two force sensors) are monitored using a 4-channels oscilloscope (Lecroy, WaveRunner 6030A). Due to the electronic noise and the discretization by the oscilloscope, the measurements of the instantaneous force suffer an uncertainty of about about 0.3 Mg. Due to an offset in the amplifier, we cannot garanty a precision better than 0.1 Mg of the mean value over time.

III. EXPERIMENTAL RESULTS

The initial conformation formed by the chain after damping of all the disturbances is close to a catenary curve. We then release the end at x_0 and monitor the signals from the three sensors.

A. Main characteristics of the monitored signals and definitions

We report in the figure 2 the typical signals from the oscilloscope after release of the end in x_0. Note here that we oriented the horizontal axis toward the free end and the vertical axis downwards so as to get positive components of the force exerted by the chain to the mount.

First, due to the release of the tip at x_0, we observe in the signal from the accelerometer (Fig. 2c), a peak in the acceleration Γ of the mount, the nickel wire is attached to, which makes it possible to determine accurately (to within 10^{-3} s) the origin of the time, $t = 0$. Then, due to the fall of the chain, both components F_x and F_y of the force $\vec{F}$ present a large maximum a few tenth of second later (Fig. 2a and 2b). We shall denote F_x^{max} and F_y^{max} the amplitude of respectively the horizontal and vertical components of the force and t_x^{max} and t_y^{max} the corresponding times. We point out here that the measurements of the horizontal force are relevant only as long as the chain remains in contact with the ring. Thus, in practice, only the first maximum in F_x is reliable (figure 2a, black curve). Apparently, in figure 2, F_x and F_y are continuous at $t = 0$. However, as we shall see later, the conclusion does not always hold true and we denote $\Delta F_x \equiv F_x(0^+) - F_x(0^-)$ and $\Delta F_y \equiv F_y(0^+) - F_y(0^-)$.

B. Time for the vertical component of the force to reach its maximum value

The time t_y^{max} for the vertical component of the force to reach its maximum exhibits a rather complex behavior as a function of the initial spacing x_0 (Fig. 3): As a function of x_0, t_y^{max} initially decreases and then increases when the initial spacing is increased, reaching thus a minimum for $x_0/L \sim 0.6$. The complex behavior cannot be accounted for analytically but we can discuss, at least, the two limiting values at $x_0 = 0$ and $x_0 = L$.

The value of t_y^{max} in the limit $x_0 \to 0$ can be estimated analytically by considering the case of the tightly folded chain[3]. In this limit, the chain tip is expected to reach its lowest position faster than a free falling weight, at $t_f \simeq 0.847213\, t_0$ where $t_0 \equiv \sqrt{2L/g}$ denotes the duration of the free fall over the length L. At $t = t_f$, the velocity of the chain tip, which is the only part of chain in motion, changes in sign. We thus expect $t_y^{max} = t_f$.

The value of t_y^{max} in the limit $x_0 \to L$ can also be estimated analytically by considering that, according to the description reported by Tomaszewski *et al.*[3], the dynamics of the vertical fall of the chain tip is identical with the dynamics of the free fall. This coincidence corresponds with the fact that, in absence of elastic contibution associated with the bending of the chain, the section of the chain, close to the free end, remains horizontal during the fall. The vertical velocity of the center of mass changes in sign when the chain tip reaches its lowest position at t_0. Thus, experimentally, the maximum value of the vertical component of the force F_y^{max} is expected to be reached at $t_y^{max} = t_0$ for $x_0 = L$, where t_0 corresponds to the time of the free fall (Fig. 3).

C. Maximum value of the vertical component of the force

The maximum value F_y^{max} of the vertical component increases when the initial spacing x_0 is decreased. The monotonic behavior of the force when the spacing x_0 is changed is in contrast with the observation of a non-monotonic behavior of the maximum acceleration of the chain tip as a function of the initial spacing. Indeed, Tomaszewski *et al.* observed the acceleration of the chain tip to be minimum for $x_0/L \simeq 0.904$[3]. However, in spite of this qualitative difference, we point out that the results are not contradictory because the force exerted by the chain to the mount images the acceleration of the center of mass of the whole chain and not that of the chain tip.

In contrast to t_y^{max}, F_y^{max}, which diverges in the framework of the tightly-folded-chain limit, cannot be predicted analytically in the limit $x_0 \to 0$. This expectation is in contradiction with the experimental observation of a finite limiting value of F_y^{max}. This is mainly due to the fact that the chain cannot be bent infinitely without loading any elastic energy and exhibits an associated minimum radius of curvature $R_{min} = (4.8 \pm 0.2)\, 10^{-3}$ m, which is not accounted for in the model. We can nevertheless obtain an estimate of F_y^{max} in the limit $x_0 \to 0$ by considering, for the tightly folded chain, the acceleration of the free tip[3] as a function of the its vertical position z:

$$a_c(z) = \frac{1}{2}\,g\Big[1+\Big(\frac{L}{L-z}\Big)^2\Big], \tag{1}$$

which predicts a_c to diverge for $z = L$. However, in this limit, the radius of curvature of the chain vanishes, which is not permitted experimentally. Due to the finite radius of curvature R_{min}, the equation (1) does not hold true up to L but only up to $L-\delta L$, where the cut-off length $\delta L = Nl$, N denoting the number of links in motion, is about R_{min}. The maximum acceleration of the chain tip is thus expected to be about $a_c^{max} = \frac{1}{2}\,g\,[1+(L/Nl)^2]$. Introducing the mass of one link, M/n, we estimate the maximum vertical force to be about, adding the contribution of the chain weight,

$$F_y^{max} \simeq Mg + \frac{1}{2}\frac{N}{n}\Big[1+\Big(\frac{L}{Nl}\Big)^2\Big]\,Mg \qquad (x_0 \to 0). \tag{2}$$

In our experimental conditions, we get, for $N = 1$, $F_y^{max}/(Mg) \simeq 115$ whereas we expect the experimental limit $F_y^{max}/(Mg) \simeq 40$ for $N \simeq 3$, associated with $\delta L \simeq 3R_{min}$. In order to validate our arguments, we also checked experimentally, by changing the length L of the chain, that F_y^{max} depends linearly on L^2, in agreement with the equation (2).

In the limit $x_0 \to L$, the maximum vertical force can be estimated by considering the motion of a solid swinging rod. Indeed, from the optical observation of the chain motion[3], we note that the chain is rotating as an almost straight, almost vertical, rod when the maximum force is measured (Fig. 5b). From the energy conservation, we can deduce the corresponding angular velocity ω by writing $\frac{1}{2}J_O\omega^2 = Mg\frac{L}{2}$, where $J_O = \frac{1}{3}ML^2$ denotes the moment of inertia of the rod in O. The associated centripetal force in O can be written $M\frac{L}{2}\omega^2$, which leads to, the weight of the chain being taken into account,

$$F_y^{max} \simeq \frac{5}{2}Mg \qquad (x_0 \to L). \tag{3}$$

This latter expectation, which applies as long as the chain does not deform significantly (solid rotation around O) close to the maximum extension, is in fair agreement with the experimental limit (Figs. 4 & 6).

D. Maximum value of the horizontal component of the force and associated time

The two arguments presented above for estimating the limiting values of F_y^{max} both fail to provide us with estimates of the limiting values of the horizontal component F_x^{max}. On the one hand, in the limit $x_0 \to 0$, the motion is strictly vertical and, thus, does not produce any horizontal component of the force in O. On the other hand, in the limit $x_0 \to L$, comparing the motion of the chain to that of a swinging rod, for which the horizontal acceleration of the center of mass vanishes around the vertical position, we predict again F_x^{max} to be zero. These expectations are in contradiction with the experimental observations of finite F_x^{max} for all x_0 (Fig. 4). However, by imaging the system, we note that the chain is almost straight and makes a finite angle with the vertical when the maximum force is measured (Fig. 5).

For small x_0, the vertical component of the force, F_y, reaches its maximum value when the chain fully extends (We remind here that the maximum force does not correspond to the lowest position of the chain). Qualitatively, one can understand that the event is associated with a maximum in the horizontal force in the following way: At $t = t_y^{max}$, the chain tip suddenly goes horizontally from one side of the chain to the other, which results in a sudden horizontal displacement of the center of mass. As a first consequence, for small values of x_0, we expect $t_x^{max} = t_y^{max}$, in agreement with the experimental measurements (Fig. 3). Quantitatively, for $t \simeq t_y^{max}$, a large portion of the chain is almost at rest (OA in Fig. 5). The tension $\vec{T}_A$ of the chain in A results from the sudden tip-acceleration which aligns with the chain tangent in A. In O, the portion OA being almost at rest, $\vec{F} \simeq M\vec{g} + \vec{T}_A$, where we estimate the mass of the segment OA to be about the total mass M of the chain. We thus expect, $F_x^{max} = (F_y^{max} - Mg)\tan\theta$, where θ denotes the angle that the tangent in A makes with the vertical. We report, as a function of the initial spacing x_0, the ratio $r \equiv F_x^{max}/(F_y^{max} - Mg)$ and, as an estimate of $\tan(\theta)$, the tangent of the angle θ_0 that the chain makes with the vertical in O at $t = t_y^{max}$, which we determined experimentally from the images (Fig. 2, inset). In spite of the uncertainty in the mass of the portion OA and in the angle θ, which differs slightly from θ_0, we observe a qualitative agreement of the experimental results with our expectation for $x_0 \leq 0.6$. For larger values of x_0, the arguments presented above do not apply. First, in spite of a loss in the accuracy due to the drastic deacrease of the maximum values of the force components (Fig. 4), we observe a significant difference between t_y^{max} and t_x^{max}. Second, in the limit $x_0 \to L$, we observe that the motion of the chain is similar to that of a swinging rod when the vertical force reaches its maximum value (Fig. 5), which does not permit to distinguish a portion of the chain at rest and a moving tip.

We can wonder why $t_x^{max} < t_y^{max}$. The difference is due to an angular effect. Indeed, considering the tension T_O of the chain and the angle θ_0 that the chain makes with the vertical in O, we expect a maximum in F_x at t_x^{max} given by $\frac{d}{dt}\left(T_O \sin\theta_0\right)|_{t=t_x^{max}} = 0$ and a maximum in F_y at t_y^{max} given by $\frac{d}{dt}\left(T_O \cos\theta_0\right)|_{t=t_y^{max}} = 0$. Let us now assume that T_O is maximum at a given time t_m and denote $\delta t_{x(,y)}^{max} \equiv t_{x(,y)}^{max} - t_m$ and $\omega \equiv -\dot{\theta}_0|_{t=t_m}$ (We remind here that $\dot{\theta}_0 < 0$). The above conditions lead, to the first order in $\omega \delta t_{x(,y)}^{max}$, to:

$$\left(\frac{1}{T_O}\left.\frac{d^2 T_O}{dt^2}\right|_{t=t_m} - \omega^2\right) \sin\theta_0\, \delta t_x^{max} = \omega \cos\theta_0 \tag{4}$$

$$\left(\frac{1}{T_O}\left.\frac{d^2 T_O}{dt^2}\right|_{t=t_m} - \omega^2\right) \cos\theta_0\, \delta t_y^{max} = -\omega \sin\theta_0. \tag{5}$$

Assuming, in addition, that the maximum in T_O corresponds to a pick having the temporal width τ, we can estimate that $-\frac{1}{T_O}\frac{d^2 T_O}{dt^2}|_{t=t_m} \simeq \frac{1}{\tau^2}$. From the equations (4) and (5), we get:

$$t_y^{max} - t_x^{max} \simeq \frac{2\omega\tau^2}{\sin(2\theta_0)}. \tag{6}$$

Thus, because of the significant change in θ_0 during τ, F_x reaches its maximum value before F_y^{max}. For instance, in the limit $x_0 \to L$, we can estimate very roughly that, from the force measurements (Fig. 6), $\tau \sim 0.1\, t_0$ and, from the images (Fig. 5), $\theta_0 \sim 0.2$. Then, taking into account the value of ω used in the section C, we obtain from equation (6) the reasonable estimate $(t_y^{max} - t_x^{max})/t_0 \sim 0.1$ (Fig. 3).

E. Discontinuities of the force components at $t = 0$

At last, we would like to point out the interesting behavior of the force components at $t = 0$. For large initial spacing x_0, one observes a jump in the components (Fig. 6).

The behavior of the vertical component at $t = 0$ can be accounted as follows. Initially, whatever the initial tension imposed by the mechanical equilibrium of the chain hanging between the point at $x = 0$ and $x = x_0$, the mount sustains half the weight at each end. At $t = 0$, the end at $x = x_0$ is released. In the limit of small x_0, about half the chain remains at rest, hanging in O, and no jump in the vertical force is observed (Fig. 2). To the contrary, in the limit $x_0 \to L$, the chain, which remains horizontal in a large part of its length, starts falling with the acceleration of the gravity. In this case the vertical force in O instantaneously vanishes, which corresponds to the sudden variation $\Delta F_y = -Mg/2$ of F_y observed at $t = 0$ (Fig. 6).

The behavior of the horizontal component at $t = 0$ can be understood in the same way, provided the knowledge of the initial horizontal component of the force in O. The initial component F_x can be determined by considering the mechanical equilibrium previous to the release of the end at $x = x_0$. At equilibrium, homogeneously subjected to the acceleration of the gravity along its length the chain takes the shape of a catenary curve which, in cartesian coordinates, can be described by:

$$y(x) = a\left[\cosh\left(\frac{x_0}{2a}\right) - \cosh\left(\frac{2x - x_0}{2a}\right)\right] \tag{7}$$

where the constant a, for a given spacing x_0, is obtained from the length L of the chain :

$$L = \int_0^{x0} \sqrt{1 + y'(x)^2}dx \qquad (x_0 \leq L). \tag{8}$$

Thus, at equilibrium, the angle α that the chain initially makes, at the fixed point O, with the horizontal satisfies :

$$\tan\alpha = y'(0) = \sinh\left(\frac{x_0}{2a}\right). \tag{9}$$

In O, the chain being assumed to be infinitely flexible, $F_y/F_x = \tan\alpha$ and, from the mechanical equilibium along the vertical axis, $F_y = Mg/2$. Thus, the horizontal component of the force:

$$F_x = \frac{Mg}{2\sinh\left(\frac{x_0}{2a}\right)}. \tag{10}$$

In the limit of small x_0, the chain is almost vertical so that $F_x \simeq 0$ at equilibrium and no jump in the horizontal force is observed (Fig. 2). To the contrary, in the limit $x_0 \to L$, the mechanical equibrium imposes a significant initial value of F_x which rapidly decreases once the end in x_0 is released. For instance, for $x_0 = 100$ cm and $L = 102.2$ cm, the equation (8) can be solved numerically and, from equation (10), we get $F_x \simeq 1.35\ Mg$. The experimental sudden decrease of the horizontal force is in quantitative agreement with this last prediction (Fig. 6).

IV. CONCLUSION

This study again illustrates the rich dynamical behavior of a rather simple mechanical system. We reported original time resolved measurements of the horizontal and vertical components of the force applied by the chain to the mount. From simple theoretical arguments, we explain the main characteristics of the force components in the limits of the tightly folded- or maximally stretched-chain. For instance, the maximum in the vertical and horizontal components are not reached at the same times when the initial spacing is large. We propose that the observed time-difference is due to the fact that the chain is not vertical when the maximum tension is reached, which, in addition, also accounts for the amplitude of the maximum in the horizontal component of the force.

We limited our study to a few features of the chain motion, as characterized by the force exerted to the mount. These measurements provide us with insights on a problem that can be described by a set of exact equations, whose analytic solution is not yet available. We are convinced that this experiment, as it, is convenient for illustrating the dynamics of a variable-mass system in a laboratory course.

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Figures

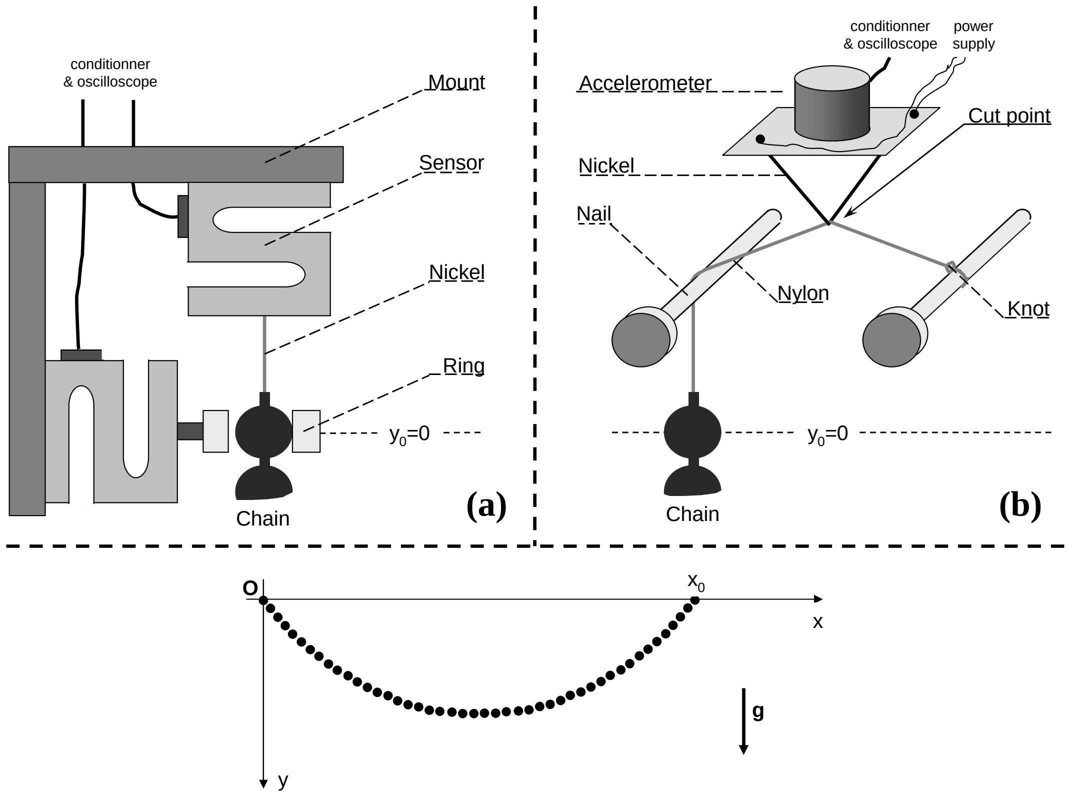

FIG. 1: **Sketch of the experimental setup**. The chain is initially stretched between two points having the same altitude $y_0 = 0$ separated by the distance x_0. (a): Detail of the fastening in O ($x = 0$), where the force is measured. (b): Detail of the fastening at x_0, where the chain is released.

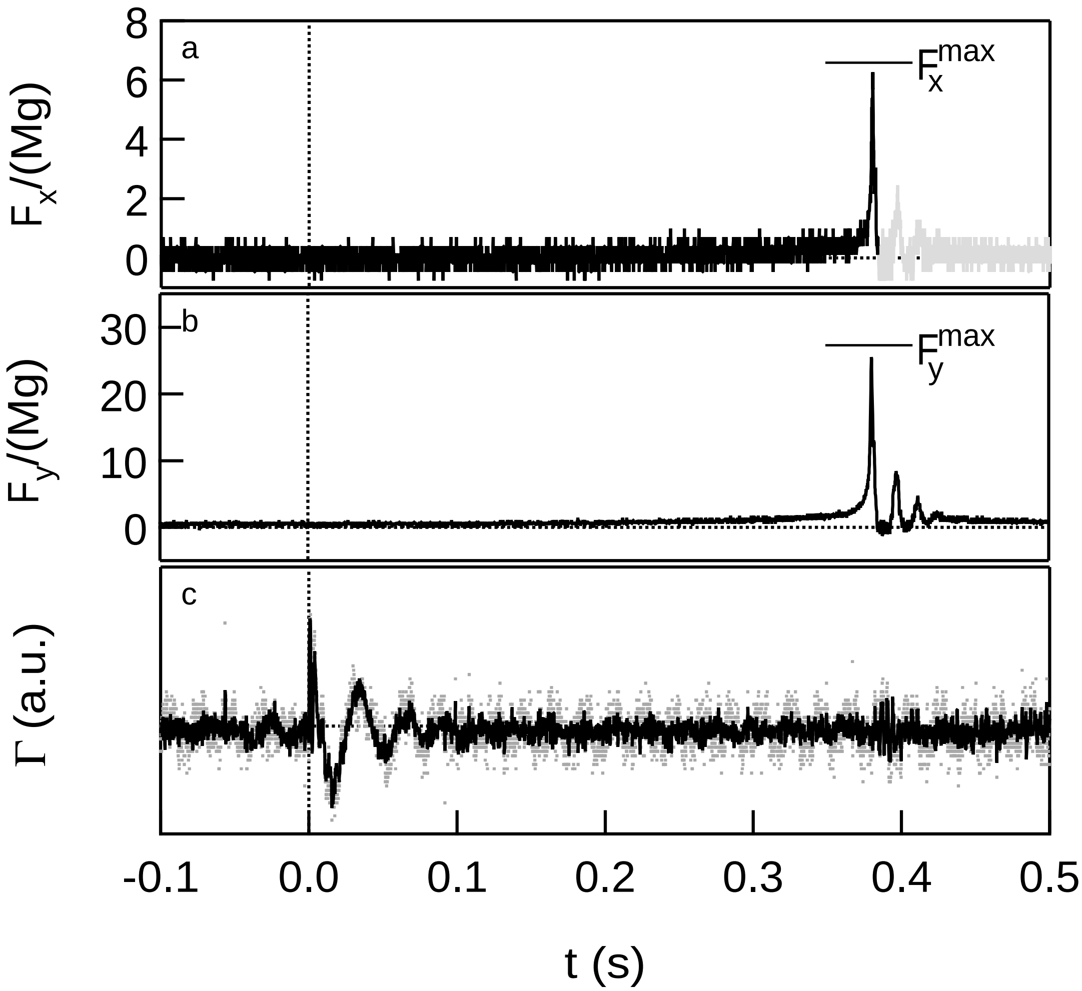

FIG. 2: **Typical signals from the accelerometer and the force sensors**. The fall of the free end produces significant variations of the horizontal (a) and vertical (b) components of the force $\vec{F}$ applied by the chain to the mount. The release of the free end produces a peak in the acceleration Γ from the accelerometer (c), which makes it possible to determine accurately the origin of time. In (a) and (b), we report the forces in units of the chain weight Mg. In (a), we point out that the measurements of the horizontal force is relevant only as long as $F_x > 0$ (black line). Grey line : The chain is likely to have lost contact with the sensor ($F_x < 0$) at least once along its trajectory (Initial spacing $x_0 = 25$ cm).

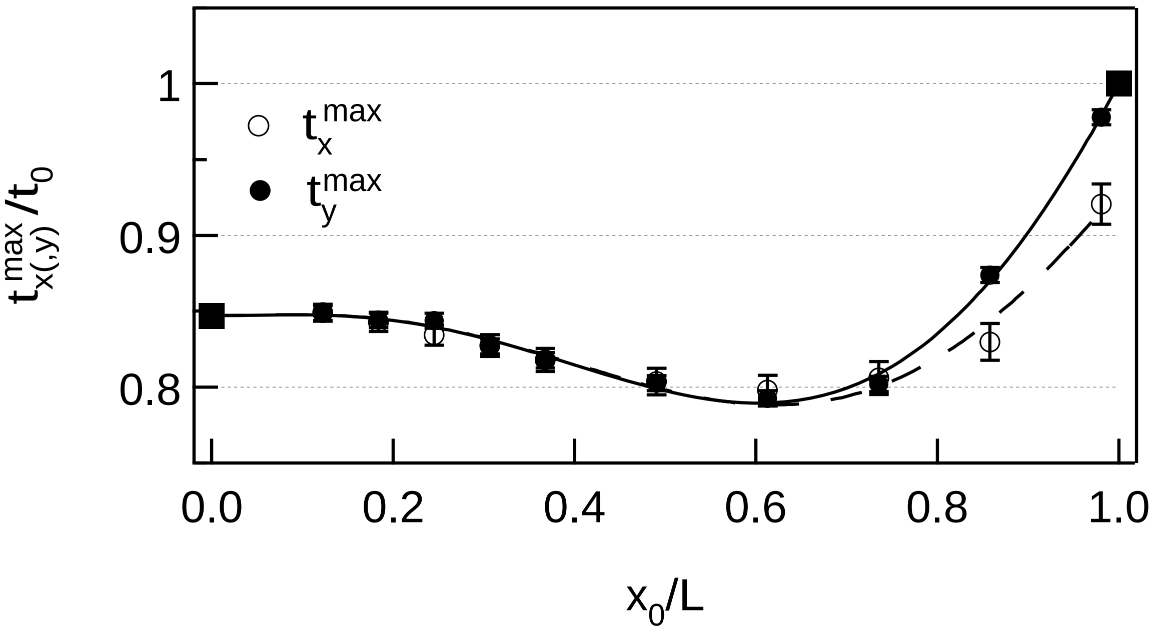

FIG. 3: **Times for the components of the force to reach their maximum value**. The times are reported in units of t_0, the characteritic time of the free fall over the distance L. The lines are polynomial interpolations helping to guide the eye. The full squares at $x_0/L = 0$ and $x_0/L = 1$ are theoretical limits of t_y^{max} (Sec. III B.)

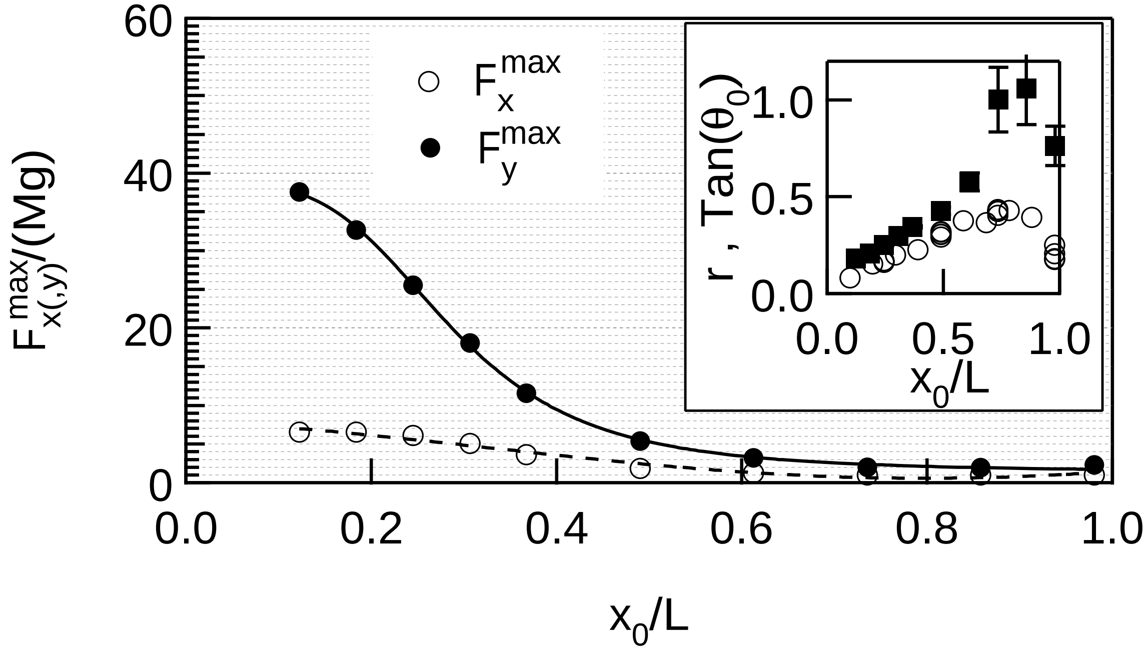

FIG. 4: **Maximum values of the force components**. The maximum values of the force components are reported in units of Mg, the total weight of the chain. The lines are polynomial interpolations helping to guide the eye. **Inset**: $\tan(\theta_0)$ (Open circles) and $r \equiv F_x^{max}/(F_y^{max} - Mg)$ (Full squares) as a function of x_0 (Sec. III D).

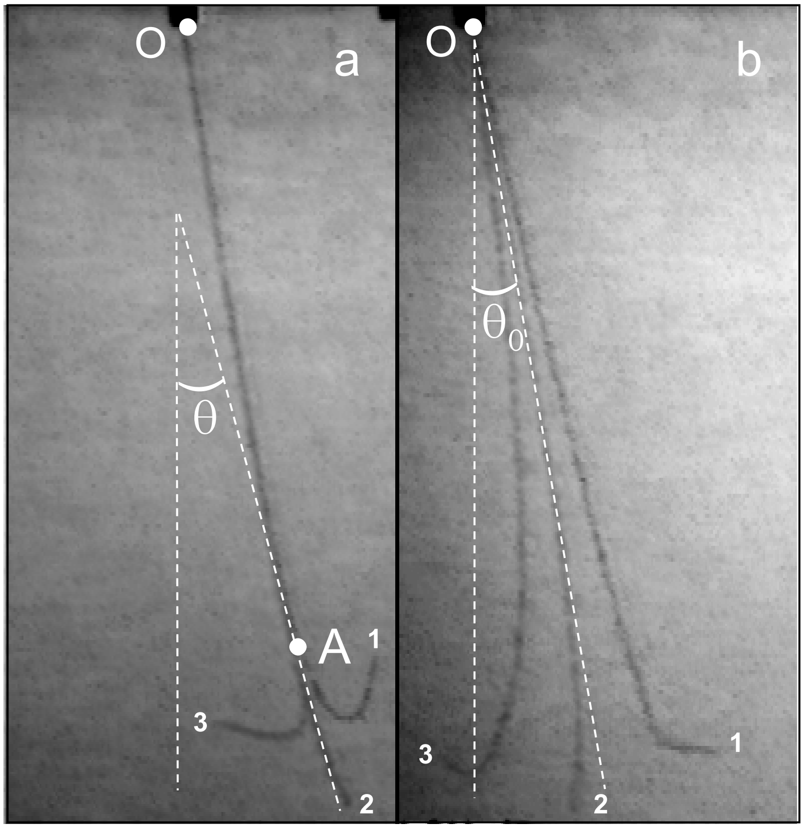

FIG. 5: **Images of the chain for** $t \simeq t_y^{max}$. (a): $x_0 = 25$ cm, (b) $x_0 = 100$ cm. We report in each case the superposition of 3 images separated by 0.02 seconds: 1: before t_y^{max}; 2: the closest to t_y^{max}; 3: after t_y^{max}. In (a), we observe that, for a small initial spacing x_0, a large portion OA of the chain is almost at rest whereas the chain tip is moving fast. We denote θ the angle that the chain makes in A with the vertical. In (b), we observe that, for a large initial spacing x_0, the motion of the chain is very similar to that of a swinging rod. We denote θ_0 the angle that the chain makes in O with the vertical.

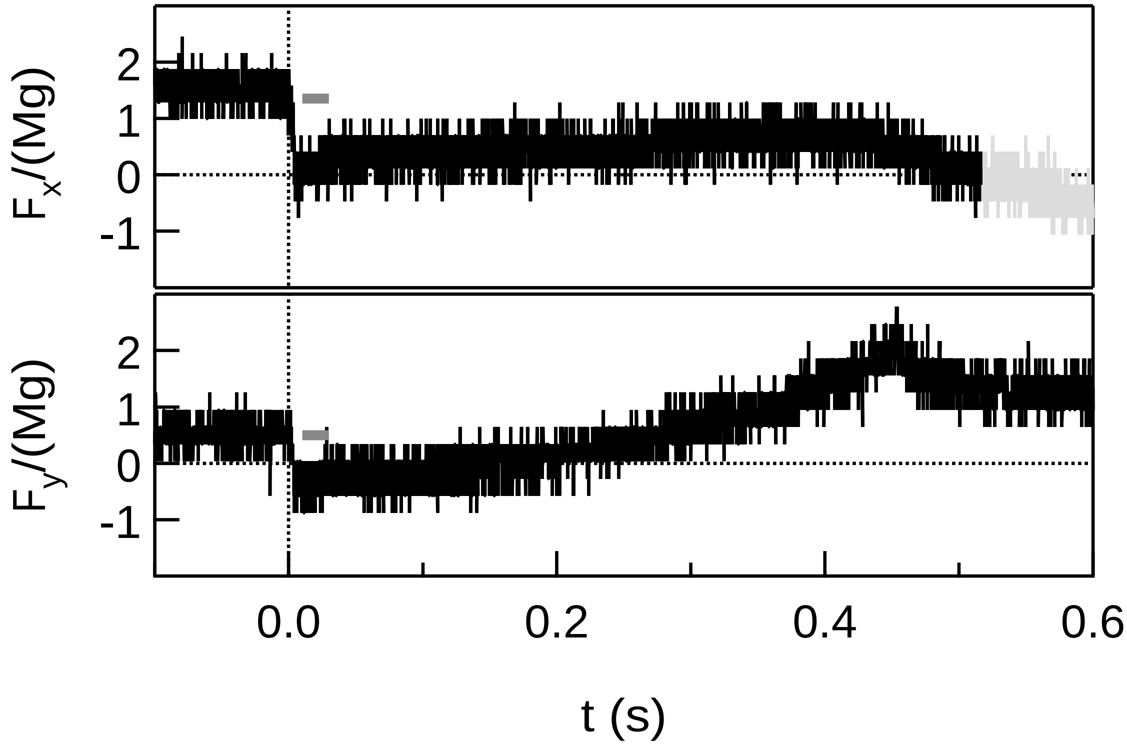

FIG. 6: **Horizontal and vertical components of the force in the limit $x_0 \to L$.** For large initial spacings, one observes at $t = 0$, jumps in the horizontal and vertical components of the force $\vec{F}$. For instance, for $x_0 = 100$ cm, the vertical component jumps from $Mg/2$ to 0 whereas the horizontal component jumps from the value 1.35 Mg, expected from equation (10), to 0 (The thick grey dashes indicate the initial values expected from the theoretical analysis presented in the section III E.) Note that the thickness of the curves is only due to the electronic noise in the signals from the force sensors and that the values right after the jumps are compatible with 0 (The experimental accuracy in the force measurements is discussed in the section II.)