On Sampling Methods for Linear Scale-Invariant Systems

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ABSTRACT

We study a class of self-similar processes that are not stationary, nor have stationary increments. They are called Euler-Cauchy (EC) processes and are built as output of linear scale-invariant parametric systems. This article study several discretization methods of EC processes which are not bandlimited processes: direct sampling, bilinear transformation and approximation on fractional B-splines. For the three different methods, we obtain theoretical formulae and compute numerical realizations and properties.

1. SCALE-INVARIANT PROCESSES

Scale invariance, or self-similarity for random processes, is now a classical property of signals acknowledged as useful to describe classes of real signals with $1/f^{\beta}$ spectrum. A prominent class rely on stationarity of the signals or their increments. Such is the case for fractional Gaussian noise, increments of the celebrated fractional Brownian motions [1], so that sampling and synthesis is straigthforward. Euler-Cauchy (EC) processes [2, 3] are output of linear scale-invariant parametric systems, in the same way as stationary processes can be seen as as outputs of linear time-shift invariant filters. EC processes are self-similar but not stationary, nor do they have stationary increments.

This road to scale invariance was followed for continuous-time processes in several works [2, 3, 4, 5, 6], but less attention has been devoted to their discrete-time formulations. For such non-stationary self-similar processes, it was proposed to work with geometric sampling, for synthesis [2, 5], or analysis [7, 8] but this is not convenient for practical and numerical applications. Another way is to study these systems by means of the Mellin spectral representation [6]. For all those techniques, a step of interpolation is required and it was never checked that the methods were stable through interpolation. Moreover, because those processes are generically not bandlimited, usual Shannon's sampling is not the best way to formulate the corresponding discrete-time system [9, 10].

This article is devoted to the synthesis of EC processes and study several discretizations of EC systems. The paper is organized as follows. Section 2 recalls basic facts about EC models. Section 3 and 4 derives discrete EC model by classical analog-to-digital correspondences: impulse invariance and bilinear transformation. Section 5 proposes a new scheme based on fractional B-splines that were defined in [11]. The results are discussed in each section.

2. CONTINOUS-TIME EULER-CAUCHY MODELS

Let us recall that self-similarity, for a Hurst exponent H, is defined as the statistical identity under dilations. The dilation operator $\mathbf{S}_{H,\lambda}$ acts on a process as $(\mathbf{S}_{H,\lambda}X)(t) = \lambda^{-H}X(\lambda t)$. The covariance $R_X(t,s)$ of a self-similar process has to satisfy: $R_X(\lambda t, \lambda s) = \lambda^{2H} R_X(t,s)$ for any $\lambda \in \mathbb{R}$.

Continuous-time EC(p,q) processes are solutions of

$$\sum_{n=0}^{p} \alpha_n t^n \mathbf{D}^n X(t) = \sum_{m=0}^{q} \beta_m t^m \mathbf{D}^m V_H(t), \qquad (1)$$

for $t > 0$ and with $V_H(t)$ a non-stationnary Gaussian white noise of variance $\mathbb{E}\{V_H(t)\overline{V_H(s)}\} = \sigma^2 t^{2H+1}\delta(t-s)$. Here we write $\mathbf{D}$ the continuous-time derivative. Note that if one considers the time deformation reducing self-similarity to stationary, called the Lamperti transformation [6], it follows immediately that EC models are, for self-similarity and scale covariance, the counterpart of what are usual ARMA models for stationarity and time-shift covariance. The correspondence is obtained by mapping $t^{1-H}\mathbf{D}$ (operator for self-similarity) to $H\mathbf{I} + \mathbf{D}$ under the Lamperti transformation. Our objective is to study the discrete-time equivalent to $t\mathbf{D} + \alpha\mathbf{I}$.

Explicitely, the first order EC model is parametrized as

$$\{t\mathbf{D} + (a-H)\mathbf{I}\}X(t) = V_H(t). \qquad (2)$$

Let us introduce the Green function $G(t,u)$ of the model, defined with initial condition $G(u,u) = 1$ and satisfying: $t\mathbf{D}G(t,u) + (a-H)G(t,u) = \delta(t-u)$. Its expression is, for $t > u$,

$$G(t,u) = (t/u)^{H-a}, \qquad (3)$$

and an expression of the process follows:

$$X(t) = G(t,t_0)X(t_0) + \int_{t_0}^{t} G(t,u)V_H(u)\frac{\mathrm{d}u}{u}. \qquad (4)$$

Let us check explicitely that the process is self-similar. The calculus of the covariance gives

$$\begin{aligned} R_X(t,s) = \; & G(\min(t,s),t_0)\mathbb{E}\left\{X(t_0)^2\right\} \\ & + \int_{t_0}^{\min(t,s)} \left(ts/u^2\right)^{H-a} \sigma^2 u^{2H-1}\mathrm{d}u. \end{aligned} \qquad (5)$$

If the initial condition $X(t_0)$ shares the equilibrium distribution of the process (a normal law with variance $\sigma^2 t^{2H}$) or asymptotically if the system is stable ($G(t, t_0) \longrightarrow 0$ if $(t - t_0) \longrightarrow +\infty$), then the covariance is not affected by the initial condition and the process is self-similar. Let $\lambda > 1$ and $s = \lambda t$, its covariance reads then $R_X(t, s) = \sigma^2 (st)^H \lambda^{-a}$. The process is self-similar with index H. Its variance growths as t^{2H} as it is characteristic for self-similarity, and the covariance decreases in a algebraic decorrelation in λ^{-a}.

Generally, EC processes are parametric models of the general linear scale-invariant models. They act by means of a multiplicative convolution [2, 3]. Higher orders models may be obtained by (multiplicatively) convolving first order EC filters. We thus mainly study this order in the rest of the article.

3. DIRECT SAMPLING OF EC SYSTEMS

In classical textbook on signal processing one learns about the impulse-invariant method as a traditionnal Analog-to-Digital conversion techniques that relies on Shannon's sampling [9]. A direct time-sampling of the continuous-time solution $X(t_k)$ at time $t_k = k\tau$ is used. Let us find the statistics of the quantities obtained for this discrete-time equation that has the form of a non-stationnary AR(1), $x_k = a_k x_{k-1} + e_k$. Using eq. (4), one has

$$a_k = G(k\tau, (k-1)\tau);\ e_k = \int_{(k-1)\tau}^{k\tau} G(k\tau, u) V_H(u) \frac{\mathrm{d}u}{u}. \tag{6}$$

The first term is given by eq. (3), so that $a_k = [k/(k-1)]^{H-a}$. As $V_H(t)$ is Gaussian with zero mean, so is also the input e_k; as $V_H(t)$ is a white noise with variance in t^{2H+1}, e_k is also a white noise and its variance is:

$$\mathbb{E}\{e_k \bar{e}_k\} = \frac{\sigma^2 (k\tau)^{2H}}{2(H+a)} (1 - [(k-1)/k]^{2a+2H}). \tag{7}$$

For this uniformely sampled process, a_k are equivalent, when k is high enough, to $1 - (a-H)/k$ which varies slowly. Note that this would be the coefficient for a backward-difference approximation of eq. (2), by changing $\mathbf{D}$ in $1 - \mathbf{B}$ ($\mathbf{B}$ is the backward operator defined so that $\mathbf{B}x_k = x_{k-1}$). On the whole, it is the non-stationary input e_k which drives the self-similarity of the process, with a variance equivalent to $\mathbb{E}\{e_k \bar{e}_k\} \sim \sigma^2 \tau^{2H} k^{2H-1}$. By combination of the recurrence equation, the covariance is given (if the system is stable so that the initial condition is forgotten), if $m > k$, as $r_x[m, k] = (m/k)^{-a} \mathbb{E}|e_k|^2$. Consequently, for $l \in \mathbb{Z}$, the covariance satisfies $r_x(lm, ln) = l^{2H} r_x(m, n)$ and the process is wide-sense self-similar. The behaviour of the random sequence and its covariance is illustrated on fig. 1.

Let us stress that the input, and not the system, drives the self-similarity of this signal. The discrete-time system is accounts for the algebraic decorrelation of the process. This is not entirely satisfactory, because we would like to have a system modeling non-stationarity. This will be achieved with a different A-to-D correspondence.

Note that for higher order, the same calculation is formally possible. For instance, a general expression of a sampled EC(p, 0) is (P_{p-2} is a polynomial of order up to $p - 2$ with time-varying coefficients):

$$(1-\mathbf{B})^p x_k + \frac{c_1}{k}(1-\mathbf{B})^{p-1} x_{k-1} + \frac{c_2}{k^2} P_{p-2}(\mathbf{B}) = e_k. \tag{8}$$

These discrete models share essentially the same behaviour as the EC(1): the coefficients are slowly varying with time and the self-similarity is mainly driven by the input noise. Another developement is to break the self-similarity by supposing discrete scale invariance (DSI), as in [13]. This is achieved by taking coefficients $a(t)$ and $\sigma(t)$ (variance of the input noise) as periodic functions in $\log t$. The same procedure leads to DSI with a log-periodic function multiplying the continous Green function that was used before. Hence the discrete coefficient a_k will be multiplied by a periodic function in $\log k$, whereas e_k is mostly unaffected. But such kind of generalization beyond simple scale invariance are minimal in fact for this discretization scheme: it comes as a small order pertubation of the mean self-similatity imposed by the noise.

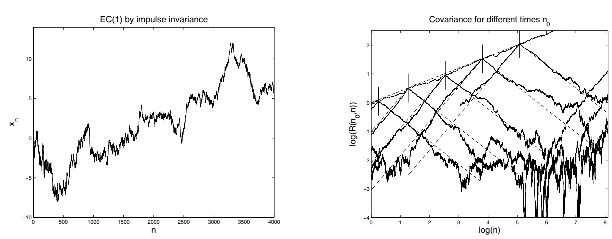

Fig. 1. **Left:** snapshot of a discrete EC(1). **Right:** covariance $r_x[n, n_0]$ of the EC(1) for several times n_0 (marked by vertical bars) and variance $r_x[n, n]$ (log-log). Averages of 1024 realizations.

4. BILINEAR TRANSFORMATION OF EC SYSTEMS

A second classical technique of A-to-D conversion is the bilinear transform that is defined via an invertible rule of correspondence between the Laplace transform (p is the Laplace variable) and the z transform.

$$p \longleftrightarrow \frac{2}{\tau}\ \frac{1 - z^{-1}}{1 + z^{-1}} \text{ and } z^{-1} \longleftrightarrow \frac{1 - p\tau/2}{1 + p\tau/2}. \tag{9}$$

In the frequency domain, with $\Omega \in \mathbb{R}$ the frequency associated to continuous time and $z = e^{i2\pi\tau\omega},\ \omega \in [-0.5\ 0.5]$, the correspondence reads $\Omega = f(\omega) = \frac{2}{\tau}\tan(\omega\tau/2)$. It was proposed in [12] to use this transform to define discrete-time dilation, then discrete-time scale invariant stationary processes which are stationary processes. Here we use the transform only for A-to-D conversion of the operator; this leads to self-similar but non-stationary sequences. A kernel representation of the transform is

$$
\begin{aligned}
&X(t) = (\mathbf{R}x_n)(t) = \textstyle\sum_{n=-\infty}^{\infty} P(t,n)x_n \\
&P(t,n) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left\{i(\Omega t - f^{-1}(\Omega)n\tau)\right\} \mathrm{d}\Omega, \\
&x_n = (\mathbf{R}^{-1}X(t))[n] = \textstyle\int_{-\infty}^{\infty} s(n,t)X(t)\mathrm{d}t \\
&s(n,t) = \frac{1}{2\pi}\int_{-1/2}^{+1/2} \exp\left\{i(\omega n\tau - f(\omega)t)\right\} \mathrm{d}\omega.
\end{aligned}
\tag{10}
$$

Using a stationary phase approximation, it is straightforward to establish an approximation of the kernels P and s, that are given as chirps with instantaneous frequency $\sqrt{(n/t-1)}/\pi$:

$$
\begin{aligned}
&P(t,n) \overset{t<n}{\simeq} \tfrac{1}{\sqrt{2\pi}} \tfrac{n^{1/2}}{t^{3/4}(n-t)^{1/4}} \cos\left(\varphi(n,t)\right). \\
&s(n,t) \overset{t<n}{\simeq} \tfrac{2}{\sqrt{2\pi}} \tfrac{t^{1/4}}{n^{1/2}(n-t)^{1/4}} \cos\left(\varphi(n,t)\right), \\
&\varphi(n,t) = 2n\mathrm{acos}\left(\sqrt{t/n}\right) - 2\sqrt{t(n-t)} - \pi/4.
\end{aligned}
\tag{11}
$$

When n is near t, a cut-off by an erf function (that we do not report here for the sake of simplicity) puts the chirp to zero. The kernels are drawn on fig 2-a. Any (non necessarily shift-invariant) linear operator is mapped from continuous time to discrete time using those kernels. For a linear operator $\mathbf{A}$ with integral representation $(\mathbf{A}\cdot Y)(t) = \int A(t,u)Y(t)\mathrm{d}t$, the discrete-time representation is $(\mathbf{a}\cdot y)[k] = \sum_k a[k,m]y_m$ such that $\mathbf{A} = \mathbf{R}\mathbf{a}\mathbf{R}^{-1}$. Then it comes

$$
a[k,m] = \int_0^k \mathrm{d}t\, s(k,t) \int_0^{\min(t,m)} A(t,u)P(u,m)\mathrm{d}u.
\tag{12}
$$

The linear kernel for an EC(1), eq. (3), is equal to $A(t,u) = (t/u)^{H-a}/u$. The discretized EC(1) is obtained here as a non-stationary mean-averaged representation. y_k is a Gaussian, nonstationary iid noise that is given by $y_k = \int s(k,t)V_H(t)\mathrm{d}t$. Its variance scales as $\mathbb{E}\left\{y_k^2\right\} \propto k^{1+2H}$. The kernel $a[k,m]$ is correctly approximated, if $k > m$, by $k^{H-a}m^{a-H-0.5}$, and 0 else; this is represented on fig. 2-b (top). The covariance follows immeadiately and it scales as $r_x[m,n] \simeq (mn)^H(m/n)^{-a}$. This scheme gives a process that shares the properties of the previous one and the realizations of the process look the same; see fig. 2-c for an illustration.

A main interest of this method of discretization is that one can use, instead of $G(t,u)$, any multiplicative kernel that is a function of $f(t/u)$. The method is not restricted to usual EC systems this allows us to study in the same framework discrete-time EC sequences of any order, or EC with non-stationary coefficients. For instance, the sequence shown in fig. 2-d has $f(t) = t^{-a}(1 + b\cos(\pi\log(t))$ and its kernel is shown on 2-b (bottom). This function to Discrete Scale Invariance. Thus this model offers a versatile discrete-time framework. The price to pay is that an numerical intregration of eq. (12) is then usually necessary to obtain the kernel. This is not very efficient for computations of large sequences.

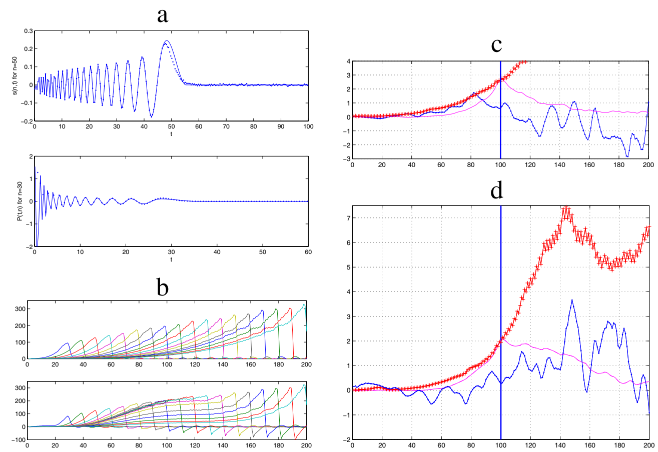

Fig. 2. Bilinear transformation. **Left:** a – approximation of eq. (11) (solid lines) and exact value (dots) for $s(50,t)$ (top) and $P(t,30)$ (bottom). b – $a[k,m]$ as function of m for $k = 30, 40, \ldots, 200$ for an EC(1) (top) and a time-varying EC (see text) (bottom). **Right:** snapshots (blue dots), variance (red crosses) and covariance $r_x[100,n]$ (solid line in magenta) of random sequences, c – EC(1), d – EC with DSI kernel. Averages on 1024 realizations.

5. DISCRETE EC MODEL BY FRACTIONAL B-SPLINE REPRESENTATION

Due to the non-bandlimited property of the continuous scale invariant signals, generalized sampling, as reviewed in [10], is an alternative solution for the problem of representation of a continuous model by a discrete sequence. For discretization, cardinal basis defined on a uniform grid are adapted. As the Green function of EC systems are usually power-laws, a class of B-splines recently introduced in [11] is relevant to the problem: the fractional B-splines. After a brief recall of their properties, a discrete EC model is developped on this basis.

Define the one-sided power functions as $(t)_+^\alpha = t^\alpha$ if $t > 0$, else 0. A fractional causal B-spline $\beta_+^\alpha(t)$ is defined by taking the fractional difference operator of the one-sided power functions. Recalling that $\Gamma(u) = \int_0^\infty x^{u-1}e^{-x}\mathrm{d}x$ and $\binom{\alpha}{k} = \Gamma(\alpha+1)/\Gamma(k+1)\Gamma(\alpha-k+1)$, we have

$$
\beta_+^\alpha(t) = \Delta_+^{\alpha+1}(t)_+^\alpha = \frac{1}{\Gamma(\alpha+1)}\sum_{k\geq 0}(-1)^k\binom{\alpha+1}{k}(t-k)_+^\alpha.
\tag{13}
$$

Fractional B-splines have a Fourier transform in $\omega^{-\alpha-1}$ and are so good candidates for approximation of self-similar signals. Any signal $X(t)$ can be approximated in the fractional spline space of order α as:

$$
X_{s.\alpha}(t) = \sum_{k\in\mathbb{Z}} c_k\beta_+^\alpha(t-k).
\tag{14}
$$

It is known that the reproduction is exact for polynomials up to order $\lceil\alpha\rceil$. More generally, the approximation order was established in [11]. Here the sequence $\{c_k\}_{k\in\mathbb{Z}}$ is used as a discrete-time representation of the signal. Because of the interpolation property, at knots k, the signal satisfies $X_{s.\alpha}(k) = X(t)|_{t=k} = x_k$. Eq.(14) is a convolution; it can be solved in the Fourier domain, using the inverse filter: $1/\beta_+^\alpha(i\omega) = \{i\omega/(1-e^{-i\omega})\}^{\alpha+1}$.

Let the process be approximated as in eq. (14) with order $\alpha - 1$. Fractional B-splines satisfy the induction equation (Prop. 2.2 in [11]):

$$\alpha\beta_+^\alpha(t) = t\beta_+^{\alpha-1}(t) + (\alpha+1-t)\beta_+^{\alpha-1}(t-1). \quad (15)$$

Using the backward operator, this reads as:

$$t(1-\mathbf{B})\beta_+^{\alpha-1}(t) + (\alpha+1)\beta_+^{\alpha-1}(t) = \alpha\beta_+^\alpha(t+1). \quad (16)$$

Combined with eq. (14) for order $\alpha - 1$, this leads to

$$\begin{aligned}&\{t(1-\mathbf{B}) + (\alpha+1)\}X_{s.(\alpha-1)}(t)\\ &= \textstyle\sum_{k\in\mathbb{Z}} c_k(\alpha\beta_+^\alpha(t-k+1) - k\beta_+^{\alpha-1}(t+k)).\end{aligned} \quad (17)$$

The l.h.s. is taken as the discretized first order EC operator in the space of representation. Note that for EC systems, one is interested in $t\mathbf{D}$ operator and not in $\mathbf{D}^\alpha$; hence there would no reason to use the fractional difference Δ_+^α and our choice appears natural. Comparing with eq. (2), the sequence c_k is obtained by the decomposition of the input white noise $V_H(t)$ on fractional B-spline. Specifically:

$$V_H(t)|_{t=k} = c_m \otimes \{\alpha\beta_+^\alpha(m+1) - m\beta_+^{\alpha-1}(m)\}[k], \quad (18)$$

where $\otimes$ stands for the convolution. Numerically, this equation is solved in the discrete Fourier domain with a stationary white random sequence ε_k because $V_H(t)|_{t=k} = k^{H+1/2}\varepsilon_k$. The process is then constructed by interpolation, $X_{s.(\alpha-1)}(t) = \sum_{k\in\mathbb{Z}} c_k\beta_+^{\alpha-1}(t-k)$. Fig. 3 shows a sample realization, its variance. The process has a variance that grows in t^{2H}, and decorrelates algebraically with exponents $\alpha + H + 1$.

An advantage of this discretization is that the sequence is synthetised from digital signal processing, with its natural interpolating function if needed. The procedure is more quicker than the one from the bilinear transformation. Moreover general tools of signal processing are easily applied to the sequence by working on c_k, according to the rules of general sampling. The sequence is obtained by the scheme with a given time resolution. A perspective would be to use the two-scale relation satisfied by fractional B-splines [11] could offer the possibility to refine the details at smaller time-scales, but this was not studied here. Another development would be to find a way to include in those models time-varying coefficients (in order to have DSI for instance) in the framework.

The three means to build discrete-time models for scale invariant Euler-Cauchy systems studied here are by now complementary depending on the refinements needed. As a final word, let us remark that an intricate property of these models is that they have no kind of stationarity. As such the wavelet methods, that transform a H-ss process with stationary increments in a stationary decomposition at each scale, is not useful to test their scale invariance, because the wavelet coefficients at one scale are nonstationary. That is why we have preferred here to show scale invariance directly from the covariances.

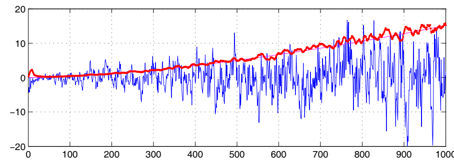

Fig. 3. Ec model on fractional B-spline. A snapshot of the process with $H = 0.8$ and $\alpha = 0.3$, superimposed with its standard deviation (dots on dashed line, estimated on 1024 realizations;).

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