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Riemann–Hilbert approach to a generalised sine kernel and applications

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Abstract

We investigate the asymptotic behaviour of a generalised sine kernel acting on a finite size interval $[\,-q\,;q\,]$. We determine its asymptotic resolvent as well as the first terms in the asymptotic expansion of its Fredholm determinant. Further, we apply our results to build the resolvent of truncated Wiener–Hopf operators generated by holomorphic symbols. Finally, the leading asymptotics of the Fredholm determinant allows us to establish the asymptotic estimates of certain oscillatory multidimensional coupled integrals that appear in the study of correlation functions of quantum integrable models.

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1 Introduction

The sine kernel

$$S(\lambda,\mu) = \frac{\sin\frac{x}{2}(\lambda-\mu)}{\pi(\lambda-\mu)}, \tag{1.1}$$

is a very important object in mathematical physics. In particular, the Fredholm determinant of the integral operator $I - S$ acting on some interval $J \subset \mathbb{R}$ appears in random matrix theory [21]. In the bulk scaling limit, $\det_J[I-S]$ stands for the probability [22] that a matrix belonging to the Gaussian unitary ensemble has no eigenvalues in xJ. The kernel (1.1) also appears in the theory of quantum integrable systems. In particular, the determinant $\det_J[I+\gamma S]$, γ being a parameter, describes various zero–temperature correlation functions of the impenetrable Bose gas [40, 33].

In all these interpretations of the sine kernel, one is interested in the large x behaviour of its Fredholm determinant. The first attempt to analyze the $x \to +\infty$ asymptotics of $\det_J[I-S]$ goes back to Gaudin and Mehta [21, 22]. In 1973, Des Cloizeaux and Mehta [18] showed that

$$\log\det_{[-1;1]}[I-S] = -\frac{x^2}{8} - \frac{1}{4}\log x + \mathrm{O}(1), \qquad x \to +\infty\,. \tag{1.2}$$

Three years later, using Widom's formula [44] for the asymptotics of Toeplitz determinants supported on an arc, Dyson [19] gave a heuristic derivation of the constant terms c_0 and proposed a recursive method to compute the subleading coefficients $c_1, c_2, \ldots$ in the asymptotic expansion:

$$\log\det_{[-1;1]}[I-S] = -\frac{x^2}{8} - \frac{1}{4}\log x + c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \ldots\,. \tag{1.3}$$

However, the forementioned results were heuristic. It was only in 1994 that Widom [45] managed to prove rigorously the first term in the asymptotics (1.2):

$$\frac{\mathrm{d}}{\mathrm{d}x}\log\det_{[-1;1]}[I-S] = -\frac{x}{4} + \mathrm{o}(1)\,. \tag{1.4}$$

One year later, this analysis was extended to the multiple interval case [46]. While Widom studied the asymptotic behaviour of the Fredholm determinant by operator techniques, Deift, Its and Zhou applied the Riemann–Hilbert problem (RHP) formulation for integrable integral operators [30] to the sine kernel acting on a union of intervals $\cup_\ell J_\ell$ and proved the existence of the asymptotic expansion (1.3). However, their method did not allow them to obtain an estimate for the constant c_0, as they inferred the asymptotic expansion of $\log\det_J[I-S]$ from that of

$$P_x = x\frac{\mathrm{d}}{\mathrm{d}x}\log\det[I-S]\,. \tag{1.5}$$

The first proofs of Dyson's heuristic formula for c_0 appeared in the independent, and based on completely different methods, works of Ehrhardt [20] and Krasovsky [38] and more recently in [15].

We would like to point out that there is a very nice connection of the sine kernel to the Painlevé V equation [33], as P_x solves this equation. The link between Painlevé V and P_x

was also investigated in [13] in the framework of RHP. It was shown that one can deduce this Painlevé equation directly from the RHP data.

This article is devoted to the study a generalisation of the sine kernel (1.1). This kernel, that we will refer to as the generalised sine kernel (GSK), is of the form

$$V(\lambda,\mu) = \frac{\gamma\sqrt{F(\lambda)F(\mu)}}{2i\pi(\lambda-\mu)}[e_+(\lambda)e_-(\mu) - e_-(\lambda)e_+(\mu)], \tag{1.6}$$

where

$$e_\pm(\lambda) = \mathrm{e}^{\pm[ixp(\lambda)+g(\lambda)]/2}. \tag{1.7}$$

We will be more specific about the functions F, p and g later on.

Various particular cases of the kernel (1.6) already appeared in the literature. These particular kernels were mostly used for the description of correlation functions of matrix models or quantum integrable models equivalent to free fermions (see e.g. [43, 41, 27, 28, 36, 9, 10, 29, 31, 8]). In the present paper we consider a rather general case, only based on the analytic properties of the functions F, p and g. The GSK (1.6) plays a crucial role in the study of correlation functions of (non free-fermion) quantum integrable systems [35]. It is also useful for the asymptotic analysis of truncated Wiener–Hopf operators with Fischer–Hartwig singularities [37].

We investigate here the large x asymptotic behaviour of the Fredholm determinant of the GSK in the framework of RHP. Our work is a natural extension of an unpublished analysis by Deift, Its and Zhou of the sine kernel $I + \gamma S$ by RHP. This kernel was also analysed by RHP in [8].

This article is organized as follows. In Section 2, we announce the main results of the paper, namely,

- the large x asymptotic behaviour of the Fredholm determinant of the integral operator $I + V$, *cf.* (1.6);
- the asymptotic resolvent of some Wiener–Hopf operators connected to (1.6);
- the asymptotic behaviour of coupled multiple integrals involving a cycle of kernels V (1.6) versus some holomorphic symmetric functions.

The proof of the asymptotic behaviour of $\log\det[I + V]$ is given in the core of the paper (Sections 3, 4, 5 and 6). More precisely, in Section 3, we recast the problem into a certain RHP. In Section 4, we transform this initial RHP into a RHP that can easily be solved asymptotically. This asymptotic solution is presented in Section 5 and used in Section 6 to obtain the leading and the first subleading terms of $\log\det[I + V]$ in the $x \to +\infty$ limit.

In Section 7, we apply these results to truncated Wiener–Hopf operators. We show how one can use the asymptotic resolvent of the generalised sine kernel to construct asymptotic resolvents of truncated Wiener–Hopf operators acting on $[-x\,;x]$, with x large. This asymptotic

resolvent is used to reproduce the low magnetic field behaviour of the so-called dressed charge arising in the theory of quantum integrable models solvable by the Bethe ansatz [6].

Section 8 is devoted to the study of the asymptotic behaviour of some particular type of coupled multiple integrals which can be obtained in terms of the GSK. This is in fact our main motivation to study the GSK: indeed, from the knowledge of the asymptotic behaviour of this type of multiple integrals one can obtain the asymptotic behaviour of quantum integrable models correlation functions, as it is done in [35].

Finally, in Section 9, we consider the case of further modifications of the GSK, in particular those useful for the correlation functions of the integrable Heisenberg spin chains [35].

Some properties of confluent hypergeometric functions and proofs of several lemmas are gathered in the appendices.

2 Problem to solve and main results

2.1 generalised sine kernel: assumptions and notations

Let $I + V$ be the integral operator with kernel (1.6) and acting on $L^2([-q\,;q\,])$.

We assume that there exists some open relatively compact neighbourhood U of $[-q\,;q\,]$ such that the functions p, F and g, as well as the parameter γ, satisfy the following properties:

- F and g are holomorphic on $\overline{U}$, the closure of U;
- p is holomorphic and injective on $\overline{U}$, $p([-q\,;q\,]) \subset \mathbb{R}$, and p stabilizes the upper half plane $\mathcal{H}_+$ (resp. the lower half plane $\mathcal{H}_-$), i.e. $p(U \cap \mathcal{H}_\pm) \subset \mathcal{H}_\pm$;
- $\gamma \in D_{0,r} = \{\lambda \in \mathbb{C} : |\lambda| < r\}$, where r is such that $|rF| < 1$ and $\arg(1 + \gamma F) \in]-\pi\,;\pi[$ on $\overline{U}$.

We study the large x expansion of the Fredholm determinant of $I + V$ under these assumptions. This will be done by asymptotically solving a certain matrix RHP. It will become clear in the next section that the assumption $p([-q\,;q\,]) \subset \mathbb{R}$ is tantamount to imposing the associated RHP to be of oscillatory nature. Moreover, the case $p(U \cap \mathcal{H}_\pm) \subset \mathcal{H}_\mp$ is obtained by the negation $(\gamma, g(\lambda)) \mapsto (-\gamma, -g(\lambda))$.

Note that γ plays here the role of a regularisation parameter; in particular it should be stressed that our method does not allow to reach the $|\gamma F| = 1$ case corresponding to (1.3) which requires a different analysis [20, 38, 15].

Before presenting the main result of this article, let us introduce some convenient notations. First, we define two auxiliary functions used in the article:

$$\nu(\lambda) = \frac{-1}{2i\pi} \log(1 + \gamma F(\lambda)) \ , \tag{2.1}$$

$$\kappa(\lambda; q) \equiv \kappa(\lambda) = \exp\left\{ \int_{-q}^{q} \frac{\nu(\lambda) - \nu(\mu)}{\lambda - \mu} \mathrm{d}\mu \right\} \ . \tag{2.2}$$

Note that κ is a function of the two parameters λ and q, although we will sometimes omit the dependence on the second parameter.

Finally, we will use the following simplified notations for the values of the functions p and ν and of their derivatives at the points $\pm q$:

$$p_\pm = p(\lambda)\Big|_{\lambda=\pm q}, \qquad p'_\pm = p'(\lambda)\Big|_{\lambda=\pm q}, \qquad etc. \tag{2.3}$$

$$\nu_\pm = \nu(\lambda)\Big|_{\lambda=\pm q}, \qquad \nu'_\pm = \nu'(\lambda)\Big|_{\lambda=\pm q}, \qquad etc. \tag{2.4}$$

2.2 The main results

We now give the asymptotic behaviour of the Fredholm determinant in the $x \to +\infty$ limit:

Theorem 2.1. *Let V be the GSK* (1.6) *with p, g, F and γ satisfying the assumptions of Section 2.1. Then, in the $x \to +\infty$ limit,* $\log\det[I+V]$ *behaves as*

$$\log\det[I+V] = \log\det[I+V]^{(0)} + \mathrm{o}(1), \tag{2.5}$$

with

$$\begin{aligned}\log\det[I+V]^{(0)} = -ix\int_{-q}^{q}\nu(\lambda)p'(\lambda)\,\mathrm{d}\lambda - (\nu_+^2+\nu_-^2)\log x - \int_{-q}^{q}\nu(\lambda)g'(\lambda)\,\mathrm{d}\lambda \\ + \log\left[\frac{G(1,\nu_+)\,G(1,\nu_-)\,\kappa^{\nu_+}(q;q)}{(2qp'_+)^{\nu_+^2}\,(2qp'_-)^{\nu_-^2}\,\kappa^{\nu_-}(-q;q)}\right] + \frac{1}{2}\int_{-q}^{q}\mathrm{d}\lambda\,\mathrm{d}\mu\,\frac{\nu'(\lambda)\nu(\mu)-\nu(\lambda)\nu'(\mu)}{\lambda-\mu},\end{aligned} \tag{2.6}$$

in which we have used the notations of Section 2.1. The Barnes G-function [3, 2] admits the integral representation:

$$G(z+1) = (2\pi)^{\frac{z}{2}}\exp\left\{-\frac{z(z-1)}{2} + \int_0^z t\psi(t)\,\mathrm{d}t\right\}, \quad \Re(z) > -1, \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{2.7}$$

and we denote $G(1,z) \equiv G(1+z)G(1-z)$.

Using the perturbation theory for singular integral equations one can refine the theorem and obtain sub-leading corrections. Although, in principle, nothing opposes to derive the next sub-leading corrections, the computations become more and more involved. We have proved the structure of the first corrections to the equation (2.6).

Proposition 2.1. *Let V be the GSK* (1.6) *with the conditions of Section 2.1. The leading asymptotics* $\log\det[I+V]^{(0)}$ *of* $\log\det[I+V]$ *as defined in Theorem 2.1 has non-oscillating and oscillating corrections.*

Let $0 < \delta < q$ be such that the disks $D_{\pm q,\delta}$ of radius δ centered at $\pm q$ fulfill $D_{\pm q,\delta} \subset \overline{U}$. Let $\overline{\varepsilon} = 2\sup_{\partial D_{q,\delta}\cup\partial D_{-q,\delta}}\left|\Re(\nu)\right|$. Then the first non-oscillating corrections are of the form

$$\frac{N_1}{x} + \mathrm{O}\left(\frac{1}{x^{2(1-\overline{\varepsilon})}}\right), \tag{2.8}$$

with

$$N_1 = i \sum_{\sigma=\pm} \frac{\nu_\sigma^2}{p'_\sigma} \left\{ 2\sigma\nu'_\sigma \log x + \sigma \frac{\mathrm{d}}{\mathrm{d}q} \log u_\sigma + p'_\sigma \frac{\mathrm{d}}{\mathrm{d}q} \left(\frac{\nu_\sigma}{p'_\sigma} \right) - \frac{\nu_{-\sigma}}{q} \right\}. \tag{2.9}$$

The first oscillating corrections are of the form

$$\frac{O_1}{x^2} + \mathrm{O}\left(\frac{1}{x^{3(1-\overline{\varepsilon})}} \right). \tag{2.10}$$

and the leading oscillating coefficient is given by

$$O_1 = \frac{\nu_- \nu_+}{(2q)^2 \; p'_+ \, p'_-} \sum_{\sigma=\pm 1} \left(\frac{u_+}{u_-} \right)^\sigma x^{2\sigma(\nu_+ + \nu_-)} \, \mathrm{e}^{i\sigma x(p_+ - p_-)} \, , \tag{2.11}$$

where we have introduced

$$u_+ = e^{g(q)} \, \frac{\Gamma(1-\nu_+)}{\Gamma(1+\nu_+)} \left\{ \frac{(2qp'_+)^{\nu_+}}{\kappa(q;q)} \right\}^2 , \tag{2.12}$$

$$u_- = e^{g(-q)} \, \frac{\Gamma(1+\nu_-)}{\Gamma(1-\nu_-)} \left\{ (2qp'_-)^{\nu_-} \kappa(-q;q) \right\}^{-2} . \tag{2.13}$$

Remark 2.1. The GSK depends only on the combination $ixp(\lambda) + g(\lambda)$ (see (1.7)). Therefore the Fredholm determinant and its asymptotics can only depend on this combination. This observation allows us to obtain the complete asymptotic expansion depending on the function $g(\lambda)$ from the asymptotic expansion of the Fredholm determinant $I + V$ corresponding to $g = 0$. Namely, it is enough to replace in the obtained formulae $p(\lambda)$ by $p(\lambda) - \frac{i}{x} g(\lambda)$ and then expand into negative powers of x.

It is quite interesting to apply the latter proposition in order to obtain the first few terms of the asymptotic expansion of $\det[I + V]$. The reason why we draw the reader's attention to these asymptotics is because they present a very interesting structure: the leading oscillating terms in the asymptotic expansion are just given by the sum of the leading asymptotics evaluated at ν shifted by 1 or -1. This structure of the asymptotics seems to restore, at least partly, the original periodicity $\nu \to \nu + n$, $n \in \mathbb{Z}$, of the Fredholm determinant of $I + V$.

Corollary 2.1. *Let I+V be the GSK as above,* $\det[I + V]^{(0)}[\nu]$ *the leading asymptotics of its Fredholm determinant just as in Theorem 2.1,* N_1 *and* O_1 *as in Proposition 2.1. Note that we have emphasized the structure of* $\det[I + V]^{(0)}[\nu]$ *as a functional of* ν*. Then the oscillating corrections* O_1 *can be reproduced from the non-oscillating part via the shift of* ν *by* ± 1*:*

$$\det[I + V]^{(0)}[\nu] \, \frac{O_1}{x^2} = \det[I + V]^{(0)}[\nu + 1] + \det[I + V]^{(0)}[\nu - 1] \; . \tag{2.14}$$

This structure of the first terms of the large x asymptotic expansion for $\det[I + V]$ leads us to raise the following conjecture on the structure of the asymptotic series :

Conjecture 2.1. *The asymptotic expansion of the Fredholm determinant* $\det[I + V]$ *of the GSK restores the periodicity* $\nu \to \nu + n$, $n \in \mathbb{Z}$, *of the determinant. In particular, this asymptotic expansion contains all the* $\mathbb{Z}$*-periodized terms with respect to* ν *of the leading asymptotics*

$\det[I+V]^{(0)}[\nu]$. *Thus, all the oscillating terms can be deduced from the non-oscillating ones. More precisely, let*

$$\mathcal{A}[\nu](x) \sim \det[I+V]^{(0)}[\nu]\left(1+\frac{C_1(\log x)[\nu]}{x}+\cdots+\frac{C_M(\log x)[\nu]}{x^M}+\dots\right) \tag{2.15}$$

stand for the formal asymptotic series corresponding to the non-oscillating part of the asymptotic series for $\log\det[I+V]$. *There* $C_k(X)[\nu]$ *are polynomials of degree* k *in* X *whose coefficients are functionals in* ν. *Moreover each of the* C_k*'s has no oscillating exponents of the type* $\mathrm{e}^{\pm ixp_\pm}$. *Then the formal asymptotic series for* $\det[I+V]$ *is given by*

$$\det[I+V] \sim \sum_{n\in\mathbb{Z}} \mathcal{A}[\nu+n](x) \ . \tag{2.16}$$

This conjecture is supported by (2.14) and also by the results of [42] where several subleading corrections to the asymptotics of the Fredholm determinant of the pure sine-kernel were computed.

The first application of the asymptotic behaviour of the GSK we consider in this article concerns the asymptotic inversion of truncated Wiener–Hopf operators. We will prove in Section 7 the following proposition:

Proposition 2.2. *Let* $I+K$ *be a truncated Wiener–Hopf operator on* $]-x\,;x[$, *acting on functions* $g\in L^2(\mathbb{R})$ *as*

$$[(I+K).g](t) = g(t) + \int_{-x}^{x} K(t-t')\,g(t')\,\mathrm{d}t' \ . \tag{2.17}$$

The kernel K *is defined by its Fourier transform* F*:*

$$K(t) = \mathcal{F}^{-1}[F](t) \ , \tag{2.18}$$

and we suppose that there exists $\delta>0$ *such that*

- F *admits an analytic continuation to* $\left\{z : \left|\Im(z)\right| \le \delta\right\}$*;*
- $\xi \mapsto F(\xi \pm i\delta) \in L^1(\mathbb{R})$*;*
- *the analytic continuation of* $1+F$ *never vanishes for* $\left|\Im(z)\right| \le \delta$.

Then the resolvent $I-R$ *of* $I+K$ *fulfills*

$$R(\lambda,\mu) = \int_{\mathbb{R}} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{4\pi^2 i}\,F(\xi)\left\{\frac{\alpha_+(\eta)}{\alpha_-(\xi)}\mathrm{e}^{ix(\xi-\eta)} - \frac{\alpha_+(\xi)}{\alpha_-(\eta)}\mathrm{e}^{-ix(\xi-\eta)}\right\}\frac{\mathrm{e}^{i(\mu\eta-\lambda\xi)}}{\xi-\eta} + \mathrm{O}\left(\mathrm{e}^{-2\delta x}\right), \tag{2.19}$$

where $\alpha(\lambda)$ *is given by*

$$\alpha(\lambda) = \exp\left\{-\frac{1}{2i\pi}\int_{\mathbb{R}} \log(1+F(\mu))\,\frac{\mathrm{d}\mu}{\mu-\lambda}\right\}. \tag{2.20}$$

Our main motivation to study the asymptotics of $\log\det[I+V]$ comes from the theory of one dimensional quantum integrable models. Indeed, the generating function of the zero temperature two-point correlation functions (at distance x) of different quantum integrable models [35] has a series expansion in terms of cycle integrals of the type

$$\mathcal{I}_n[\mathcal{F}_n] = \oint_{\Gamma([-q;q])} \frac{\mathrm{d}^n z}{(2i\pi)^n} \int_{-q}^{q} \frac{\mathrm{d}^n \lambda}{(2i\pi)^n}\, \mathcal{F}_n\begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} \prod_{j=1}^{n} \frac{\mathrm{e}^{ix(p(z_j)-p(\lambda_j))}}{\big(z_j-\lambda_j\big)\big(z_j-\lambda_{j+1}\big)}\,. \tag{2.21}$$

Here the function $\mathcal{F}_n$ is holomorphic in some open neighbourhood of $[-q;q]^{2n}$ and symmetric in the n variables $\{\lambda\}$ (we set $\lambda_{n+1} \equiv \lambda_1$) and in the n variables $\{z\}$; $\Gamma([-q;q])$ is a counter clockwise closed contour around $[-q;q]$ inside this neighbourhood.

In Section 8, using the above results for the GSK, we prove the following asymptotic expansion of $\mathcal{I}_n[\mathcal{F}_n]$ in the $x \to +\infty$ limit :

Proposition 2.3. *Let $\mathcal{F}_n$ and $\mathcal{I}_n[\mathcal{F}_n]$ be as above. Then for $x \to +\infty$,*

$$\begin{aligned}
\mathcal{I}_n[\mathcal{F}_n] &= \frac{1}{2i\pi}\int_{-q}^{q} \mathrm{d}\lambda\, \{ixp'(\lambda)+\partial_\epsilon\}\mathcal{F}_n\begin{pmatrix} \{\lambda\}^n \\ \{\lambda+\epsilon\},\{\lambda\}^{n-1} \end{pmatrix}\Big|_{\epsilon=0} \\
&+ \sum_{\sigma=\pm}\big(b_n - c_n\log(2qp'_\sigma x)\big)\,\mathcal{F}_n\begin{pmatrix} \{\sigma q\}^n \\ \{\sigma q\}^n \end{pmatrix} \\
&+ \frac{n}{(2\pi)^2}\sum_{\sigma=\pm}\sum_{p=1}^{n-1}\int_{-q}^{q}\mathrm{d}\lambda \frac{\mathcal{F}_n\begin{pmatrix} \{\sigma q\}^n \\ \{\sigma q\}^n \end{pmatrix} - \mathcal{F}_n\begin{pmatrix} \{\sigma q\}^p,\{\lambda\}^{n-p} \\ \{\sigma q\}^p,\{\lambda\}^{n-p} \end{pmatrix}}{p(n-p)(q-\sigma\lambda)} \\
&+ \frac{n}{2(2\pi)^2}\sum_{p=1}^{n-1}\int_{-q}^{q}\frac{\mathrm{d}\lambda\mathrm{d}\mu}{(n-p)(\lambda-\mu)}\Big\{\partial_\epsilon\mathcal{F}_n\begin{pmatrix} \{\lambda+\epsilon\},\{\lambda\}^{p-1},\{\mu\}^{n-p} \\ \{\lambda+\epsilon\},\{\lambda\}^{p-1},\{\mu\}^{n-p} \end{pmatrix} \\
&\qquad\qquad -\partial_\epsilon\mathcal{F}_n\begin{pmatrix} \{\mu+\epsilon\},\{\mu\}^{p-1},\{\lambda\}^{n-p} \\ \{\mu+\epsilon\},\{\mu\}^{p-1},\{\lambda\}^{n-p} \end{pmatrix}\Big\}\Big|_{\epsilon=0} + \mathrm{o}(1)\,,
\end{aligned} \tag{2.22}$$

with

$$c_n = \frac{(-1)^{n-1}}{(n-1)!}\,\frac{\partial^n \nu_0^2}{\partial\gamma^n}\bigg|_{\gamma=0}\,, \quad b_n = \frac{(-1)^{n-1}}{(n-1)!}\,\frac{\partial^n \log G(1,\nu_0)}{\partial\gamma^n}\bigg|_{\gamma=0}\,, \quad \nu_0 = \frac{i}{2\pi}\log(1+\gamma), \tag{2.23}$$

and where $\{\lambda\}^n$ denotes the set formed by n copies of the same parameter λ.

Moreover, in Section 8 we will also describe the form of the sub-leading corrections to this result.

2.3 Comparison with known results

There are several results in the literature concerning the asymptotic behaviour of the Fredholm determinant $\det[I+\gamma S]$. This determinant corresponds to the GSK with $p = \mathrm{id}$, $F = 1$ and $g =$

0. It is clear that we reproduce the answer concerning the leading asymptotics of $\det[I+\gamma S]$ analyzed in [7] and [4].

As observed in [33], $x\dfrac{\mathrm{d}}{\mathrm{d}x}\log\det[I+\gamma S]$ satisfies the fifth Painlevé equation. The authors of [33] used this property to obtain an asymptotic expansion of $\log\det[I+\gamma S]$. This fact was also exploited by the authors of [42] in order to derive the first few terms in the sub-leading asymptotics of the latter quantity. Their result reads

$$x\frac{\mathrm{d}}{\mathrm{d}x}\log\det[I+\gamma S] = -4ix\nu_0 - 2\nu_0^2 - i\frac{\nu_0^3}{x} + i\frac{\nu_0^2}{4x}\left\{\left(\frac{\Gamma(-\nu_0)}{\Gamma(\nu_0)}\right)^2 (4x)^{4\nu_0}\,\mathrm{e}^{4ix} - \left(\frac{\Gamma(-\nu_0)}{\Gamma(\nu_0)}\right)^2 \frac{\mathrm{e}^{-4ix}}{(4x)^{4\nu_0}}\right\}, \quad (2.24)$$

with ν_0 given in (2.23) and $q=2$. It is straightforward to see that in such a limit $N_1 = i\nu_0^3$ and

$$O_1 \to \frac{\nu_0^2}{(2q)^2}\left\{\mathrm{e}^{2iqx}\,(2qx)^{4\nu_0}\left(\frac{\Gamma(-\nu_0)}{\Gamma(\nu_0)}\right)^2 + \frac{\mathrm{e}^{-2iqx}}{(2qx)^{4\nu_0}}\left(\frac{\Gamma(\nu_0)}{\Gamma(-\nu_0)}\right)^2\right\}, \quad (2.25)$$

which reproduces the oscillating terms (2.24) after setting $q=2$ and taking the q derivative.

3 The initial Riemann–Hilbert problem

The GSK (1.6) belongs to a special algebra of integral operators, the so-called integrable integral operators. This algebra was first singled out in [30] and then studied more thoroughly in [13]. It is well known that many properties of these integrable operators can be obtained from the solution of a certain RHP.

In this section, we formulate our problem in terms of a RHP that we then asymptotically solve.

3.1 Notations

An important property of completely integrable integral operators is that their resolvent still lies in the same algebra. However, before presenting the formula for the resolvent we introduce some quite useful vector notations. Namely, let

$$|\,E^R(\lambda)\,\rangle = \frac{\gamma\sqrt{F(\lambda)}}{2i\pi}\begin{pmatrix} e_+(\lambda)\\ e_-(\lambda)\end{pmatrix}, \quad \langle\,E^L(\lambda)\,| = \sqrt{F(\lambda)}\begin{pmatrix}-e_-(\lambda)\,,\, e_+(\lambda)\end{pmatrix}, \quad (3.1)$$

so that the kernel V has a simple expression in terms of $|\,E^R(\lambda)\,\rangle$ and $\langle\,E^L(\lambda)\,|$:

$$V(\lambda,\mu) = \frac{\langle E^L(\lambda)\,|\,E^R(\mu)\rangle}{\lambda-\mu}\,. \quad (3.2)$$

Observe that

$$\langle E^L(\lambda)\,|\,E^R(\lambda)\rangle = 0, \quad (3.3)$$

and, hence, the kernel V is not singular at $\lambda = \mu$.

Let $|\, F^R(\lambda)\,\rangle$ be the solution to the integral equation:

$$|\, F^R(\mu)\,\rangle + \int_{-q}^{q} V(\lambda,\mu)\,|\, F^R(\lambda)\,\rangle \mathrm{d}\lambda = |\, E^R(\mu)\,\rangle\,, \tag{3.4}$$

and $\langle\, F^L(\lambda)\,|$ be the solution to the corresponding dual equation. It is convenient to write $|\, F^R(\lambda)\,\rangle$ as well as its dual $\langle\, F^L(\lambda)\,|$ in a form similar to $|\, E^R(\lambda)\,\rangle$ and $\langle\, E^L(\lambda)\,|$:

$$|\, F^R(\lambda)\,\rangle = \frac{\gamma\sqrt{F(\lambda)}}{2i\pi}\begin{pmatrix} f_+(\lambda) \\ f_-(\lambda)\end{pmatrix}, \quad \langle\, F^L(\lambda)\,| = \sqrt{F(\lambda)}\Big(-f_-(\lambda)\,,\, f_+(\lambda)\Big). \tag{3.5}$$

Then the resolvent of the kernel V defined by $I - R = (I+V)^{-1}$ reads:

$$R(\lambda,\mu) = \frac{\langle F^L(\lambda)\,|\,F^R(\mu)\rangle}{\lambda-\mu} = \frac{\gamma\sqrt{F(\lambda)F(\mu)}}{2i\pi\,(\lambda-\mu)}\left[f_+(\lambda)f_-(\mu) - f_+(\mu)f_-(\lambda)\right]. \tag{3.6}$$

3.2 The Riemann–Hilbert problem associated to the GSK

Proposition 3.1. *Let V be the GSK* (1.6) *understood as acting on $L^2([\,-q\,;q\,])$, and such that* $\det[I+V] \neq 0$. *Then, there exists a 2×2 matrix $\chi(\lambda)$ such that*

$$|\, F^R(\lambda)\,\rangle = \chi(\lambda)\,|\, E^R(\lambda)\,\rangle, \qquad \langle\, F^L(\lambda)\,| = \langle\, E^L(\lambda)\,|\chi^{-1}(\lambda)\,. \tag{3.7}$$

The matrix $\chi(\lambda)$ is the unique solution of the RHP:

- *χ is analytic on $\mathbb{C}\setminus[\,-q\,;q\,]$;*
- $\chi(\lambda) = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}\log\left|\lambda^2 - q^2\right|$ *for $\lambda \to \pm q$;*
- $\chi(\lambda) \underset{\lambda\to\infty}{\to} I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$;
- $\chi_+(\lambda)\,G_\chi(\lambda) = \chi_-(\lambda)$ *for $\lambda \in\,]\,-q\,;q\,[$* .

The jump matrix G_χ for this RHP reads

$$G_\chi(\lambda) = \begin{pmatrix} 1-\gamma F(\lambda) & \gamma F(\lambda)\,e_+^2(\lambda) \\ -\gamma F(\lambda)\,e_-^2(\lambda) & 1+\gamma F(\lambda)\end{pmatrix} = I + 2i\pi|\, E^R(\lambda)\,\rangle\langle\, E^L(\lambda)\,|\,. \tag{3.8}$$

Finally, χ and its inverse can be expressed in terms of $|\, F^R(\lambda)\,\rangle$ and of its dual $\langle\, F^L(\lambda)\,|$:

$$\chi(\lambda) = I_2 - \int_{-q}^{q} \frac{|\, F^R(\mu)\,\rangle\langle\, E^L(\mu)\,|}{\mu-\lambda}\,\mathrm{d}\mu, \quad \chi^{-1}(\lambda) = I_2 + \int_{-q}^{q} \frac{|\, E^R(\mu)\,\rangle\langle\, F^L(\mu)\,|}{\mu-\lambda}\,\mathrm{d}\mu. \tag{3.9}$$

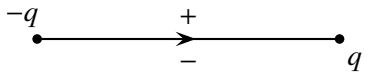

Figure 1: Original contour for the RHP.

We emphasize that the big O symbol, $\mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, is to be understood entrywise. Moreover, $\chi_{\pm}(\mu)$ stands for the non-tangential limit of $\chi(\lambda)$ when λ approaches a point μ belonging to the jump curve from the left, resp. right, side of the contour (see Fig. 1).

Proof — The unicity of the solution to this RHP is proved along the same line as in [39]. The proof of existence of the solution is based on the equivalence between RHP and singular integral equations which, in the case of the above RHP, implies

$$\chi(\lambda) = I_2 + \int_{-q}^{q} \frac{\mathrm{d}\mu}{\lambda-\mu} \chi_+(\mu) \,|\, E^R(\mu) \,\rangle\langle\, E^L(\mu) \,|\quad,\quad \lambda \in \mathbb{C} \setminus [\,-q\,;q\,]\ . \tag{3.10}$$

The solution to this equation can be expressed in terms of the resolvent kernel $I - R$ of $I + V$

$$\chi(\lambda) = I_2 + \int_{-q}^{q} \frac{\mathrm{d}\mu}{\lambda-\mu} \Big\{ |\, E^R \,\rangle . (I-R) \Big\}(\mu) \,\langle\, E^L(\mu) \,|\ . \tag{3.11}$$

In its turn, the resolvent kernel exists as $\det[I+V] \neq 0$. Moreover, the explicit construction of the resolvent through a Fredholm series shows that $(\lambda,\mu) \mapsto R(\lambda,\mu)$ is analytic in $\overline{U} \times \overline{U}$. Hence, so is $|\, F^R(\mu) \,\rangle = \Big[|\, E^R \,\rangle . (I-R) \Big](\mu)$. The estimate $|\chi| = O\Big(\log\big|\lambda^2 - q^2\big|\Big)$, $\lambda \to \pm q$ follows from the integral representation (3.11) supplemented with the fact that both $\langle\, E^L \,|$ and $|\, F^R \,\rangle$ are smooth on $[\,-q\,;q\,]$.

Applying (3.9) to $|E^R(\lambda)\rangle$ and $\langle E^L(\lambda)|$ we obtain the equations (3.7). Hereby one can easily check that due to the orthogonality condition (3.3) the transform (3.7) is continuous across $[-q,q]$. □

It is also possible to express logarithmic derivatives of $\det[I+V]$ either in terms of the resolvent R of $I+V$ or in terms of χ. Indeed, we have the

Lemma 3.1. *The derivative of* $\log\det[I+V]$ *with respect to x is related to the following trace involving the matrix* χ

$$\partial_x \log\det[I+V] = \oint_{\Gamma([\,-q\,;q\,])} \frac{\mathrm{d}\lambda}{4\pi} p(\lambda)\,\mathrm{tr}\Big[\partial_\lambda\chi(\lambda)\,\sigma_3\chi^{-1}(\lambda)\Big], \tag{3.12}$$

with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $\Gamma([\,-q\,;q\,])$ *defined as in (2.21), whereas its derivatives with respect to*

γ and q are expressed in terms of the resolvent as

$$\partial_\gamma \log\det\left[I+V\right] = \int_{-q}^{q} \frac{\mathrm{d}\lambda}{\gamma} R\left(\lambda,\lambda\right), \quad \partial_q \log\det\left[I+V\right] = R\left(q,q\right) + R\left(-q,-q\right). \tag{3.13}$$

Proof — The last two equations are easily proved by the multiple integral series expansion of $\log\det\left[I+V\right]$. We shall only focus on the equation relating the x derivative of $\log\det\left[I+V\right]$ to χ. Clearly

$$\partial_x \log\det\left[I+V\right] = \int_{-q}^{q} \left[\partial_x V.\left(I-R\right)\right]\left(\lambda,\lambda\right)\mathrm{d}\lambda\ , \tag{3.14}$$

with

$$\partial_x V\left(\lambda,\mu\right) = -\oint_{\Gamma\left(\left[-q\,;q\right]\right)} \frac{\mathrm{d}z}{4\pi} \frac{p\left(z\right)}{\left(z-\lambda\right)\left(z-\mu\right)} \langle\, E^L\left(\lambda\right)\, |\sigma_3|\, E^R\left(\mu\right)\,\rangle. \tag{3.15}$$

So that, using the representation (3.6) of the resolvent R in terms of $\langle\, F^L\,|$ and $|\, F^R\,\rangle$ and the fact that $\langle\, F^L(\mu)\,|\,\sigma_3\,|\,F^R(\lambda)\,\rangle = \mathrm{tr}\left[\sigma_3|\,F^R(\lambda)\,\rangle\langle\, F^L(\mu)\,|\right]$, we get

$$\begin{aligned}\partial_x \log\det\left[I+V\right] = &-\oint_{\Gamma\left(\left[-q\,;q\right]\right)} \frac{\mathrm{d}z}{4\pi}\, p\left(z\right) \int_{-q}^{q} \mathrm{d}\lambda\, \frac{\langle\, E^L\left(\lambda\right)\,|\sigma_3|\, E^R\left(\lambda\right)\,\rangle}{\left(z-\lambda\right)^2} \\ &+ \mathrm{tr}\left\{ \oint_{\Gamma\left(\left[-q\,;q\right]\right)} \frac{\mathrm{d}z}{4\pi}\, p\left(z\right) \int_{-q}^{q} \mathrm{d}\lambda\mathrm{d}\mu\, |\, F^R\left(\lambda\right)\,\rangle\langle\, E^L\left(\lambda\right)\,| \right. \\ &\left. \times \left(\frac{1}{\lambda-z} - \frac{1}{\lambda-\mu}\right)\sigma_3 \frac{|\, E^R\left(\mu\right)\,\rangle\langle\, F^L\left(\mu\right)\,|}{\left(\mu-z\right)^2} \right\}. \end{aligned} \tag{3.16}$$

Using the integral expressions (3.9) for χ and χ^{-1}, we obtain

$$\begin{aligned}\partial_x \log\det\left[I+V\right] &= -\oint_{\Gamma\left(\left[-q\,;q\right]\right)} \frac{\mathrm{d}z}{4\pi}\, p\left(z\right) \int_{-q}^{q} \mathrm{d}\lambda\, \frac{\langle\, E^L\left(\lambda\right)\,|\sigma_3|\, E^R\left(\lambda\right)\,\rangle}{\left(z-\lambda\right)^2} \\ &\quad + \mathrm{tr}\left\{ \oint_{\Gamma\left(\left[-q\,;q\right]\right)} \frac{\mathrm{d}z}{4\pi}\, p(z) \int_{-q}^{q} \mathrm{d}\mu\, \left(\chi(\mu)-\chi(z)\right)\sigma_3 \frac{|\, E^R(\mu)\,\rangle\langle\, F^L(\mu)\,|}{\left(\mu-z\right)^2} \right\} \\ &= \oint_{\Gamma\left(\left[-q\,;q\right]\right)} \frac{\mathrm{d}z}{4\pi}\, p\left(z\right) \mathrm{tr}\left\{\partial_z\chi\left(z\right)\sigma_3\chi^{-1}\left(z\right)\right\}, \end{aligned} \tag{3.17}$$

where we used (3.7). □

It is worth noticing that formula (3.12) is particularly effective when p is a rational function as then the contour of integration can be deformed to the poles of p (including the pole at ∞). The integrals can be then easily calculated. In particular, in the case $p(\lambda) = \lambda$, we have the following result:

Corollary 3.1. *Let χ_1 be the first non-trivial coefficient of the expansion of χ around ∞, i.e.*

$$\chi(\lambda) = I_2 + \frac{\chi_1}{\lambda} + \mathrm{o}\left(\frac{1}{\lambda}\right). \tag{3.18}$$

Then

$$\partial_x \log\det[I+V]\,|_{p=\mathrm{id}} = -\frac{i}{2}\mathrm{tr}\{\chi_1\sigma_3\}. \tag{3.19}$$

Proof — $-\mathrm{tr}(\sigma_3\chi_1)$ is the residue of the pole at infinity of $\lambda \mapsto \lambda\,\mathrm{tr}\left\{\partial_\lambda\chi(\lambda)\,\sigma_3\chi^{-1}(\lambda)\right\}$. □

In this way, we recover one of the formulae derived for the sine kernel [13], but also for more general kernels as in [27, 28, 31]. We emphasize that (3.19) is valid not only for the sine kernel as it was originally derived, but also for the generalised sine kernel with $p = \mathrm{id}$.

4 Transformations of the original RHP

In this section we perform several transformations on the RHP for χ so as to implement Deift–Zhou's steepest descent method [14]. The first substitution maps the RHP for the matrix χ into a RHP for a matrix Ξ whose jump matrix has 1 on its lower diagonal entry. This jump matrix is then easily factorized into upper/lower triangular matrices. This factorization allows us to define another RHP for an unknown matrix Υ whose jump matrices are already exponentially close to identity uniformly away from the endpoints $\pm q$. It remains to construct the parametrices at q and $-q$. These parametrices enable us to define a matrix Π satisfying a RHP with jump matrices uniformly $I_2 + \mathrm{o}(1)$ when $x \to +\infty$.

4.1 The first step $\chi \to \Xi$

Let

$$\alpha(\lambda) = \exp\left\{\int_{-q}^{q} \frac{\nu(\mu)}{\mu-\lambda}\,\mathrm{d}\mu\right\} = \kappa(\lambda)\left(\frac{\lambda-q}{\lambda+q}\right)^{\nu(\lambda)}. \tag{4.1}$$

Then clearly, $\alpha(\lambda)$ solves the scalar RHP

$$\alpha_-(\lambda) = \alpha_+(\lambda)(1+\gamma F(\lambda)), \qquad \lambda \in [-q\,;q\,], \quad \alpha(\lambda) \to 1 \quad \text{at} \quad \lambda \to \infty\,. \tag{4.2}$$

The functions $\kappa(\lambda)$ and $\nu(\lambda)$ were already introduced in (2.2) and (2.1). In the following, we shall also use another representation for the function $\alpha(\lambda)$:

$$\alpha(\lambda) = \kappa_p(\lambda)\left[\frac{p(\lambda)-p_+}{p(\lambda)-p_-}\right]^{\nu(\lambda)}, \tag{4.3}$$

where κ_p is defined as

$$\log \kappa_p(\lambda; q) \equiv \log \kappa_p(\lambda) = \int_{-q}^{q} \left(\nu(\lambda) \frac{p'(\mu)}{p(\lambda) - p(\mu)} - \frac{\nu(\mu)}{\lambda - \mu} \right) \mathrm{d}\mu \ . \tag{4.4}$$

We specify that we chose the principal branch of the logarithm, i.e. $\arg \in]-\pi\,;\pi[$. Due to our assumptions on γ, F and p, Morera's theorem implies that the functions ν, $\log \kappa$ and $\log \kappa_p$ are holomorphic on $\overline{U}$. Moreover we have $|\Re(\nu(\lambda))| < 1/2$, $\forall \lambda \in \overline{U}$. Indeed

$$\nu(\lambda) = \frac{i}{2\pi} \log |1 + \gamma F(\lambda)| - \frac{1}{2\pi} \arg(1 + \gamma F(\lambda)) \ , \tag{4.5}$$

and we have assumed that $\arg(1 + \gamma F) \in]-\pi\,;\pi[$.

We use the function α to transform the RHP for χ. Let us define the matrix $\Xi(\lambda)$ according to

$$\Xi(\lambda) = \chi(\lambda) \begin{pmatrix} \alpha(\lambda) & 0 \\ 0 & \alpha^{-1}(\lambda) \end{pmatrix}. \tag{4.6}$$

This new matrix $\Xi(\lambda)$ satisfies the following RHP:

- Ξ is analytic on $\mathbb{C} \setminus [-q\,;q]$;
- $|\Xi(\lambda)| = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \left|\lambda^2 - q^2\right|^{\pm\sigma_3 \Re(\nu_\pm)} \log\left|\lambda^2 - q^2\right| \quad$ for $\lambda \to \pm q$;
- $\Xi(\lambda) \underset{\lambda\to\infty}{\to} I_2$;
- $\Xi_+(\lambda)\, G_\Xi(\lambda) = \Xi_-(\lambda) \quad$ for $\lambda \in [-q\,;q]$.

Here the new jump matrix G_Ξ reads

$$G_\Xi(\lambda) = \begin{pmatrix} 1 + P(\lambda)Q(\lambda) & P(\lambda)\mathrm{e}^{ixp(\lambda)} \\ Q(\lambda)\mathrm{e}^{-ixp(\lambda)} & 1 \end{pmatrix}, \tag{4.7}$$

and

$$P(\lambda) = \frac{\gamma F(\lambda)}{1 + \gamma F(\lambda)} \alpha_+^{-2}(\lambda)\, \mathrm{e}^{g(\lambda)} = -2i\mathrm{e}^{i\pi\nu(\lambda)} \frac{\sin \pi\nu(\lambda)}{\alpha_+^2(\lambda)} \mathrm{e}^{g(\lambda)}, \tag{4.8}$$

$$Q(\lambda) = -\frac{\gamma F(\lambda)}{1 + \gamma F(\lambda)} \alpha_-^2(\lambda)\, \mathrm{e}^{-g(\lambda)} = 2i\mathrm{e}^{i\pi\nu(\lambda)} \frac{\sin \pi\nu(\lambda)}{\mathrm{e}^{g(\lambda)}} \alpha_-^2(\lambda) \,. \tag{4.9}$$

The solution of this RHP for Ξ exists as it can be constructed from χ. Moreover it is unique as seen by arguments similar to those providing uniqueness of the solution to the RHP for χ.

4.2 The second step $\Xi \to \Upsilon$

As already mentioned, the jump matrix G_Ξ admits an explicit factorization into a product of upper and lower triangular matrices:

$$G_\Xi = M_+ M_- \,. \tag{4.10}$$

The matrices $M_\pm$ are given by

$$M_+(\lambda) = \begin{pmatrix} 1 & P(\lambda)\,\mathrm{e}^{ixp(\lambda)} \\ 0 & 1 \end{pmatrix}, \qquad M_-(\lambda) = \begin{pmatrix} 1 & 0 \\ Q(\lambda)\,\mathrm{e}^{-ixp(\lambda)} & 1 \end{pmatrix}, \tag{4.11}$$

and can be continued to $U \cap \mathcal{H}_+$, resp. $U \cap \mathcal{H}_-$, where we recall that $\mathcal{H}_\pm$ is the upper/lower half plane and $\overline{U}$ is the domain of holomorphy of all the functions appearing in the RHP. Then we draw two new contours $\Gamma_\pm$ in $p(U)$ and define a new matrix $\Upsilon(\lambda)$ according to Fig. 2.

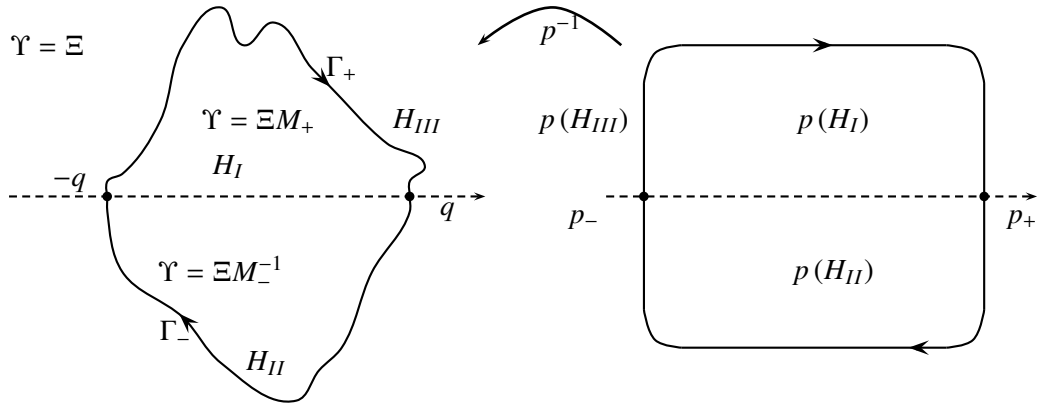

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Figure 2: Contours Γ_+ and Γ_- associated with the RHP for Υ.

As readily checked, $\Upsilon(\lambda)$ is continuous across $]-q\,;q[$ and thus holomorphic in the interior of $\Gamma_+ \cup \Gamma_-$. We have thus removed the cut along $[-q\,;q\,]$ and replaced it with cuts along $\Gamma_+ \cup \Gamma_-$.

The matrix Υ solves the following RHP:

- Υ is analytic in $\mathbb{C} \setminus \Gamma_+ \cup \Gamma_-$;
- $\Upsilon(\lambda) = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} |\lambda \mp q|^{\pm\Re(\nu_\pm)} & |\lambda \mp q|^{\mp\Re(\nu_\pm)} \\ 0 & |\lambda \mp q|^{\mp\Re(\nu_\pm)} \end{pmatrix}\log|\lambda \mp q|\,, \quad \lambda \underset{\lambda\in H_I}{\longrightarrow} \pm q;$
- $\Upsilon(\lambda) = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} |\lambda \mp q|^{\pm\Re(\nu_\pm)} & 0 \\ |\lambda \mp q|^{\pm\Re(\nu_\pm)} & |\lambda \mp q|^{\mp\Re(\nu_\pm)} \end{pmatrix}\log|\lambda \mp q|\,, \quad \lambda \underset{\lambda\in H_{II}}{\longrightarrow} \pm q;$
- $\Upsilon(\lambda) = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}|\lambda \mp q|^{\pm\sigma_3\Re(\nu_\pm)}\log|\lambda \mp q|\,, \quad \lambda \underset{\lambda\in H_{III}}{\longrightarrow} \pm q;$

- $\Upsilon(\lambda) \underset{\lambda\to\infty}{\to} I_2$;
- $\begin{cases} \Upsilon_+(\lambda) M_+(\lambda) = \Upsilon_-(\lambda) & \text{for } \lambda \in \Gamma_+, \\ \Upsilon_+(\lambda) M_-^{-1}(\lambda) = \Upsilon_-(\lambda) & \text{for } \lambda \in \Gamma_-, \end{cases}$

where the domains H_I, H_{II}, H_{III} are shown on the Figure 2.

Clearly, the solution of the RHP for Υ exists and is unique. Hence, the matrices Υ and χ are in a one to one correspondence.

Note that, except in some vicinities of q and $-q$, the jump matrices M_+ and M_-^{-1} for Υ are exponentially close to the identity matrix. Therefore, to study the asymptotic solution of the RHP, it is enough to study the local problems in the vicinities of q and $-q$.

4.3 Parametrix around $-q$

We first present the parametrix $\mathcal{P}$ on a small disk $D_{-q,\delta} \subset U$ of radius δ and centered at $-q$, that is an exact solution of the RHP:

- $\mathcal{P}$ is analytic on $D_{-q,\delta} \setminus \{\Gamma_+ \cup \Gamma_-\}$;
- $\mathcal{P}(\lambda) = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} |\lambda+q|^{-\Re(\nu_-)} & |\lambda+q|^{\Re(\nu_-)} \\ 0 & |\lambda+q|^{\Re(\nu_-)} \end{pmatrix}\log|\lambda+q|\,,\ \lambda \underset{\lambda\in H_I}{\longrightarrow} -q$;
- $\mathcal{P}(\lambda) = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} |\lambda+q|^{-\Re(\nu_-)} & 0 \\ |\lambda+q|^{-\Re(\nu_-)} & |\lambda+q|^{\Re(\nu_-)} \end{pmatrix}\log|\lambda+q|\,,\ \lambda \underset{\lambda\in H_{II}}{\longrightarrow} -q$;
- $\mathcal{P}(\lambda) = \mathrm{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}|\lambda+q|^{-\sigma_3\Re(\nu_-)}\log|\lambda+q|\,, \quad \lambda \underset{\lambda\in H_{III}}{\longrightarrow} -q$;
- $\mathcal{P}(\lambda) = I_2 + \mathrm{O}\left(\dfrac{1}{x^{1-\varepsilon}}\right), \quad$ uniformly for $\lambda \in \partial D_{-q,\delta}$,
- $\begin{cases} \mathcal{P}_+(\lambda) M_+(\lambda) = \mathcal{P}_-(\lambda) & \text{for } \lambda \in \Gamma_+ \cap D_{-q,\delta}, \\ \mathcal{P}_+(\lambda) M_-^{-1}(\lambda) = \mathcal{P}_-(\lambda) & \text{for } \lambda \in \Gamma_- \cap D_{-q,\delta}. \end{cases}$

Here $\varepsilon = 2\sup\limits_{\partial D_{-q,\delta}}\left|\Re(\nu)\right| < 1$. The canonically oriented contour $\partial D_{-q,\delta}$ is depicted in Fig. 3. The RHP for $\mathcal{P}$ admits a class of solutions. Each element of this class is related to another one through a left multiplication by a holomorphic matrix that is uniformly $I_2 + O\left(1/x^{1-\varepsilon}\right)$ on $\partial D_{-q,\delta}$. In order to construct the solution $\mathcal{P}$ to this problem, we first focus on the simpler case where the functions F, g and κ_p are constant. Then the solution to the RHP for $\mathcal{P}_{const}$ can be obtained by the differential equation method [26, 11, 12]. This leads to the solution

$$\mathcal{P}_{const}(\lambda) = \Psi(\lambda) L(\lambda) \left[\zeta_{-q}\right]^{-\nu\sigma_3} \mathrm{e}^{\frac{i\pi\nu}{2}} \ . \tag{4.12}$$

Here $\zeta_{-q} = x(p(\lambda) - p_-)$, $\nu = i\log(1+\gamma F)/2\pi$,

$$\Psi(\lambda) = \begin{pmatrix} \Psi\left(-\nu, 1; -i\zeta_{-q}\right) & ib_{12}\Psi\left(1+\nu, 1; i\zeta_{-q}\right) \\ -ib_{21}\Psi\left(1-\nu, 1; -i\zeta_{-q}\right) & \Psi\left(\nu, 1; i\zeta_{-q}\right) \end{pmatrix}, \tag{4.13}$$

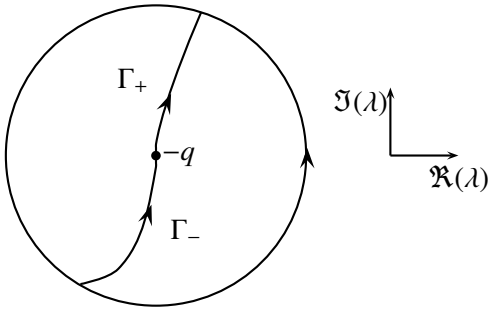


Figure 3: Contours in the RHP for $\mathcal{P}$.

and finally

$$b_{12}(\lambda) = -i\frac{\sin[\pi\nu]\,\Gamma^2(1+\nu)}{\pi\kappa_p^2\left[x(p_+ - p(\lambda))\right]^{2\nu}}\mathrm{e}^{ixp_- + g}, \tag{4.14}$$

$$b_{21}(\lambda) = -i\frac{\pi\kappa_p^2\left[x(p_+ - p(\lambda))\right]^{2\nu}}{\sin[\pi\nu]\,\Gamma^2(\nu)}\mathrm{e}^{-ixp_- - g}\ . \tag{4.15}$$

$\Psi(a,c;z)$ denotes Tricomi confluent hypergeometric function (CHF) of the second kind (see Appendix A). It solves the differential equation

$$zy'' + (c-z)\,y' - ay = 0\,. \tag{4.16}$$

Recall that Ψ has a cut along $\mathbb{R}^-$. Note that this choice for the cut of Ψ implies the use of the principal branch of the logarithm: $-\pi < \arg(z) < \pi$. The expression for the piecewise constant matrix L depends on the region of the complex plane. Namely,

$$L(\lambda) = \begin{cases} I_2 & -\pi/2 < \arg\left[p(\lambda) - p_-\right] < \pi/2, \\ \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-2i\pi\nu} \end{pmatrix} & \pi/2 < \arg\left[p(\lambda) - p_-\right] < \pi, \\ \begin{pmatrix} \mathrm{e}^{-2i\pi\nu} & 0 \\ 0 & 1 \end{pmatrix} & -\pi < \arg\left[p(\lambda) - p_-\right] < -\pi/2. \end{cases} \tag{4.17}$$

The reader can check using the monodromy properties of Tricomi CHF (A.4) and (A.5) that the jump condition for constant functions F and g are satisfied by the matrix $\mathcal{P}_{const}$. Moreover the asymptotic expansion for $\Psi(a,c;z)$ at $z \to \infty$ allows one to check that $\mathcal{P}_{const}$ has the correct behaviour at infinity. We remind that this parametrix also appeared recently in the work [32].

In order to extend this result to the case of arbitrary holomorphic functions $F(\lambda)$, $g(\lambda)$ and $\kappa_p(\lambda)$, it is enough to add the λ dependency in all places where these functions appear. One ends up with the following solution to the RHP for $\mathcal{P}$:

$$\mathcal{P}(\lambda) = \Psi(\lambda)\,L(\lambda)\left[\zeta_{-q}\right]^{-\nu(\lambda)\sigma_3}\mathrm{e}^{\frac{i\pi\nu(\lambda)}{2}}\ . \tag{4.18}$$

Here $\zeta_{-q} = x\,(p\,(\lambda) - p_-)$,

$$\Psi\,(\lambda) = \begin{pmatrix} \Psi\left(-\nu\,(\lambda)\,, 1; -i\zeta_{-q}\right) & ib_{12}(\lambda)\,\Psi\left(1+\nu\,(\lambda)\,, 1; i\zeta_{-q}\right) \\ -ib_{21}(\lambda)\,\Psi\left(1-\nu\,(\lambda)\,, 1; -i\zeta_{-q}\right) & \Psi\left(\nu\,(\lambda)\,, 1; i\zeta_{-q}\right) \end{pmatrix}, \tag{4.19}$$

with

$$b_{12}\,(\lambda) = -i\frac{\sin\left[\pi\nu\,(\lambda)\right]\Gamma^2\,(1+\nu\,(\lambda))}{\pi\kappa_p^2\,(\lambda)\left[x\,(p_+ - p\,(\lambda))\right]^{2\nu(\lambda)}}\mathrm{e}^{g(\lambda)+ixp_-} = -i\nu\,(\lambda)\,u\,(\lambda; x)\,, \tag{4.20}$$

$$b_{21}\,(\lambda) = -i\frac{\pi\kappa_p^2\,(\lambda)\left[x\,(p_+ - p\,(\lambda))\right]^{2\nu(\lambda)}}{\sin\left[\pi\nu\,(\lambda)\right]\Gamma^2\,(\nu\,(\lambda))}\mathrm{e}^{-ixp_- - g(\lambda)} = -i\frac{\nu\,(\lambda)}{u\,(\lambda; x)}, \tag{4.21}$$

and finally

$$u\,(\lambda; x) = \frac{\Gamma\,(1+\nu\,(\lambda))}{\Gamma\,(1-\nu\,(\lambda))}\left\{\kappa_p(\lambda)\,x^{\nu(\lambda)}\left[p_+ - p\,(\lambda)\right]^{\nu(\lambda)}\right\}^{-2} e^{ixp_- + g(\lambda)}. \tag{4.22}$$

In the above formulae we have explicitly stressed the dependence of the functions b_{12}, b_{21} and ν on λ. Finally, the matrix $L\,(\lambda)$ is given by (4.17) with ν replaced by the function $\nu\,(\lambda)$.

This construction originates from the observation that the replacements $F \mapsto F\,(\lambda)$, $g \mapsto g\,(\lambda)$ and $\kappa_p \mapsto \kappa_p\,(\lambda)$ preserve the jump conditions as the latter hold pointwise. Of course, once the parametrix $\mathcal{P}$ is guessed it is not a problem to check directly that it solves the RHP in question. The asymptotic behaviour is inferred from (A.6), whereas the jump conditions can be verified thanks to (A.4) and (A.5). Furthermore, due to the definition of the matrix L, the solution is continuous across the line $\arg\left[p\,(\lambda) - p_-\right] = \pi$ and thus analytic in the whole domain $\{\lambda \in \mathbb{C};\, \Re\left[p\,(\lambda) - p_-\right] < 0\}$.

4.4 Parametrix around q

The RHP for the parametrix $\widetilde{\mathcal{P}}$ around q reads

- $\widetilde{\mathcal{P}}$ is analytic on $D_{q,\delta} \setminus \{\Gamma_+ \cup \Gamma_-\}$;
- $\widetilde{\mathcal{P}}(\lambda) = O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} |\lambda-q|^{+\Re(\nu_+)} & |\lambda-q|^{-\Re(\nu_+)} \\ 0 & |\lambda-q|^{-\Re(\nu_+)} \end{pmatrix}\log|\lambda-q|\,, \quad \lambda \underset{\lambda\in H_I}{\longrightarrow} q$;
- $\widetilde{\mathcal{P}}(\lambda) = O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} |\lambda-q|^{+\Re(\nu_+)} & 0 \\ |\lambda-q|^{+\Re(\nu_+)} & |\lambda-q|^{-\Re(\nu_+)} \end{pmatrix}\log|\lambda-q|\,, \quad \lambda \underset{\lambda\in H_{II}}{\longrightarrow} q$;
- $\widetilde{\mathcal{P}}(\lambda) = O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}|\lambda-q|^{\sigma_3\Re(\nu_+)}\log|\lambda-q|\,, \quad \lambda \underset{\lambda\in H_{III}}{\longrightarrow} q$;
- $\widetilde{\mathcal{P}}(\lambda) = I_2 + O\left(\dfrac{1}{x^{1-\tilde{\varepsilon}}}\right)$ uniformly for $\lambda \in \partial D_{q,\delta}$;
- $\begin{cases} \widetilde{\mathcal{P}}_+\,(\lambda)\,M_+\,(\lambda) = \widetilde{\mathcal{P}}_-\,(\lambda) & \text{for } \lambda \in \Gamma_+ \cap D_{q,\delta}, \\ \widetilde{\mathcal{P}}_+\,(\lambda)\,M_-^{-1}\,(\lambda) = \widetilde{\mathcal{P}}_-\,(\lambda) & \text{for } \lambda \in \Gamma_- \cap D_{q,\delta}. \end{cases}$

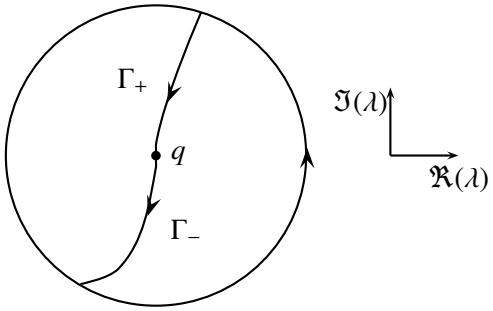


Figure 4: Contours in the RHP for $\widetilde{\mathcal{P}}$.

and $\widetilde{\varepsilon} = 2\sup_{\partial D_{q,\delta}} \left|\Re(\nu)\right| < 1$. The solution of the RHP for the parametrix $\widetilde{\mathcal{P}}$ around q can be formally obtained from the one at $-q$ through the transformation $q \to -q$ and $\nu \to -\nu$ on the solution to the RHP for $\mathcal{P}$. Indeed, the two RHP are identical modulo this negation.

Just as for the parametrix around $-q$, we focus on the solution

$$\widetilde{\mathcal{P}}(\lambda) = \widetilde{\Psi}(\lambda)\,\widetilde{L}(\lambda)\,\zeta_q^{\nu(\lambda)\sigma_3}\mathrm{e}^{-\frac{i\pi\nu(\lambda)}{2}}, \tag{4.23}$$

where $\zeta_q = x\left[p(\lambda) - p_+\right]$, and

$$\widetilde{\Psi}(\lambda) = \begin{pmatrix} \Psi\left(\nu(\lambda), 1; -i\zeta_q\right) & i\tilde{b}_{12}(\lambda)\,\Psi\left(1-\nu(\lambda), 1; i\zeta_q\right) \\ -i\tilde{b}_{21}(\lambda)\,\Psi\left(1+\nu(\lambda), 1; -i\zeta_q\right) & \Psi\left(-\nu(\lambda), 1; i\zeta_q\right) \end{pmatrix}. \tag{4.24}$$

Here

$$\tilde{b}_{12}(\lambda) = i\frac{\sin[\pi\nu(\lambda)]\,\Gamma^2(1-\nu(\lambda))}{\pi\kappa_p^2(\lambda)}[x(p(\lambda)-p_-)]^{2\nu(\lambda)}\mathrm{e}^{g(\lambda)+ixp_+} = i\nu(\lambda)\tilde{u}(\lambda;x),$$

$$\tilde{b}_{21}(\lambda) = i\frac{\pi\kappa_p^2(\lambda)\,\mathrm{e}^{-g(\lambda)-ixp_+}}{\sin[\pi\nu(\lambda)]\,\Gamma^2(-\nu(\lambda))\,[x(p(\lambda)-p_-)]^{2\nu(\lambda)}} = i\frac{\nu(\lambda)}{\tilde{u}(\lambda;x)},$$

and

$$\tilde{u}(\lambda;x) = \frac{\Gamma(1-\nu(\lambda))}{\Gamma(1+\nu(\lambda))}\left\{\frac{x^{\nu(\lambda)}\left[p(\lambda)-p_-\right]^{\nu(\lambda)}}{\kappa_p(\lambda)}\right\}^2 e^{ixp_+ + g(\lambda)}. \tag{4.25}$$

Just as for the parametrix around $-q$, the matrix $\tilde{L}(\lambda)$ depends on the quadrant of the complex plane:

$$\tilde{L}(\lambda) = \begin{cases} I_2 & -\pi/2 < \arg\left[p(\lambda)-p_+\right] < \pi/2, \\ \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2i\pi\nu(\lambda)} \end{pmatrix} & \pi/2 < \arg\left[p(\lambda)-p_+\right] < \pi, \\ \begin{pmatrix} \mathrm{e}^{2i\pi\nu(\lambda)} & 0 \\ 0 & 1 \end{pmatrix} & -\pi < \arg\left[p(\lambda)-p_+\right] < -\pi/2. \end{cases} \tag{4.26}$$

4.5 The last transformation $\Upsilon \to \Pi$

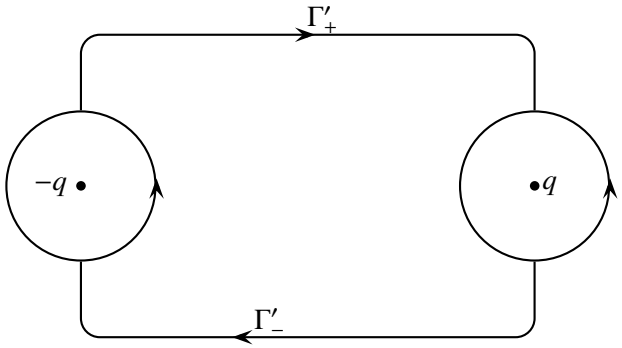
Let

$$\Pi(\lambda) = \begin{cases} \Upsilon(\lambda)\widetilde{\mathcal{P}}^{-1}(\lambda) & \text{for } \lambda \in D_{q,\delta}, \\ \Upsilon(\lambda)\mathcal{P}^{-1}(\lambda) & \text{for } \lambda \in D_{-q,\delta}, \\ \Upsilon(\lambda) & \text{for } \lambda \in \mathbb{C} \setminus \left\{\overline{D}_{q,\delta} \cup \overline{D}_{-q,\delta}\right\}. \end{cases} \tag{4.27}$$

Introduce the curve $\mathscr{C} = \{\Gamma_+ \cup \Gamma_-\} \cap \left\{\overline{D}_{q,\delta} \cup \overline{D}_{-q,\delta}\right\}$. Then Π is continuous across $\mathscr{C} \setminus \{q, -q\}$. Since Π is holomorphic in a vicinity of $\mathscr{C}$, we have that Π is holomorphic in $D_{q,\delta} \cup D_{-q,\delta} \setminus \{q, -q\}$. This, in turn, due to the estimates for $\mathcal{P}$, $\widetilde{\mathcal{P}}$ and Υ around the points $\pm q$, ensures that the singularities at these points are of a removable type. Hence Π is holomorphic on the disks $D_{q,\delta} \cup D_{-q,\delta}$. Finally, we see that Π satisfies the following RHP:

- Π is analytic in $\mathbb{C} \setminus \Sigma_\Pi$ (*cf* Fig. 5) ;
- $\Pi(\lambda) = I_2 + O(1/\lambda) \quad$ for $\lambda \to \infty$;
- $$\begin{cases} \Pi_+(\lambda) M_+(\lambda) = \Pi_-(\lambda) & \text{for } \lambda \in \Gamma'_+, \\ \Pi_+(\lambda) M_-^{-1}(\lambda) = \Pi_-(\lambda) & \text{for } \lambda \in \Gamma'_-, \\ \Pi_+(\lambda)\mathcal{P}(\lambda) = \Pi_-(\lambda) & \text{for } \lambda \in \partial D_{-q,\delta}, \\ \Pi_+(\lambda)\widetilde{\mathcal{P}}(\lambda) = \Pi_-(\lambda) & \text{for } \lambda \in \partial D_{q,\delta}. \end{cases}$$

The solution to the RHP for Π, exits and is unique as seen by standard arguments.



$$\Sigma_\Pi = \Gamma'_- \cup \Gamma'_+ \cup \partial D_{-q,\delta} \cup \partial D_{q,\delta}$$

Figure 5: Contour Σ_Π appearing in the RHP for Π.

The jump matrices for Π are uniformly exponentially close to I_2 in x on $\Gamma'_- \cup \Gamma'_+$ and uniformly $I_2 + \mathrm{O}\left(x^{\overline{\varepsilon}-1}\right)$ on $\partial D_{q,\delta} \cup \partial D_{-q,\delta}$, with $\overline{\varepsilon} = 2 \sup_{\partial D_{q,\delta} \cup \partial D_{-q,\delta}} \left|\Re(\nu)\right|$. As a consequence, I_2 is the unique solution of the RHP, up to uniformly $\mathrm{O}\left(x^{\overline{\varepsilon}-1}\right)$ corrections. In addition, using the equivalence between singular integral equations and RHP, the asymptotic expansion of Π can be obtained by a Neumann series. This will be done in the upcoming section.

5 Asymptotic solution of the RHP

In this Section we asymptotically solve the above RHP for Π.

We derive an asymptotic expansion into negative powers of x for the jump matrices for Π, and use it to prove the existence of an asymptotic series for Π. The corresponding asymptotic series for χ follows readily. One can finally infer the asymptotic behaviour of the resolvent of the GSK up to any order in $1/x$.

5.1 Asymptotics of the jump matrices

Denote the jump matrices for Π by $I_2+\Delta(\lambda)$. Then the matrix $\Delta(\lambda)$ has the asymptotic expansion in the limit $x \to +\infty$:

$$\Delta(\lambda) = \sum_{m=1}^{M} \frac{\Delta^{(m)}(\lambda; x)}{x^m} \quad + \quad \mathrm{o}\left(x^{-M-1+\overline{\varepsilon}}\right), \tag{5.1}$$

with $\overline{\varepsilon} = 2\sup_{\partial D_{q,\delta}\cup\partial D_{-q,\delta}} \left|\Re(\nu)\right|$.

The explicit form of the matrices $\Delta^{(n)}(\lambda; x)$ depends on the position of λ in the contour Σ_Π: they vanish to any order in $1/x$ on $\Gamma'_+ \cup \Gamma'_-$, whereas the asymptotic expansion for Δ on $\partial D_{q,\delta} \cup D_{-q,\delta}$ follows promptly from the asymptotic expansion of Tricomi CHF (A.6). More explicitly, for any $n \in \mathbb{N}^*$,

$$\Delta^{(n)}(\lambda; x) = \begin{cases} \dfrac{\Delta^{(n)}_{(-)}(\lambda; x)}{\left[p(\lambda) - p_-\right]^n} & \text{for } \lambda \in \partial D_{-q,\delta}, \\ \dfrac{\Delta^{(n)}_{(+)}(\lambda; x)}{\left[p(\lambda) - p_+\right]^n} & \text{for } \lambda \in \partial D_{q,\delta}, \\ 0 & \text{for } \lambda \in \Gamma'_+ \cup \Gamma'_-. \end{cases} \tag{5.2}$$

We have separated the jump matrices into their pole parts $\left[p(\lambda) - p_\pm\right]^{-n}$ and regular parts $\Delta^{(n)}_{(-)}$ and $\Delta^{(n)}_{(+)}$, with

$$\Delta^{(n)}_{(-)}(\lambda; x) = \frac{i^n}{n!} \begin{pmatrix} 1 & -\dfrac{n\,u(\lambda; x)}{\nu(\lambda)} \\ \dfrac{n}{\nu(\lambda)u(\lambda; x)} & 1 \end{pmatrix} \begin{pmatrix} (-1)^n(-\nu(\lambda))_n^2 & 0 \\ 0 & (\nu(\lambda))_n^2 \end{pmatrix} \tag{5.3}$$

for $\lambda \in \partial D_{-q,\delta}$, and

$$\Delta^{(n)}_{(+)}(\lambda; x) = \frac{i^n}{n!} \begin{pmatrix} 1 & \dfrac{n\,\tilde{u}(\lambda; x)}{\nu(\lambda)} \\ -\dfrac{n}{\nu(\lambda)\tilde{u}(\lambda; x)} & 1 \end{pmatrix} \begin{pmatrix} (-1)^n(\nu(\lambda))_n^2 & 0 \\ 0 & (-\nu(\lambda))_n^2 \end{pmatrix} \tag{5.4}$$

for $\lambda \in \partial D_{q,\delta}$. Here we use the standard notation $(\nu)_n = \Gamma(\nu+n)/\Gamma(\nu)$ and $u(\lambda; x)$, resp. $\tilde{u}(\lambda; x)$, have been defined in (4.22), resp. (4.25). Thus, the matrices $\Delta^{(n)}$ depend on x, but their entries are a $\mathrm{O}(x^{\overline{\varepsilon}})$.

5.2 Asymptotic expansion for Π

Using the equivalence between RHP and singular integral equations we can express Π in terms of its boundary value from the "+" side of the contour Σ_Π

$$\Pi(\lambda) = I_2 + \frac{1}{2i\pi} \int_{\Sigma_\Pi} \Pi_+(s)\,\Delta(s)\,\frac{\mathrm{d}s}{\lambda - s}\ . \tag{5.5}$$

In its turn $\Pi_+(\lambda)$ belongs to $L^2(\Sigma_\Pi)$ and fulfills the linear singular integral equation of Cauchy type

$$\Pi_+(z) = I_2 + C^+_{\Sigma_\Pi}[\Pi_+\Delta](z)\ . \tag{5.6}$$

Recall that the Cauchy operator on $L^2(\Sigma_\Pi)$ is defined as

$$C^+_{\Sigma_\Pi}[g](z) = \lim_{t\to z^+} C_{\Sigma_\Pi}[g](t) \qquad \text{and} \qquad C_{\Sigma_\Pi}[g](t) = \frac{1}{2i\pi}\int_{\Sigma_\Pi} \frac{g(s)\,\mathrm{d}s}{t-s},\ \ t \notin \Sigma_\Pi\ . \tag{5.7}$$

The notation $t \to z^+$ stands for the non-tangential limit of t approaching z from the " + " side of the contour Σ_Π. Recall that the Cauchy operator is bounded: i.e. there exists a constant c_2 such that, for any function $g \in L^2(\Sigma_\Pi)$, one has $\left\|C^+_{\Sigma_\Pi}[g]\right\| \le c_2\,\|g\|$, where $\|.\|$ is the canonical $L^2(\Sigma_\Pi)$ norm.

The matrix Π_+ can be asymptotically approximated by the following series:

Proposition 5.1. *Let $\Pi^{(k)}_+$ be defined recursively according to*

$$\Pi^{(k)}_+ = \sum_{p=1}^{k} C^+_{\Sigma_\Pi}\left[\Pi^{(k-p)}_+\,\Delta^{(p)}\right] \quad \textit{with} \quad \Pi^{(0)}_+ = I_2. \tag{5.8}$$

Then, for any integer $M > 0$, there exists a constant $C(M) > 0$ such that

$$\left\|\Pi_+ - \sum_{p=0}^{M-1} x^{-p}\,\Pi^{(p)}_+\right\| \le \frac{C(M)}{x^{M(1-\overline{\varepsilon})}}. \tag{5.9}$$

Proof — Let us prove this statement by induction on M. For $M = 1$ we have that

$$\|\Pi_+ - I_2\| = \left\|C^+_{\Sigma_\Pi}\left[(\Pi_+ - I_2)\,\Delta\right] + C^+_{\Sigma_\Pi}[\Delta]\right\| \le c_2\,\|\Pi_+ - I_2\|\,\|\Delta\| + c_2\,\|\Delta\|\,. \tag{5.10}$$

Therefore, for x large,

$$\|\Pi_+ - I_2\| \le \frac{c_2\,\|\Delta\|}{1 - c_2\,\|\Delta\|} \le \frac{C(1)}{x^{1-\overline{\varepsilon}}}\ . \tag{5.11}$$

Let us now suppose that the result holds up to M. Then,

$$\begin{aligned}
\left\| \Pi_+ - \sum_{k=0}^{M} x^{-k}\, \Pi_+^{(k)} \right\| &= \left\| C^+_{\Sigma_\Pi} \left[\Pi_+ \Delta - \sum_{k=1}^{M} x^{-k} \sum_{p=1}^{k} \Pi_+^{(k-p)} \Delta^{(p)} \right] \right\| \\
&= \left\| C^+_{\Sigma_\Pi} \left[\Pi_+ \left(\Delta - \sum_{k=1}^{M} x^{-k} \Delta^{(k)} \right) + \sum_{p=1}^{M} \left(\Pi_+ - \sum_{k=0}^{M-p} \Pi_+^{(k)} x^{-k} \right) \Delta^{(p)} x^{-p} \right] \right\| \\
&\leq c_2 \, \|\Pi_+\| \, C_{\Delta_M} \, x^{-M-1+\overline{\varepsilon}} + \sum_{p=1}^{M} c_2 \left\| \Delta^{(p)} \right\| \, x^{-p} \, C(M-p) \, x^{(p-M-1)(1-\overline{\varepsilon})} \\
&\leq \frac{C(M+1)}{x^{(M+1)(1-\overline{\varepsilon})}} \,, \qquad (5.12)
\end{aligned}$$

for some constants C_{Δ_M} and $C(M+1)$. We used the fact that all $\Delta^{(p)}$ are in $L^2(\Sigma_\Pi)$ and that $\|\Pi_+\|$ is bounded in virtue of (5.11). □

Let us now extend this result for points λ being uniformly away from the contour Σ_Π. Define the matrices

$$\Pi^{(0)}(z) = I_2, \qquad \Pi^{(p)}(z) = \sum_{k=1}^{p} C_{\Sigma_\Pi} \left[\Pi_+^{(p-k)} \Delta^{(k)} \right](z), \quad p > 0, \qquad (5.13)$$

$$\Pi(z; M) = \sum_{p=0}^{M} x^{-p}\, \Pi^{(p)}(z), \qquad (5.14)$$

which are analytic away from Σ_Π. Then we have the following result:

Proposition 5.2. *Let K be any compact subset of $\mathbb{C} \setminus \Sigma_\Pi$. Then, $\forall k \in \mathbb{N}$, $\forall M \in \mathbb{N}^\star$,*

$$\left| \partial_\lambda^k \Pi(\lambda) - \partial_\lambda^k \Pi(\lambda; M-1) \right| \leq \frac{k!\, C(M)\, \mathrm{lgth}(\Sigma_\Pi)}{d(K, \Sigma_\Pi)^{k+1} x^{M(1-\overline{\varepsilon})}}, \quad \lambda \in K. \qquad (5.15)$$

Here $|.|$ denotes the usual max norm $|\Pi| \equiv \max_{i,j} \left|\Pi_{i,j}\right|$, $d(.,.)$ is any distance on $\mathbb{C}$ and $\mathrm{lgth}(\Sigma_\Pi)$ is the length of the curve Σ_Π.

Proof — Let $k \in \mathbb{N}$, $M \in \mathbb{N}^\star$, then

$$\begin{aligned}
&\left| \partial_\lambda^k \Pi(\lambda) - \sum_{p=0}^{M-1} x^{-p}\, \partial_\lambda^k \Pi^{(p)}(\lambda) \right| \\
&\qquad \leq \left| \frac{k!}{2i\pi} \int_{\Sigma_\Pi} \frac{\mathrm{d}s}{(\lambda - s)^{k+1}} \left\{ \Pi_+(s)\, \Delta(s) - \sum_{p=1}^{M-1} x^{-p} \sum_{l=1}^{p} \Pi_+^{(p-l)}(s)\, \Delta^{(l)}(s) \right\} \right| \\
&\qquad \leq \frac{k!\, C(M)\, \mathrm{lgth}(\Sigma_\Pi)}{x^{M(1-\overline{\varepsilon})}\, d^{k+1}\, (K, \Sigma_\Pi)}
\end{aligned}$$

due to (5.9). □

5.3 The functions $f_\pm$ to the leading order

We now perform the transformations from Π back to χ.

The solution to the RHP of Proposition 3.1 reads

$$\chi(\lambda) = \Pi(\lambda)\,\chi^{(0)}(\lambda)\,. \tag{5.16}$$

We call $\chi^{(0)}$ the zero$^{\text{th}}$ order solution (i.e. obtained for $\Pi = I_2$). In the vicinities of the endpoints of $[\,-q\,;q\,]$, $\chi^{(0)}$ is given as

$$\chi^{(0)}(\lambda) = \begin{cases} \mathcal{P}(\lambda)\,M_+^{-1}(\lambda)\,\alpha(\lambda)^{-\sigma_3}, & \lambda \in D_{-q,\delta} \cap \{0 < \arg[p(\lambda) - p_-] < \pi/2\}\,, \\ \widetilde{\mathcal{P}}(\lambda)\,M_+^{-1}(\lambda)\,\alpha(\lambda)^{-\sigma_3}, & \lambda \in D_{q,\delta} \cap \{\pi/2 < \arg[p(\lambda) - p_+] < \pi\}\,. \end{cases}$$

Similarly, on $[\,-q\,;q\,]$, and uniformly away from the endpoints,

$$\chi^{(0)}(\lambda) = M_+^{-1}(\lambda)\,\alpha_+(\lambda)^{-\sigma_3}, \qquad \lambda \in \,]\,-q+\delta\,;q-\delta\,[\ .$$

In the $\Im\,(\lambda) = 0^+$ limit and for $\Re\,(\lambda) \in [\,-q\,;q\,]$,

$$M_+^{-1}\alpha_+^{-\sigma_3}\begin{pmatrix} e_+\,(\lambda) \\ e_-\,(\lambda) \end{pmatrix} = (\alpha_+ e_-)^{-\sigma_3}\begin{pmatrix} \mathrm{e}^{2i\pi\nu} \\ 1 \end{pmatrix}, \tag{5.17}$$

so that, for $\lambda \in \,]\,-q+\delta\,;q+\delta\,[$,

$$\begin{pmatrix} f_+^{(0)}\,(\lambda) \\ f_-^{(0)}\,(\lambda) \end{pmatrix} = \mathrm{e}^{i\pi\nu(\lambda)}\left[\alpha_+^{-1}\,(\lambda)\,e_+\,(\lambda)\,\mathrm{e}^{i\pi\nu(\lambda)}\right]^{\sigma_3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{5.18}$$

where we have explicitly written all the dependence on λ.

When $\lambda \in [\,-q\,;-q+\delta\,]$, we should multiply the latter expression by $\mathcal{P}$. Using the decomposition (A.7) of Humbert CHF into a sum of two Tricomi CHF we get

$$\begin{pmatrix} f_+^{(0)}\,(\lambda) \\ f_-^{(0)}\,(\lambda) \end{pmatrix} = \mathrm{e}^{\frac{i\pi\nu}{2}}\left(\frac{e_+(\lambda)}{\kappa_p\,(\lambda)\,\zeta_q^\nu}\right)^{\sigma_3}\begin{pmatrix} \Gamma\,(1+\nu)\,\Phi\left(-\nu,1;-i\zeta_{-q}\right) \\ \Gamma\,(1-\nu)\,\Phi\left(\nu,1;i\zeta_{-q}\right) \end{pmatrix}, \tag{5.19}$$

with $\zeta_q = x\,[p_+ - p\,(\lambda)]$ and $\zeta_{-q} = x\,[p\,(\lambda) - p_-]$.

Analogously, for $\lambda \in [\,q-\delta\,;q\,]$,

$$\begin{pmatrix} f_+^{(0)}\,(\lambda) \\ f_-^{(0)}\,(\lambda) \end{pmatrix} = \mathrm{e}^{\frac{i\pi\nu}{2}}\left(\frac{e_+(\lambda)\zeta_{-q}^\nu}{\kappa_p\,(\lambda)}\right)^{\sigma_3}\begin{pmatrix} \Gamma\,(1-\nu)\,\Phi\left(\nu,1;i\zeta_q\right) \\ \Gamma\,(1+\nu)\,\Phi\left(-\nu,1;-i\zeta_q\right) \end{pmatrix}. \tag{5.20}$$

Note that the piecewise expressions for the functions $f_\pm^{(0)}\,(\lambda)$ are in fact analytic in a vicinity of their respective domain of validity, although they have been obtained by taking the limit of λ approaching a point in $[\,-q\,;q\,]$ from the upper half plane. More precisely, the formula (5.19) holds on $D_{-q,\delta}$, (5.20) on $D_{q,\delta}$, and (5.18) on the connected component of the interior of Σ_π containing $[\,\delta-q\,;q-\delta\,]$. This observation follows from (3.3), but of course it can be checked by a direct computation based on the expression for the matrix χ in the lower half plane.

5.4 Integral bounds for the resolvent

We now introduce a function $R^{(0)}(\lambda,\mu)$ and show that it is a good approximation of the resolvent in the sense that

$$\operatorname{tr}\left(R - R^{(0)}\right) = \mathrm{O}\left(x^{\overline{\varepsilon}-1}\right) . \tag{5.21}$$

Such estimates are necessary for the integration of the γ-derivative of $\log\det[I+V]$.

Definition 5.1. *Let $\Pi_\tau(\lambda)$ denote the solution of the RHP given in Subsection 4.5 whose jumps are on circles of radius τ and on the corresponding curves Γ'_+ and Γ'_-.*

We can then write the solution of the RHP for χ as $\chi(\lambda) = \Pi_\tau \chi_\tau^{(0)}$. There $\chi_\tau^{(0)}$ do not depend explicitly on τ. The radius τ only determines which patch we should use for the definition of the matrix $\chi_\tau^{(0)}$. Moreover the whole combination $\Pi_\tau\chi_\tau^{(0)}$ does not depend on the radius τ at all. Hence, we can represent the exact resolvent as

$$R(\lambda,\mu) = \langle\, E^L(\lambda) \,|\left[\chi_\tau^{(0)}(\lambda)\right]^{-1} \frac{\Pi_\tau^{-1}(\lambda)\,\Pi_\tau(\mu)}{\lambda-\mu}\chi_\tau^{(0)}(\mu)\,|\, E^R(\mu)\,\rangle\ . \tag{5.22}$$

There, without altering the value of $R(\lambda,\mu)$, we can chose different values of τ depending on the point we consider. This is quite useful as we can take one value of the radius τ in order to have estimates around $\pm q$ and another one to perform estimates in the bulk $[\,\delta-q\,;q-\delta\,]$. This will become clearer during the proof of the proposition below.

Definition 5.2. *Let us fix δ, $q > \delta > 0$ and define what we call the diagonal zero$^{\text{th}}$ order resolvent*

$$R^{(0)}(\lambda,\lambda) = \frac{\gamma F(\lambda)}{2i\pi}\left(\partial_\lambda f_+^{(0)}(\lambda)\, f_-^{(0)}(\lambda) - \partial_\lambda f_-^{(0)}(\lambda)\, f_+^{(0)}(\lambda)\right), \tag{5.23}$$

where the functions $f_\pm^{(0)}(\lambda)$ are given by (5.18) *for $\lambda\in[\,\delta-q\,;q-\delta\,]$,* (5.19) *for $\lambda\in[\,-q\,;\delta-q\,[$ and* (5.20) *for $\lambda\in\,]\,q-\delta\,;q\,]$. Similarly, $|\,F^{R;(0)}(\lambda)\,\rangle$ and $\langle\, F^{L;(0)}(\lambda)\,|$ are defined in terms of the same functions $f_\pm^{(0)}$.*

We stress that the radius τ previously introduced to build the exact solution $\Pi_\tau(\lambda)\chi_\tau^{(0)}(\lambda)$ and δ appearing in the definition are, a priori, unrelated.

Proposition 5.3. *Let $R(\lambda,\mu)$ be the exact resolvent of the generalised sine kernel. Then*

$$\operatorname{tr}\left(R - R^{(0)}\right) = \mathrm{O}\left(x^{\overline{\varepsilon}-1}\right), \tag{5.24}$$

where the O *is uniform in $\gamma\in D_{0,r}$.*

Proof — According to the preceding observations we have, for $\lambda\in[\,-q\,;-q+\delta\,[\,\cup\,]\,q-\delta\,;q\,]$,

$$R(\lambda,\lambda) = R^{(0)}(\lambda,\lambda) + \langle\, F^{L;(0)}(\lambda)\,|\Pi_{2\delta}(\lambda)\,\partial_\lambda\Pi_{2\delta}(\lambda)\,|\,F^{R;(0)}(\lambda)\,\rangle, \tag{5.25}$$

and

$$R(\lambda,\lambda) = R^{(0)}(\lambda,\lambda) + \langle\, F^{L;(0)}(\lambda)\,|\Pi_{\delta/2}(\lambda)\,\partial_\lambda\Pi_{\delta/2}(\lambda)\,|\,F^{R;(0)}(\lambda)\,\rangle, \tag{5.26}$$

for $\lambda \in [\,\delta - q\,;q - \delta\,]$. The advantage of using two different matrices Π for the corrections of $R(\lambda,\lambda)$ with respect to the zero$^{\text{th}}$ order resolvent $R^{(0)}(\lambda,\lambda)$ is that the corrections are always analytic on the whole domain where they are considered. One does not need to take into account that $\Pi_\delta(\lambda)$ has a jump across $\lambda = \pm(q-\delta)$. This might be problematic as, for instance, the integral of $\partial_\lambda \Pi(\lambda)$ on $[\,-q\,;\delta - q\,]$ might be ill-defined. Moreover the uniform estimates that we have derived for the matrix $\Pi(\lambda)$ only hold uniformly away from the jump contour. As we will only integrate the terms containing $\Pi_{2\delta}$ on $[\,-q\,;\delta - q\,[\,\cup\,]\,q-\delta\,;q\,]$, we will be in this situation. The same holds for the terms involving $\Pi_{\delta/2}$. However, we would not be able to use the uniform estimates (5.2) for $\partial_\lambda \Pi_\delta$ when integrating it on $[\,-q\,;\delta - q\,]$, as we would not always be uniformly away from the boundary of the jump contour for Π_δ.

With this way of understanding the corrections we have

$$\operatorname{tr}\left(R - R^{(0)}\right) = \left(\int_{-q}^{\delta-q} + \int_{q-\delta}^{q}\right) \mathrm{d}\lambda \,\langle\, F^{L;(0)}(\lambda)\, |\Pi_{2\delta}(\lambda)\,\partial_\lambda \Pi_{2\delta}(\lambda)\,|\, F^{R;(0)}(\lambda)\,\rangle$$
$$+ \int_{\delta-q}^{q-\delta} \mathrm{d}\lambda \,\langle\, F^{L;(0)}(\lambda)\, |\Pi_{\delta/2}(\lambda)\,\partial_\lambda \Pi_{\delta/2}(\lambda)\,|\, F^{R;(0)}(\lambda)\,\rangle. \quad (5.27)$$

Let us start by the bulk part of integral, i.e. the part on $[\,\delta - q\,;q-\delta\,]$. From the explicit form for $f_\pm^{(0)}$ on $[\,\delta - q\,;q-\delta\,]$ given in (5.18) we see that these functions are uniformly $\mathrm{O}(1)$. Moreover, the uniform estimates for the matrices $\Pi_{\delta/2}(\lambda)$ for λ uniformly away from the jump contour guarantee that

$$\langle\, F^{L;(0)}(\lambda)\, |\Pi_{\delta/2}(\lambda)\,\partial_\lambda \Pi_{\delta/2}(\lambda)\,|\, F^{R;(0)}(\lambda)\,\rangle = \mathrm{O}\left(x^{\overline{\varepsilon}-1}\right), \quad (5.28)$$

the $\mathrm{O}\left(x^{\overline{\varepsilon}-1}\right)$ being uniform in γ, at least for γ small enough.

The situation at the boundaries is a little more complex. We only consider the right boundary as the other case is treated similarly. We still have that $\Pi_{2\delta}(\lambda) = I_2 + \mathrm{O}\left(x^{\overline{\varepsilon}-1}\right)$ and $\partial_\lambda \Pi_{2\delta}(\lambda) = \mathrm{O}\left(x^{\overline{\varepsilon}-1}\right)$ uniformly on $[\,q-\delta\,;q\,]$. However the functions $f_\pm^{(0)}(\lambda)$ are no longer uniformly a $\mathrm{O}(1)$ on this interval. We should thus estimate the following integral

$$\sum_{\sigma,\sigma'=\pm} \int_{q-\delta}^{q} f_\sigma^{(0)}(\lambda)\, f_{\sigma'}^{(0)}(\lambda)\, G_{\sigma,\sigma'}(\lambda)\,\mathrm{d}\lambda \quad (5.29)$$

with $G_{\sigma,\sigma'}(\lambda) = \mathrm{O}\left(x^{\overline{\varepsilon}-1}\right)$ being related to the entries of $\Pi_{\delta/2}(\lambda)\,\partial_\lambda \Pi_{\delta/2}(\lambda)$. The situation being similar for all the possible choices of σ and σ', we explain the mechanism for $(\sigma,\sigma') = (+,+)$. The asymptotics of Humbert CHF guarantees that

$$\Phi(a,1;\pm it) = \frac{c_\pm}{|t|^a}\,(1 + \mathrm{o}(1)) \quad t \to +\infty \quad (5.30)$$

for some computable constants $c_\pm$ depending on a. These constants are continuous with respect to a belonging to an open neighbourhood of $\nu([\,q-\delta\,;q\,])$, and so is the $\mathrm{o}(1)$ term. Hence, there exist an a independent constant C such that

$$\left|(1+|t|)^{\Re(a)}\,\Phi(a,1;\pm it)\right| \leq C \quad . \quad (5.31)$$

Indeed the latter function is continuous on $\mathbb{R}$ and has a finite limit at ∞. Moreover the constant C can be chosen in such a way that the estimate holds for a belonging to some small vicinity of $\nu([\,q-\delta\,;q\,])$. Hence, by explicitly extracting the $x^{\overline{\varepsilon}-1}$ factor coming from $G_{+,+}(\lambda)$ we get that, for some constant C',

$$\left|f_+^{(0)}(\lambda)\, f_+^{(0)}(\lambda)\, G_{+,+}(\lambda)\right| \leq C' x^{\overline{\varepsilon}-1} \varphi_x\left(p(\lambda)-p_+\right) \ , \tag{5.32}$$

with $\varphi_x(t) = x^{2\Re(\nu(\lambda))}\left(1+x|t|\right)^{-2\Re(\nu(\lambda))}$. The function $\varphi_x(t)$ fulfills

$$|\varphi_x(p(\lambda)-p_+)| \leq \tilde{C}\,|p(\lambda)-p_+|^{-2\Re(\nu(\lambda))} \tag{5.33}$$

as, for any $\alpha \in \mathbb{R}$, $t \mapsto t^\alpha/(1+t^\alpha)$ is bounded. The latter function is integrable on $[\,q-\delta\,;q\,]$ (we consider the case $\left|\Re(\nu)\right| < 1/2$). Thus the integrals in (5.29) do eventually yield $\mathrm{O}\left(x^{\overline{\varepsilon}-1}\right)$ contributions. □

One can prove, in a very similar way, the estimates for the Hilbert–Schmidt norm of the resolvent. Namely,

Proposition 5.4. *Under the assumptions of the previous proposition,*

$$\left\|R-R^{(0)}\right\|_2 = \mathrm{O}\left(x^{\overline{\varepsilon}-1}\right) \tag{5.34}$$

with $\|.\|_2$ being the Hilbert–Schmidt norm.

5.5 Asymptotic expansion of the resolvent

We now prove that the asymptotic expansion for Π can be used to obtain an asymptotics expansion for the diagonal of the resolvent $R(\lambda,\lambda)$. We derive point-wise bounds for the latter as this quantity appears in the q-derivative of the Fredholm determinant:

$$\partial_q \log\det\left[I+V\right] = R(q,q)+R(-q,-q) \ . \tag{5.35}$$

We need to estimate the error when we replace the exact resolvent R by the approximate one $R^{(0)}$. The magnitude for the error term follows from the following result:

Proposition 5.5. *Let $\chi^{(0)}$ be the solution of the RHP for χ up to the leading order in x, that is to say the one obtained from $\Pi = I_2$ and corresponding to the contour Σ_Π with disks $D_{\pm q,\delta}$ having radius δ. Define the leading vectors $\langle\, F^{L;(0)}\,|$ and $|\, F^{R;(0)}\,\rangle$ as*

$$\langle\, F^{L;(0)}(\lambda)\,| = \langle\, E^L(\lambda)\,|\chi^{(0)}(\lambda)^{-1}, \qquad |\, F^{R;(0)}(\lambda)\,\rangle = \chi^{(0)}(\lambda)\,|\, E^R(\lambda)\,\rangle\,, \tag{5.36}$$

and the leading order of the resolvent by

$$R^{(0)}(\lambda,\mu) = \frac{\langle F^{L;(0)}(\lambda)\,|\,F^{R;(0)}(\mu)\,\rangle}{\lambda-\mu} \ . \tag{5.37}$$

Then

$$R(\lambda,\lambda) = R^{(0)}(\lambda,\lambda) + \sum_{p=1}^{k}\frac{R^{(p)}(\lambda,\lambda)}{x^p} + \mathrm{O}\left(\frac{x^{\overline{\varepsilon}}}{x^{(k+1)(1-\overline{\varepsilon})}}\right)\,, \tag{5.38}$$

for λ uniformly away from Σ_Π and belonging to $[\,-q\,;q\,]$. Here,

$$R^{(0)}(\lambda,\lambda) = -\langle F^{L;(0)}(\lambda) \mid \partial_\lambda F^{R;(0)}(\lambda)\rangle, \tag{5.39}$$

$$R^{(p)}(\lambda,\lambda) = -\langle\, F^{L;(0)}(\lambda)\,|\hat{\Pi}^{(p)}(\lambda)|\,F^{R;(0)}(\lambda)\,\rangle, \quad p>0, \tag{5.40}$$

in which

$$\Pi^{-1}(\lambda;k)\,\partial_\lambda\Pi(\lambda;k) = \sum_{p=1}^{k}\hat{\Pi}^{(p)}(\lambda)\,x^{-p} + \mathrm{O}\left(\frac{1}{x^{(k+1)(1-\overline{\varepsilon})}}\right). \tag{5.41}$$

Proof — Clearly,

$$R(\lambda,\mu) = \frac{\langle F^{L;(0)}(\lambda) \mid F^{R;(0)}(\mu)\rangle}{\lambda-\mu} + \langle\, F^{L;(0)}(\lambda)\,|\frac{\Pi^{-1}(\lambda)\,\Pi(\mu)-I_2}{\lambda-\mu}|\,F^{R;(0)}(\mu)\,\rangle,$$
$$\underset{\lambda\to\mu}{\to} -\langle F^{L;(0)}(\lambda) \mid \partial_\lambda F^{R;(0)}(\lambda)\rangle - \langle\, F^{L;(0)}(\lambda)\,|\Pi^{-1}(\lambda)\,\partial_\lambda\Pi(\lambda)|\,F^{R;(0)}(\mu)\,\rangle.$$

The corrections to the leading order for the resolvent are given here by the second term.

The inversion operator on $\mathcal{M}_2(\mathbb{C})$: $u\mapsto u^{-1}$ is continuously differentiable around the identity I_2. Thus there exists an open neighbourhood W of the identity matrix I_2 and a constant $C>0$ such that, $\forall\ A,\ B\ \in W$, one has $\left\|A^{-1}-B^{-1}\right\| \le C\,\|A-B\|$. Here $\|.\|$ denotes any matrix norm. The matrices $\Pi(\lambda)$ and $\Pi(\lambda;k)$ belong to W for x sufficiently large, as they both go to I_2 in the $x\to+\infty$ limit for λ uniformly away from Σ_Π, and we get, from Proposition 5.2,

$$\begin{aligned}\left\|\Pi^{-1}(\lambda)\,\partial_\lambda\Pi(\lambda) - \Pi^{-1}(\lambda;k)\,\partial_\lambda\Pi(\lambda;k)\right\| &\\ \le C\,\|\Pi(\lambda)-\Pi(\lambda;k)\|\,\|\partial_\lambda\Pi(\lambda)\| &\\ + C\,(\|\Pi(\lambda)\| + \|\Pi(\lambda)-\Pi(\lambda;k)\|)\,\|\partial_\lambda\Pi(\lambda)-\partial_\lambda\Pi(\lambda;k)\| &\\ \le \frac{\widetilde{C}(k)}{x^{(k+1)(1-\overline{\varepsilon})}},&\end{aligned}$$

for some constant $\widetilde{C}(k)$. Thus, uniformly away from Σ_Π and on the real axis, one has

$$\left|\langle\, F^{L;(0)}(\lambda)\,|\Pi^{-1}(\lambda)\,\partial_\lambda\Pi(\lambda) - \Pi^{-1}(\lambda;k)\,\partial_\lambda\Pi(\lambda;k)|\,F^{R;(0)}(\lambda)\,\rangle\right| = \mathrm{O}\left(\frac{x^{\overline{\varepsilon}}}{x^{(k+1)(1-\overline{\varepsilon})}}\right). \tag{5.42}$$

In the last equality, we used the fact that $f_\pm^{(0)}$ are at most of order $\mathrm{O}(x^{\overline{\varepsilon}})$ on the real axis, as follows from their behaviour around $\pm q$. □

6 Leading asymptotic behaviour of $\log\det[I+V]$

In this Section, we prove the result of Theorem 2.1; that is to say, we compute the leading asymptotic behaviour $\det[I+V]^{(0)}$ of $\det[I+V]$ up to $\mathrm{o}(1)$ corrections in the $x\to+\infty$ limit.

More precisely, we show that

$$\log\det\left[I+V\right]^{(0)} = 2\int_{-q}^{q} \mathrm{d}\lambda\, \nu(\lambda)\log'[e_-(\lambda)] + \sum_{\sigma=\pm}\log\left[\frac{G(1,\nu_\sigma)\,\kappa^{\sigma\nu_\sigma}(\sigma q;q)}{(2qp'_\sigma x)^{\nu_\sigma^2}}\right] + \frac{1}{2}\int_{-q}^{q}\mathrm{d}\lambda\,\mathrm{d}\mu\,\frac{\nu'(\lambda)\,\nu(\mu)-\nu(\lambda)\,\nu'(\mu)}{\lambda-\mu}\,. \quad (6.1)$$

This result will be obtained by two different methods based on the integration of equations (3.13). The first one, which uses the derivative of the Fredholm determinant over γ, is based on the uniformness of the asymptotic expansion for the resolvent for γ small enough. It is worth mentioning that this way is technically quite involved. The second method deals with the derivative of the Fredholm determinant over q. Although we have not been able to provide a full rigorous proof for it, we would like to draw the reader's attention to this method as it is much more direct and simple.

6.1 The leading asymptotics from the γ-derivative method

Due to Proposition 5.3, the proof of the leading asymptotics of the Fredholm determinant from the first equation (3.13),

$$\partial_\gamma \log\det\left[I+V\right] = \int_{-q}^{q}\frac{\mathrm{d}\lambda}{\gamma}R\left(\lambda,\lambda\right), \quad (6.2)$$

only necessitates the use of $R^{(0)}\left(\lambda,\lambda\right)$ defined in (5.37). Recall that $R^{(0)}\left(\lambda,\lambda\right)$ has different leading asymptotics in the bulk $\left]-q\,;q\right[$ and near the boundary. Let $\delta>0$ be sufficiently small. Then

$$\frac{R^{(0)}(\lambda,\lambda)}{\gamma} = \begin{cases} R_q^{(0)}\left(\lambda,\lambda\right) & \lambda\in\left[\,q-\delta\,;q\,\right],\\ R_{\mathrm{bk}}^{(0)}\left(\lambda,\lambda\right) & \lambda\in\left[\,-q+\delta\,;q-\delta\,\right],\\ R_{-q}^{(0)}\left(\lambda,\lambda\right) & \lambda\in\left[\,-q\,;-q+\delta\,\right], \end{cases} \quad (6.3)$$

where

$$R_{\mathrm{bk}}^{(0)}(\lambda,\lambda) = \frac{F(\lambda)}{2i\pi\left(1+\gamma F(\lambda)\right)}\left\{2\partial_\lambda\log e_+(\lambda) - 2\partial_\lambda\log\left[\kappa_p(\lambda)\left(\frac{p_+-p(\lambda)}{p(\lambda)-p_-}\right)^{\nu(\lambda)}\right]\right\},$$

$$R_{-q}^{(0)}\left(\lambda,\lambda\right) = -\nu\varphi\left(\nu;x\left[p-p_-\right]\right)\Big\{2\nu'\log x - 2\partial_\lambda\Big[\nu\log\left(p_+-p\right)-2\partial_\lambda\log\kappa_p\Big] + \nu'\left[\psi\left(1+\nu\right)+\psi\left(1-\nu\right)\right]+g'\Big\} + ix\nu p'\tau\left(\nu;x\left[p-p_-\right]\right)+\nu\nu'\rho\left(\nu;x\left[p-p_-\right]\right),$$

$$R_{q}^{(0)}\left(\lambda,\lambda\right) = -\nu\varphi\left(\nu;x\left[p_+-p\right]\right)\Big\{2\nu'\log x - 2\partial_\lambda\Big[\nu\log\left(p-p_-\right)-2\partial_\lambda\log\kappa_p\Big] + \partial_\lambda\nu\left[\psi\left(1+\nu\right)+\psi\left(1-\nu\right)\right]+g'\Big\} + ix\nu p'\tau\left(\nu;x\left[p_+-p\right]\right)+\nu\nu'\rho\left(\nu;x\left[p_+-p\right]\right).$$

Here $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ and we have introduced the shorthand notations

$$\begin{aligned}
\varphi(\nu;t) &= \Phi(-\nu,1;-it)\,\Phi(\nu,1;it)\,,\\
\rho(\nu;t) &= (\partial_1\Phi)(\nu,1;it)\,\Phi(-\nu,1;-it) + (\partial_1\Phi)(-\nu,1;-it)\,\Phi(\nu,1;it)\,,\\
\tau(\nu;t) &= -\Phi(-\nu,1;-it)\,\Phi(\nu,1;it) + (\partial_z\Phi)(-\nu,1;-it)\,\Phi(\nu,1;it)\\
&\qquad + \Phi(-\nu,1;-it)\,(\partial_z\Phi)(\nu,1;it)\,.
\end{aligned}$$

Moreover, in order to lighten the above expressions and similar ones in the following, we omit the explicit dependence on the argument λ of the different functions involved (like ν, p, and their derivatives ν', p', etc.).

We can now split the integration contour into three parts

$$\int_{-q}^{q} R^{(0)}(\lambda,\lambda)\,\frac{\mathrm{d}\lambda}{\gamma} = \int_{-q}^{-q+\delta} \frac{\mathrm{d}\lambda}{\gamma} R^{(0)}_{-q}(\lambda,\lambda) + \int_{-q+\delta}^{q-\delta} \frac{\mathrm{d}\lambda}{\gamma} R^{(0)}_{\mathrm{bk}}(\lambda,\lambda) + \int_{q-\delta}^{q} \frac{\mathrm{d}\lambda}{\gamma} R^{(0)}_{q}(\lambda,\lambda)\,. \tag{6.4}$$

The bulk integral is carried out straightforwardly. The integrals over the vicinities of the endpoints are more involved. Consider, for instance, the integration over $[\,-q\,;-q+\delta\,]$.

Using the asymptotic series for Humbert CHF Φ (A.9) and the equations (A.10), (A.11) we get that

$$\begin{aligned}
&\varphi(a;t) - \mathrm{e}^{i\pi a}\,,\\
&\rho(a;t) + \frac{\mathrm{e}^{i\pi a}}{\Gamma(1-a)\,\Gamma(1+a)}\left\{2\log t - \psi(1-a) - \psi(1+a) - \frac{4ia}{1+t}\right\}\,,\\
&\tau(a;t) + \mathrm{e}^{i\pi a}\left(1 - \frac{2ia}{1+t}\right)\,,
\end{aligned}$$

are uniformly Riemann integrable on $\mathbb{R}^+$ in the sense of the definition of Lemma B.1 (See Appendix B). Using the integration Lemma B.1 as well as the estimates for the integrals of τ and φ (A.10), (A.11), we find

$$\begin{aligned}
\int_{-q}^{-q+\delta} R^{(0)}(\lambda,\lambda)\,\mathrm{d}\lambda = &-\int_{-q}^{-q+\delta} \frac{\mathrm{d}\lambda\;\mathrm{e}^{i\pi\nu}}{\Gamma(\nu)\,\Gamma(1-\nu)}\left\{2\nu'\log x - 2\partial_\lambda\left[\nu\log(p_+ - p)\right]\right.\\
&\qquad\qquad \left. +\nu'\left(\psi(\nu) + \psi(-\nu)\right) + g'\right\}\\
&+\int_{-q}^{-q+\delta} \frac{\mathrm{d}\lambda\;\mathrm{e}^{i\pi\nu}\nu'}{\Gamma(\nu)\,\Gamma(1-\nu)}\left\{2\log\left[x(p-p_-)\right] - \psi(-\nu) - \psi(\nu) + \frac{4i\nu}{1+x(p-p_-)}\right\}\\
&+ix\int_{-q}^{-q+\delta} \mathrm{d}\lambda\;p'\nu\left\{\frac{2i\nu}{x(p-p_-)+1} - 1\right\}\\
&\qquad\qquad + \frac{\nu_-\mathrm{e}^{i\pi\nu_-}}{\Gamma(1-\nu_-)\,\Gamma(\nu_-)}\left\{2 - \psi(\nu_-) - \psi(-\nu_-)\right\} + \mathrm{o}(1)\;.
\end{aligned} \tag{6.5}$$

Here the $\mathrm{o}(1)$ is with respect to the successive limits $x\delta \to +\infty$ and $\delta \to 0$. The two terms proportional to $\log x$ compensate each other. The remaining part of the first three lines of (6.5) is an $\mathrm{O}(\delta)$ and can thus be dropped. The integral in the last two lines of (6.5) is evaluated thanks to the second integration Lemma B.2 (see Appendix B). We get,

$$\int_{-q}^{-q+\delta} R^{(0)}(\lambda,\lambda)\,\mathrm{d}\lambda = -i\int_{-q}^{-q+\delta} \frac{\mathrm{e}^{i\pi\nu}p'}{\Gamma(\nu)\,\Gamma(1-\nu)}\mathrm{d}\lambda$$
$$+\frac{\mathrm{e}^{i\pi\nu_-}\nu_-}{\Gamma(\nu_-)\,\Gamma(1-\nu_-)}\left\{-2\log\left[x\left(p\left(\delta-q\right)-p_-\right)\right]+2-\psi(\nu_-)-\psi(-\nu_-)\right\}+\mathrm{o}(1)\ . \quad (6.6)$$

The integration over $[\,q-\delta\,;q\,]$ can be treated similarly. The result reads

$$\int_{q-\delta}^{q} R^{(0)}(\lambda,\lambda)\,\mathrm{d}\lambda = -i\int_{q-\delta}^{q} \frac{\mathrm{e}^{i\pi\nu}p'}{\Gamma(\nu)\,\Gamma(1-\nu)}$$
$$+\frac{\nu_+\mathrm{e}^{i\pi\nu_+}}{\Gamma(\nu_+)\,\Gamma(1-\nu_+)}\left\{-2\log\left[x\left(p_+-p\left(q-\delta\right)\right)\right]+\left[2-\psi(\nu_+)-\psi(-\nu_+)\right]\right\}+\mathrm{o}(1)\ . \quad (6.7)$$

So that,

$$\int_{-q}^{q} R^{(0)}(\lambda,\lambda)\ \mathrm{d}\lambda = \int_{-q}^{q}\frac{\gamma F}{2i\pi\,(1+\gamma F)}\left[ixp'+g'-2\partial_\lambda\log\kappa_p\right]\mathrm{d}\lambda$$
$$+\int_{\delta-q}^{q-\delta}\nu\log\left(\frac{p_+-p}{p-p_-}\right)\partial_\lambda\left\{\frac{\gamma F}{1+\gamma F}\right\}\frac{\mathrm{d}\lambda}{i\pi}-\left[\frac{\gamma\nu F}{i\pi\,(1+\gamma F)}\log\left(\frac{p_+-p}{p-p_-}\right)\right]_{\delta-q}^{q-\delta}$$
$$+\frac{\gamma\nu_-F_-}{2i\pi\,(1+\gamma F_-)}\left\{2\log\left[x\left(p\left(\delta-q\right)-p_-\right)\right]-2+\psi(\nu_-)+\psi(-\nu_-)\right\}$$
$$+\frac{\gamma\nu_+F_+}{2i\pi\,(1+\gamma F_+)}\left\{2\log\left[x\left(p\left(q-\delta\right)-p_+\right)\right]-2+\psi(\nu_+)+\psi(-\nu_+)\right\}+\mathrm{o}(1)\,, \quad (6.8)$$

where we used $\dfrac{\mathrm{e}^{i\pi\nu}}{\Gamma(\nu)\,\Gamma(1-\nu)} = -\dfrac{\gamma F}{2i\pi\,(1+\gamma F)}$. Using the integral representation for the Barnes G-function (2.7), it is not a problem to see that

$$\int_{-q}^{q} R^{(0)}(\lambda,\lambda)\,\frac{\mathrm{d}\lambda}{\gamma} = \partial_\gamma\left\{\int_{-q}^{q}\nu\,\partial_\lambda\log e_-\,\mathrm{d}\lambda+\log\left(\frac{G(1,\nu_+)\,G(1,\nu_-)}{\left[x\left(p_+-p_-\right)\right]^{\nu_+^2+\nu_-^2}}\right)\right\}$$
$$+2\int_{-q}^{q}\partial_\gamma\nu\,\partial_\lambda\log\kappa_p\,\mathrm{d}\lambda-2\int_{-q}^{q}\left[\partial_\lambda\partial_\gamma\nu\right]\nu\log\left(\frac{p_+-p}{p-p_-}\right)\mathrm{d}\lambda\ . \quad (6.9)$$

Now we should recast the last line as a derivative with respect to γ. We have

$$
\begin{aligned}
&2\int_{-q}^{q}\left[\partial_\gamma \nu \partial_\lambda \log \kappa_p - \left[\partial_\lambda \partial_\gamma \nu\right] \nu \log\left(\frac{p_+ - p}{p - p_-}\right)\right] \mathrm{d}\lambda \\
&= \partial_\gamma \log\left\{\frac{\kappa^{\nu_+}(q;q)}{\kappa^{\nu_-}(-q;q)}\left(\frac{p_+ - p_-}{2qp'_+}\right)^{2\nu_+^2}\left(\frac{p_+ - p_-}{2qp'_-}\right)^{2\nu_-^2}\right\} \\
&+ \sum_{\sigma=\pm}\left[\sigma\partial_\gamma \nu_\sigma \log \kappa(\sigma q;q) - \sigma \nu_\sigma \partial_\gamma \log \kappa(\sigma q;q)\right] \\
&\qquad\qquad - 2\int_{-q}^{q}\left[\partial_\gamma\partial_\lambda \nu\right]\left[\log \kappa + \nu \log\left(\frac{q-\lambda}{q+\lambda}\right)\right] \mathrm{d}\lambda \ . \qquad (6.10)
\end{aligned}
$$

It remains to apply the identity

$$
\begin{aligned}
\partial_\gamma \int_{-q}^{q} \frac{\nu'(\lambda)\nu(\mu) - \nu'(\mu)\nu(\lambda)}{2(\lambda-\mu)} \mathrm{d}\lambda\, \mathrm{d}\mu &= -2\int_{-q}^{q}\left[\partial_\gamma\partial_\lambda \nu\right]\left[\log \kappa + \nu \log\left(\frac{q-\lambda}{q+\lambda}\right)\right] \mathrm{d}\lambda \\
&+ \sum_{\sigma=\pm}\left[\sigma\partial_\gamma \nu_\sigma \log \kappa(\sigma q;q) - \sigma \nu_\sigma \partial_\gamma \log \kappa(\sigma q;q)\right]. \qquad (6.11)
\end{aligned}
$$

Indeed, we have for the r.h.s. of (6.11)

$$
\begin{aligned}
RHS \ &= \int_{-q}^{q}\left(\frac{\nu_+\partial_\gamma \nu - \nu\,\partial_\gamma \nu_+}{q-\lambda} + \frac{\nu_-\partial_\gamma \nu - \nu\partial_\gamma \nu_-}{q+\lambda}\right) \mathrm{d}\lambda \\
&\qquad + \int_{-q}^{q} \nu(\mu)\,\partial_\gamma\partial_\lambda \nu(\lambda)\left\{\frac{1}{\lambda-\mu+i0} + \frac{1}{\lambda-\mu-i0}\right\} \mathrm{d}\lambda\, \mathrm{d}\mu \\
&= \frac{1}{2}\int_{-q}^{q}\sum_{\sigma=\pm}\left(\nu_\sigma\partial_\gamma \nu + \nu\partial_\gamma \nu_\sigma\right)\left\{\frac{1}{q-\sigma\lambda+i0} + \frac{1}{q-\sigma\lambda-i0}\right\} \mathrm{d}\lambda \\
&\qquad + \int_{-q}^{q} \nu(\mu)\,\partial_\gamma \nu(\lambda)\left\{\frac{1}{(\lambda-\mu+i0)^2} + \frac{1}{(\lambda-\mu-i0)^2}\right\} \mathrm{d}\lambda\, \mathrm{d}\mu\,. \qquad (6.12)
\end{aligned}
$$

There we have regularized all the integrands and then performed an integration by parts. On the other hand, one has for the l.h.s. of (6.11)

$$
LHS \ = \int_{-q}^{q}\left[\partial_\gamma \nu(\mu)\,\partial_\lambda \nu(\lambda) + \nu(\mu)\,\partial_\gamma\partial_\lambda \nu(\lambda)\right]\left\{\frac{1}{\lambda-\mu+i0} + \frac{1}{\lambda-\mu-i0}\right\} \mathrm{d}\lambda\, \mathrm{d}\mu\,. \qquad (6.13)
$$

Taking the last integral by parts we arrive at (6.12).

Thus, the l.h.s. of (6.8) is presented as a derivative with respect to γ. Since the asymptotic expansion is uniform in γ we can integrate this result from 0 to γ. As $\log\det[I+V]\,|_{\gamma=0}=0$ we get the desired result.

6.2 The leading asymptotics from the q-derivative method

The method we use here is based on the second equation in (3.13),

$$\partial_q \log\det[I+V] = R(q,q) + R(-q,-q)\,. \tag{6.14}$$

For the purpose of this sub-section, we assume that $\left|\Re\left(\nu(\lambda)\right)\right| < 1/4$. Indeed we are then able to use the pointwise estimates for the resolvent established in Proposition 5.5. Such a restriction on $\left|\Re\left(\nu(\lambda)\right)\right|$ could be relaxed by much more refined estimates. Recall that one has for λ uniformly away from the boundary Σ_Π corresponding to disks of radius δ,

$$\left|R(\lambda,\lambda) - R^{(0)}(\lambda,\lambda)\right| \leq \frac{C(q)}{x^{1-2\overline{\varepsilon}}} \leq \frac{C(q)}{x^{1-2\mathfrak{e}}} \quad , \text{ with } \mathfrak{e} = 2\sup_{\overline{U}}\left|\Re\nu\right| \quad , \tag{6.15}$$

so that the q anti-derivative of $R(q,q)+R(-q,-q)-R^{(0)}(q,q)-R^{(0)}(-q,-q)$ will be a $o(1)$ in the $x\to+\infty$ limit.

Equation (6.3) allows us to determine the value of

$$\begin{aligned} R^{(0)}(\lambda,\lambda) &= -\langle F^{L;(0)}(\lambda) \mid \partial_\lambda F^{R;(0)}(\lambda)\rangle \\ &= \frac{\gamma F(\lambda)}{2i\pi} f_+^{(0)}(\lambda)\, f_-^{(0)}(\lambda)\, \{\partial_\lambda \log f_+ - \partial_\lambda \log f_-\} \end{aligned} \tag{6.16}$$

at both endpoints q and $-q$.

Consider, for instance, $R^{(0)}(-q,-q)$. We have, for $\lambda \in D_{-q,\delta}$,

$$\begin{aligned} R^{(0)}(\lambda,\lambda) = -\nu(\lambda)\,\Phi\left(-\nu,1;-i\zeta_{-q}\right)\Phi\left(\nu,1;i\zeta_{-q}\right)\Big\{ & 2\partial_\lambda \log e_+(\lambda) \\ & - 2\partial_\lambda[\log\kappa_p(\lambda) + \nu(\lambda)\,\log\zeta_q] + \nu'(\lambda)[\psi(1+\nu)+\psi(1-\nu)] \\ & - ix\,p'(\lambda)\left[\frac{(\partial_z\Phi)\left(\nu,1;i\zeta_{-q}\right)}{\Phi\left(\nu,1;i\zeta_{-q}\right)} + \frac{(\partial_z\Phi)\left(-\nu,1;-i\zeta_{-q}\right)}{\Phi\left(-\nu,1;-i\zeta_{-q}\right)}\right] \\ & - \nu'(\lambda)\left[\frac{(\partial_1\Phi)\left(\nu,1;i\zeta_{-q}\right)}{\Phi\left(\nu,1;i\zeta_{-q}\right)} + \frac{(\partial_1\Phi)\left(-\nu,1;-i\zeta_{-q}\right)}{\Phi\left(-\nu,1;-i\zeta_{-q}\right)}\right]\Big\}, \end{aligned} \tag{6.17}$$

where, so as to lighten the formula, we have omitted the argument λ of $\nu(\lambda)$ when ν appears as an argument of another function (here ψ or Φ). The symbol ∂_z stands for the derivative of a CHF with respect to its variable, whereas ∂_1 stands for the derivative with respect to its first argument. Recall also that $\zeta_{-q} = x[p(\lambda)-p_-]$ and $\zeta_q = x[p_+ - p(\lambda)]$.

It is remarkable that the last two terms involving derivatives of CHF vanish in the $\lambda\to -q$ limit. The resulting expression can be further simplified thanks to the identities:

$$\log\kappa_p(\lambda) = \log\kappa(\lambda) + \nu(\lambda)\left\{\log\left(\frac{q-\lambda}{p_+-p(\lambda)}\right) - \log\left(\frac{\lambda+q}{p(\lambda)-p_-}\right)\right\}, \tag{6.18}$$

$$\nu(\lambda)\,\nu'(\lambda)\,[\psi(1+\nu)+\psi(1-\nu)] = \partial_\lambda \log G(1,\nu) + 2\nu(\lambda)\,\nu'(\lambda), \tag{6.19}$$

Thus, we obtain

$$R^{(0)}(-q,-q) = -2\nu_-[\partial_\lambda \log e_+(\lambda)]\Big|_{\lambda=-q} + 2\nu'_-\nu_- \log x + 2\nu'_-\nu_- \log(2qp'_-) - \frac{\nu_-^2}{q} + \nu_-^2\frac{p''_-}{p'_-} - 2\nu_-\nu'_- + 2\nu_-[\partial_\lambda \log \kappa(\lambda)]\Big|_{\lambda=-q}, \quad (6.20)$$

where we have used the notations (2.3), (2.4).

The final aim is to integrate (6.14) over the variable q. One should keep in mind that the function $\kappa(\lambda) \equiv \kappa(\lambda; q)$ is actually a function of the two parameters λ and q. Therefore, one should replace partial λ derivatives at $\lambda = \pm q$ by total q derivatives thanks to

$$\frac{\mathrm{d}}{\mathrm{d}q}[\log \kappa(-q;q)] = -\partial_\lambda \log \kappa(\lambda;q)\Big|_{\lambda=-q} + \partial_q \log \kappa(\lambda;q)\Big|_{\lambda=-q}. \quad (6.21)$$

Then $R^{(0)}(-q,-q)$ is almost a total q derivative:

$$R^{(0)}(-q,-q) = -2\nu_-[\partial_\lambda \log e_+(\lambda)]\Big|_{\lambda=-q} + \frac{\mathrm{d}}{\mathrm{d}q}\log\left[\frac{G(1,\nu_-)}{(2qp'_-x)^{\nu_-^2}}\right] - 2\nu_-\frac{\mathrm{d}}{\mathrm{d}q}[\log \kappa(-q;q)] + \nu_-\frac{(\nu_+-\nu_-)}{q}. \quad (6.22)$$

Similar calculations based on the expressions (5.20) for $f_\pm^{(0)}$ around q lead to

$$R^{(0)}(q,q) = -2\nu_+[\partial_\lambda \log e_+(\lambda)]\Big|_{\lambda=q} + \frac{\mathrm{d}}{\mathrm{d}q}\log\left[\frac{G(1,\nu_+)}{(2qp'_+x)^{\nu_+^2}}\right] + 2\nu_+\frac{\mathrm{d}}{\mathrm{d}q}[\log \kappa(q;q)] - \nu_+\frac{(\nu_+-\nu_-)}{q}. \quad (6.23)$$

Hence, we have

$$\partial_q \log\det[I+V] = 2\sum_{\sigma=\pm}\nu_\sigma[\partial_\lambda \log e_-(\lambda)]\Big|_{\lambda=\sigma q} + \frac{\mathrm{d}}{\mathrm{d}q}\log\left[\frac{G(1,\nu_+)G(1,\nu_-)}{(2qp'_-x)^{\nu_-^2}(2qp'_+x)^{\nu_+^2}}\right] + 2\sum_{\sigma=\pm}\sigma\nu_\sigma\frac{\mathrm{d}\log\kappa(\sigma q;q)}{\mathrm{d}q} - \frac{(\nu_+-\nu_-)^2}{q} + \mathrm{o}(1). \quad (6.24)$$

It remains to express the last line as a total q-derivative thanks to Lemma B.3. After an integration with respect to q of (6.24) we arrive to

$$\log\det[I+V] = 2\int_{-q}^{q}\mathrm{d}\lambda\,\nu(\lambda)\log'[e_-(\lambda)] + \log\left[\frac{G(1,\nu_+)G(1,\nu_-)\kappa^{\nu_+}(q;q)}{(2qp'_+x)^{\nu_+^2}(2qp'_-x)^{\nu_-^2}\kappa^{\nu_-}(-q;q)}\right] + \frac{1}{2}\int_{-q}^{q}\mathrm{d}\lambda\mathrm{d}\mu\,\frac{\nu'(\lambda)\nu(\mu)-\nu(\lambda)\nu'(\mu)}{\lambda-\mu} + C + \mathrm{o}(1), \quad (6.25)$$

where C is a q-independent integration constant still to be determined.

One can give arguments that this constant should be also γ-independent. Indeed, the asymptotic expansion of the Fredholm determinant, being a functional of the holomorphic function $\gamma F(\lambda)$ in $\overline{U}$, can depend on this function either in the integral form with integration over $[-q; q]$, or through the values of γF and of its derivatives at the ends of the integration contour $-q$ and q. In both cases the result should depend on q. Hence, the q-independent constant C can not depend on $\gamma F(\lambda)$ and, thus, it is γ-independent. We can then fix the constant C by setting $\gamma = 0$ in the asymptotic formula. This yields $C = 0$. A rigourous proof of this equality within this q-derivative method is however still missing. Indeed, although the above statement (about the functional form of the asymptotic expansion of the Fredholm determinant) is clear in the case of one-dimensional oscillatory integrals without saddle point, its generalisation to the needed series of multiple oscillatory integrals would require additional work.

6.3 The first corrections to the leading asymptotics of the Fredholm determinant

We close this section by deriving the sub-leading corrections from the x-derivative (3.19) of $\log\det[I + V]$. This will constitute the proof of Proposition 2.1.

In order to prove the claim of the Proposition 2.1, one has to derive the first two sub-leading corrections for the matrix Π. As one might expect the computations are, by far, simpler that those necessary to fix the constant. We also would like to point out that one can obtain the sub-leading asymptotic of $\det[I + V]$ by the q-derivative method. However, the computations are quite involved, so we omit the presentation of this method.

We derive the first term in the $1/x$ expansion of $\log\det[I + V]$ thanks to (3.12):

$$\partial_x \log\det[I + V] = \oint_{\Gamma([-q;q])} \frac{\mathrm{d}\lambda}{4\pi}\, p(\lambda)\, \mathrm{tr}\left[\partial_\lambda \chi(\lambda)\, \sigma_3 \chi^{-1}(\lambda)\right], \tag{6.26}$$

where we chose the contour $\Gamma([-q\,;q\,])$ to lie outside of the contour Σ_Π but still in the region of holomorphy for p. There the solution for the RHP for χ has a simple form:

$$\chi(\lambda) = \Pi(\lambda)\ \alpha^{-\sigma_3}(\lambda). \tag{6.27}$$

In order to derive the first correction to the leading asymptotics, it is enough to consider the first two terms in the asymptotic expansion for $\Pi(\lambda)$:

$$\Pi(\lambda) = I_2 + \frac{\Pi^{(1)}(\lambda)}{x} + \frac{\Pi^{(2)}(\lambda)}{x^2} + \mathrm{O}\left(x^{3(\overline{\varepsilon}-1)}\right). \tag{6.28}$$

There, as follows from (5.15), the O is uniform on the whole contour $\Gamma([-q\,;q\,])$. Thus

$$\begin{aligned}\partial_x \log\det[I + V] = &-\oint_{\Gamma([-q;q])} \frac{\mathrm{d}\lambda}{2\pi}\, p(\lambda)\, \frac{\partial_\lambda \alpha(\lambda)}{\alpha(\lambda)} + \frac{1}{x}\oint_{\Gamma([-q;q])} \frac{\mathrm{d}\lambda}{4\pi}\, p(\lambda)\, \mathrm{tr}\left[\sigma_3\, \partial_\lambda \Pi^{(1)}(\lambda)\right] \\ &+ \frac{1}{x^2}\oint_{\Gamma([-q;q])} \frac{\mathrm{d}\lambda}{4\pi}\, p(\lambda)\, \mathrm{tr}\left\{\sigma_3\left[\partial_\lambda \Pi^{(2)}(\lambda) - \Pi^{(1)}(\lambda)\partial_\lambda \Pi^{(1)}(\lambda)\right]\right\} + \mathrm{O}\left(x^{3(\overline{\varepsilon}-1)}\right).\end{aligned} \tag{6.29}$$

The first term in this expansion will yield the leading correction. Indeed,

$$-\oint\limits_{\Gamma([\,-q\,;q\,])}\frac{\mathrm{d}\lambda}{2\pi}\,p\,(\lambda)\,\frac{\partial_\lambda\alpha\,(\lambda)}{\alpha\,(\lambda)}=\oint\limits_{\Gamma([\,-q\,;q\,])}\frac{\mathrm{d}\lambda}{2\pi}\,p'\,(\lambda)\log\alpha\,(\lambda)=-i\int\limits_{-q}^{q}\mathrm{d}\lambda\,p'\,(\lambda)\,\nu\,(\lambda)\ . \tag{6.30}$$

Here we shrunk the contour to $[\,-q\,;q\,]$ and used the jump condition for α.

In order to evaluate the higher order corrections in (6.29) we need to derive the expressions for the matrices $\Pi^{(1)}=C_{\Sigma_\Pi}[\Delta^{(1)}]$ and $\Pi^{(2)}=C_{\Sigma_\Pi}[\Pi^{(1)}_+\Delta^{(1)}+\Delta^{(2)}]$ outside of Σ_Π. An elementary computation of residues yields:

$$\partial_\lambda\Pi^{(1)}\,(\lambda)=-\frac{\Delta^{(1)}_{(+)}\,(q;x)}{(\lambda-q)^2\,p'_+}-\frac{\Delta^{(1)}_{(-)}\,(-q;x)}{(\lambda+q)^2\,p'_-}, \tag{6.31}$$

as well as

$$\begin{aligned}
&\partial_\lambda\Pi^{(2)}\,(\lambda)-\Pi^{(1)}\,(\lambda)\,\partial_\lambda\Pi^{(1)}\,(\lambda)\\
&\quad=-\sum_{\sigma=\pm}\frac{\left(\partial_\lambda\Delta^{(2)}_{(\sigma)}\right)(\sigma q;x)+\Delta^{(1)}_{(\sigma)}\,(\sigma q;x)\left(\partial_\lambda\Delta^{(1)}_{(\sigma)}\right)(\sigma q;x)}{\left[(\lambda-\sigma q)\,p'_\sigma\right]^2}\\
&\quad+\sum_{\sigma=\pm}\frac{2\Delta^{(2)}_{(\sigma)}\,(\sigma q;x)+\left[\Delta^{(1)}_{(\sigma)}\,(\sigma q;x)\right]^2}{2\,(p'_\sigma)^2}\left\{\frac{p''_\sigma}{p'_\sigma\,(\lambda-\sigma q)^2}-\frac{2}{(\lambda-\sigma q)^3}\right\}\\
&\qquad\qquad+\frac{\left[\Delta^{(1)}_{(+)}\,(q;x)\,,\Delta^{(1)}_{(-)}\,(-q;x)\right]}{2q\,p'_+p'_-(\lambda-q)(\lambda+q)}.
\end{aligned} \tag{6.32}$$

Thus the $1/x$ term in (6.29) gives the coefficient of $\log x$ in (2.6). Indeed,

$$\begin{aligned}
\oint\limits_{\Gamma([\,-q\,;q\,])}\frac{\mathrm{d}\lambda}{4\pi}\,p\,(\lambda)\,\mathrm{tr}\left[\sigma_3\,\partial_\lambda\Pi^{(1)}\,(\lambda)\right]&=\frac{1}{2i}\mathrm{tr}\left[\sigma_3\Delta^{(1)}_{(-)}\,(-q;x)+\sigma_3\Delta^{(1)}_{(+)}\,(q;x)\right]\\
&=-\left(\nu_+^2+\nu_-^2\right).
\end{aligned} \tag{6.33}$$

We now focus on the last term in (6.29). It yields, after an x integration, the first correction to (2.6). A straightforward computation leads to

$$\begin{aligned}
&\oint\limits_{\Gamma([\,-q\,;q\,])}\frac{\mathrm{d}\lambda}{2i\pi}\,p\,(\lambda)\,\mathrm{tr}\left\{\sigma_3\left[\partial_\lambda\Pi^{(2)}\,(\lambda)-\Pi^{(1)}\,(\lambda)\,\partial_\lambda\Pi^{(1)}\,(\lambda)\right]\right\}\\
&\quad=\frac{p_+-p_-}{4q^2p'_-p'_+}\mathrm{tr}\left\{\sigma_3\left[\Delta^{(1)}_{(+)}\,(q;x)\ ,\ \Delta^{(1)}_{(-)}\,(-q;x)\right]\right\}\\
&\qquad\qquad-\sum_{\sigma=\pm}\frac{1}{p'_\sigma}\mathrm{tr}\left\{\sigma_3\left[\left(\partial_\lambda\Delta^{(2)}_{(\sigma)}\right)(\sigma q;x)+\Delta^{(1)}_{(\sigma)}\,(\sigma q;x)\left(\partial_\lambda\Delta^{(1)}_{(\sigma)}\right)(\sigma q;x)\right]\right\}.
\end{aligned} \tag{6.34}$$

The first term corresponds to the oscillating correction:

$$\mathrm{tr}\left\{\sigma_3\left[\Delta^{(1)}_{(+)}\,(q;x)\ ,\ \Delta^{(1)}_{(-)}\,(-q;x)\right]\right\}=2\nu_+\nu_-\left\{\frac{\tilde u\,(q;x)}{u\,(-q;x)}-\frac{u\,(-q;x)}{\tilde u\,(q;x)}\right\}. \tag{6.35}$$

The last term gives the non-oscillating one:

$$\mathrm{tr}\left\{\sigma_3\left(\partial_\lambda \Delta^{(2)}_{(\sigma)}\right)(\sigma q;x)\right\} = -2\frac{\mathrm{d}\nu_\sigma^3}{\mathrm{d}q}, \tag{6.36}$$

and

$$\begin{aligned}\sum_{\sigma=\pm}\frac{1}{p'_\sigma}\mathrm{tr}\left\{\sigma_3\ \Delta^{(1)}_{(\sigma)}(\sigma q;x)\left(\partial_\lambda\Delta^{(1)}_{(\sigma)}\right)(\sigma q;x)\right\}&\\ = \frac{2}{p'_+}\nu_+^2\partial_\lambda\left(\log\tilde{u}\,(\lambda;x)\right)\Big|_{\lambda=q} + \frac{2}{p'_-}\nu_-^2\partial_\lambda\left(\log u\,(\lambda;x)\right)\Big|_{\lambda=-q}&\\ = \sum_{\sigma=\pm}\frac{2\nu_\sigma^2}{p'_\sigma}\left\{2\sigma\nu'_\sigma\log x + \sigma\frac{\mathrm{d}}{\mathrm{d}q}\log u_\sigma + \frac{p'_\sigma}{\nu_\sigma}\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\nu_\sigma^2}{p'_\sigma}\right) - \frac{\nu_{-\sigma}}{p'_-}\right\}\ ,&\end{aligned} \tag{6.37}$$

with

$$u_+ = e^{g(q)}\,\frac{\Gamma(1-\nu_+)}{\Gamma(1+\nu_+)}\left\{\frac{(2qp'_+)^{\nu_+}}{\kappa(q;q)}\right\}^2, \tag{6.38}$$

$$u_- = e^{g(-q)}\,\frac{\Gamma(1+\nu_-)}{\Gamma(1-\nu_-)}\left\{(2qp'_-)^{\nu_-}\kappa(-q;q)\right\}^{-2}. \tag{6.39}$$

Putting all this together we obtain

$$\begin{aligned}\partial_x\log\det\left[I+V\right] = -i\int_{-q}^{q}\mathrm{d}\lambda\ \nu(\lambda)\,p'(\lambda) - \frac{\nu_+^2+\nu_-^2}{x} - 2i\frac{\log x-1}{x^2}\sum_{\sigma=\pm}\frac{\nu_\sigma^2}{p'_\sigma}\frac{\mathrm{d}\nu_\sigma}{\mathrm{d}q}&\\ - \frac{i}{x^2}\sum_{\sigma=\pm}\frac{\nu_\sigma^2}{p'_\sigma}\left\{\sigma\frac{\mathrm{d}}{\mathrm{d}q}\log u_\sigma + p'_\sigma\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\nu_\sigma}{p'_\sigma}\right) - \frac{\nu_{-\sigma}}{q}\right\}&\\ + \frac{i\,(p_+-p_-)\,\nu_-\nu_+}{(2q)^2\,p'_+p'_-x^2}\left\{\frac{u_+}{u_-}x^{2(\nu_++\nu_-)}\mathrm{e}^{ix(p_+-p_-)} - \frac{u_-}{u_+}x^{-2(\nu_++\nu_-)}\mathrm{e}^{ix(p_--p_+)}\right\}&\\ + \mathrm{O}\left(\frac{\mathrm{e}^{ix(p_+-p_-)}}{x^{3(1-\overline{\varepsilon})}}, \frac{1}{x^{3(1-\overline{\varepsilon})}}\right).&\end{aligned} \tag{6.40}$$

The first two terms reproduce the already known answer for the leading asymptotics. The remaining ones reproduce the first oscillating and non-oscillating corrections as given in Proposition 2.1. Note that for the oscillating terms, one only should integrate the exponent with respect to x as all the other terms will give subdominant contributions.

7 Applications to truncated Wiener–Hopf operators

Truncated Wiener–Hopf operators appear in many domains of mathematical physics such as scattering or diffusion processes. Moreover, many observables (dressed energy, momentum or dressed charge) related to quantum integrable models are solutions of integral equations of truncated Wiener–Hopf type (2.17).

Let us recall that a truncated Wiener–Hopf operator can be interpreted as an integral operator $I + K$ on $L^2(\mathbb{R})$ such that it acts on $L^2(\mathbb{R})$ functions according to

$$(I + K).\varphi(t) = \varphi(t) + \int_0^x \mathrm{d}t' K(t - t')\varphi(t')\,\mathrm{d}t'. \tag{7.1}$$

The kernel K is characterized in terms of its Fourier transform F:

$$K(t) = \mathcal{F}^{-1}[F](t), \quad \text{with} \quad \mathcal{F}^{-1}[F](t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}\xi\, F(\xi)\,\mathrm{e}^{-it\xi}, \quad \forall F \in L^1(\mathbb{R}). \tag{7.2}$$

The study of truncated Wiener–Hopf operators is equivalent to a 2×2 matrix RHP. Another facet of this equivalence is the correspondence between a truncated Wiener–Hopf operator and the GSK acting on $\mathbb{R}$ in which $p = \mathrm{id}$ and $g = 0$. Indeed, it is easy to see that

$$K.\varphi = \mathcal{F}^{-1} \circ \widetilde{V} \circ \mathcal{F}[\varphi], \tag{7.3}$$

where $\widetilde{V}$ acts in $L^2(\mathbb{R})$ with a kernel

$$\widetilde{V}(\xi, \eta) = F(\xi)\frac{\mathrm{e}^{ix(\xi-\eta)} - 1}{2i\pi(\xi - \eta)}. \tag{7.4}$$

The operator identity

$$I + K = \mathcal{F}^{-1}.\left(I + \widetilde{V}\right)\mathcal{F}, \tag{7.5}$$

together with the facts that $\widetilde{V}$ is trace-class and $\mathcal{F}^{\pm 1}$ are continuous, ensures the equality between the Fredholm determinants:

$$\det[I + K] = \det\left[I + \widetilde{V}\right]. \tag{7.6}$$

The kernel $\widetilde{V}$ is related to

$$V(\xi, \eta) = \sqrt{F(\xi)F(\eta)}\,\frac{\sin(x(\xi - \eta)/2)}{\pi(\xi - \eta)} \tag{7.7}$$

by a similarity transformation. Hence,

$$\det[I + K]_{L^2(0,x)} = \det[I + V]_{L^2(\mathbb{R})}. \tag{7.8}$$

7.1 The Akhiezer–Kac formula

Our study of the generalised sine kernel allows us to recover the Akhiezer–Kac formula describing the large x behaviour of Fredholm determinants of truncated Wiener–Hopf operators:

Theorem 7.1 (Akhiezer–Kac [1, 34])**.** *Let $I + K$ be a truncated Wiener–Hopf operator as above and such that*

- *F is analytic in an open neighbourhood U of $\mathbb{R}$;*
- *F goes sufficiently fast to 0 at $\pm\infty$;*
- *$1 + F(\xi)$ does not vanish on U.*

Then

$$\log\det[I+K] = x\,\tau(0) + E[F] + \mathrm{o}(1), \quad \textit{with} \quad E[F] = \int_0^\infty t\,\tau(t)\,\tau(-t)\,\mathrm{d}t, \tag{7.9}$$

in which

$$\tau(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \log(F(\xi)+1)\,\mathrm{e}^{-it\xi}\,\mathrm{d}\xi\ . \tag{7.10}$$

Proof — The large x asymptotics of $\det[I+K]$ follows from (7.8) after taking the $q \to +\infty$ limit in the leading asymptotics for the corresponding generalised sine kernel (2.6). This limit may seem a little heuristic as we did not specify any estimates in q for the small o terms with respect to the leading asymptotics. However, the validity of such a limit may either be seen by refining all the estimates obtained in the previous section or by considering the RHP for χ (3.1) on the whole real line from the very beginning. We shall make the second approach more explicit in the forthcoming subsection 7.2. Here we formally take the $q \to +\infty$ limit in the leading asymptotics of Theorem 2.1.

One should notice that, in the asymptotic formula (2.6), all the terms evaluated at the end-points vanish due to the fact that $\nu(\pm q)\log q \to 0$ when $q \to +\infty$, which is a consequence of the sufficiently fast decrease of F at infinity. Hence, the only constant contribution $E[F]$ to the asymptotics of $\log\det[I+K]$ is given by the integral

$$E[F] = \lim_{q\to+\infty} \frac{1}{2}\int_{-q}^{q} \frac{\nu'(\lambda)\,\nu(\mu) - \nu(\lambda)\,\nu'(\mu)}{\lambda-\mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu\ . \tag{7.11}$$

Let us recast the constant $E[F]$ in a more standard form. We have

$$\begin{aligned} E[F] &= -\frac{1}{8\pi^2}\int_{\mathbb{R}} \mathrm{d}\xi\,\mathrm{d}\eta\,\frac{\log'(F(\xi)+1)\log(F(\eta)+1) - \log'(F(\eta)+1)\log(F(\xi)+1)}{\xi-\eta} \\ &= \frac{i}{16\pi^2}\int_{\mathbb{R}} \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x\,\mathrm{d}y\,\left\{\frac{1}{\xi-\eta+i0} + \frac{1}{\xi-\eta-i0}\right\}\tau(x)\tau(y)\,(x-y)\,\mathrm{e}^{ix\eta+iy\xi}\ . \end{aligned}$$

Let H be the Heaviside function, then

$$\begin{aligned} E[F] &= -\frac{1}{8\pi}\int_{\mathbb{R}} \mathrm{d}\eta\,\mathrm{d}x\,\mathrm{d}y\ \tau(x)\,\tau(y)\,(x-y)\left(\mathrm{e}^{i\eta(x+y)}H(y)-\mathrm{e}^{i\eta(x+y)}H(-y)\right) \\ &= -\frac{1}{4}\int_{0}^{+\infty}\mathrm{d}y\,\tau(y)\,\tau(-y)\ (-2y)+\frac{1}{4}\int_{-\infty}^{0}\mathrm{d}y\,\tau(y)\,\tau(-y)\ (-2y) \\ &= \int_{0}^{+\infty}\mathrm{d}y\,\tau(y)\,\tau(-y)\,y\ , \end{aligned}$$

which ends the proof of Theorem 7.1. □

It happens that this correspondence between truncated Wiener–Hopf operators and generalised sine kernels can be pushed further so as to obtain the asymptotic behaviour of Fredholm determinants of truncated Wiener–Hopf operators with symbols having Fischer–Hartwig type discontinuities. Considering the GSK for finite q corresponds to the asymptotic behaviour of a determinant whose symbol has two jumps. The case of symbols having general Fischer–Hartwig type singularities is studied in [37, 16, 17]. The results for the case of Toeplitz, Hankel and Toeplitz + Hankel determinants with Fisher–Hartwig singularities appeared recently in [16, 17].

7.2 The resolvent of truncated Wiener–Hopf operators

Proposition 7.1. *Let $I+K$ be a truncated Wiener–Hopf operator on $]-x\,;x[$,*

$$[(I+K).g](t) = g(t)+\int_{-x}^{x}K(t-t')\,g(t')\,\mathrm{d}t', \quad \textit{with} \quad K(t)=\mathcal{F}^{-1}[F](t). \tag{7.12}$$

Suppose that there exists $\delta>0$ such that

- *F admits an analytic continuation to $\left\{z:\left|\Im(z)\right|\leq\delta\right\}$;*
- *$\xi\mapsto F(\xi\pm i\delta)\in L^1(\mathbb{R})$;*
- *the analytic continuation of $1+F$ does not vanish on U.*

Then the resolvent $I-R$ of $I+K$ fulfills

$$R(\lambda,\mu)=\int_{\mathbb{R}}\frac{\mathrm{d}\xi\mathrm{d}\eta}{4i\pi^2}F(\xi)\left\{\frac{\alpha_+(\eta)}{\alpha_-(\xi)}\mathrm{e}^{ix(\xi-\eta)}-\frac{\alpha_+(\xi)}{\alpha_-(\eta)}\mathrm{e}^{-ix(\xi-\eta)}\right\}\frac{\mathrm{e}^{i(\mu\eta-\lambda\xi)}}{\xi-\eta}+\mathrm{O}(\mathrm{e}^{-2\delta x}), \tag{7.13}$$

where

$$\alpha(\lambda)=\exp\left\{\int_{\mathbb{R}}\frac{\nu(\mu)}{\mu-\lambda}\mathrm{d}\mu\right\}, \qquad \textit{and} \quad \nu(\lambda)=\frac{i}{2\pi}\log\left(1+F(\lambda)\right). \tag{7.14}$$

Proof — The GSK associated to $I + K$ through the transformation $\mathcal{F}^{-1} \circ [I + V] \circ \mathcal{F} = I + K$ acts on the whole real axis with the kernel

$$V(\xi, \eta) = F(\xi) \frac{e^{i(\xi-\eta)x} - e^{i(\eta-\xi)x}}{2i\pi(\xi - \eta)} \ . \tag{7.15}$$

Just as for the leading asymptotics of $\log\det[I + K]$ (see Section 7.2), one can obtain the leading asymptotic of the resolvent of V just by taking formally the limit $q \to +\infty$ in all the expressions derived in the first part of the article. Note that in this process all power law corrections vanish: they are computed as contour integrals around $\pm q$ and, since F approaches 0 sufficiently fast at infinity, the residues at $\pm q$ vanish in the $q \to +\infty$ limit. However, in order to justify this limit, one should also check that all the uniform estimates still hold for $q \to +\infty$.

An alternative way is to consider from the very beginning a RHP for χ on the whole real axis $\mathbb{R}$. This is actually much simpler, than the RHP on a finite interval. Then it is enough to perform the first two transformations described in Section 4 so as to obtain jump matrices that are already uniformly close to I_2 up to exponentially small corrections in x. Moreover, the jump matrices for this RHP are given by M_+ and M_-^{-1} (4.11), so that they approach the identity matrix at $\lambda \to \infty$ just as fast as F goes to zero at infinity. As expected, there is no need for parametrices, and the corrections are immediately exponentially decreasing with x. It means that, up to uniformly exponentially small corrections, the resolvent R_V of V is given by

$$R_V^{(0)}(\xi, \eta) = \frac{F(\xi)}{2i\pi(\xi - \eta)} \left\{ \frac{\alpha_+(\eta)}{\alpha_-(\xi)} e^{ix(\xi-\eta)} - \frac{\alpha_+(\xi)}{\alpha_-(\eta)} e^{-ix(\xi-\eta)} \right\}, \tag{7.16}$$

where as usual $\alpha(\lambda)$ is given by (7.14). Note that the integral in (7.14) is well defined in virtue of the assumptions made on F.

We should now take the Fourier/inverse Fourier of R_V in order to get R. To this end, we must justify that the sub-leading corrections do admit a Fourier transform in two variables. Recall the exact expression for the resolvent:

$$R_V(\lambda, \mu) = R_V^{(0)}(\lambda, \mu) + \langle\, F^{L;(0)}(\lambda) \,| \frac{\Pi^{-1}(\lambda)\Pi(\mu) - I_2}{\lambda - \mu} | \, F^{R;(0)}(\mu) \,\rangle \ . \tag{7.17}$$

Here, the matrix Π is defined in terms of $\Pi_+(\lambda)$, the limiting value of Π on Σ_Π when λ approaches a point of Σ_Π from the "+" side of the contour:

$$\Pi(\lambda) = I_2 + \int_{\Gamma_+} \frac{dz}{\lambda - z} \Pi_+(z) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{F(z)\alpha_+^{-2}(z)}{1 + F(z)} e^{2ixz} + \int_{\Gamma_-} \frac{dz}{\lambda - z} \Pi_+(z) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{F(z)\alpha_-^{2}(z)}{1 + F(z)} e^{2ixz} \ . \tag{7.18}$$

The L^1 integrability of F as well as the asymptotic condition $\Pi(\lambda) \underset{\lambda\to+\infty}{\longrightarrow} I_2$ guarantee that the integrals are well defined. Thus one readily infers from (7.18) the asymptotics of Π on the real axis:

$$\Pi(\lambda) = I_2 + \frac{e^{-2\delta x} C}{\lambda} + O\left(\frac{e^{-2\delta x}}{\lambda^2}\right), \tag{7.19}$$

where C is some constant matrix and where we have explicitly extracted the exponential decay in x of the matrix C. Hence using the boundedness of $f_{\pm}^{(0)}$ on the real axis we obtain that

$$R_V(\lambda,\mu) = R_V^{(0)}(\lambda,\mu) + \mathrm{e}^{-2\delta x} C \frac{F(\lambda)}{\lambda\mu} + \mathrm{o}\left(\frac{\mathrm{e}^{-2\delta x} F(\lambda)}{\lambda\mu}\right). \tag{7.20}$$

Hence the corrections admit a Fourier transform in λ and an inverse Fourier transform in μ as oscillatory integrals. Therefore, taking the Fourier transform does not change the nature of the corrections. □

It is clear that, up to a similarity transformation, a Wiener–Hopf operator on $[\,-a\,;b\,]$ has the same generalised sine kernel as the same operator acting on $[\,0\,;a+b\,]$. Therefore our method works for any interval, of course up to a similarity transformation on the resolvent (7.16) of V. We chose here to present this less standard form of Wiener–Hopf operators as it fits better the forthcoming application.

We apply our asymptotic inversion formula for truncated Wiener–Hopf operators acting on a symmetric interval $[\,-x\,;x\,]$ to re-derive some formulas concerning the low magnetic field behaviour of the so-called dressed charge [6]. This function, traditionally denoted $Z(\lambda)$, describes the intrinsic magnetic moment of an elementary excitation above the ground state in the XXZ spin-1/2 model. It satisfies the following integral equation:

$$Z(\lambda) + \int_{-x}^{x} \mathrm{d}\mu\, K(\lambda-\mu)\, Z(\mu) = 1, \quad \text{with} \quad K(\lambda) = \frac{\sin 2\zeta}{2\pi \sinh(\lambda+i\zeta)\, \sinh(\lambda-i\zeta)}\,. \tag{7.21}$$

K is often called the Lieb kernel and $\zeta \in\,]\,0\,;\pi\,[$ is some real parameter describing the coupling of the model. The large parameter x is a function of the external longitudinal magnetic field; it goes to infinity when the magnetic field vanishes.

For the study of Z, one should distinguish two domains in the interval $[\,-x\,;x\,]$: the bulk, i.e. the region $|\lambda| \ll x$, and the boundaries $\lambda \sim \pm x$. While the asymptotic value of $Z(\lambda)$ in the bulk ($|\lambda| \ll x$) is enough to describe the intrinsic magnetic moment of elementary excitations, the value of Z at the boundaries ($\pm x$) determines the critical exponents of the two-point functions of the model [23, 24, 25, 5]. As we will see, the bulk and the boundary behaviour of the dressed charge differ fundamentally.

First, let us note that, setting directly $x = +\infty$ in (7.21), one can solve explicitly the integral equation for Z by taking the Fourier transform: one obtains in this case that $Z(\lambda)$ is equal to a constant value $Z(\lambda) = \pi/\,[2\,(\pi-\zeta)]$ on the whole real axis.

Let us now consider the limit $x \to \infty$ in (7.21) in a more accurate way, namely, taking x large but finite, and use the method described above. Let $\widehat{K}$ be the Fourier transform of K,

$$\widehat{K}(\xi) \equiv \mathcal{F}[K](\xi) = \frac{\sinh\left[\xi\,(\zeta-\pi/2)\right]}{\sinh(\xi\zeta/2)}\,. \tag{7.22}$$

Then in virtue of Proposition 7.1,

$$Z(\lambda) = 1 - \int_{-x}^{x} R(\lambda,\mu)\,\mathrm{d}\mu \tag{7.23}$$

$$= 1 - \int_{\mathbb{R}} \frac{\mathrm{d}\xi}{2i\pi} \frac{\widehat{K}(\xi)}{\xi} \left\{ \frac{\alpha_+(0)}{\alpha_-(\xi)} \mathrm{e}^{i(x-\lambda)\xi} - \frac{\alpha_+(\xi)}{\alpha_-(0)} \mathrm{e}^{-i(x+\lambda)\xi} \right\}. \tag{7.24}$$

First let us study the bulk limit i.e. $|\lambda| \ll x$. Using the jump equation satisfied by $\alpha_\pm$: $[1 + \widehat{K}(\lambda)]\alpha_+(\lambda) = \alpha_-(\lambda)$, we recast the integrand as

$$\begin{aligned} Z(\lambda) &= 1 - \int_{\mathbb{R}} \frac{\mathrm{d}\xi}{2i\pi} \frac{\widehat{K}(\xi)}{[1+\widehat{K}(\xi)](\xi - i0^+)} \left\{ \frac{\alpha_+(0)}{\alpha_+(\xi)} \mathrm{e}^{i(x-\lambda)\xi} - \frac{\alpha_-(\xi)}{\alpha_-(0)} \mathrm{e}^{-i(x+\lambda)\xi} \right\} \\ &= \frac{1}{1+\widehat{K}(0)} - \int_{\mathbb{R}} \frac{\mathrm{d}\xi}{2i\pi} \left\{ \frac{\widehat{K}(\xi + i\zeta/2)}{1+\widehat{K}(\xi+i\zeta/2)} \frac{\alpha_+(0)}{\alpha_+(\xi+i\zeta/2)} \frac{\mathrm{e}^{(x-\lambda)(i\xi-\zeta/2)}}{\xi+\zeta/2} \right. \\ &\qquad \left. - \frac{\widehat{K}(\xi - i\zeta/2)}{1+\widehat{K}(\xi-i\zeta/2)} \frac{\alpha_-(0)}{\alpha_-(\xi-i\zeta/2)} \frac{\mathrm{e}^{-(x+\lambda)(i\xi+\zeta/2)}}{\xi-\zeta/2} \right\}. \end{aligned} \tag{7.25}$$

Here we have separated the integral into two parts and then moved the contour to the upper/lower half-plane. This gives a pole contribution from $\xi = i0$. The integral appearing in (7.25) is clearly a $\mathrm{O}\left(\mathrm{e}^{-(x-|\lambda|)\zeta/2}\right)$. So that, in the bulk,

$$Z(\lambda) \sim \frac{1}{1+\widehat{K}(0)} = \frac{\pi}{2(\pi-\zeta)}\,, \tag{7.26}$$

up to exponentially small corrections. As expected, we recover the value of Z obtained in the case of an infinite interval. Note that the corrections become larger and larger as we approach any of the endpoints $\pm x$.

Let us now study the behaviour of the dressed charge at the boundaries. Since the kernel K is even, so is Z. We can thus focus on a single boundary, say $\lambda = x$. We have,

$$Z(x) = 1 - \int_{\mathbb{R}} \frac{\mathrm{d}\xi}{2i\pi(\xi - i0^+)} \left\{ \widehat{K}(\xi) \frac{\alpha_+(0)}{\alpha_-(\xi)} - \frac{\widehat{K}(\xi)}{1+\widehat{K}(\xi)} \frac{\alpha_-(\xi)}{\alpha_-(0)} \mathrm{e}^{-2ix\xi} \right\}. \tag{7.27}$$

As before, the integral of the second term gives an exponentially small contribution $\mathrm{O}\left(\mathrm{e}^{-x\zeta}\right)$. The integral of the first term is explicitly computable. Using once again the jump equation satisfied by $\alpha_\pm(\xi)$, we have,

$$\begin{aligned} Z(x) &= 1 - \alpha_+(0) \int_{\mathbb{R}} \frac{\mathrm{d}\xi}{2i\pi} \frac{\alpha_+^{-1}(\xi) - 1 + 1 - \alpha_-^{-1}(\xi)}{\xi - i0} + \mathrm{O}\left(\mathrm{e}^{-x\zeta}\right) \\ &= 1 - \alpha_+(0) \int_{\mathbb{R}} \frac{\mathrm{d}\xi}{2i\pi} \frac{\alpha_+^{-1}(\xi) - 1}{\xi - i0} + \mathrm{O}\left(\mathrm{e}^{-x\zeta}\right) \\ &= \alpha_+(0) + \mathrm{O}\left(\mathrm{e}^{-x\zeta}\right)\,. \end{aligned} \tag{7.28}$$

We have computed the remaining integral by residues, since $\alpha_{\pm}^{-1}(\xi) - 1 = \mathrm{O}\left(\xi^{-1}\right)$ for $\xi \to \infty$ in the respective half plane of holomorphy.

For an even kernel like the Lieb one $\widehat{K}(\xi) = \widehat{K}(-\xi)$, and then it follows from the integral representation (7.14) of α that $\alpha_+(\xi) = \alpha_-^{-1}(-\xi)$. This means that $1 + \widehat{K}(0) = \alpha_+^{-2}(0)$. Hence, for x large enough

$$Z(x) \sim \sqrt{\frac{1}{1 + \widehat{K}(0)}} = \sqrt{\frac{\pi}{2\,(\pi - \zeta)}}, \tag{7.29}$$

and the value of $Z(\lambda)$ at the boundary is the square root of its value in the bulk up to exponentially small corrections. In the limit $x \to +\infty$ this correspondence becomes exact.

8 Asymptotics of multiple integrals

We have already mentioned that the asymptotic expansion of the Fredholm determinant of the GSK can be used for the asymptotic analysis of correlation functions of quantum integrable models. For a relatively wide class of integrable systems, the correlation functions can be presented as series of multiple integrals of a special type [35]. These series can be summed up to Fredholm determinants for the models equivalent to free fermions. In the general case, such a reduction to determinants is not known. However, the asymptotic behaviour of individual terms of the series can be derived from the asymptotics of the Fredholm determinant of the GSK. In the present section we consider this problem.

More precisely, our purpose is to derive the large x asymptotic behaviour of the following type of integrals (cycle integrals):

$$\mathcal{I}_n\left[\mathcal{F}_n\right] = \oint\limits_{\Gamma([\,-q\,;q\,])} \frac{\mathrm{d}^n z}{(2i\pi)^n} \int\limits_{-q}^{q} \frac{\mathrm{d}^n \lambda}{(2i\pi)^n}\; \mathcal{F}_n\begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} \prod_{j=1}^{n} \frac{\mathrm{e}^{ix(p(z_j)-p(\lambda_j))}}{\big(z_j - \lambda_j\big)\big(z_j - \lambda_{j+1}\big)}\,. \tag{8.1}$$

In this expression, $\mathcal{F}_n$ is a holomorphic function of $2n$ variables $\lambda_1, \ldots, \lambda_n, z_1, \ldots, z_n$ in $U^n \times W^n$, in which U and W are open neighbourhoods of $[\,-q\,;q\,]$, and $\Gamma\left([\,-q\,;q\,]\right)$ denotes a closed counter clock-wise contour in W surrounding $[\,-q\,;q\,]$ with index 1. We moreover assume that $\mathcal{F}_n$ is symmetric separately in the n variables $\lambda_1, \ldots, \lambda_n$ and in the n variables $z_1, \ldots, z_n$. Finally, we agree upon $\lambda_{n+1} \equiv \lambda_1$.

8.1 Leading asymptotic behaviour of $\mathcal{I}_n\left[\mathcal{F}_n\right]$

Let us first suppose that the function $\mathcal{F}_n$ is of the special (factorized) type

$$\mathcal{F}_n^{(\varphi,\phi)}\begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} = \prod_{i=1}^{n} \left[\varphi(\lambda_i)\,\phi(z_i)\right], \tag{8.2}$$

where φ is a one-variable holomorphic function in U and ϕ is a one-variable holomorphic function in W, non-vanishing on W. For two such functions φ and ϕ we introduce the associated

GSK $V^{(\varphi,\phi)}$ given by (1.6) provided the identification $F(\lambda) = \varphi(\lambda)\phi(\lambda)$ and $e^{g(z)} = \phi(z)$ is made. Then, the integral (8.1) can be expressed in terms of $\log\det\left[I + V^{(\varphi,\phi)}\right]$ as

$$\begin{aligned}\mathcal{I}_n[\mathcal{F}_n^{(\varphi,\phi)}] &= \int_{-q}^{q} \mathrm{d}^n\lambda \prod_{k=1}^{n} V^{(\varphi,\phi)}(\lambda_k,\lambda_{k+1}) \\ &= \frac{(-1)^{n-1}}{(n-1)!}\,\partial_\gamma^n \log\det\left[I + V^{(\varphi,\phi)}\right]\Big|_{\gamma=0}. \end{aligned} \tag{8.3}$$

In this specific case, it is straightforward to obtain the leading asymptotic behaviour of the multiple integral (8.3) in the large x limit thanks to the results of the previous sections.

This remark leads us to the following definition:

Definition 8.1. *Let U, W be two open neighbourhoods of $[\,-q\,;q\,]$, and let $\mathcal{H}(U)$ (resp. $\mathcal{H}(W)$) be the set of holomorphic functions on U (resp. on W). Let also*

$$\widetilde{\mathcal{S}}_n^{U,W} = \left\{ \sum_{\ell=1}^{p} \mathcal{F}_n^{(\varphi_\ell,\phi_\ell)} \;;\; p\in\mathbb{N},\ (\varphi_\ell,\phi_\ell)\in\mathcal{H}(U)\times\mathcal{H}(W) \text{ and } \phi_\ell\big|_{[\,-q\,;q\,]} \neq 0 \right\}, \tag{8.4}$$

in which $\mathcal{F}_n^{(\varphi,\phi)}$ denotes a pure factor function of $2n$ variables defined in terms of φ and ϕ as in (8.2)*. We define the linear functional $I_n^{(0)}$ on $\widetilde{\mathcal{S}}_n^{U,W}$ as*

$$I_n^{(0)}[\mathcal{F}_n^{(\varphi,\phi)}] = \frac{(-1)^{n-1}}{(n-1)!}\,\partial_\gamma^n \log\det\left[I + V^{(\varphi,\phi)}\right]^{(0)}\Big|_{\gamma=0}, \tag{8.5}$$

and by imposing linearity on functions $\sum_{\ell=1}^{p}\mathcal{F}_n^{(\varphi_\ell,\phi_\ell)}$. Here $V^{(\varphi,\phi)}$ denotes the generalised sine kernel (1.6) *with $F(\lambda) = \varphi(\lambda)\phi(\lambda)$ and $\mathrm{e}^{g(z)} = \phi(z)$, and $\log\det\left[I + V^{(\varphi,\phi)}\right]^{(0)}$ denotes the leading asymptotics of the Fredholm determinant $\log\det\left[I + V^{(\varphi,\phi)}\right]$ as in Theorem 2.1.*

It is easy, using the expression (2.6) of $\log\det\left[I + V^{(\varphi,\phi)}\right]^{(0)}$, to obtain an explicit expression for $I_n^{(0)}[\mathcal{F}_n^{(\varphi,\phi)}]$:

$$\begin{aligned} I_n^{(0)}[\mathcal{F}_n^{(\varphi,\phi)}] &= \int_{-q}^{q} \frac{\mathrm{d}\lambda}{2i\pi}\,\varphi^n(\lambda)\,\phi^{n-1}(\lambda)\,(ixp'(\lambda)\,\phi(\lambda) + \phi'(\lambda)) \\ &+ \sum_{\sigma=\pm} (b_n - c_n\log(2qp'_\sigma x))\,[\varphi(\sigma q)\,\phi(\sigma q)]^n \\ &+ \frac{n}{4\pi^2}\sum_{\sigma=\pm}\sum_{p=1}^{n-1}\int_{-q}^{q}\mathrm{d}\lambda\,\frac{[\varphi(\sigma q)\,\phi(\sigma q)]^n - [\varphi(\sigma q)\,\phi(\sigma q)]^p\,[\varphi(\lambda)\,\phi(\lambda)]^{n-p}}{p\,(n-p)\,(q-\sigma\lambda)} \\ &+ \frac{n}{8\pi^2}\sum_{p=1}^{n-1}\int_{-q}^{q}\frac{\mathrm{d}\lambda\,\mathrm{d}\mu}{(n-p)\,(\lambda-\mu)}\Big\{\partial_\lambda[\varphi(\lambda)\,\phi(\lambda)]\;[\varphi(\lambda)\,\phi(\lambda)]^{p-1}\,[\varphi(\mu)\,\phi(\mu)]^{n-p} \\ &\qquad - \partial_\mu[\varphi(\mu)\,\phi(\mu)]\;[\varphi(\mu)\,\phi(\mu)]^{p-1}\,[\varphi(\lambda)\,\phi(\lambda)]^{n-p}\Big\}, \end{aligned} \tag{8.6}$$

where b_n and c_n are given by (2.23). The $x \to +\infty$ asymptotics of the Fredholm determinant are uniform in γ to any fixed order n in ∂_γ^n. This means that

$$\mathcal{I}_n[\mathcal{F}_n^{(\varphi,\phi)}] = I_n^{(0)}[\mathcal{F}_n^{(\varphi,\phi)}] + \mathrm{o}(1)\,. \tag{8.7}$$

In the next proposition we show that $I_n^{(0)}$ can be extended into a linear functional on the space of holomorphic functions $\mathcal{F}_n$ (not necessarily of the form $\mathcal{F}_n^{(\varphi,\phi)}$) that are symmetric in n variables $\lambda_1, \dots, \lambda_n$ and n variables $z_1, \dots, z_n$ separately. This extension of $I_n^{(0)}$, as we prove below, is the good way to evaluate cycle integrals (8.1) with such arbitrary symmetric functions $\mathcal{F}_n$.

Proposition 8.1. *Let U and W be open neighbourhoods of $[\,-q\,;q\,]$, and let $\mathcal{S}\mathrm{ym}_n(U,W)$ be the set of holomorphic functions $\mathcal{F}_n$ on $U^n \times W^n$ of $2n$ variables $\lambda_1, \dots, \lambda_n, z_1, \dots, z_n$, symmetric in the n variables $\lambda_1, \dots, \lambda_n$ and in the n variables $z_1, \dots, z_n$ separately. Then $I_n^{(0)}$ extends to a continuous linear functional on $\mathcal{S}\mathrm{ym}_n(U,W)$ endowed with the topology of the sup norm convergence on compact sets.*

Proof — $I_n^{(0)}[\mathcal{F}_n^{(\varphi,\phi)}]$ contains at most first order derivatives of the functions φ and ϕ. Now recall that, for any compacts K, P such that[1] $K \subset \overset{\circ}{P}$ and $P \subset U$

$$\forall\, k \in \mathbb{N},\ \exists\, c_k \text{ such that } \forall\, \phi \in \mathcal{H}(U)\,,\ \left\|\phi^{(k)}\right\|_{0;K} \le c_k \left\|\phi\right\|_{0;P}\ , \tag{8.8}$$

where $\|.\|_{0;K} = \sup_{z\in K} |.|$ is the sup norm with support on the compact K. In consequence, $I_n^{(0)}$ is continuous on $\widetilde{\mathcal{S}}_n^{U,W}$. The latter is dense in $\mathcal{S}_n^{U,W}$, with

$$\mathcal{S}_n^{U,W} = \left\{ \sum_{\ell=1}^{p} \mathcal{F}_n^{(\varphi_\ell,\phi_\ell)}\ ;\ p \in \mathbb{N},\ (\varphi_\ell,\phi_\ell) \in \mathcal{H}(U)\times\mathcal{H}(W) \right\}. \tag{8.9}$$

Hence $I_n^{(0)}$ extends by density to a continuous linear functional on $\mathcal{S}_n^{U,W}$. Due to the density Theorem C.1 (See Appendix C), we have that $\mathcal{S}_n^{U,W}$ is dense in $\mathcal{S}\mathrm{ym}_n(U,W)$. Therefore $I_n^{(0)}$ extends to a linear functional on $\mathcal{S}\mathrm{ym}_n(U,W)$. □

Corollary 8.1. *Let U and W be open neighbourhoods of $[\,-q\,;q\,]$, and let $\mathcal{F}_n \in \mathcal{S}\mathrm{ym}_n(U,W)$.*

[1] Here $\overset{\circ}{P}$ is the interior of P

Then,

$$I_n^{(0)}[\mathcal{F}_n] = \frac{1}{2i\pi}\int_{-q}^{q} \mathrm{d}\lambda\, \{ixp'(\lambda) + \partial_\epsilon\}\mathcal{F}_n\begin{pmatrix} \{\lambda\}^n \\ \{\lambda+\epsilon\}, \{\lambda\}^{n-1} \end{pmatrix}\Big|_{\epsilon=0}$$
$$+ \sum_{\sigma=\pm} (b_n - c_n \log(2qp'_\sigma x))\,\mathcal{F}_n\begin{pmatrix} \{\sigma q\}^n \\ \{\sigma q\}^n \end{pmatrix}$$
$$+ \frac{n}{(2\pi)^2}\sum_{\sigma=\pm}\sum_{p=1}^{n-1}\int_{-q}^{q}\mathrm{d}\lambda \frac{\mathcal{F}_n\begin{pmatrix} \{\sigma q\}^n \\ \{\sigma q\}^n \end{pmatrix} - \mathcal{F}_n\begin{pmatrix} \{\sigma q\}^p, \{\lambda\}^{n-p} \\ \{\sigma q\}^p, \{\lambda\}^{n-p} \end{pmatrix}}{p\,(n-p)\,(q-\sigma\lambda)}$$
$$+ \frac{n}{2\,(2\pi)^2}\sum_{p=1}^{n-1}\int_{-q}^{q}\frac{\mathrm{d}\lambda\mathrm{d}\mu}{(n-p)\,(\lambda-\mu)}\Big\{\partial_\epsilon\mathcal{F}_n\begin{pmatrix} \{\lambda+\epsilon\}, \{\lambda\}^{p-1}, \{\mu\}^{n-p} \\ \{\lambda+\epsilon\}, \{\lambda\}^{p-1}, \{\mu\}^{n-p} \end{pmatrix}$$
$$-\partial_\epsilon\mathcal{F}_n\begin{pmatrix} \{\mu+\epsilon\}, \{\mu\}^{p-1}, \{\lambda\}^{n-p} \\ \{\mu+\epsilon\}, \{\mu\}^{p-1}, \{\lambda\}^{n-p} \end{pmatrix}\Big\}\Big|_{\epsilon=0}. \quad (8.10)$$

There $\{\lambda\}^n$ denotes the set formed by n copies of the same parameter λ.

Proof — Apply Theorem C.1 to (8.6). □

Finally, we have the following large x asymptotic behaviour for integrals of the form (8.1) (which seems hardly attainable through a direct analysis of the multiple integrals):

Theorem 8.1. *Let U and W be open neighbourhoods of $[\,-q\,;q\,]$, and let $\mathcal{F}_n \in \mathcal{S}\mathrm{ym}_n(U,W)$. Then, when $x \to +\infty$, the integral $\mathcal{I}_n[\mathcal{F}_n]$ (8.1) behaves as*

$$\mathcal{I}_n[\mathcal{F}_n] = I_n^{(0)}[\mathcal{F}_n] + \mathrm{O}\left(\frac{\log^n x}{x}\right), \quad (8.11)$$

the explicit expression of $I_n^{(0)}[\mathcal{F}_n]$ being given in Corollary 8.1.

The whole difficulty of the proof is to show that the small $\mathrm{o}(1)$ in (2.5) is preserved by the density procedure formulated in Theorem C.1. This is nontrivial since the series converging to $\mathcal{F}_n$ may not converge absolutely. We need therefore, so as to prove this theorem, to study more precisely the sub-leading corrections and to see how they pass through all the steps described above. This will be done in the next subsection.

8.2 Study of sub-leading corrections

In this subsection, we study the behaviour of the sub-leading corrections to $\log\det[I+V]^{(0)}$ when the above procedure is applied. In particular, we will show that they indeed remain subleading, which will prove Theorem 8.1. In fact we will prove an even more general result:

Theorem 8.2. *Let U and W be open neighbourhoods of $[\,-q\,;q\,]$, and let $\mathcal{F}_n \in \mathcal{S}\mathrm{ym}_n(U,W)$. For any positive integer M, there exists a continuous linear functional $I_n^{(M)}$ such that*

$$\mathcal{I}_n[\mathcal{F}_n] = I_n^{(M)}[\mathcal{F}_n] + \mathrm{O}\left(\frac{\log^n x}{x^{M+1}}\right) \quad \text{when } x \to +\infty. \tag{8.12}$$

The explicit expression for $I_n^{(M)}$ can be obtained by some perturbative computations that become more and more involved with the growth of M. We will nevertheless obtain the general structure for $I_n^{(M)}$, showing that it can be decomposed in terms of non-oscillating and oscillating contributions, with oscillating factors of the form $e^{imx(p_+-p_-)}$, $m \in \mathbb{Z}^*$:

$$I_n^{(M)}[\mathcal{F}_n] = I_n^{(0)}[\mathcal{F}_n] + \sum_{N=1}^{M} \frac{1}{x^N}\, I_n^{(N;\,\mathrm{nosc})}[\mathcal{F}_n] + \sum_{N=2}^{M} \frac{1}{x^N}\, I_n^{(N;\,\mathrm{osc})}[\mathcal{F}_n]\,, \tag{8.13}$$

$$= I_n^{(0)}[\mathcal{F}_n] + \sum_{N=1}^{M} \frac{1}{x^N}\, I_n^{(N;\,\mathrm{nosc})}[\mathcal{F}_n] + \sum_{\substack{m\in\mathbb{Z}^* \\ |m|\le M/2}} e^{ixm(p_+-p_-)} \sum_{N=2|m|}^{M} \frac{1}{x^N}\, I_n^{(N;\,m)}[\mathcal{F}_n]\,, \tag{8.14}$$

$I_n^{(N;\,\mathrm{nosc})}[\mathcal{F}_n]$ (resp. $I_n^{(N;\,\mathrm{osc})}[\mathcal{F}_n]$) being given in terms of the function $\mathcal{F}_n$ and of its derivatives up to order N (resp. up to order $N-2$) evaluated at $\pm q$ or integrated from $-q$ to q.

8.2.1 General strategy

In the previous subsection, we have defined the functional $I_n^{(0)}$ from the leading asymptotic part $\log\det[I+V]^{(0)}$ (2.6) of the GSK. More precisely, we have seen in Corollary 8.1 that $\partial_\gamma^n \log\det[I+V]^{(0)}\,|_{\gamma=0}$ yields the functional $(-1)^{n-1}\,(n-1)!\,I_n^{(0)}[\mathcal{F}_n]$ after the density procedure, as explained in Proposition 8.1 and Theorem C.1, is applied. In order to estimate the corrections to $I_n^{(0)}[\mathcal{F}_n]$ for the large x behaviour of cycle integrals $\mathcal{I}_n[\mathcal{F}_n]$ of length n (8.1), we have to take into account the corrections $\log\det[I+V]^{\mathrm{sub}}$ to $\log\det[I+V]^{(0)}$,

$$\log\det[I+V] = \log\det[I+V]^{(0)} + \log\det[I+V]^{\mathrm{sub}}\,, \tag{8.15}$$

and to analyze the effect of the density procedure on the n-th γ-derivative of the subleading part $\partial_\gamma^n \log\det[I+V]^{\mathrm{sub}}\,|_{\gamma=0}$. We will show in particular that it preserves the small o(1) with respect to the $x\to+\infty$ limit, i.e. that $\partial_\gamma^n \log\det[I+V]^{\mathrm{sub}}\,|_{\gamma=0}$ can only generate o(1) corrections.

In the spirit of Definition 8.1, we therefore introduce the:

Definition 8.2. *Let U, W be two open neighbourhoods of $[\,-q\,;q\,]$. We define the linear functional I_n^{sub} on $\widetilde{\mathcal{S}}_n^{U,W}$ as*

$$I_n^{\mathrm{sub}}[\mathcal{F}_n^{(\varphi,\phi)}] = \frac{(-1)^{n-1}}{(n-1)!}\,\partial_\gamma^n \log\det\left[I+V^{(\varphi,\phi)}\right]^{\mathrm{sub}}\Big|_{\gamma=0}, \tag{8.16}$$

and by imposing linearity on functions $\sum_{\ell=1}^{p}\mathcal{F}_n^{(\varphi_\ell,\phi_\ell)}$. Here, as in Definition 8.1, $\mathcal{F}_n^{(\varphi,\phi)}$ denotes a factorized function of $2n$ variables defined in terms of φ and ϕ as in (8.2)*, and $V^{(\varphi,\phi)}$ denotes the generalised sine kernel* (1.6) *with $F(\lambda)=\varphi(\lambda)\phi(\lambda)$ and $\mathrm{e}^{g(z)}=\phi(z)$.*

According to the scheme presented in the previous subsection, the next steps will be:

- to obtain a convenient representation for $I_n^{\rm sub}[\mathcal{F}_n^{(\varphi,\phi)}]$: this means in particular to obtain the form of n-th γ-derivatives of $\log\det[I+V]^{\rm sub}$ in terms of the functions F and g, to set $g(z)=\log\phi(z)$ and $F(z)=\varphi(z)\,\phi(z)$, and to estimate this result in the large x limit;
- to apply the density procedure: one should first extend by density and continuity the functional $I_n^{\rm sub}$ to the space $\mathcal{S}_n^{U,W}$; then, for any holomorphic function $\mathcal{F}_n$ in $2n$ variables $\lambda_1,\ldots,\lambda_n,z_1,\ldots,z_n$, symmetric separately in the variables λ and in the variables z, one has to consider a sequence (φ_k,ϕ_k) such that $\sum_{k=1}^N \mathcal{F}_n^{(\varphi_k,\phi_k)}\to\mathcal{F}_n$ so as to be able to define and characterize $I_n^{\rm sub}[\mathcal{F}_n]$ and to see how it behaves in the large x limit;
- to refine the procedure in order to get an asymptotic expansion of $I_n^{\rm sub}[\mathcal{F}_n]$.

8.2.2 γ-derivatives of $\log\det[I+V]^{\rm sub}$

As in Section 6.3, we will obtain the corrections to $\log\det[I+V]^{(0)}$ through the x-derivative path, starting from formula (3.12) that we recast as

$$\partial_x\log\det\left[I+V\right]=-i\int_{-q}^{q}\mathrm{d}\lambda\,p'(\lambda)\,\nu\,(\lambda)+\oint_{\Gamma([\,-q\,;q\,])}\frac{\mathrm{d}\lambda}{4\pi}\,p(\lambda)\,\mathrm{tr}\left\{[\partial_\lambda\Pi(\lambda)]\;\sigma_3\,\Pi^{-1}(\lambda)\right\}. \quad (8.17)$$

Here, as in Section 6.3, we have chosen the counter clock-wise contour $\Gamma\left([\,-q\,;q\,]\right)$ to lie in U and to encircle Σ_Π, which means that $\chi(\lambda)=\Pi(\lambda)\,\alpha^{-\sigma_3}(\lambda)$ on $\Gamma\left([\,-q\,;q\,]\right)$. Integrating this equation with respect to x, we obtain

$$\log\det\left[I+V\right]^{\rm sub}=\int_{+\infty}^{x}\mathrm{d}x'\oint_{\Gamma([\,-q\,;q\,])}\frac{\mathrm{d}\lambda}{4\pi}\,p(\lambda)\left\{\mathrm{tr}\left\{\left[\partial_\lambda\Pi\left(\lambda\right)\right]\,\sigma_3\,\Pi^{-1}\left(\lambda\right)\right\}\right.$$
$$\left.+\frac{1}{x'}\int_{\Sigma_\Pi}\frac{\mathrm{tr}\left\{\Delta^{(1)}\left(z;x'\right)\,\sigma_3\right\}}{2i\pi\left(\lambda-z\right)^2}\,\mathrm{d}z\right\}. \quad (8.18)$$

The convergence of this integral will be proved later on. We recall that the second term in (8.17) produces also, when integrated over x, the $\log x$ term appearing in the definition (2.6) of $\log\det\left[I+V\right]^{(0)}$ (see (6.33)). We have therefore substracted the corresponding contribution (second term of (8.18)) in the definition of $\log\det\left[I+V\right]^{\rm sub}$.

In order to obtain the n-th γ derivatives of this expression, we have to compute the γ-derivatives of $\partial_\lambda\Pi(\lambda)$ and of $\Pi^{-1}(\lambda)$, which in their turn follow from those of the jump matrix $\Delta(\lambda)$.

- **γ-derivatives of Δ**

In order to determine the n-th γ-derivative at $\gamma=0$ of the jump matrix $\Delta\left(z\right)$, it is convenient to express it in the following form:

$$\Delta(z)=\kappa^{-\sigma_3}(z)\,\widetilde{\Delta}(z)\,\kappa^{\sigma_3}(z)\,. \quad (8.19)$$

Here, the matrix $\widetilde{\Delta}(z)$ depends on γ only through the combination $\gamma F(z)$, whereas $\kappa^{\pm\sigma_3}(z)$ depends on γ through the combination $\int_{-q}^{q} \mathrm{d}\mu \, [\nu(z) - \nu(\mu)]/(z-\mu)$.

It is easy to compute the multiple γ-derivative of $\widetilde{\Delta}(z)$ at $\gamma = 0$. It is given as

$$\partial_\gamma^n \widetilde{\Delta}(z)\Big|_{\gamma=0} = \mathrm{Ad}_{\mathrm{e}^{\sigma_3 \, g(z)/2}} \left[\partial_\gamma^n \Delta_0(z)\right]\Big|_{\gamma=0} \cdot F^n(z). \tag{8.20}$$

In this expression, $\mathrm{Ad}_X[Y]$ stands for the usual adjoint action of the matrix X on the matrix Y, and Δ_0 denotes the jump matrix Δ at $F \equiv 1$ and $g \equiv 0$.

It remains to compute the γ-derivatives of $\kappa^{\pm\sigma_3}(z)$. They follow from the Faa-di-Bruno formula:

$$\partial_\gamma^n \kappa^{\pm\sigma_3}(z)\Big|_{\gamma=0} = \sum_{\substack{p_1,\dots,p_n=0 \\ \sum_{s=1}^n s p_s = n}} \frac{n! \, (\pm\sigma_3)^{\sum_{s=1}^n p_s}}{\prod_{s=1}^n p_s!} \prod_{s=1}^n \left[\frac{(-1)^s}{2i\pi s} \int_{-q}^{q} \frac{F^s(z) - F^s(\mu)}{z-\mu} \mathrm{d}\mu \right]^{p_s}. \tag{8.21}$$

Therefore, gathering these informations and applying Leibnitz's rule, we obtain that

$$\begin{aligned}
\partial_\gamma^n \Delta(z)\Big|_{\gamma=0} = & \sum_{\substack{p+p_0+q=n \\ p_0 \geq 1}} \sum_{\substack{p_1,\dots,p_n=0 \\ \sum_{s=1}^n s p_s = p}} \sum_{\substack{q_1,\dots,q_n=0 \\ \sum_{s=1}^n s q_s = q}} \frac{C_n^p \, C_{n-p}^q \, p! \, q!}{\prod_{s=1}^n (p_s)! \, (q_s)!} \\
& \times \mathrm{Ad}_{\mathrm{e}^{\sigma_3 \, g(z)/2}} \left[(-\sigma_3)^{\sum_{s=1}^n p_s} \cdot \partial_\gamma^{p_0} \Delta_0(z)\Big|_{\gamma=0} \cdot (\sigma_3)^{\sum_{s=1}^n q_s} \right] \\
& \times F^{p_0}(z) \prod_{s=1}^n \left[\frac{(-1)^s}{2i\pi s} \int_{-q}^{q} \frac{F^s(z) - F^s(\mu)}{z-\mu} \mathrm{d}\mu \right]^{p_s+q_s} \\
= & \sum_{\substack{p_0+\Sigma s p_s = n \\ p_0 \geq 1}} \delta^{(\{p_i\})}(z;x) \, F^{p_0}(z) \prod_{s=1}^n \left[\int_{-q}^{q} \frac{F^s(z) - F^s(\mu)}{z-\mu} \mathrm{d}\mu \right]^{p_s}.
\end{aligned} \tag{8.22}$$

Note that, in the first line, we have extended for convenience the sum over parameters p_s and q_s up to n, since anyway, due to the constraint, $p_s \leq p$ and $p_s = 0$ if $s > p$ (resp. $q_s \leq q$ and $q_s = 0$ if $s > q$).

In the last line, we have changed the order of summations and incorporated all the F independent prefactors into the definition of the matrix $\delta^{(\{p_i\})}(z;x)$. More precisely,

$$\begin{aligned}
\delta^{(\{p_i\})}(z;x) = & \prod_{s=1}^n \left(\frac{(-1)^s}{2i\pi s} \right)^{p_s} \sum_{p+q=n-p_0} \sum_{\sum_{s=1}^n s q_s = q} \frac{C_n^p \, C_{n-p}^q \, p! \, q!}{\prod_{s=1}^n (q_s)! \, (p_s - q_s)!} \\
& \times \mathrm{Ad}_{\mathrm{e}^{\sigma_3 \, g(z)/2}} \left[(-\sigma_3)^{\sum_{s=1}^n (p_s - q_s)} \cdot \partial_\gamma^{p_0} \Delta_0(z)\Big|_{\gamma=0} \cdot (\sigma_3)^{\sum_{s=1}^n q_s} \right].
\end{aligned} \tag{8.23}$$

From the properties of the jump matrix $\Delta_0(z)$, it is easy to see that the diagonal entries of the matrices $\delta^{(\{p_i\})}(z;x)$ are a $\mathrm{O}(x^{-1})$, whereas their off-diagonal ones are a $\mathrm{O}(\log^{p_0} x/x)$ uniformly on the contours Σ_Π.

• **γ-derivatives of $\partial_\lambda \Pi$**

Let us recall the integral representation for $\partial_\lambda \Pi(\lambda)$, which is a direct consequence of (5.5),

$$\partial_\lambda \Pi(\lambda) = -\frac{1}{2i\pi} \int_{\Sigma_\Pi} \frac{\mathrm{d}z}{(\lambda - z)^2} \Pi_+(z) \Delta(z) . \tag{8.24}$$

Therefore, the γ-derivatives of $\partial_\lambda \Pi(\lambda)$ can be directly obtained from the ones of $\Delta(\lambda)$ and of $\Pi_+(\lambda)$.

Recall that $\Pi_+(\lambda)$ satisfies the following integral equation on $\mathcal{M}_2\left(L^2(\Sigma_\Pi)\right)$:

$$\left(I - C^\Delta_{\Sigma_\Pi}\right)[\Pi_+] = I_2 , \tag{8.25}$$

where the operator $C^\Delta_{\Sigma_\Pi}$ is defined by

$$C^\Delta_{\Sigma_\Pi}[M] \equiv C^+_{\Sigma_\Pi}[M\Delta] , \quad \forall M \in \mathcal{M}_2\left(L^2(\Sigma_\Pi)\right) . \tag{8.26}$$

This matrix Cauchy operator is invertible, at least for x large enough. Indeed, using the continuity of the scalar Cauchy operator:

$$\exists\, c_2 > 0 \text{ such that, } \forall g \in L^2(\Sigma_\Pi) , \quad \left\| C^+_{\Sigma_\Pi}[g] \right\|_{L^2(\Sigma_\Pi)} \leq c_2 \|g\|_{L^2(\Sigma_\Pi)} , \tag{8.27}$$

one gets that the operator norm $\left|\!\left|\!\left| C^\Delta_{\Sigma_\Pi} \right|\!\right|\!\right|$ fulfills:

$$\left|\!\left|\!\left| C^\Delta_{\Sigma_\Pi} \right|\!\right|\!\right| \leq c_2 \|\Delta\|_{\mathcal{M}_2(L^2(\Sigma_\Pi))} \underset{x \to +\infty}{\longrightarrow} 0. \tag{8.28}$$

Moreover, $C^\Delta_{\Sigma_\Pi}$ being a holomorphic function of γ we have that, for x large enough, $(I - C^\Delta_{\Sigma_\Pi})$ is invertible and that its inverse is also a holomorphic function of γ. In particular, (8.25) implies

$$\partial_\gamma \Pi_+ = \left(I - C^\Delta_{\Sigma_\Pi}\right)^{-1} \circ \left(\partial_\gamma C^\Delta_{\Sigma_\Pi}\right)[\Pi_+] \ . \tag{8.29}$$

A straightforward induction shows that there exist some coefficients $\mathfrak{c}_r^{(\{p_i\})} \in \mathbb{Z}$ such that

$$\begin{aligned} \partial^n_\gamma \Pi_+ = \sum_{r=1}^{n} \sum_{\Sigma_{i=1}^r p_i = n} \mathfrak{c}_r^{(\{p_i\})} \ & \left(I - C^\Delta_{\Sigma_\Pi}\right)^{-1} \circ \left(\partial^{p_1}_\gamma C^\Delta_{\Sigma_\Pi}\right) \\ & \circ \left(I - C^\Delta_{\Sigma_\Pi}\right)^{-1} \circ \left(\partial^{p_2}_\gamma C^\Delta_{\Sigma_\Pi}\right) \circ \cdots \circ \left(I - C^\Delta_{\Sigma_\Pi}\right)^{-1} \circ \left(\partial^{p_r}_\gamma C^\Delta_{\Sigma_\Pi}\right)[\Pi_+] . \end{aligned} \tag{8.30}$$

This expression simplifies at $\gamma = 0$ as $\Delta_{|_{\gamma=0}} = 0$ and $\Pi_{+\,|_{\gamma=0}} = I_2$. Hence,

$$\partial^n_\gamma \Pi_+ \Big|_{\gamma=0} = \sum_{r=1}^{n} \sum_{\Sigma_{i=1}^r p_i = n} \mathfrak{c}_r^{(\{p_i\})} \left(\partial^{p_1}_\gamma C^\Delta_{\Sigma_\Pi}\right) \circ \left(\partial^{p_2}_\gamma C^\Delta_{\Sigma_\Pi}\right) \circ \cdots \circ \left(\partial^{p_r}_\gamma C^\Delta_{\Sigma_\Pi}\right)[I_2] . \tag{8.31}$$

We can slightly deform the different contours Σ_Π, so as to regularize the explicit integral representation for the above chain of operators. Namely, recalling the construction of the jump

contours for the matrices occurring in the different transformations applied to the RHP for χ, we are able to write

$$\partial_\gamma^n \Pi_+\big|_{\gamma=0} = \sum_{r=1}^{n} \sum_{\Sigma_{i=1}^r p_i = n} \mathfrak{c}_r^{(\{p_i\})} \left(\partial_\gamma^{p_1} C^\Delta_{\Sigma_\Pi^{(1)}}\right) \circ \left(\partial_\gamma^{p_2} C^\Delta_{\Sigma_\Pi^{(2)}}\right) \circ \cdots \circ \left(\partial_\gamma^{p_r} C^\Delta_{\Sigma_\Pi^{(r)}}\right)[I_2]\,. \tag{8.32}$$

There, the contours $\Sigma_\Pi^{(i)}$ are such that the $-$ side of $\Sigma_\Pi^{(i-1)}$ is at small but non vanishing distance from the $+$ side of $\Sigma_\Pi^{(i)}$, with the exception of a finite number of points of intersection (cf. Fig. 6). The matrix Δ corresponding to the contour $\Sigma_\Pi^{(i)}$ is equal to $M_\pm^{\pm 1} - I_2$ on $\Gamma_\pm^{\prime(i)}$, to $\mathcal{P} - I_2$ on $\partial D_{-q,\delta_i}$, and to $\widetilde{\mathcal{P}} - I_2$ on $\partial D_{q,\delta_i}$. We emphasize that, already in (8.32), one can use the integral representation for the Cauchy operators without turning to boundary values. Indeed, the integrand appearing in (8.32) is already integrable on $\Sigma_\Pi^{\times(r)} \equiv \Sigma_\Pi^{(1)} \times \cdots \times \Sigma_\Pi^{(r)}$.

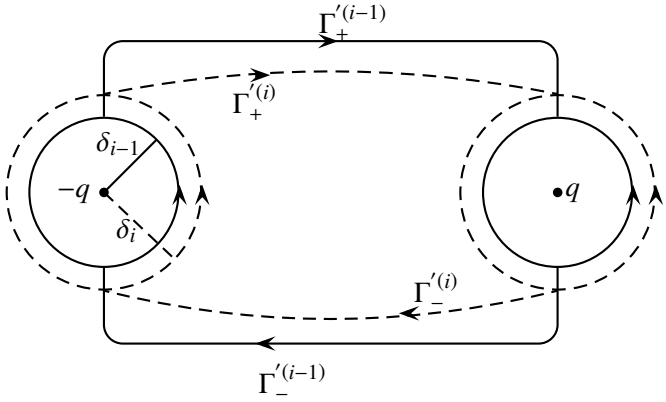


Figure 6: Encased Contours $\Sigma_\Pi^{(i)}$ (in the case $p = \mathrm{id}$).

Finally, one infers from (8.32), from (8.22) and from the integral representation for $\partial_\lambda \Pi$ (8.24) that, when $\lambda \in \Gamma([-q\,;q\,])$, there exist some recursively computable coefficients $\tilde{\mathfrak{c}}_r^{(\{p_{\ell i}\})}$ such that

$$\partial_\gamma^n \partial_\lambda \Pi(\lambda)\big|_{\gamma=0} = -\sum_{r=1}^{n} \sum_{\substack{p_{\ell 0}\geq 1,\, p_{\ell 1},\ldots,p_{\ell n}\geq 0 \\ \Sigma_{\ell=1}^r \bar{p}_\ell = n}} \tilde{\mathfrak{c}}_r^{(\{p_{\ell i}\})} \int_{\Sigma_\Pi^{\times(r)}} \frac{\mathrm{d}^r z}{(2i\pi)^r} \, \frac{\delta^{(\{p_{ri}\})}(z_r;x)\ldots\delta^{(\{p_{1i}\})}(z_1;x)}{(\lambda - z_1)^2 \prod_{\ell=2}^{r}(z_{\ell-1}-z_\ell)}$$
$$\times \prod_{\ell=1}^{r} \left\{ F^{p_{\ell 0}}(z_\ell) \prod_{m=1}^{n} \left[\int_{-q}^{q} \frac{F^m(z_\ell) - F^m(\mu)}{z_\ell - \mu} \right]^{p_{\ell m}} \right\}. \tag{8.33}$$

In this expression, the integration is performed over the skeleton $\Sigma_\Pi^{\times(r)} = \Sigma_\Pi^{(1)} \times \cdots \times \Sigma_\Pi^{(r)}$, and the second summation is performed over integers $p_{\ell j}$, $1 \leq \ell \leq r$, $1 \leq j \leq n$, with $1 \leq p_{\ell 0} \leq n$ and $0 \leq p_{\ell j} \leq n$ for $j \geq 1$, and such that $\Sigma_{\ell=1}^r \bar{p}_\ell = n$, in which we have introduced the notation $\bar{p}_\ell = p_{\ell 0} + \Sigma_{j=1}^n j p_{\ell j}$.

• **γ-derivatives of Π^{-1}**

All the above observations also hold for the inverse matrix $\Pi^{-1}(\lambda)$. Indeed Π_+^{-1} satisfies the integral equation

$$\Pi_+^{-1}(\lambda) = I_2 + {}^tC_{\Sigma_\Pi}^{\nabla}\left[\Pi_+^{-1}\right], \tag{8.34}$$

in which

$${}^tC_{\Sigma_\Pi}^{\nabla}[M] \equiv C_{\Sigma_\Pi}^{+}[\nabla M], \qquad \forall M \in \mathcal{M}_2(L^2(\Sigma_\Pi)), \tag{8.35}$$

and the matrix ∇ is defined[2] by the equation $I_2 + \nabla = (I_2 + \Delta)^{-1}$. In other words, ∇ is the adjugate of Δ (we remind that we consider 2×2 matrices and that $\det[I + \Delta] = 1$). Hence, one easily sees that, for $n \geq 1$,

$$\partial_\gamma^n\left[\Pi^{-1}\right](\lambda)\Big|_{\gamma=0} = \sum_{t=1}^{n} \sum_{\substack{\Sigma_{\ell=1}^{t}\tilde{p}_\ell=n \\ p_{\ell 0}\geq 1}} \tilde{\mathfrak{c}}_t^{(\{p_{\ell i}\})} \int_{\Sigma_\Pi^{\times(t)}} \frac{\mathrm{d}^t z}{(2i\pi)^t} \, \frac{\varrho^{(\{p_{1i}\})}(z_1;x)\ldots\varrho^{(\{p_{ti}\})}(z_t;x)}{(\lambda - z_1)\prod_{\ell=2}^{t}(z_{\ell-1}-z_\ell)} \times \prod_{\ell=1}^{t}\left\{F^{p_{\ell 0}}(z_\ell)\prod_{m=1}^{n}\left[\int_{-q}^{q}\frac{F^m(z_\ell)-F^m(\mu)}{z_\ell-\mu}\right]^{p_{\ell m}}\right\}, \tag{8.36}$$

where $\varrho^{(\{p_{\ell i}\})}(z;x)$ is the adjugate matrix of $\delta^{(\{p_{\ell i}\})}(z;x)$.

• **γ-derivatives of** $\log\det[I+V]^{\mathrm{sub}}$

From the expressions (8.18), (8.33) and (8.36), it is easy to see that there exist some combinatorial coefficients $\mathfrak{C}_{r,t}^{(\{p_{\ell i}\})} \in \mathbb{Z}$ (with $\mathfrak{C}_{1,0}^{(\{p_{1i}\})} = -\delta_{p_{10},n}$) such that

$$\partial_\gamma^n \log\det[I+V]^{\mathrm{sub}}\Big|_{\gamma=0} = \sum_{\substack{1\leq r+t\leq n \\ r\geq 1,\ t\geq 0}} \sum_{\substack{p_{\ell 0}\geq 1,\ p_{\ell 1},\ldots,p_{\ell r+t}\geq 0 \\ \Sigma_{\ell=1}^{r+t}\tilde{p}_\ell=n}} \mathfrak{C}_{r,t}^{(\{p_{\ell i}\})} \oint_{\Gamma([-q\,;q])} \frac{\mathrm{d}\lambda}{4\pi}\, p(\lambda) \int_{\Sigma_\Pi^{\times(r+t)}} \frac{\mathrm{d}^{r+t}z}{(2i\pi)^{r+t}} \times \frac{\int_{+\infty}^{x}\mathrm{d}x'\ \mathrm{tr}_{r,t}^{(\{p_{\ell j}\})}(\{z_j\};x')[g]}{(\lambda-z_1)^2(\lambda-z_{r+1})\prod_{\ell=2}^{r}(z_{\ell-1}-z_\ell)\prod_{\ell=r+2}^{r+t}(z_{\ell-1}-z_\ell)} \times \prod_{\ell=1}^{r+t}\left\{F^{p_{\ell 0}}(z_\ell)\prod_{m=1}^{n}\left[\int_{-q}^{q}\frac{F^m(z_\ell)-F^m(\mu)}{z_\ell-\mu}\,\mathrm{d}\mu\right]^{p_{\ell m}}\right\}, \tag{8.37}$$

[2]We stress that this matrix ∇ has nothing to do with the differential operator usually denoted by ∇.

in which, in the terms $t = 0$, the empty product $(\lambda - z_{r+1}) \prod_{\ell=r+2}^{r+t} (z_{\ell-1} - z_\ell)$ should be understood as 1. In this expression, $\mathrm{tr}_{r,t}^{(\{p_{\ell j}\})}(\{z_j\}; x')[g]$ corresponds to the following trace:

$$\mathrm{tr}_{1,0}^{(\{p_{1j}\})}(z; x')[g] = \mathrm{tr}\left\{\sigma_3\, \partial_\gamma^n\Big[\Delta_0(z; x') - \frac{1}{x'}\Delta_0^{(1)}(z; x')\Big]_{\big|_{\gamma=0}}\right\}, \tag{8.38}$$

$$\mathrm{tr}_{r,t}^{(\{p_{\ell j}\})}(\{z_j\}; x')[g] = \mathrm{tr}\Big\{\delta^{(\{p_{ri}\})}(z_r; x') \dots \delta^{(\{p_{1i}\})}(z_1; x')\, \sigma_3 \times \varrho^{(\{p_{r+1i}\})}(z_{r+1}; x') \dots \varrho^{(\{p_{r+ti}\})}(z_{r+t}; x')\Big\} \quad \text{if } r + t > 1. \tag{8.39}$$

In (8.38), $\Delta_0^{(1)}(z; x')$ corresponds to the first term in the asymptotic expansion (5.1) of $\Delta_0(z)$.

Remark 8.1. We have gathered in the term $r = 1, t = 0$ the contribution of the second term in (8.18), as well as the term that would correspond to the contribution of only one jump matrix Δ in the first term of (8.18). Note that, in this term $r = 1, t = 0$, the only non-zero contribution comes from the diagonal elements of Δ, hence from the sequence $p_{10} = n$, $p_{1j} = 0$ for $j \geq 1$ (indeed we have $\mathfrak{C}_{1,0}^{(\{p_{1i}\})} = -\delta_{p_{10},n}$).

Remark 8.2. It is easy to see from these expressions that the integrals over x' are convergent. Indeed, it follows from the asymptotic expansion of the matrices Δ_0 that

$$\partial_\gamma^n\Big[\Delta_0(z; x) - \frac{1}{x}\Delta_0^{(1)}(z; x)\Big]_{\big|_{\gamma=0}} = \mathrm{O}\left(\frac{\log^n x}{x^2}\right) \tag{8.40}$$

uniformly on the integration contour, so that an integration of the trace (8.38) is convergent. We emphasize that the trace (8.39) is at least $\mathrm{O}\left((\log x)^n/x^2\right)$ uniformly on the integration contour: indeed, each of the matrices $\delta^{(\{p_{\ell j}\})}$ or $\varrho^{(\{p_{\ell j}\})}$ is uniformly a $\mathrm{O}\left((\log x)^{p_{\ell 0}}/x\right)$; in addition, the trace (8.39) involves a product of at least two such matrices since $r + t \geq 2$. These estimates guarantee that the integrals over x' in (8.37) are well defined.

8.2.3 Application of the density procedure and proof of Theorem 8.1

In order to be able to apply the density procedure, we should express more explicitly the functional dependence of $\partial_\gamma^n \log\det\left[I + V^{(\varphi,\phi)}\right]^{\mathrm{sub}}|_{\gamma=0}$ on $\mathcal{F}_n^{(\varphi,\phi)}$.

The F-dependence of $\partial_\gamma^n \log\det\left[I+V\right]^{\mathrm{sub}}|_{\gamma=0}$ has already been explicitly extracted in (8.37), and all the g-dependence is contained in the traces $\mathrm{tr}_{r,t}^{(\{p_{\ell j}\})}(\{z_j\}; x')[g]$. Using the structure of the matrices $\delta^{(\{p_{\ell i}\})}(z; x)$ and $\rho^{(\{p_{\ell i}\})}(z; x)$, one can be more precise concerning this g-dependence. Indeed, it follows from (8.23) that there exist some coefficients $D_{r,t}^{[\{p_{\ell j}\},\{\epsilon_j\}]}(\{z_j\}; x)$ which are piecewise smooth on the integration contour such that

$$\mathrm{tr}_{r,t}^{(\{p_{\ell j}\})}(\{z_j\}; x)[g] = \sum_{\substack{\epsilon_1,\dots,\epsilon_{r+t}\in\{\pm1,0\}\\ \Sigma\epsilon_i=0}} D_{r,t}^{[\{p_{\ell j}\},\{\epsilon_j\}]}(\{z_j\}; x)\ \exp\left\{\sum_{\ell=1}^{r+t} \epsilon_\ell\, g(z_\ell)\right\}. \tag{8.41}$$

Note that these coefficients $D_{r,t}^{[\{p_{\ell j}\},\{\epsilon_j\}]}(\{z_j\};x)$ are at least $\mathrm{O}\left((\log x)^n/x^2\right)$ uniformly on the integration contour. Integrating these coefficients with respect to x, and defining

$$\widetilde{D}_{r,t}^{[\{p_{\ell j}\},\{\epsilon_j\}]}(\{z_j\};x) = \mathfrak{C}_{r,t}^{(\{p_{\ell j}\})} \int_{+\infty}^{x} \mathrm{d}x' D_{r,t}^{[\{p_{\ell j}\},\{\epsilon_j\}]}(\{z_j\};x'), \tag{8.42}$$

which are at least $\mathrm{O}\left((\log x)^n/x\right)$ uniformly on the integration contours $\Sigma_\Pi^{(i)}$, one gets

$$\begin{aligned}\partial_\gamma^n \log\det\left[I+V\right]^{\mathrm{sub}}\Big|_{\gamma=0} = &\sum_{\substack{1\le r+t\le n\\ r\ge 1,\ t\ge 0}} \sum_{\substack{\Sigma_{\ell=1}^{r+t}\bar{p}_\ell=n\\ p_{\ell 0}\ge 1}} \oint_{\Gamma([-q\,;q])} \frac{\mathrm{d}\lambda}{4\pi}\, p(\lambda) \int_{\Sigma_\Pi^{\times(r+t)}} \frac{\mathrm{d}^{r+t}z}{(2i\pi)^{r+t}}\\ \times \sum_{\substack{\epsilon_1,\dots,\epsilon_{r+t}\in\{\pm1,0\}\\ \sum\epsilon_i=0}} &\frac{\widetilde{D}_{r,t}^{[\{p_{\ell j}\},\{\epsilon_j\}]}(\{z_j\};x)}{(\lambda-z_1)^2(\lambda-z_{r+1})\prod_{\ell=2}^{r}(z_{\ell-1}-z_\ell)\prod_{\ell=r+2}^{r+t}(z_{\ell-1}-z_\ell)}\\ &\times\prod_{\ell=1}^{r+t}\left\{F^{p_{\ell 0}}(z_\ell)\,\mathrm{e}^{\epsilon_\ell g(z_\ell)}\prod_{m=1}^{n}\left[\int_{-q}^{q}\frac{F^m(z_\ell)-F^m(\mu)}{z_\ell-\mu}\,\mathrm{d}\mu\right]^{p_{\ell m}}\right\}. \end{aligned}\tag{8.43}$$

We stress that each $\mathrm{e}^{\pm g(z_\ell)}$ may only appear in combination with at least one $F(z_\ell)$ (as $p_{\ell 0}\ge 1$): $F(z_\ell)\,\mathrm{e}^{\pm g(z_\ell)}$. This guarantees that the functional above is continuous with respect to the sup norm on the space of symmetric functions in n variables z and n variables λ.

Before applying the density procedure, let us introduce one more useful notation. Define the finite difference operator $\eth_z^{(m)}(\mu)$ by its action on pure product functions

$$\eth_z^{(m)}(\mu)\cdot F^k(z) = F^k(z) - F^{k-m}(z)\,F^m(\mu)\,. \tag{8.44}$$

This action naturally extends to symmetric functions of n variables

$$\eth_{z_\ell}^{(m)}(\mu)\cdot\mathcal{F}_n\left[\left\{\{z_i\}^{\bar{p}_i}\right\}_{i=1,\dots,r}\right] = \mathcal{F}_n\left[\left\{\{z_i\}^{\bar{p}_i}\right\}_{i=1,\dots,r}\right] - \mathcal{F}_n\left[\left\{\{z_k\}^{\bar{p}_k}\right\}_{k\neq\ell},\{\mu\}^m,\{z_\ell\}^{\bar{p}_\ell-m}\right]. \tag{8.45}$$

We remind here that $\bar{p}_\ell = p_{\ell 0} + \Sigma_{s=1}^n s p_{\ell s}$, with $\sum_{\ell=1}^{r+t}\bar{p}_\ell = n$. We have moreover used the notation $\{z_\ell\}^{\bar{p}_\ell}$, which means that the variable z_ℓ is repeated $\bar{p}_\ell$ times, and $\left\{\{z_i\}^{\bar{p}_i}\right\}_{i=1,\dots,r}$, which means that the variable z_1 is repeated $\bar{p}_1$ times, z_2 is repeated $\bar{p}_2$ times, …, z_r is repeated $\bar{p}_r$ times. The purpose of introducing such finite difference operator is to recast products of functions $F^m(z_\ell)-F^m(\mu)$ appearing in (8.43) into a more compact form. Namely,

$$\begin{aligned}\prod_{m=1}^{n}\left[\int_{-q}^{q}\frac{F^m(z_\ell)-F^m(\mu)}{z_\ell-\mu}\,\mathrm{d}\mu\right]^{p_{\ell m}} &= \prod_{m=1}^{n}\left[\int_{-q}^{q}\frac{\eth_{z_\ell}^{(m)}(\mu)\cdot F^m(z_\ell)}{z_\ell-\mu}\mathrm{d}\mu\right]^{p_{\ell m}}\\ &= \prod_{m=1}^{n}\prod_{j=1}^{p_{\ell m}}\int_{-q}^{q}\frac{\mathrm{d}\mu_{\ell,m,j}}{z_\ell-\mu_{\ell,m,j}}\left\{\prod_{m=1}^{n}\prod_{j=1}^{p_{\ell m}}\eth_{z_\ell}^{(m)}(\mu_{\ell,m,j})\right\}\cdot F^{\bar{p}_\ell-p_{\ell 0}}(z_\ell)\,.\end{aligned}$$

Therefore, setting $e^{g(z)} = \phi(z)$ and $F(z) = \varphi(z)\,\phi(z)$, we get

$$\partial_\gamma^n \log\det\left[I + V^{(\varphi,\phi)}\right]^{\mathrm{sub}}\Big|_{\gamma=0} = \sum_{\substack{1\le r+t\le n\\ r\ge 1,\ t\ge 0}} \sum_{\substack{\Sigma_{\ell=1}^{r+t}\bar{p}_\ell = n\\ p_{\ell 0}\ge 1}} \oint_{\Gamma([-q\,;q\,])} \frac{\mathrm{d}\lambda}{4\pi}\, p(\lambda) \int_{\Sigma_\Pi^{\times(r+t)}} \frac{\mathrm{d}^{r+t}z}{(2i\pi)^{r+t}}$$
$$\times \sum_{\substack{\epsilon_1,\ldots,\epsilon_{r+t}\in\{\pm 1,0\}\\ \sum \epsilon_i = 0}} \frac{\widetilde{D}_{r,t}^{[\{p_{\ell j}\},\{\epsilon_j\}]}(\{z_j\};x)}{(\lambda - z_1)^2\,(\lambda - z_{r+1})\prod_{\ell=2}^{r}(z_{\ell-1}-z_\ell)\prod_{\ell=r+2}^{r+t}(z_{\ell-1}-z_\ell)}$$
$$\times \prod_{\ell=1}^{r+t}\prod_{m=1}^{n}\prod_{j=1}^{p_{\ell m}}\left\{\int_{-q}^{q}\frac{\mathrm{d}\mu_{\ell,m,j}}{z_\ell - \mu_{\ell,m,j}}\,\mathfrak{d}_{z_\ell}^{(m)}(\mu_{\ell,m,j})\right\}\cdot\prod_{\ell=1}^{r+t}\left\{\varphi^{\bar{p}_\ell}(z_\ell)\,\phi^{\bar{p}_\ell+\epsilon_\ell}(z_\ell)\right\}. \quad (8.46)$$

It follows immediately from the density procedure formulated in Theorem C.1 that I_n^{sub} can be extended into a linear functional on $\mathcal{S}\mathrm{ym}_n(U,W)$. Its action on a holomorphic function $\mathcal{F}_n \in \mathcal{S}\mathrm{ym}_n(U,W)$ is given as

$$I_n^{\mathrm{sub}}[\mathcal{F}_n] = \frac{(-1)^{n-1}}{(n-1)!} \sum_{\substack{1\le r+t\le n\\ r\ge 1,\ t\ge 0}} \sum_{\substack{\Sigma_{\ell=1}^{r+t}\bar{p}_\ell = n\\ p_{\ell 0}\ge 1}} \oint_{\Gamma([-q\,;q\,])} \frac{\mathrm{d}\lambda}{4\pi}\, p(\lambda) \int_{\Sigma_\Pi^{\times(r+t)}} \frac{\mathrm{d}^{r+t}z}{(2i\pi)^{r+t}}$$
$$\times \sum_{\substack{\epsilon_1,\ldots,\epsilon_{r+t}\in\{\pm 1,0\}\\ \sum \epsilon_i = 0}} \frac{\widetilde{D}_{r,t}^{[\{p_{\ell j}\},\{\epsilon_j\}]}(\{z_j\};x)}{(\lambda - z_1)^2\,(\lambda - z_{r+1})\prod_{\ell=2}^{r}(z_{\ell-1}-z_\ell)\prod_{\ell=r+2}^{r+t}(z_{\ell-1}-z_\ell)}$$
$$\times \prod_{\ell=1}^{r+t}\prod_{m=1}^{n}\prod_{j=1}^{p_{\ell m}}\left\{\int_{-q}^{q}\frac{\mathrm{d}\mu_{\ell,m,j}}{z_\ell - \mu_{\ell,m,j}}\,\mathfrak{d}_{z_\ell}^{(m)}(\mu_{\ell,m,j})\right\}\cdot\mathcal{F}_n\begin{pmatrix}\{\{z_\ell\}^{\bar{p}_\ell}\}_{1\le\ell\le r+t}\\ \{\{z_\ell\}^{\bar{p}_\ell+\epsilon_\ell}\}_{1\le\ell\le r+t}\end{pmatrix}. \quad (8.47)$$

The sum appearing in (8.47) is finite, and since each integrand is a $\mathrm{O}\left(\log^n x/x\right)$, $I_n^{\mathrm{sub}}[\mathcal{F}_n]$ is itself a $\mathrm{O}\left(\log^n x/x\right)$. Hence Theorem 8.1 follows directly, since

$$\mathcal{I}_n[\mathcal{F}_n] = I_n^{(0)}[\mathcal{F}_n] + I_n^{\mathrm{sub}}[\mathcal{F}_n]\ . \quad (8.48)$$

8.2.4 Asymptotic expansion of $I_n^{\mathrm{sub}}[\mathcal{F}_n]$ and proof of Theorem 8.2

In order to prove the existence of an asymptotic series of for $\mathcal{I}_n[\mathcal{F}_n]$, i.e. for $I_n^{\mathrm{sub}}[\mathcal{F}_n]$, we should be more precise on the structure of the coefficients $\widetilde{D}_{r,t}$, i.e. show that they themselves admit an asymptotic expansion. Let us recall that these coefficients are obtained from the traces (8.41) involving the matrices $\delta^{(\{p_{\ell j}\})}(z_\ell;x)$ and $\varrho^{(\{p_{\ell j}\})}(z_\ell;x)$. The latter being obtained from the jump matrix Δ_0.

Clearly, all terms corresponding to an integration on the contours $\Gamma_\pm^{'(i)}$ yield exponentially small corrections. Thus in what concerns the proof of an asymptotic expansion we can only focus on integrations along the contours $\partial D_{\pm q,\delta_i}$. We decompose the relevant contour $[\partial D_{q,\delta}\cup\partial D_{-q,\delta}]^{\times(r+t)}$ into sums of elementary skeletons $\partial D_{\sigma q,\delta}^{\times(r+t)} \equiv \partial D_{\sigma_1 q,\delta_1}\times\cdots\times\partial D_{\sigma_{r+t}q,\delta_{r+t}}$, where

each σ_i takes values in $\{\pm 1\}$:

$$\int\limits_{\Sigma_\Pi^{(r+t)}} \frac{\mathrm{d}^{r+t}z}{(2i\pi)^{r+t}} = \sum_{\sigma_i=\pm} \int\limits_{\partial D_{\sigma q,\delta}^{(r+t)}} \frac{\mathrm{d}^{r+t}z}{(2i\pi)^{r+t}} + \mathrm{O}\left(x^{-\infty}\right) . \tag{8.49}$$

The matrices $\delta^{(\{p_j\})}(z;x)$ (8.23) admit an asymptotic expansion into inverse powers of x on $\partial D_{q,\delta} \cup \partial D_{-q,\delta}$. This fact follows from the asymptotic expansion of $\Delta_0(z)$. The latter is obtained by taking adequate a-derivatives at $a = 0$ or 1 of the asymptotics series (A.6) for $\Psi(a,1;z)$ when $z \to \infty$. This is licit as, for fixed M, the $\mathrm{O}(z^{-M-1})$ estimate in the asymptotic series (A.6) is uniform with respect to a and since we perform a finite number of derivative with respect to a. This asymptotic expansion takes the following form:

$$\delta^{(\{p_j\})}(z;x) = \begin{cases} \sum\limits_{k=1}^{M} \dfrac{\mathrm{Ad}_{\mathrm{e}^{\sigma_3\,[ixp_+ + g(z)]/2}}\left[\delta_+^{(\{p_j\},k)}(z;X_+)\right]}{x^k\,(p(z)-p_+)^k} + \mathrm{O}\left(\dfrac{\log^{p_0} x}{x^{M+1}}\right), & z \in \partial D_{q,\delta}\,, \\ \sum\limits_{k=1}^{M} \dfrac{\mathrm{Ad}_{\mathrm{e}^{\sigma_3\,[ixp_- + g(z)]/2}}\left[\delta_-^{(\{p_j\},k)}(z;X_-)\right]}{x^k\,(p(z)-p_-)^k} + \mathrm{O}\left(\dfrac{\log^{p_0} x}{x^{M+1}}\right), & z \in \partial D_{-q,\delta}\,, \end{cases} \tag{8.50}$$

the corrections being uniform on the contours. There the diagonal entries of the matrices $\delta_\pm^{(\{p_j\},k)}(z;X_\pm)$ are some constants (i.e. x and z independent), whereas the off-diagonal ones are polynomials of degree p_0 in the variable $X_\pm = \log\left[\pm x\,(p(z)-p_\mp)\right]$.

An exactly similar structure holds for $\varrho^{(\{p_j\})}(z;x)$ as it is adjunct to $\delta^{(\{p_j\})}(z;x)$. Hence, on the skeleton $\partial D_{\sigma q,\delta} = \partial D_{\sigma_1 q,\delta_1} \times \cdots \times \partial D_{\sigma_{r+t} q,\delta_{r+t}}$, the trace (8.41) can be expanded in the following form:

$$\begin{aligned} \mathrm{tr}_{r,t}^{(\{p_{\ell j}\})}(\{z_j\};x)[g] = \sum_{N=2}^{M+1} \frac{1}{x^N} \sum_{\substack{k_1,\ldots,k_{r+t}=1 \\ \Sigma k_i = N}}^{N} \sum_{\substack{\epsilon_1,\ldots,\epsilon_{r+t}\in\{\pm 1,0\} \\ \Sigma\epsilon_i = 0}} \mathfrak{D}_{r,t}\begin{bmatrix}\{p_{\ell j}\},\{\epsilon_i\} \\ \{\sigma_i\},\{k_i\}\end{bmatrix}(\{z_i\}\,;\log x) \\ \times \frac{\exp\left\{\sum\limits_{\ell=1}^{r+t} \epsilon_\ell\left[g(z_\ell) + ixp_{\sigma_\ell}\right]\right\}}{\prod\limits_{\ell=1}^{r+t}\left(p(z_\ell) - p_{\sigma_\ell}\right)^{k_\ell}} + \mathrm{O}\left(\frac{\log^n x}{x^{M+2}}\right). \end{aligned} \tag{8.51}$$

There we have explicitly factored out the dependence on the oscillating factor $\exp\{ix\Sigma_\ell \epsilon_\ell p_{\sigma_\ell}\}$. In this expression, the coefficients $\mathfrak{D}_{r,t}$ are piecewise smooth on the integration contour and are polynomials of degree $\Sigma_\ell\,|\epsilon_\ell|\,p_{\ell 0}$ in $\log x$.

We set, for $N \geq 2$,

$$x^{1-N}\widetilde{\mathfrak{D}}_{r,t}^{(0)}\begin{bmatrix}\{p_{\ell j}\},\{\epsilon_i\} \\ \{\sigma_i\},\{k_i\}\end{bmatrix}(\{z_i\}\,;\log x) = \mathfrak{C}_{r,t}^{(\{p_{\ell j}\})} \int\limits_{+\infty}^{x} \frac{\mathrm{d}x'}{(x')^N} \mathfrak{D}_{r,t}\begin{bmatrix}\{p_{\ell j}\},\{\epsilon_i\} \\ \{\sigma_i\},\{k_i\}\end{bmatrix}(\{z_i\}\,;\log x') \tag{8.52}$$

when $\sum \epsilon_\ell p_{\sigma_\ell} = 0$, and

$$\mathrm{e}^{ix\sum p_{\sigma_\ell}\epsilon_\ell} \sum_{k=N}^{M} x^{-k}\, \widetilde{\mathfrak{D}}^{(k)}_{r,t}\begin{bmatrix}\{p_{\ell j}\},\{\epsilon_i\}\\ \{\sigma_i\},\{k_i\}\end{bmatrix}(\{z_i\}\,;\log x) + \mathrm{O}\left(\frac{\log^n x}{X^{M+1}}\right)$$
$$= \mathfrak{C}^{(\{p_{\ell j}\})}_{r,t} \int_{+\infty}^{x} \frac{\mathrm{d}x'}{(x')^N} \mathfrak{D}_{r,t}\begin{bmatrix}\{p_{\ell j}\},\{\epsilon_i\}\\ \{\sigma_i\},\{k_i\}\end{bmatrix}(\{z_i\}\,;\log x)\, \mathrm{e}^{ix'\sum p_{\sigma_\ell}\epsilon_\ell} \quad (8.53)$$

otherwise. Note at this stage that, due to the constraint $\sum \epsilon_i = 0$, there exist some integer $m \neq 0$ such that $\mathrm{e}^{ix\sum p_{\sigma_\ell}\epsilon_\ell} = \mathrm{e}^{ixm(p_+ - p_-)}$.

We then insert the result of integration into the expression for $I_n^{\mathrm{sub}}[\mathcal{F}_n]$, rearrange the asymptotic expansion into decreasing powers of x and separate the oscillating and non-oscillating parts. We obtain

$$I_n^{\mathrm{sub}}[\mathcal{F}_n] = \sum_{N=1}^{M} \frac{1}{x^N}\, I_n^{(N;\,\mathrm{nosc})}[\mathcal{F}_n] + \sum_{N=2}^{M} \frac{1}{x^N}\, I_n^{(N;\,\mathrm{osc})}[\mathcal{F}_n] + \mathrm{O}\left(\frac{\log^n x}{x^{M+1}}\right), \quad (8.54)$$

where

$$I_n^{(N;\,\mathrm{nosc})}[\mathcal{F}_n] = \sum_{\substack{r,t\\ \{p_{\ell j}\},\{\epsilon_i\}}} \sum_{\substack{\sigma_1,\ldots,\sigma_{r+t}=\pm\\ \sum \epsilon_\ell p_{\sigma_\ell}=0}} \sum_{\substack{k_1,\ldots,k_{r+t}=1\\ \sum k_i = N+1}}^{N+1} \mathfrak{I}^{(0)}_{r,t}\begin{bmatrix}\{p_{\ell j}\},\{\epsilon_i\}\\ \{\sigma_i\},\{k_i\}\end{bmatrix}(\log x)\,[\mathcal{F}_n]\,, \quad (8.55)$$

$$I_n^{(N;\,\mathrm{osc})}[\mathcal{F}_n] = \sum_{\substack{r,t\\ \{p_{\ell j}\},\{\epsilon_i\}}} \sum_{\substack{\sigma_1,\ldots,\sigma_{r+t}=\pm\\ \sum \epsilon_\ell p_{\sigma_\ell}\neq 0}} \sum_{s=2}^{N} \sum_{\substack{k_1,\ldots,k_{r+t}=1\\ \sum k_i = s}}^{s} \mathrm{e}^{ix\sum_{\ell=1}^{r+t}\epsilon_\ell p_{\sigma_\ell}}\, \mathfrak{I}^{(s)}_{r,t}\begin{bmatrix}\{p_{\ell j}\},\{\epsilon_i\}\\ \{\sigma_i\},\{k_i\}\end{bmatrix}(\log x)\,[\mathcal{F}_n]\,. \quad (8.56)$$

In these expressions,

$$\sum_{\substack{r,t\\ \{p_{\ell j}\},\{\epsilon_i\}}} \equiv \sum_{\substack{1\leq r+t\leq n\\ r\geq 1,\, t\geq 0}} \sum_{\substack{p_{\ell 0},\ldots,p_{\ell n}\\ p_{\ell 0}\geq 1,\, \sum_{\ell=1}^{r+t}\bar{p}_\ell = n}} \sum_{\substack{\epsilon_1,\ldots,\epsilon_{r+t}\in\{\pm 1,0\}\\ \sum \epsilon_i = 0}} \quad (8.57)$$

and the functional $\mathfrak{I}_{r,t}$ is given by

$$\mathfrak{I}^{(s)}_{r,t}\begin{bmatrix}\{p_{\ell j}\},\{\epsilon_i\}\\ \{\sigma_i\},\{k_i\}\end{bmatrix}(\log x)\,[\mathcal{F}_n] = \frac{(-1)^{n-1}}{(n-1)!} \oint_{\Gamma([-q\,;q])} \frac{\mathrm{d}\lambda}{4\pi}\, p(\lambda) \int_{\partial D^{\times(r+t)}_{\sigma q,\delta}} \frac{\mathrm{d}^{r+t}z}{(2i\pi)^{r+t}}$$
$$\times \frac{\widetilde{\mathfrak{D}}^{(s)}_{r,t}\begin{bmatrix}\{p_{\ell j}\},\{\epsilon_i\}\\ \{\sigma_i\},\{k_i\}\end{bmatrix}(\{z_i\}\,;\log x)}{(\lambda - z_1)^2\,(\lambda - z_{r+1})\prod_{\ell=2}^{r}(z_{\ell-1}-z_\ell)\prod_{\ell=r+2}^{r+t}(z_{\ell-1}-z_\ell)} \prod_{\ell=1}^{r+t}\frac{1}{(p(z_\ell)-p_{\sigma_\ell})^{k_\ell}}$$
$$\times \prod_{\ell=1}^{r+t}\prod_{m=1}^{n}\prod_{j=1}^{p_{\ell m}}\left\{\int_{-q}^{q}\frac{\mathrm{d}\mu_{\ell,m,j}}{z_\ell - \mu_{\ell,m,j}}\,\mathfrak{d}^{(m)}_{z_\ell}(\mu_{\ell,m,j})\right\}\cdot \mathcal{F}_n\begin{pmatrix}\{\{z_\ell\}^{\bar{p}_\ell}\}_{1\leq\ell\leq r+t}\\ \{\{z_\ell\}^{\bar{p}_\ell+\epsilon_\ell}\}_{1\leq\ell\leq r+t}\end{pmatrix}. \quad (8.58)$$

The coefficients $\widetilde{\mathfrak{D}}_{r,t}$ being polynomials of degree $\sum_\ell |\epsilon_\ell|\, p_{\ell 0}$ in $\log x$, this ends the proof of Theorem 8.2 concerning the existence of the asymptotic expansion of cyclic integrals to any order in $1/x$.

As it is presented, the form of this asymptotic expansion may look quite involved. Note however that the integrals over the contours $\partial D^{\times(r+t)}_{\sigma q,\delta}$ in (8.58) can be computed; they are expressible in terms of partial derivatives of the function $\mathcal{F}_n$ at $\pm q$ (see Appendix D). It is proved in Appendix D that the non-oscillating term $I_n^{(N;\,\mathrm{nosc})}[\mathcal{F}_n]$ of order N can be expressed in terms of derivatives of $\mathcal{F}_n$ of total order not higher than N, whereas the order of such derivatives does not exceed $N-2$ in the case of $I_n^{(N;\,\mathrm{osc})}[\mathcal{F}_n]$. This property is useful in [35], when we sum up the asymptotic behaviour of a whole class of cycle integrals of the form (8.1) to obtain the asymptotic behaviour of correlation functions. To perform this summation we use the knowledge of the number of partial derivatives applied to $\mathcal{F}_n$. We finally point out that the integral over λ produces derivatives of the function $p(\lambda)$ evaluated at $\pm q$.

9 More general kernels

In the applications to quantum integrable models, one sometimes needs to use some modified versions of the GSK.

Consider the operator $I + V_\theta$ acting on $[\,-q\,;q\,]$ with kernel

$$V_\theta(\lambda,\mu) = \sqrt{F(\lambda)\,F(\mu)\,\theta'(\lambda)\,\theta'(\mu)}\;\frac{e_+(\lambda)\,e_-(\mu) - e_-(\lambda)\,e_+(\mu)}{2i\pi\,[\theta(\lambda)-\theta(\mu)]}, \tag{9.1}$$

where $e_\pm$ and F are defined as in (1.7). We assume in addition that θ is a biholomorphism of U onto its image, that $\theta([\,-q\,;q\,]) \subset \mathbb{R}$ and $\theta(U\cap\mathcal{H}_\pm)\subset\mathcal{H}_\pm$.

Then the asymptotic behaviour of $\log\det[I+V_\theta]$ when $x\to\infty$ follows from Theorem 2.1. More precisely, we have the following corollary:

Corollary 9.1. *Let V_θ be as above. Then*

$$\log\det[I+V_\theta] = 2\int_{-q}^{q}\mathrm{d}\lambda\,\nu(\lambda)\log'[e_-(\lambda)] + \sum_{\sigma=\pm}\log\left[\frac{G(1,\nu_\sigma)\;\theta'(\sigma q)^{\nu_\sigma^2}\;\mathcal{K}^{\sigma\nu_\sigma}(\sigma q;q)}{[(\theta(q)-\theta(-q))\,p'_\sigma x]^{\nu_\sigma^2}}\right]$$
$$+\frac{1}{2}\int_{-q}^{q}\mathrm{d}\lambda\,\mathrm{d}\mu\;\frac{\nu'(\lambda)\,\theta'(\mu)\,\nu(\mu)-\nu(\lambda)\,\theta'(\lambda)\,\nu'(\mu)}{\theta(\lambda)-\theta(\mu)} + \mathrm{o}(1), \tag{9.2}$$

where

$$\nu(\lambda) = \frac{-1}{2i\pi}\log(1+\gamma F(\lambda)), \quad \mathcal{K}(\lambda;q) = \exp\left\{\int_{-q}^{q}\frac{\nu(\lambda)-\nu(\mu)}{\theta(\lambda)-\theta(\mu)}\,\theta'(\mu)\,\mathrm{d}\mu\right\}, \tag{9.3}$$

and, as before, $p'_\pm = [\partial_\lambda p(\lambda)]\,|_{\lambda=\pm q}$, $\nu_\pm = \nu(\pm q)$.

Proof — The change of variables $\theta(\lambda) = \xi$ maps the kernel V_θ on the one of the GSK

$$V(\xi,\eta) = \sqrt{F\circ\theta^{-1}(\xi)\; F\circ\theta^{-1}(\eta)}\;\frac{e_+\circ\theta^{-1}(\xi)\; e_-\circ\theta^{-1}(\eta) - e_-\circ\theta^{-1}(\xi)\; e_+\circ\theta^{-1}(\eta)}{2i\pi\,(\xi-\eta)}.$$

This kernel acts on $[\,\theta(-q)\,;\theta(q)\,]$ which is, a priori, a non symmetric interval. However, it is enough to apply the transformation $\lambda \mapsto \lambda - (\theta(q)+\theta(-q))/2$ so as to recover the symmetry of the interval. Then, it remains to enforce the inverse transformations on the asymptotic formula for the Fredholm determinant of V. □

Let us write explicitly the asymptotics (9.2) in the case of the kernel

$$V_{\rm sh}(\lambda,\mu) = \gamma\sqrt{F(\lambda)\,F(\mu)}\;\frac{e_+(\lambda)\,e_-(\mu) - e_-(\lambda)\,e_+(\mu)}{2i\pi\sinh(\lambda-\mu)},\qquad e_\pm(\lambda) = \mathrm{e}^{\pm[ixp(\lambda)+g(\lambda)]/2},$$

as it plays a crucial role in the analysis of the asymptotic behaviour of the two-point functions in the massless phase of the XXZ Heisenberg chain [35]. In this case, equation (9.2) reads

$$\begin{aligned}\log\det\left[I+V_{\rm sh}\right] = {}& 2\int_{-q}^{q}\mathrm{d}\lambda\,\nu(\lambda)\log'[e_-(\lambda)] + \frac{1}{2}\int_{-q}^{q}\mathrm{d}\lambda\,\mathrm{d}\mu\;\frac{\nu'(\lambda)\,\nu(\mu)-\nu(\lambda)\,\nu'(\mu)}{\tanh(\lambda-\mu)}\\ &+\sum_{\sigma=\pm}\left\{\log\frac{G(1,\nu_\sigma)}{\left[\sinh(2q)\,p'_\sigma x\right]^{\nu_\sigma^2}} + \sigma\nu_\sigma\int_{-q}^{q}\mathrm{d}\lambda\;\frac{\nu_\sigma-\nu(\lambda)}{\tanh(\sigma q-\lambda)}\right\} + \mathrm{o}(1)\,. \qquad (9.4)\end{aligned}$$

It is clear that the last equation can be used in order to obtain an analog of the asymptotic expansion for multiple integrals of the type (8.1) where the rational functions $z-\lambda$ are replaced by the hyperbolic $\sinh(z-\lambda)$. Namely, let

$$\mathcal{I}_n^{\rm sh}\left[\mathcal{F}_n\right] = \oint_{\Gamma([\,-q\,;q\,])}\frac{\mathrm{d}^n z}{(2i\pi)^n}\int_{-q}^{q}\frac{\mathrm{d}^n\lambda}{(2i\pi)^n}\;\mathcal{F}_n\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix}\prod_{j=1}^{n}\frac{\mathrm{e}^{ix(p(z_j)-p(\lambda_j))}}{\sinh\left(z_j-\lambda_j\right)\sinh\left(z_j-\lambda_{j+1}\right)}\,. \qquad (9.5)$$

Then under the conditions of Corollary 8.1 one has the following asymptotic estimate

$$\mathcal{I}_n^{\mathrm{sh}}[\mathcal{F}_n] = \frac{1}{2i\pi}\int_{-q}^{q} \mathrm{d}\lambda\, \{ixp'(\lambda)+\partial_\epsilon\}\mathcal{F}_n\begin{pmatrix} \{\lambda\}^n \\ \{\lambda+\epsilon\},\{\lambda\}^{n-1}\end{pmatrix}\Big|_{\epsilon=0}$$
$$+\sum_{\sigma=\pm}\left(b_n - c_n\log\left(\sinh(2q)p'_\sigma x\right)\right)\mathcal{F}_n\begin{pmatrix}\{\sigma q\}^n\\ \{\sigma q\}^n\end{pmatrix}$$
$$+\frac{n}{(2\pi)^2}\sum_{\sigma=\pm}\sum_{p=1}^{n-1}\int_{-q}^{q}\mathrm{d}\lambda\frac{\mathcal{F}_n\begin{pmatrix}\{\sigma q\}^n\\ \{\sigma q\}^n\end{pmatrix}-\mathcal{F}_n\begin{pmatrix}\{\sigma q\}^p,\{\lambda\}^{n-p}\\ \{\sigma q\}^p,\{\lambda\}^{n-p}\end{pmatrix}}{p\,(n-p)\tanh\left(q-\sigma\lambda\right)}$$
$$+\frac{n}{2\,(2\pi)^2}\sum_{p=1}^{n-1}\int_{-q}^{q}\frac{\mathrm{d}\lambda\mathrm{d}\mu}{(n-p)\tanh\left(\lambda-\mu\right)}\left\{\partial_\epsilon\mathcal{F}_n\begin{pmatrix}\{\lambda+\epsilon\},\{\lambda\}^{p-1},\{\mu\}^{n-p}\\ \{\lambda+\epsilon\},\{\lambda\}^{p-1},\{\mu\}^{n-p}\end{pmatrix}\right.$$
$$\left.-\partial_\epsilon\mathcal{F}_n\begin{pmatrix}\{\mu+\epsilon\},\{\mu\}^{p-1},\{\lambda\}^{n-p}\\ \{\mu+\epsilon\},\{\mu\}^{p-1},\{\lambda\}^{n-p}\end{pmatrix}\right\}\Big|_{\epsilon=0}+\mathrm{o}(1)\ . \quad (9.6)$$

Conclusion

We have obtained in this article the leading asymptotic expansion of the Fredholm determinant of the GSK. As we have mentioned, our main motivation is to apply this result to the asymptotic analysis of the correlation functions of quantum integrable models, using in particular the asymptotic study of multiple integrals performed in Section 8. This is done in [35].

Another development is to extend the above analysis so as to handle truncated Wiener–Hopf operators with symbols having Fischer–Hartwig type discontinuities. The corresponding results are published in [37]. The results for the case of Toeplitz, Hankel and Toeplitz + Hankel determinants with Fisher-Hartwig singularities appeared recently in [16, 17].

Let us also point out some unsolved problems. One of them concerns the derivation of the asymptotics of the Fredholm determinant of the GSK via the method based on its derivative over endpoint q. It would be important to obtain a complete justification of this method, since it is rather powerful and at the same time relatively simple.

Another problem is to prove the conjecture on the $\mathbb{Z}-\nu$ periodicity for the asymptotic expansion of the Fredholm determinant. If this property does hold, then all oscillating corrections can be obtained from the non-oscillating ones via a simple shift of ν by integer numbers. This could lead to a much simpler way to compute sub-leading corrections for such determinants.

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A Some properties of confluent hypergeometric function

For generic parameters (a, c) the Tricomi confluent hypergeometric function $\Psi(a, c; z)$ is one of the solutions to the differential equation

$$zy'' + (c - z)\, y' - ay = 0\ . \tag{A.1}$$

It satisfies the properties:

- Differentiation:

$$\begin{aligned}\Psi'(a, c; z) &= \frac{a}{z}\Big[(a - c + 1)\Psi(a + 1, c; z) - \Psi(a, c; z)\Big] \\ &= \frac{1}{z}\Big[(a - c + z)\Psi(a, c; z) - \Psi(a - 1, c; z)\Big].\end{aligned} \tag{A.2}$$

- Monodromy:

$$\begin{aligned}\Psi(a, 1; ze^{2mi\pi}) = \Psi(a, 1; z)\left(1 - me^{i\pi a(\epsilon+1)} + me^{i\pi a(\epsilon-1)}\right) \\ + \frac{2\pi ime^{i\pi a\epsilon+z}}{\Gamma^2(a)}\Psi(1 - a, 1; -z),\end{aligned} \tag{A.3}$$

where $\epsilon = \operatorname{sgn}(\Im(z))$. In particular,

$$\Psi(a, 1; ze^{2i\pi}) = \Psi(a, 1; z)e^{-2i\pi a} + \frac{2\pi ie^{-i\pi a+z}}{\Gamma^2(a)}\Psi(1 - a, 1; -z), \qquad \Im(z) < 0, \tag{A.4}$$

$$\Psi(a, 1; ze^{-2i\pi}) = \Psi(a, 1; z)e^{2i\pi a} - \frac{2\pi ie^{i\pi a+z}}{\Gamma^2(a)}\Psi(1 - a, 1; -z), \qquad \Im(z) > 0. \tag{A.5}$$

- Asymptotic expansion:

$$\Psi(a, c; z) \sim \sum_{n=0}^{\infty}(-1)^n \frac{(a)_n(a - c + 1)_n}{n!}z^{-a-n}, \quad z \to \infty, \quad -\frac{3\pi}{2} < \arg(z) < \frac{3\pi}{2}, \tag{A.6}$$

with $(a)_n = \Gamma(a + n)/\Gamma(a)$.

We have the following recombination between the Tricomi CHF $\Psi(a, c; z)$ and the Humbert CHF $\Phi(a, c; z)$

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(c - a)}\mathrm{e}^{i\epsilon a\pi}\Psi(a, c; z) + \frac{\Gamma(c)}{\Gamma(a)}\mathrm{e}^{i\epsilon\pi(a-c)+z}\Psi(c - a, c; -z)\,, \tag{A.7}$$

where $\epsilon = \operatorname{sgn}(\Im(z))$, and

$$\Phi(a, c; z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}. \tag{A.8}$$

Such a recombination formula allows to obtain the asymptotic expansion of the Humbert CHF:

$$\Phi(a,c;z) = \frac{\Gamma(c)}{\Gamma(c-a)}\left(\frac{e^{i\pi\epsilon}}{z}\right)^a \sum_{n=0}^{M} \frac{(a)_n (a-c+1)_n}{n!\,(-z)^n} + O\left(|z|^{-a-M-1}\right) + \frac{\Gamma(c)}{\Gamma(a)} e^z z^{a-c} \sum_{n=0}^{N} \frac{(c-a)_n (1-a)_n}{n! z^n} + O\left(\left|e^z z^{a-1-c-N}\right|\right). \quad (A.9)$$

One can estimate integrals involving a product of two CHF as below, either by using Laplace–type integral representations for the functions $\Phi(a,c;z)$ and $\Psi(a,c;z)$ or applying the method given in [37]. The latter uses Erdelyi's representation of Laplace transforms of products of CHF in terms of Lauricella function adjoint to some asymptotic expansion of Lauricella function. In any case, the result reads:

$$\int_0^{+\infty} dt \left\{e^{-i\pi a}\varphi(a;t) - 1\right\} = -2ia, \quad (A.10)$$

$$\int_0^{+\infty} dt \left\{e^{-i\pi a}\tau(a;t) + 1 + \frac{2ia}{t+1}\right\} = 2ia - a\left[\psi(a) + \psi(-a)\right], \quad (A.11)$$

and the Riemann integrability of the integrands is part of the conclusion. We remind the definition of the functions $\tau(a;t)$ and $\varphi(a;t)$:

$$\varphi(\nu;t) = \Phi(-\nu,1;-it)\,\Phi(\nu,1;it), \quad (A.12)$$

$$\tau(\nu;t) = -\Phi(-\nu,1;-it)\,\Phi(\nu,1;it) + (\partial_z\Phi)(-\nu,1;-it)\,\Phi(\nu,1;it) + \Phi(-\nu,1;-it)\,(\partial_z\Phi)(\nu,1;it). \quad (A.13)$$

B Three preparatory Lemmas

Here we prove three preparatory integration lemmas used in Section 6.

Lemma B.1. *Let $\mathcal{R}(u,t)$ be a function of two variables defined on $I \times \mathbb{R}^+$, where I is an open interval of $\mathbb{R}$ containing 0. Suppose that the partial applications $u \to \mathcal{R}(u,t)$ are $\mathscr{C}^1(I)$ for all but finitely many t's and that $t \to \mathcal{R}(u,t)$ is Riemann integrable uniformly in u, i.e.:*

$$\forall \rho > 0,\ \forall M > 0,\ \forall u_0 \in I,\ \exists \upsilon > 0 \quad \text{such that}$$

$$u \in\ ]-\upsilon + u_0\,;\upsilon + u_0\,[\,\cap I, k \in \{0,1\} \Rightarrow \left|\int_M^{+\infty} dt\left[\partial_1^k \mathcal{R}(u,t) - \partial_1^k \mathcal{R}(u_0,t)\right]\right| \leq \rho. \quad (B.1)$$

Then for $g \in \mathscr{C}^1(I)$

$$\int_0^{\delta} x g(t)\,\mathcal{R}(t,xt)\,dt = g(0)\int_0^{+\infty} \mathcal{R}(0,t)\,dt + o(1) \quad (B.2)$$

where the small $o(1)$ is with respect to the successive limit $x\delta \to +\infty$ and $\delta \to 0$.

Proof — One has

$$\int_0^\delta x(g(t)\mathcal{R}(t,xt) - g(0)\mathcal{R}(0,xt))\,\mathrm{d}t = \int_0^\delta \mathrm{d}y \int_{xy}^{x\delta} \partial_y \left[g(y)\mathcal{R}(y,t)\right]\mathrm{d}t\ . \tag{B.3}$$

Consider a function

$$g : (y,a,b) \mapsto \int_a^b \mathrm{d}t\partial_y \left[g(y)\mathcal{R}(y,t)\right] \tag{B.4}$$

on the compact set $[\,0\,;\delta\,] \times \overline{\mathbb{R}^+} \times \overline{\mathbb{R}^+}$. g is clearly continuous on the interior and the uniform Riemann-integrability of $\mathcal{R}(y,t)$ guarantees that it is continuous in an neighbourhood of $(*,+\infty,*)$, $(*,*,+\infty)$ and $(*,+\infty,+\infty)$. Hence, $|g|$ is bounded, say by B, as continuous function on a compact set. Thus,

$$\left|\int_0^\delta \mathrm{d}y \int_{xy}^{x\delta} \partial_y \left[g(y)\mathcal{R}(y,t)\right]\mathrm{d}t\right| \leq \delta B, \tag{B.5}$$

which ends the proof of Lemma B.1. □

Lemma B.2. *Let $g \in \mathscr{C}^1(I)$ for some open interval I containing 0, then*

$$\int_0^\delta \frac{g(t)\,x\mathrm{d}t}{1+xt} = g(0)\log x\delta + \mathrm{o}(1)\,, \tag{B.6}$$

where $\mathrm{o}(1)$ *stands with respect to the successive limits $x\delta \to +\infty$ and $\delta \to 0$.*

Proof — We have

$$\int_0^\delta g(t)\,\frac{x\mathrm{d}t}{1+xt} = g(0)\log(x\delta+1) + \int_0^\delta \mathrm{d}t \int_0^t \mathrm{d}y \frac{g'(y)\,x}{1+xt} \tag{B.7}$$

$$= g(0)\log\delta x + \mathrm{o}(1) + \int_0^\delta \mathrm{d}y g'(y)\log\left(\frac{x\delta+1}{xy+1}\right)\,. \tag{B.8}$$

But,

$$\left|\int_0^\delta \mathrm{d}y g'(y)\log\left(\frac{\delta+1/x}{y+1/x}\right)\right| \leq \sup_{[\,0\,;\delta\,]}\left|g'\right| \times (\delta - \log(\delta+1/x)/x) \underset{\delta\to 0}{\longrightarrow} 0\,, \tag{B.9}$$

which ends the proof of Lemma B.2. □

Lemma B.3. *Let κ be defined in terms of ν as in* (2.2)*, and set*

$$H(q) = \frac{1}{2}\int_{-q}^{q} \mathrm{d}\lambda \mathrm{d}\mu \frac{\nu'(\lambda)\,\nu(\mu) - \nu'(\mu)\,\nu(\lambda)}{\lambda - \mu} + \sum_{\epsilon=\pm} \epsilon \nu_\epsilon \, \log \kappa(\epsilon q; q). \tag{B.10}$$

Then,

$$2\nu_+ \frac{\mathrm{d}}{\mathrm{d}q}\left[\log \kappa\left(q;q\right)\right] - 2\nu_- \frac{\mathrm{d}}{\mathrm{d}q}\left[\log \kappa\left(-q;q\right)\right] - \frac{(\nu_+ - \nu_-)^2}{q} = \frac{\mathrm{d}H\left(q\right)}{\mathrm{d}q}. \tag{B.11}$$

Proof — Using (2.2), one can express the derivative of $H\left(q\right)$ as

$$\frac{\mathrm{d}H\left(q\right)}{\mathrm{d}q} = \sum_{\epsilon=\pm} \nu\left(\epsilon q\right) \int_{-q}^{q} \mathrm{d}\mu \frac{\nu'\left(\epsilon q\right) - \nu'\left(\mu\right)}{\epsilon q - \mu} + \sum_{\epsilon=\pm} \epsilon \nu\left(\epsilon q\right) \frac{\mathrm{d}}{\mathrm{d}q}\left[\log \kappa\left(\epsilon q; q\right)\right]. \tag{B.12}$$

Thus, proving (B.11) amounts to establishing the equality

$$\sum_{\epsilon=\pm} \epsilon \nu\left(\epsilon q\right) \frac{\mathrm{d}}{\mathrm{d}q}\left[\log \kappa\left(\epsilon q; q\right)\right] - \frac{(\nu_+ - \nu_-)^2}{q} = \sum_{\epsilon=\pm} \nu\left(\epsilon q\right) \int_{-q}^{q} \mathrm{d}\mu \frac{\nu'\left(\epsilon q\right) - \nu'\left(\mu\right)}{\epsilon q - \mu}\,. \tag{B.13}$$

The latter follows from an integration by parts:

$$\begin{aligned}
&\sum_{\epsilon=\pm} \epsilon \nu\left(\epsilon q\right) \frac{\mathrm{d}}{\mathrm{d}q}\left[\log \kappa\left(\epsilon q; q\right)\right] \\
&\qquad = \sum_{\epsilon=\pm} \epsilon \nu_\epsilon \left\{ \nu'_\epsilon + \frac{\nu_+ - \nu_-}{2q} + \epsilon \int_{-q}^{q} \mathrm{d}\mu \frac{\nu\left(\mu\right) - \nu_\epsilon - \nu'_\epsilon\left(\mu - \epsilon q\right)}{\left(\mu - \epsilon q\right)^2} \right\} \\
&\qquad = \sum_{\epsilon=\pm} \epsilon \nu_\epsilon \left\{ \nu'_\epsilon + \frac{\nu_+ - \nu_-}{2q} + \frac{\nu_{-\epsilon} - \nu_\epsilon + 2\epsilon q \nu'_\epsilon}{-2\epsilon q} + \epsilon \int_{-q}^{q} \mathrm{d}\mu \frac{\nu'\left(\mu\right) - \nu'_\epsilon}{\mu - \epsilon q} \right\} \\
&\qquad = \frac{(\nu_+ - \nu_-)^2}{q} + \sum_{\epsilon=\pm} \nu_\epsilon \int_{-q}^{q} \mathrm{d}\mu \frac{\nu'\left(\mu\right) - \nu'_\epsilon}{\mu - \epsilon q}.
\end{aligned}$$

This ends the proof. □

C The density Theorem

Theorem C.1. *Let U, W be two open neighbourhoods of $[\,-q\,;q\,]$, and let $\mathcal{F}_n\begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix}$ be a holomorphic function on $U^n \times W^n$, symmetric separately in the n variables λ and in the n*

variables z. Then, for any compact subsets K (resp. P) of U (resp.W) there exists a sequence $\left(\varphi_p, \phi_p\right)_{p\in\mathbb{N}}$ *in* $\mathcal{H}(K)\times\mathcal{H}(P)$ *such that*

$$\mathcal{F}_n\left(\begin{array}{c}\{\lambda\}\\ \{z\}\end{array}\right)=\sum_{p=0}^{+\infty}\prod_{i=1}^{n}\varphi_p(\lambda_i)\,\phi_p(z_i)\qquad \textit{uniformly on } K^n\times P^n\,. \tag{C.1}$$

Proof — Let K and P be as above.

Let $X = K^n\times P^n/\sim$, where the relation $\sim$ is defined as follows: $(\lambda,z)\sim(\lambda',z')$ if there exists a couple of permutations $(\sigma,\pi)\in\mathfrak{G}_n\times\mathfrak{G}_n$ such that $(\lambda^\sigma,z^\pi)=(\lambda',z')$, where λ^σ stands for $(\lambda_{\sigma(1)},\ldots,\lambda_{\sigma(n)})$. Since $\mathfrak{G}_n\times\mathfrak{G}_n$ is a discrete group, its action on $K^n\times P^n$ is by definition proper, i.e. $\forall L\supset K^n\times P^n$

$$\{(\sigma,\pi)\in\mathfrak{G}_n\times\mathfrak{G}_n : L^{\sigma,\pi}\cap L=\emptyset\}\quad \text{is discrete.} \tag{C.2}$$

This ensures that X is a compact Hausdorff topological space. Moreover the space $\mathscr{C}(X,\mathbb{C})$ of continuous functions on X is canonically identified with the space of continuous functions on $K^n\times P^n$ that are symmetric in the first or the last n variables.

Define the subspace S of $\mathscr{C}(X,\mathbb{C})$ as the subset of functions $\mathcal{F}_n^{(\varphi,\phi)}$ of the form

$$\mathcal{F}_n^{(\varphi,\phi)}\left(\begin{array}{c}\{\lambda\}\\ \{z\}\end{array}\right)=\prod_{i=1}^{n}\varphi(\lambda_i)\,\phi(z_i)\,, \tag{C.3}$$

where $(\varphi,\phi)\in\mathcal{H}(K)\times\mathcal{H}(P)$, and let $\mathcal{S}$ be the C^*-algebra generated by S. We have that S and hence $\mathcal{S}$ separates points in X. Indeed, let (λ,z) and (μ,y) be any two representatives in $K^n\times P^n$ of two distinct points in X. Thus

- there exists $\lambda_i\in K$ such that exactly p of the n coordinates of the n-tuple λ are equal to λ_i, whereas exactly q of the n coordinates of the n-tuple μ are equal to λ_i, with $p\neq q$;
- or there exists $z_i\in P$ such that exactly p of the n coordinates of the n-tuple z are equal to z_i, whereas exactly q of the n coordinates of the n-tuple y are equal to z_i, with $p\neq q$.

The situation is similar in the case of the first n and last n variables, therefore we only treat the first case. By Lagrange interpolation there exist a polynomial Q such that, for any coordinate λ_k of λ and any coordinate μ_k of μ satisfying $\lambda_k\neq\lambda_i$ and $\mu_k\neq\lambda_i$,

$$Q(\lambda_k)=Q(\mu_k)=1\quad\text{and}\quad Q(\lambda_i)=2. \tag{C.4}$$

The function

$$\mathcal{F}_n^{(Q,1)}\left(\begin{array}{c}\{\lambda\}\\ \{z\}\end{array}\right)=\prod_{p=1}^{n}Q\left(\lambda_p\right)\in\mathcal{S} \tag{C.5}$$

separates the projections of (λ,z) and (μ,y) on X. Thus $\mathcal{S}$ is a C^*-subalgebra of $\mathscr{C}(X;\mathbb{C})$ that separates points. It then follows by the Stone-Weierstrass theorem that $\mathcal{S}=\mathscr{C}(X;\mathbb{C})$.

Let $\mathcal{F}_n$ be holomorphic on $U^n \times W^n$ and symmetric in the first and in the last n variables. There exists compact sets $K_\epsilon \subset U$ and $P_\epsilon \subset W$ such that $K \subset \overset{\circ}{K}_\epsilon$ and $P \subset \overset{\circ}{P}_\epsilon$. Here, $\overset{\circ}{K}_\epsilon$ stands for the interior of K_ϵ. Thus the restriction of $\mathcal{F}_n$ to $K_\epsilon^n \times P_\epsilon^n$ also belongs to $\mathscr{C}(X_\epsilon;\mathbb{C})$, with $X_\epsilon = K_\epsilon^n \times P_\epsilon^n / \sim$, and therefore there exists $(\widetilde{\varphi}_p, \widetilde{\phi}_p)_{p\in\mathbb{N}}$ in $\mathscr{C}(K_\epsilon;\mathbb{C}) \times \mathscr{C}(P_\epsilon;\mathbb{C})$ such that

$$\mathcal{F}_n\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix} = \sum_{p=0}^{+\infty}\prod_{i=1}^{n}\widetilde{\varphi}_p(\lambda_i)\,\widetilde{\phi}_p(z_i) \qquad \text{uniformly on} \quad K_\epsilon^n \times P_\epsilon^n. \tag{C.6}$$

In particular the sequence converges uniformly to $\mathcal{F}_n$ on $(\partial K_\epsilon)^n \times (\partial P_\epsilon)^n$, the latter set being compact. Therefore we have

$$\begin{aligned}\sum_{p=0}^{N}\int_{\partial K_\epsilon}\frac{\mathrm{d}^n\mu}{(2i\pi)^n}\int_{\partial P_\epsilon}\frac{\mathrm{d}^n y}{(2i\pi)^n}\prod_{i=1}^{n}\frac{\widetilde{\varphi}_p(\mu_i)\,\widetilde{\phi}_p(y_i)}{(\mu_i-\lambda_i)(y_i-z_i)} &= \sum_{p=0}^{N}\prod_{i=1}^{n}\varphi_p(\lambda_i)\,\phi(z_i)\\ \underset{N\to+\infty}{\longrightarrow}\int_{\partial K_\epsilon}\frac{\mathrm{d}^n\mu}{(2i\pi)^n}\int_{\partial P_\epsilon}\frac{\mathrm{d}^n y}{(2i\pi)^n}\frac{\mathcal{F}_n(\{\mu\}\mid\{y\})}{\prod_{i=1}^{n}(\mu_i-\lambda_i)(y_i-z_i)} &= \mathcal{F}_n(\{\lambda\}\mid\{z\}),\end{aligned} \tag{C.7}$$

uniformly in $(\lambda, z) \in K^n \times P^n$. Moreover,

$$\varphi_p(\lambda) = \int_{\partial K_\epsilon}\frac{\mathrm{d}\mu}{2i\pi}\frac{\widetilde{\varphi}_p(\mu)}{\mu-\lambda} \quad \text{and} \quad \phi_p(z) = \int_{\partial P_\epsilon}\frac{\mathrm{d}y}{2i\pi}\frac{\widetilde{\phi}_p(y)}{y-z} \tag{C.8}$$

are holomorphic in K, resp. P. □

D Form of the sub-leading terms in I_n^{sub}

In this appendix, we focus on the general structure of the sub-leading asymptotics of cyclic integrals. We show that the $1/x^N$ term in the non-oscillating part can be obtained as an action of at most N partial derivatives of the function $\mathcal{F}_n$ followed by an evaluation at $\pm q$ or by an integration over $[-q;q]$.

In principle, the contour integrals defining (8.58) can be computed to the end. However, the result is quite intricate, and we do not need, for the further applications, the formula in its whole generality. Indeed, we are interested in a particular sub-class of such integrals. More precisely we shall focus on the sub-class that is susceptible to produce the highest possible derivatives of the function $\mathcal{F}_n$. Here, by highest derivative we mean the total degree of all the partial derivatives that might act on the integrand. This subclass is identified in the upcoming lemma.

Lemma D.1. *Let $r,t \in \mathbb{N}$ with $r+t \geq 1$ label negations $\sigma_1,\dots,\sigma_{r+t} \in \{\pm\}$. Also introduce sufficiently small numbers $0 < \delta_1 < \cdots < \delta_{r+t} < q$ as well as positive integers $k_1,\dots,k_{r+t}$. Finally, let $G \in \mathcal{H}\left(\overline{D}_{\sigma_1 q,\delta_1} \times \cdots \times \overline{D}_{\sigma_{r+t} q,\delta_{r+t}}\right)$ and*

$$\mathcal{G}_{r,t}^{(\{\sigma_i\},\{k_i\})}[G] = \int_{\partial D_{\sigma q,\delta}^{\times(r+t)}}\frac{\mathrm{d}^{r+t}z}{(2i\pi)^{r+t}}\prod_{\ell=2}^{r}\frac{1}{z_{\ell-1}-z_\ell}\prod_{\ell=r+2}^{r+t}\frac{1}{z_{\ell-1}-z_\ell}\prod_{\ell=1}^{r+t}\frac{1}{(z_\ell-\sigma_\ell q)^{k_\ell}}\,G(\{z\}). \tag{D.1}$$

with $\partial D_{\sigma q,\delta}^{\times(r+t)} \equiv \partial D_{\sigma_1 q,\delta_1} \times \cdots \times \partial D_{\sigma_{r+t} q,\delta_{r+t}}$.

Then, the integral $\mathcal{G}_{r,t}^{(\{\sigma_i\},\{k_i\})}[G]$ *can be computed as some combinatorial sum involving derivatives of* G *at the points* $\sigma_i q$*, the maximal order of such derivatives being equal to*

$$\sum_{i=1}^{r+t} k_i - n_r - n_t + \delta_{r,0} + \delta_{t,0} - 2. \tag{D.2}$$

Here n_r*, resp.* n_t*, is the number of times the sequence* $(\sigma_1, \ldots, \sigma_r)$*, resp.* $(\sigma_{r+1}, \ldots, \sigma_{r+t})$*, changes sign, and* $\delta_{r,0}$*,* $\delta_{t,0}$ *denote the usual Kronecker symbols.*

Proof — Let us prove the claim by induction on $r + t$.

First, for $r + t = 1$, (D.2) is obviously satisfied. Indeed,

$$\mathcal{G}_{1,0}^{(\sigma,k)}[G] = \mathcal{G}_{0,1}^{(\sigma,k)}[G] = \frac{1}{(k-1)!}\left(\partial_z^{k-1} G\right)(\sigma q). \tag{D.3}$$

Let us now assume that the result holds for any function G up to some value of $r + t$. We will prove that it also holds for $r + t + 1$.

Note first that $\mathcal{G}_{r,t}^{(\{\sigma_i\},\{k_i\})}[G] = \mathcal{G}_{t,r}^{(\{\tilde{\sigma}_i\},\{\tilde{k}_i\})}[\tilde{G}]$, in which $\tilde{G}$, $\{\tilde{\sigma}_i\}$, $\{\tilde{k}_i\}$ are obtained from G, $\{\sigma_i\}$, $\{k_i\}$ by a reordering of the variables. Hence, it is enough to prove the claim for $\mathcal{G}_{r+1,t}$. We will have to distinguish two cases, depending on whether $r + 1 = 1$ or $r + 1 > 1$.

In the case $r + 1 = 1$, it is easy to see that

$$\mathcal{G}_{1,t}^{(\{\sigma_i\},\{k_i\})}[G] = \frac{1}{(k_1-1)!}\,\mathcal{G}_{0,t}^{(\sigma_2,\ldots,\sigma_{t+1},k_2,\ldots,k_{t+1})}\left[\partial_{z_1}^{k_1-1} G\left(\sigma_1 q, \{z_i\}_{i=2}^{t+1}\right)\right], \tag{D.4}$$

which means that $\mathcal{G}_{1,t}^{(\{\sigma_i\},\{k_i\})}[G]$ can be expressed in terms of derivatives of G of maximal order $\left(\sum_{i=2}^{t+1} k_i - 1 - n_t + \delta_{t,0}\right) + (k_1 - 1)$, hence the result.

Let us now consider the case $r + 1 > 1$. We have

$$\begin{aligned}
\mathcal{G}_{r+1,t}^{(\{\sigma_i\},\{k_i\})}[G] = \frac{1}{(k_1-1)!} \int\limits_{\partial D_{\sigma q,\delta}^{\times(r+t)}} \prod_{\ell=2}^{r+t+1} \frac{\mathrm{d}z_\ell}{2i\pi} \prod_{\ell=3}^{r+1} \frac{1}{z_{\ell-1}-z_\ell} \prod_{\ell=r+3}^{r+t+1} \frac{1}{z_{\ell-1}-z_\ell} \prod_{\ell=2}^{r+t+1} \frac{1}{(z_\ell - \sigma_\ell q)^{k_\ell}} \\
\times \partial_{z_1}^{k_1-1}\left(\frac{G(\{z\})}{z_1 - z_2}\right)\Big|_{z_1=\sigma_1 q} \\
= -\int\limits_{\partial D_{\sigma q,\delta}^{\times(r+t)}} \prod_{\ell=2}^{r+t+1} \frac{\mathrm{d}z_\ell}{2i\pi} \prod_{\ell=3}^{r+1} \frac{1}{z_{\ell-1}-z_\ell} \prod_{\ell=r+3}^{r+t+1} \frac{1}{z_{\ell-1}-z_\ell} \prod_{\ell=2}^{r+t+1} \frac{1}{(z_\ell - \sigma_\ell q)^{k_\ell}} \\
\times \sum_{k=0}^{k_1-1} \frac{1}{k!} \frac{1}{(z_2 - \sigma_1 q)^{k_1-k}} \left(\partial_{z_1}^k G\right)\left(\sigma_1 q, \{z_i\}_{i=2}^{r+t+1}\right).
\end{aligned} \tag{D.5}$$

At this point one should distinguish between the two possible cases: $\sigma_1\sigma_2 = 1$ or $\sigma_1\sigma_2 = -1$. We first assume $\sigma_1\sigma_2 = 1$ (i.e. that there is no change of sign between σ_1 and σ_2), and set $\hat{G}_k^+(z_2, \ldots, z_{r+t+1}) = \partial_{z_1}^k G(z_1, \ldots, z_{r+t+1})\,|_{z_1=\sigma_1 q}$. Then

$$\mathcal{G}_{r+1,t}^{(\{\sigma_i\},\{k_i\})}[G] = -\sum_{k=0}^{k_1-1} \frac{1}{k!}\,\mathcal{G}_{r,t}^{(\{\sigma_i\}_2^{r+t+1},\{k_2+k_1-k\}\cup\{k_i\}_3^{r+t+1})}\left[\hat{G}_k^+\right]. \tag{D.6}$$

The latter can be expressed in terms of derivatives of G of maximal order $k + (k_2 + k_1 - k + \sum_{i=3}^{r+t+1} k_i - n_r - n_t + \delta_{t,0} - 2) = \sum_{i=1}^{r+t+1} k_i - 2 - n_{r+1} - n_t + \delta_{t,0}$.

We now assume that $\sigma_1\sigma_2 = -1$. This leads to

$$\mathcal{G}_{r+1,t}^{(\{\sigma_i\},\{k_i\})}[G] = -\sum_{k=0}^{k_1-1} \frac{1}{k!}\, \mathcal{G}_{r,t}^{(\{\sigma_i\}_2^{r+t+1},\{k_i\}_2^{r+t+1})}\left[\hat{G}_k^-\right], \tag{D.7}$$

where the function

$$\hat{G}_k^-(z_2,\dots,z_{r+t+1}) = \frac{\partial_{z_1}^k G(z_1,\dots,z_{r+t+1})\,|_{z_1=\sigma_1 q}}{(z_2+\sigma_2 q)^{k_1-k}} \tag{D.8}$$

is holomorphic inside the integration contour $\partial D_{\sigma_2 q,\delta_2} \times \cdots \times \partial D_{\sigma_{r+t} q,\delta_{r+t}}$. Once again, the result will be expressed in terms of derivatives of G and the maximal order of these derivatives will be $k_1 - 1 + (\sum_{i=2}^{r+t+1} k_i - n_r - n_t + \delta_{t,0} - 2) = \sum_{i=1}^{r+t+1} k_i - n_{r+1} - n_t + \delta_{t,0} - 2$, which ends the proof of Lemma D.1. □

Remark D.1. The integral can be explicitly computed using the recurrence formulas (D.6) and (D.7). In particular, in the simplest case $\sigma_1 = \cdots = \sigma_r$ and $\sigma_{r+1} = \cdots = \sigma_{r+t}$, we have

$$\mathcal{G}_{r,t}^{(\{\sigma_i\},\{k_i\})}[G] = (-1)^{r+t-\delta_{r,0}-\delta_{t,0}} \sum_{\substack{u_1,\dots,u_{r+t}\\ u_\ell\in\Gamma_\ell}} \prod_{\ell=1}^{r+t} \frac{1}{u_\ell!}\, \partial_{z_{r+t}}^{u_{r+t}} \dots \partial_{z_1}^{u_1} G(\{z\})\Big|_{z_i=\sigma_i q}, \tag{D.9}$$

in which the parameters u_ℓ are summed over sets Γ_ℓ defined as

$$\Gamma_\ell = \left\{0,\dots,\sum_{j=1}^{\ell} k_j - \sum_{j=1}^{\ell-1} u_j - 1\right\}, \quad \Gamma_{r+\ell} = \left\{0,\dots,\sum_{j=r+1}^{r+\ell} k_j - \sum_{j=r+1}^{r+\ell-1} u_j - 1\right\}, \quad (1 \le \ell < r), \tag{D.10}$$

$$\Gamma_r = \left\{\sum_{j=1}^{r} k_j - \sum_{j=1}^{r-1} u_j - 1\right\}, \qquad \Gamma_{r+t} = \left\{\sum_{j=r+1}^{r+t} k_j - \sum_{j=r+1}^{r+t-1} u_j - 1\right\}. \tag{D.11}$$

Corollary D.1. *The subleading terms of order N in the asymptotic expansion* (8.54) *for the cycle integral $\mathcal{I}_n[\mathcal{F}_n]$ are obtained in terms of derivatives of the function $\mathcal{F}_n$. More precisely, the non-oscillating term $I_n^{(N;\,\mathrm{nosc})}[\mathcal{F}_n]$ involves derivatives of $\mathcal{F}_n$ of total order at most equal to N, whereas the oscillating one $I_n^{(N;\,\mathrm{osc})}[\mathcal{F}_n]$ involves derivatives of $\mathcal{F}_n$ of total order at most equal to $N-2$.*

Proof — In order to apply Lemma D.1 to the integral over $\partial D_{\sigma q,\delta}^{\times(r+t)}$ in (8.58), let us set

$$G(\{z\}) = \frac{\widetilde{\mathfrak{D}}_{r,t}^{(s)}\begin{bmatrix}\{p_{\ell j}\},\{\epsilon_i\}\\ \{\sigma_i\},\{k_i\}\end{bmatrix}(\{z_i\}\,;\log x)}{(\lambda-z_1)^2\,(\lambda-z_{r+1})} \prod_{\ell=1}^{r+t}\left(\frac{z_\ell-\sigma_\ell q}{p(z_\ell)-p_{\sigma_\ell}}\right)^{k_\ell}$$
$$\times \prod_{\ell=1}^{r+t}\prod_{m=1}^{n}\prod_{j=1}^{p_{\ell m}}\left\{\int_{-q}^{q}\frac{\mathrm{d}\mu_{\ell,m,j}}{z_\ell-\mu_{\ell,m,j}}\,\eth_{z_\ell}^{(m)}(\mu_{\ell,m,j})\right\}\cdot\mathcal{F}_n\begin{pmatrix}\{\{z_\ell\}^{\bar{p}_\ell}\}_{1\le\ell\le r+t}\\ \{\{z_\ell\}^{\bar{p}_\ell+\epsilon_\ell}\}_{1\le\ell\le r+t}\end{pmatrix}. \tag{D.12}$$

The poles at $z_1 = \lambda$ and $z_{r+1} = \lambda$ being outside of the skeleton $\partial D_{\sigma q,\delta}^{\times(r+t)}$, this function is indeed holomorphic in a vicinity of the polydisc $D_{\sigma q,\delta}^{\times(r+t)}$.

Applying the result of Lemma D.1 to this function and using the fact that $\sum k_i = N + 1$ in (8.55), it follows immediately that the expression of $I_n^{(N;\,\mathrm{nosc})}[\mathcal{F}_n]$ cannot involve derivatives of the function $\mathcal{F}_n$ of order higher than N. This maximal order of derivatives corresponds to $t = 0$ and $\forall\, i\ \sigma_i = \sigma$ with $\sigma = \pm$ in (8.55).

Similarly, as in (8.56) $\sum k_i \leq N$, $I_n^{(N;\,\mathrm{osc})}[\mathcal{F}_n]$ cannot involve derivatives of $\mathcal{F}_n$ of order higher than $N - 1$. Moreover, due to the constraints $\sum \epsilon_\ell = 0$ and $\sum \epsilon_\ell p_{\sigma_\ell} \neq 0$, it follows that the variables σ_i have to take both values + and −, which means that either $t \geq 1$ or $n_r \geq 1$ in (D.2) (we recall that $r \geq 1$). Hence $I_n^{(N;\,\mathrm{osc})}[\mathcal{F}_n]$ cannot involve derivatives of $\mathcal{F}_n$ of order higher than $N - 2$. □

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